Записки научных
семинаров ПОМИ
Том 307, 2004 г.

V. N. Ivanov

INTERPOLATION ANALOGUES OF SCHUR Q–FUNCTIONS

Abstract. We introduce interpolation analogues of the Schur Q-functions – the multiparameter Schur Q-functions. We obtain for them several results: a combinatorial formula, generating functions for one-row and two-rows functions, vanishing and characterization properties, a Pieri-type formula, a Nimmo-type formula (a relation of two Pfaffians), a Giambelli–Schur-type Pfaffian formula, a determinantal formula for the transition coefficients between multiparameter Schur Q-functions with different parameters. We write an explicit Pfaffian expression for the dimension of a skew shifted Young diagram. This paper is a continuation of the author's paper math.CO/0303169 and is a partial projective analogue of the paper q-alg/9605042 by A. Okounkov and G. Olshanski, and of the paper math.CO/0110077 by G. Olshanski, A. Regev, and A. Vershik.

1. Introduction

This text is an announcement of results. In most cases we give only the main ideas of the proofs. The detailed proofs will be presented later.

In 1911 in the paper [31] I. Schur introduced the Q-functions in order to study the projective representations of the symmetric groups. The projective representations of the symmetric group $S(n)$ are linearized by the spin-symmetric group $\widetilde{S}(n)$ (one of two possible central $\mathbb{Z}_2$-extensions of the group $S(n)$). The Schur Q-functions Q_λ, where λ runs over partitions of a number n into distinct parts, play the same role in the representation theory of the spin-symmetric groups ([31, 6, 33, 10]) as the conventional Schur S-functions in the representation theory of the symmetric groups ([17, Ch. I, §7]). A "projective" analogue of the Schur–Weyl duality for $gl(N)$ and $S(n)$ ([36]) was obtained by A. Sergeev in [32]. A. Sergeev introduced a new group (now it is called the Sergeev group) instead of $S(n)$ and replaced the Lie algebra $gl(N)$ by the Lie superalgebra $Q(N)$. Thus the Schur Q-functions, up to scalar factors, are equal to the characters of the irreducible representations of the Lie superalgebra $Q(N)$ (see also [35]). Besides the Schur Q-functions, it is convenient to use the Schur P-functions P_λ, which differ from the Q-functions Q_λ by simple scalar factors (Definition 2.13). The Schur P-functions are $t=-1$ specializations of the Hall–Littlewood polynomials and constitute a linear basis in the algebra of supersymmetric functions Γ; see [17, §8; 29, 6] for more details. For a partition with distinct parts $\lambda=(\lambda_1>\lambda_2>\dots>\lambda_l>0)$ the Schur P-function P_λ is the symmetric polynomial (suppose the number of variables n is greater than l)

$$P_\lambda(x_1,\dots,x_n)=\frac{1}{(n-l)!}\sum_{\omega\in S(n)}\prod_{i=1}^{l}x_{\omega(i)}^{\lambda_i}\prod_{i\leqslant l,\, i\leqslant j\leqslant n}\frac{x_{\omega(i)}+x_{\omega(j)}}{x_{\omega(i)}-x_{\omega(j)}}; \quad (1.1)$$

$$Q_\lambda(x_1,\dots,x_n)=2^l P_\lambda(x_1,\dots,x_n); \quad (1.2)$$

see Definition 2.8 for details.

We replace the ordinary powers x^k by the generalized powers $(x-a_1)(x-a_2)\dots(x-a_k)$ (we suppose $a_1=0$) in these definitions of the Schur P- and Q-functions and we obtain (Definition 2.10) new supersymmetric functions $P_{\lambda;a}$ and $Q_{\lambda;a}$. We call them the *multiparameter Schur P- and Q-functions*. The classical Schur Q-function Q_λ is the leading homogeneous term of the function $Q_{\lambda;a}$. As the ordinary Schur Q-functions Q_λ, the multiparameter Schur Q-functions $Q_{\lambda;a}$ constitute a linear basis in the algebra of supersymmetric functions. When all a_j are distinct, the function $Q_{\lambda;a}$ can be viewed as a solution of some multivariate interpolation problem (see Sec. 5). For this reason we call these functions the interpolation analogues of the Schur Q-functions.

Earlier interpolation analogues for many other symmetric functions were found and studied: for analogues of the classical Schur S-functions, see [1, 22, 25, 26] and references in [25]; in the case of the supersymmetric Schur S-functions, see [18, 28]; in the case of the Jack functions, see [13, 11, 27]; for the case of the Macdonald polynomials, excepting the Hall–Littlewood polynomials, see [12, 30, 23, 24] and references therein.

We obtain the following main results about the multiparameter Schur Q-functions:

- A Nimmo-type formula expressing $P_{\lambda;a}$ as a relation of two Pfaffians (Sec. 3).
- A combinatorial formula expressing $Q_{\lambda;a}$ in terms of marked shifted tableaux (Sec. 4). This formula has an additional symmetry as compared to its counterpart for the conventional Schur Q-functions. Also we rewrite our formula in terms of unmarked shifted tableaux.
- The generating functions for the one-row $Q_{(r);a}$ and two-row $P_{(r,s);a}$ multiparameter functions (Sec. 8). (Note that even in the ordinary case

the formula for the generating function for the two-row function $P_{(r,s)}$ seems to be new.)

• A Giambelli–Schur-type Pfaffian formula that expresses an arbitrary multiparameter Schur Q-function $Q_{\lambda;a}$ as a Pfaffian of the two-row multiparameter Schur Q-functions (Sec. 9).

Also we find explicitly the transition coefficients between two bases $\{Q_{\lambda;a}\}$ and $\{Q_{\lambda;b}\}$ for two sequences (a_k) and (b_k) (Sec. 10). In particular, we can express each of the multiparameter Schur Q-functions $Q_{\lambda;a}$ as a linear combination of the ordinary Schur Q-functions Q_μ.

Suppose $a_k = k-1$; then we obtain an important particular case of the multiparameter Schur Q-functions. We call them the *factorial Schur Q-functions*. In Sec. 7 we calculate the number of shifted standard tableaux of a given skew shape in terms of the factorial P-functions (called the *dimension of a skew shifted Young diagram*). We obtain a simple Pfaffian expression for this dimension (Theorem 7.5). This dimension can be also rewritten in terms of the multiplicity of the restriction of the irreducible representations of the spin-symmetric group or the Sergeev group to a smaller subgroup; for details, see, for example, [6, 9]. Using this formula with the factorial P-functions in [7] we obtain a new proof of Nazarov's theorems about the characters of the infinite spin-symmetric group ([19]). Note also that the factorial Schur P-functions can possibly be obtained from the Schubert polynomials ([14, 15, 16, 5, 2]), but we did not verify this fact. Certain problems related to the factorial Schur Q-functions are considered in [20, 9].

The author is very grateful to G. Olshanski for his constant attention to this work and many valuable remarks, to A. Okounkov for the main Definition 2.9, and to A. Borodin for the statement of Theorem 8.4 (the generating function for the two-row multiparameter P-functions).

2. Notation and definitions

Definition 2.1. A polynomial $f(x_1, \dots, x_n)$ is called *supersymmetric* if the following conditions hold:

(1) $f(x_1, \dots, x_n)$ is symmetric in the variables $x_1, \dots, x_n$;

(2) for all $1 \leqslant i < j \leqslant n$ the polynomial

$$f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{j-1}, -t, x_{j+1}, \dots, x_n)$$

does not depend on t.

Supersymmetric polynomials in n variables form an algebra. Denote this algebra by $\Gamma(n)$. The algebra $\Gamma(n)$ is graded by the degree of polynomials. The specialization $x_{n+1} = 0$ is a morphism of graded algebras

$$\Gamma(n+1) \to \Gamma(n)\,. \tag{2.1}$$

Definition 2.2. *The algebra Γ of supersymmetric functions* is the projective limit

$$\Gamma = \varprojlim \Gamma(n), \quad n \to \infty,$$

in the category of graded algebras, taken with respect to morphisms (2.1). In other words, an element $f \in \Gamma$ is a sequence $(f_n)_{n \geqslant 1}$ such that

(1) $f_n \in \Gamma(n)$, $n = 1, 2, \ldots$,
(2) $f_{n+1}(x_1, \ldots, x_n, 0) = f_n(x_1, \ldots, x_n)$ (the stability condition),
(3) $\sup_n \deg f_n < \infty$.

Definition 2.3. A finite sequence $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_k \geqslant 0)$ of decreasing positive integers is called *a partition*. The number of nonzero parts of λ is called *the length of the partition* λ and is denoted by $\ell(\lambda)$. Let us denote by $|\lambda|$ the sum of the parts of the partition λ:

$$|\lambda| = \sum_{k=1}^{\ell(\lambda)} \lambda_k\,.$$

If $|\lambda| = n$ we will also write $\lambda \vdash n$. Also put $m_i(\lambda) = \#\{k \mid \lambda_k = i\}$ for $i \geqslant 1$. Obviously, we have

$$\ell(\lambda) = \sum_{i \geqslant 1} m_i(\lambda), \quad |\lambda| = \sum_{i \geqslant 1} i m_i(\lambda).$$

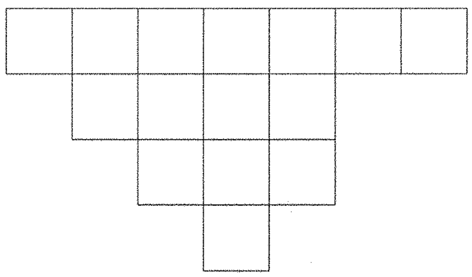
Definition 2.4. A partition is called *strict* if any two nonzero parts of it are distinct. We will denote by DP_n the set of strict partitions of the number n. A partition is called *odd* if all its nonzero parts are odd. We will denote by OP_n the set of odd partitions of the number n. Also put

$$DP = \bigcup_{k \geqslant 0} DP_k, \quad OP = \bigcup_{k \geqslant 0} OP_k. \tag{2.2}$$

Definition 2.5. For a strict partition one may define the corresponding shifted Young diagram besides the simple Young diagram [17, Ch. I, §1]. If λ is a strict partition, then the set

$$\{(i,j) \in \mathbb{Z}^2 \mid i \leqslant j \leqslant \lambda_i + i - 1, 1 \leqslant i \leqslant \ell(\lambda)\}$$

is called *the shifted Young diagram of the partition* λ and is denoted by $D(\lambda)'$. It is useful to draw unit squares instead of points of $\mathbb{Z}^2$. We assume that the first coordinate axis is directed downwards, the second coordinate axis is directed to the right, and the point $(0,0)$ is in the left upper corner of the figure. For example, if $\lambda = (7,4,3,1)$, then $D(\lambda)' =$



Proposition 2.6. *Let $r(x_1,\dots,x_l)$ be a polynomial in variables $x_1,\dots,x_l$. For arbitrary $n \geqslant l$ we put*

$$R_n(x_1,\dots,x_n) = r(x_1,\dots,x_l) \prod_{i \leqslant l, i<j \leqslant n} \frac{x_i + x_j}{x_i - x_j} \tag{2.3}$$

and

$$\widetilde{R}_n(x_1,\dots,x_n) = \sum_{\omega \in S(n)} R_n(x_{\omega(1)},\dots,x_{\omega(n)}). \tag{2.4}$$

Then

a) *$\widetilde{R}_n$ is a polynomial in $x_1,\dots,x_n$ and*

$$\deg \widetilde{R}_n \leqslant \deg r;$$

b) *$\widetilde{R}_n$ is a supersymmetric polynomial;*

c) *if for some $i \neq j$ the polynomial $r(x_1,\dots,x_l)$ is symmetric in x_i, x_j, then $\widetilde{R}_n = 0$;*

d) *if $r(x_1,\dots,x_l)$ is divisible by $x_1 x_2 \dots x_l$, then*

$$\widetilde{R}_{n+1}(x_1,\dots,x_n,0) = (n+1-l)\widetilde{R}_n(x_1,\dots,x_n).$$

Proof. See [7, Proposition 1.1]. $\square$

Next we consider three special cases of Proposition 2.6 (Definitions 2.8, 2.9, and 2.10). First let us introduce generalizations of the ordinary power x^n.

Definition 2.7. Put

$$(x \downarrow n) = \prod_{k=1}^{n} (x - k + 1)$$

for $n \geqslant 1$. Also put $(x \downarrow 0) = 1$. Suppose $(a_k)_{k \geqslant 1}$ is an arbitrary sequence of complex numbers. Put

$$(x \mid a)^n = \prod_{k=1}^{n} (x - a_k), \quad (x \mid a)^0 = 1.$$

Definition 2.8. Suppose λ is a partition, $\ell(\lambda) = l < n$. Put

$$r(x_1, \ldots, x_l) = \frac{\prod_{i=1}^{l} x_i^{\lambda_i}}{(n-l)!}$$

in Proposition 2.6. Namely, put

$$P_{\lambda|n}(x_1, \ldots, x_n) = \frac{1}{(n-l)!} \sum_{\omega \in S(n)} \prod_{i=1}^{l} x_{\omega(i)}^{\lambda_i} \prod_{i \leqslant l,\, i \leqslant j \leqslant n} \frac{x_{\omega(i)} + x_{\omega(j)}}{x_{\omega(i)} - x_{\omega(j)}}.$$

If $\lambda \in DP$, then the polynomials $P_{\lambda|n}$ are the specializations of the Hall–Littlewood polynomials when the parameter $t = -1$ (see [17, Ch. III, §2, §8]). If $\ell(\lambda) > n$, then we put $P_{\lambda|n} = 0$. From Proposition 2.6 it follows that the sequence $(P_{\lambda|n})_{n \geqslant 1}$ defines a supersymmetric function, which is denoted by P_λ. The functions P_λ are called *the Schur P-functions*. Note that if ν is not a strict partition, then from Proposition 2.6c) it follows that $P_\nu = 0$. So our definition differs at this point from the traditional definition of the Schur P-functions (see [17, Ch. III, §2, (2.2)]).

The next definition is due to A. Okounkov.

Definition 2.9. Suppose $l = \ell(\lambda) \leqslant n$, $\lambda \in DP$. Put

$$F_{\lambda|n}(x_1, \ldots, x_n) = \prod_{i=1}^{l} (x_i \downarrow \lambda_i) \prod_{i \leqslant l, i < j \leqslant n} \frac{x_i + x_j}{x_i - x_j}.$$

By definition, we put

$$P^*_{\lambda|n} = \frac{1}{(n-l)!} \sum_{\omega \in S(n)} F_{\lambda|n}(x_{\omega(1)}, \dots, x_{\omega(n)}).$$

This corresponds to the case

$$r(x_1, \dots, x_l) = \frac{\prod_{i=1}^{l} (x_i \downarrow \lambda_i)}{(n-l)!}$$

in Proposition 2.6. If $\ell(\lambda) > n$, then, by definition, put $P^*_{\lambda|n} = 0$. From Proposition 2.6 it follows that the sequence $(P^*_{\lambda|n})_{n \geqslant 1}$ defines a supersymmetric function P^*_λ. We call these functions *the factorial Schur P-functions.*

The next definition generalizes both Definition 2.8 and Definition 2.9.

Definition 2.10. Suppose $a = (a_k)_{k \geqslant 1}$ is an arbitrary sequence of complex numbers, $\lambda \in DP$, $\ell(\lambda) = l < n$. Then put

$$P_{\lambda;a|n}(x_1, \dots, x_n) = \frac{1}{(n-l)!} \sum_{\omega \in S(n)} \prod_{i=1}^{l} (x_{\omega(i)} \mid a)^{\lambda_i} \prod_{i \leqslant l,\, i \leqslant j \leqslant n} \frac{x_{\omega(i)} + x_{\omega(j)}}{x_{\omega(i)} - x_{\omega(j)}}.$$

If $\ell(\lambda) > n$, then we put $P_{\lambda;a|n} = 0$. From Proposition 2.6 it follows that if $a_1 = 0$, then the sequence $(P_{\lambda;a|n})_{n \geqslant 1}$ defines a supersymmetric function, which is denoted by $P_{\lambda;a}$. The functions $P_{\lambda;a}$ are called *the multiparameter Schur P-functions*. Further in the text we suppose $a_1 = 0$.

Proposition 2.11. *Suppose $\lambda \in DP$, $\ell(\lambda) \leqslant n$; then*

$$P_{\lambda;a}(x_1, \dots, x_n) = P_\lambda(x_1, \dots, x_n) + g(x_1, \dots, x_n),$$

where $g(x_1, \dots, x_n)$ is a supersymmetric polynomial such that $\deg g < |\lambda|$.

Proof. Follows from Proposition 2.6, Definition 2.8, and Definition 2.10. $\square$

Proposition 2.12.

a) *The set $\{P_\lambda \mid \lambda \in DP\}$ is a linear basis of the algebra Γ.*

b) *The set $\{P_{\lambda;a} \mid \lambda \in DP\}$ is a linear basis of the algebra Γ. In particular, the set $\{P^*_\lambda \mid \lambda \in DP\}$ is a linear basis of the algebra Γ.*

Proof.

a) Follows from [29, Theorem 2.11].
b) Follows from assertion a) and Proposition 2.11. □

Suppose $\nu \notin DP$; then from Proposition 2.6 it follows that $P_{\nu;a} \equiv P^*_\nu \equiv 0$.

Definition 2.13. For arbitrary partition λ we put

$$Q_\lambda = 2^{\ell(\lambda)} P_\lambda, \quad Q^*_\lambda = 2^{\ell(\lambda)} P^*_\lambda, \quad Q_{\lambda;a} = 2^{\ell(\lambda)} P_{\lambda;a}.$$

The supersymmetric functions Q_λ were introduced by I. Schur in [31]. They are called *the Schur* Q*-functions.* The supersymmetric functions Q^*_λ ($Q_{\lambda;a}$) are called *the factorial (multiparameter) Schur* Q*-functions.*

3. Nimmo-type formula

In this section we obtain a formula for $P_{\lambda;a}$, which is an analogue of the formula for the ordinary Schur P-functions obtained by Nimmo ([21]).

Recall the definition of the Pfaffian of a skew-symmetric matrix.

Definition 3.1. Suppose A is a skew-symmetric $2n \times 2n$ matrix. Put

$$\mathrm{Pf}(A) = \sum_{\omega \in \tilde{S}(2n)} \mathrm{sgn}(\omega) \prod_{i=1}^{n} a_{\omega(2i-1)\omega(2i)},$$

where the sum is over $\omega \in \tilde{S}(2n) \subset S(2n)$ such that

$$\omega(2i-1) < \omega(2i) \quad \text{and} \quad \omega(1) < \omega(3) < \cdots < \omega(2n-3) < \omega(2n-1).$$

Theorem 3.2. *Suppose $n \geqslant \ell(\lambda) = l$, $\lambda \in DP$, $a = (a_k)_{k \geqslant 1}$ is an arbitrary sequence of complex numbers, $a_1 = 0$. Let $A_0(x_1, \dots, x_n)$ denote the skew-symmetric $n \times n$ matrix*

$$\left(\frac{x_i - x_j}{x_i + x_j} \right)_{1 \leqslant i,j \leqslant n},$$

and let B_λ denote the $n \times l$ matrix

$$\left((x_i|a)^{\lambda_{l+1-j}} \right)_{i \leqslant n,\, j \leqslant l}.$$

Let $A_\lambda(x_1, \dots, x_n)$ be the skew-symmetric $(n+l) \times (n+l)$ matrix

$$\begin{pmatrix} A_0(x_1, \dots, x_n) & B_\lambda(x_1, \dots, x_n) \\ -B_\lambda(x_1, \dots, x_n)^t & 0 \end{pmatrix}$$

Put

$$\mathrm{Pf}_0(x_1,\dots,x_n)=\begin{cases}\mathrm{Pf}(A_0(x_1,\dots,x_n)) & \text{if } n \text{ is even;}\\ \mathrm{Pf}(A_0(x_1,\dots,x_n,0)) & \text{if } n \text{ is odd.}\end{cases}$$

Also put

$$\mathrm{Pf}_\lambda(x_1,\dots,x_n)=\begin{cases}\mathrm{Pf}(A_\lambda(x_1,\dots,x_n)) & \text{if } n+l \text{ is even;}\\ \mathrm{Pf}(A_\lambda(x_1,\dots,x_n,0)) & \text{if } n+l \text{ is odd.}\end{cases}$$

Then

$$P_{\lambda;a}(x_1,\dots,x_n)=\frac{\mathrm{Pf}_\lambda(x_1,\dots,x_n)}{\mathrm{Pf}_0(x_1,\dots,x_n)}.$$

Proof. The proof follows Nimmo's method, see [21] or [17, Ch. III, §8, Example 13]. □

4. Combinatorial formulas

Definition 4.1. Consider the ordered alphabet $\mathbb{P}_n=\{1'<1<2'<2<\cdots<n'<n\}$. By definition, put

$$|k'|=|k|=k,\quad \mathrm{sgn}(k)=-\mathrm{sgn}(k')=1$$

for an arbitrary natural number k.

Definition 4.2. Let λ be an arbitrary strict partition. *A marked shifted Young tableaux of shape λ and length n* is a mapping $T: D'_\lambda \to \mathbb{P}_n$ such that the following conditions hold:

(1) $T(i,j)\leqslant T(i+1,j)$, $T(i,j)\leqslant T(i,j+1)$;

(2) for each natural number k there is at most one k in the image of each column and at most one k' in the image of each row of D'_λ.

Let us denote by $\mathrm{MSTab}(\lambda,n)$ the set of all shifted marked Young tableaux of shape λ and length n.

Here is an example of a shifted Young tableau of shape $(7,4,3,1)$ and length 6:

1	1	2	$3'$	$4'$	4	$6'$
	$2'$	$3'$	3	$4'$		
		3	$4'$	4		
			$4'$			

Theorem 4.3 (Combinatorial formula for the multiparameter Schur Q-functions).

$$Q_{\lambda;a}(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathrm{MSTab}(\lambda,n)} \prod_{(i,j)\in D'_\lambda} (x_{|T(i,j)|} - \operatorname{sgn}(T(i,j)) a_{j-i+1}).$$

Proof. The combinatorial formula for the factorial Schur Q-functions (see Corollary 4.4 below) is proved in [8]. We may use the same method in the case of the multiparameter Schur Q-functions with evident changes. $\square$

Corollary 4.4 (Combinatorial formula for the factorial Schur Q-functions).

$$Q^*_\lambda(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathrm{MSTab}(\lambda,n)} \prod_{(i,j)\in D'_\lambda} (x_{|T(i,j)|} - \operatorname{sgn}(T(i,j))(j-i)).$$

Combinatorial formulas for other interpolation analogues of the symmetric functions were obtained earlier in [25, §11; 23, 18, 28].

We may reformulate Theorem 4.3 in terms of unmarked shifted tableaux.

Definition 4.5. Suppose μ and λ are strict partitions, $D(\mu)' \subseteq D(\lambda)'$; then we write $\mu \subset \lambda$. In this case let us consider *the skew shifted diagram* $D(\lambda)' \setminus D(\mu)'$. For each box $b \in D(\lambda)' \setminus D(\mu)'$ let us denote by $i(b)$ and $j(b)$ its first and second coordinates, respectively. The skew shifted diagram $\nu = D(\lambda)' \setminus D(\mu)'$ is called *a border strip* if the following conditions hold:

(1) The diagram ν has no 2×2 block of squares.

(2) The set $I(\nu) = \{j(b) - i(b) \mid b \in \nu\} \subset \mathbb{Z}_+ \cup \{0\}$ is an interval of integers, i.e., $I(\nu) = [\min I(\nu), \max I(\nu)] \cap \mathbb{Z}$. When this condition is satisfied we say that ν is *connected.*

Following [28, §4], consider *interior sides* of the squares of the shape ν: *an interior side* is adjacent to two squares of ν; the total number of interior sides is equal to $|\nu| - 1$. To each interior side s we attach the coordinates (ε, δ) of its midpoint, and we write $s = (\varepsilon, \delta)$. Note that one of the coordinates is always half-integer while another coordinate is integer.

Suppose $a = (a_n)_{n \geqslant 1}$ is a sequence of complex numbers, ν is a border strip; then put

$$f(\nu; a; x) = 2x \prod{}' (x + a_{\delta-\varepsilon+1/2}) \prod{}'' (x - a_{\delta-\varepsilon+1/2}),$$

where $\prod'$ is taken over the horizontal interior sides $s = (\varepsilon, \delta)$ of ν, and $\prod''$ is taken over the vertical interior sides $s = (\varepsilon, \delta)$ of ν.

Suppose $\nu = D(\lambda)' \setminus D(\mu)'$ has no 2×2 block of squares; then we may represent ν as the disjoint union of minimal number of border strips $\nu = \cup_{j=1}^{k} \nu_j$, i.e., such that $\forall i, j$ the set $\nu_i \cup \nu_j$ is not connected in the sense of condition 2. In this case we put

$$f(\nu; a; x) = \prod_{j=1}^{k} f(\nu_j; a; x).$$

An (unmarked) shifted tableau T of shape λ and length n is a sequence of strict partitions $\varnothing = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \lambda^{(n)} = \lambda$ such that each $D(\lambda^{(j)})' \setminus D(\lambda^{(j-1)})'$ has no 2×2 block of squares. In this case we put

$$f(T; a; x_1, \ldots, x_n) = \prod_{j=1}^{n} f((D(\lambda^{(j)})' \setminus D(\lambda^{(j-1)})'); a; x_j).$$

Theorem 4.6. *As always we suppose that $a = (a_n)_{n \geqslant 1}$ is a sequence of complex numbers such that $a_1 = 0$; then we have*

$$Q_{\lambda;a}(x_1, \ldots, x_n) = \sum_{T} f(T; a; x_1, \ldots, x_n),$$

where the sum is over all shifted tableaux of shape λ and length n.

Proof. We can easily deduce this theorem from Theorem 4.3. □

5. Characterization properties

Definition 5.1. Suppose μ is a strict partition, $a = (a_n)_{n\geqslant 1}$ is a sequence of complex numbers; then, by definition, we put

$$H_a(\mu) = \prod_{k=1}^{\ell(\mu)} (a_{\mu_k+1} \mid a)^{\mu_k} \prod_{i<j} \frac{a_{\mu_i+1} + a_{\mu_j+1}}{a_{\mu_i+1} - a_{\mu_j+1}} \tag{5.1}$$

$$= \prod_{i<j} \left((a_{\mu_i+1} + a_{\mu_j+1}) \prod_{k=\mu_j+2}^{\mu_i} (a_{\mu_i+1} - a_k) \right). \tag{5.2}$$

In the "factorial" case, i.e., when $a_k = k - 1$ the expression $H_a(\mu)$ becomes the "shifted product of hook lengths" (see [17, Ch. III, §7, examples]):

$$H(\mu) = \prod_{t=1}^{\ell(\mu)} \mu_t! \prod_{i<j} \frac{\mu_i + \mu_j}{\mu_i - \mu_j}.$$

Definition 5.2. Suppose λ is a strict partition, $a = (a_n)_{n\geqslant 1}$ is a sequence of complex numbers; then let us define a collection of variables $x(\lambda)$ by means of the formula

$$x(\lambda)_i = a_{\lambda_i+1}.$$

Theorem 5.3 (Vanishing property). *Assume that the numbers a_j are pairwise distinct and $a_1 = 0$.*

a) *If $\mu, \lambda \in DP$, $\mu \not\subset \lambda$, then $Q_{\mu;a}(x(\lambda)) = P_{\mu;a}(x(\lambda)) = 0$.*

b) *$P_{\mu;a}(x(\mu)) = H_a(\mu)$.*

Proof. a) This claim follows from Definition 2.10.

b) This formula can be readily obtained by a direct computation. □

Next we write explicitly Theorem 5.3 in the "factorial" case.

Corollary 5.4. a) *If $\mu, \lambda \in DP$, $\mu \not\subset \lambda$, then $Q^*_\mu(\lambda_1, \ldots, \lambda_{\ell(\lambda)}) = P^*_\mu(\lambda_1, \ldots, \lambda_{\ell(\lambda)}) = 0$.*

b) *$P^*_\mu(\mu) = H(\mu)$.*

Following the method of [22, 25], we can obtain characterization theorems for the multiparameter Schur P-functions.

Theorem 5.5 (Characterization Theorem I). *Assume that the numbers a_j are pairwise distinct. Suppose $f \in \Gamma$ satisfies the following conditions:*

(1) *The top degree homogeneous component of f is equal to*

$$\sum_{\mu \in DP_n} c_\mu P_\mu.$$

(2) *For every $\lambda \in DP$ such that $|\lambda| < n$ we have $f(x(\lambda)) = 0$.*

Then

$$f \equiv \sum_{\mu \in DP_n} c_\mu P_{\mu;a}.$$

Proof. We can use Proposition 2.11 and Theorem 5.3. □

Theorem 5.6 (Characterization Theorem II). *Assume that the numbers a_j are pairwise distinct. Suppose $f \in \Gamma$ satisfies the following conditions for some strict partition μ:*

(1) $f(x(\mu)) = H_a(\mu)$.
(2) $\deg f \leqslant |\mu|$.
(3) *For every $\lambda \in DP$ such that $|\lambda| \leqslant |\mu|$ and $\lambda \neq \mu$ we have $f(x(\lambda)) = 0$.*

Then $f \equiv P_{\mu;a}$.

Proof. Again we use Proposition 2.11 and Theorem 5.3. □

Note that these characterization theorems can be easily reformulated in the particular "factorial" case.

Theorem 5.5 and Theorem 5.6 show that the multiparameter Schur P- and Q-functions can be considered as solutions of multivariate interpolation problems. Earlier such interpolation analogues were obtained for other classical symmetric functions, see [22, 25, 26, 18, 28, 13, 27, 12, 23, 30]. The general scheme of such an interpolation for the symmetric polynomials was considered in [24].

6. Pieri-type formula

Definition 6.1. For $\mu, \lambda \in DP$ we will write $\mu \nearrow \lambda$ if $|\mu| + 1 = |\lambda|$ and $\mu \subset \lambda$ (Definition 4.5).

Recall that

$$P_{(1);a}(x) = P^*_{(1)}(x) = P_{(1)}(x) = \sum_j x_j.$$

Theorem 6.2. *As usual we assume $a_1 = 0$. Then for an arbitrary $\mu \in DP$ we have*

$$P_{\mu;a}(P_{(1)} - \sum_{j=1}^{\ell(\mu)} a_{\mu_j+1}) = \sum_{\mu \nearrow \lambda} P_{\lambda;a}. \tag{6.1}$$

Proof. First we consider the case when all a_j are pairwise distinct. Then (6.1) may be easily deduced from Theorem 5.5. In the general case we use the continuity of both sides of (6.1). □

Corollary 6.3. *For arbitrary $\mu \in DP$ we get*

$$P^*_\mu(P_{(1)} - |\mu|) = \sum_{\mu \nearrow \lambda} P^*_\lambda.$$

7. Dimensions of skew shifted Young diagrams

Definition 7.1. Suppose μ and λ are strict partitions and $\mu \subset \lambda$ (Definition 4.5). *A standard shifted Young tableau T of shape λ/μ* is a bijection $T : D(\lambda)' \setminus D(\mu)' \to \{1, 2, \ldots, |\lambda| - |\mu|\}$ such that $T(i,j) < T(i+1,j)$; $T(i,j) < T(i,j+1)$. The number of standard shifted tableaux of shape λ/μ is called *the dimension of the skew shifted Young diagram* λ/μ and is denoted by $g_{\lambda/\mu}$.

Now we give another description of this dimension. First we define the Schur graph (first considered in [4, 3]).

Definition 7.2. The set of vertices of the Schur graph is labelled by the set DP. The Schur graph is an acyclic directed graph; there is an edge directed from $\mu \in DP$ to $\lambda \in DP$ if and only if $\mu \nearrow \lambda$ (Definition 6.1).

Proposition 7.3. *There exists a directed path in the Schur graph from $\mu \in DP$ to $\lambda \in DP$ if and only if $\mu \subset \lambda$. In this case the number of these directed paths is equal to $g_{\lambda/\mu}$.*

Proof. The proof is trivial. □

An explicit formula for $g_{\lambda/\varnothing}$ is well-known ([17, Ch. III, §7, Examples]):

$$g_{\lambda/\varnothing} = \frac{|\lambda|!}{\prod_{k=1}^{\ell(\lambda)} \lambda_k!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

In the next theorem we obtain an explicit expression for $g_{\lambda/\mu}$.

Theorem 7.4. *Suppose $\mu, \lambda \in DP$, $\mu \subset \lambda$; then*

$$g_{\lambda/\mu} = g_{\lambda/\varnothing} \frac{P^*_\mu(\lambda_1, \dots, \lambda_{\ell(\lambda)})}{(|\lambda| \downarrow |\mu|)}. \tag{7.1}$$

Proof. This theorem is proved in [7] by a direct computation. Following [25, §9], we can also deduce this theorem from the Pieri-type formula (Corollary 6.3). □

Earlier in [25] A. Okounkov and G. Olshanski studied the case of the Young graph, which is connected with the Schur S-functions and linear representations of the symmetric groups. Theorem 7.4 may be viewed as a projective analogue of [25, Theorem 8.1] (see also [28]). An analogous result for the more general case of the Jack graph was obtained in [27, Section 5].

Now we rewrite the expression (7.1) in the Pfaffian form. In the ordinary case the analogous formula has a determinantal form (see, for example, [28, Proof of Proposition 1.2]).

Theorem 7.5. *Suppose $\mu, \lambda \in DP$, $\mu \subset \lambda$. Let $X(\lambda_1, \dots, \lambda_k)$ denote the skew-symmetric $k \times k$ matrix*

$$\left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \cdot \frac{1}{\lambda_i! \lambda_j!} \right)_{1 \leqslant i,j \leqslant k},$$

and let $Y(\lambda_1, \dots, \lambda_k; \mu)$ denote the $k \times \ell(\mu)$ matrix

$$\left(\frac{1}{(\lambda_i - \mu_j)!} \right)_{i \leqslant k,\, j \leqslant \ell(\mu)},$$

where we suppose $\frac{1}{m!} = 0$ if $m < 0$. Put $s = \ell(\lambda) + \ell(\mu)$. If s is even, then let $A_{\lambda/\mu}$ be the skew-symmetric $s \times s$ matrix

$$\begin{pmatrix} X(\lambda_1, \dots, \lambda_{\ell(\lambda)}) & Y(\lambda_1, \dots, \lambda_{\ell(\lambda)}; \mu) \\ -Y(\lambda_1, \dots, \lambda_{\ell(\lambda)}; \mu)^t & 0 \end{pmatrix}.$$

If s is odd, then let $A_{\lambda/\mu}$ be the skew-symmetric $(s+1) \times (s+1)$ matrix

$$\begin{pmatrix} X(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0) & Y(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0; \mu) \\ -Y(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0; \mu)^t & 0 \end{pmatrix}.$$

Then

$$g_{\lambda/\mu} = (|\lambda| - |\mu|)! \operatorname{Pf}(A_{\lambda/\mu}). \tag{7.2}$$

Proof. The formula (7.2) can be easily deduced from Theorem 3.2 and Theorem 7.4. □

8. Generating functions

It is known (see, e.g., [17, III, §2, (2.10)]) that

$$\sum_{r=0}^{\infty} Q_{(r)}(x_1, x_2, \dots) y^r = \prod_{j=1}^{\infty} \frac{1 + x_j y}{1 - x_j y}.$$

Definition 8.1. Let $a = (a_n)_{n \geqslant 1}$ be a sequence of complex numbers; then denote by (τa) the result of shifting the sequence a to the left:

$$(\tau a)_n = a_{n+1}.$$

Thus,

$$(x|\tau a)^k = \prod_{j=2}^{k+1} (x - a_j).$$

Theorem 8.2.

$$\sum_{r=0}^{\infty} \frac{Q_{(r);a}(x_1, x_2, \dots)}{(u|\tau a)^r} = \prod_{j=1}^{\infty} \frac{u + x_j}{u - x_j}.$$

Proof. By our assumption (Definition 2.10) $a_1 = 0$. Then the following identity holds:

$$1 + 2 \sum_{r=1}^{\infty} \frac{(x|a)^r}{(u|\tau a)^r} = \frac{u + x}{u - x}.$$

Then we can reason as in the proof of [17, III, §2, (2.10)]. □

Corollary 8.3.

$$\sum_{r=0}^{\infty} \frac{Q^*_{(r)}(x_1, x_2, \dots)}{(u \downarrow r)} = \prod_{i=1}^{\infty} \frac{u + 1 + x_i}{u + 1 - x_i}.$$

The next formula is due to A. Borodin.

Theorem 8.4. *If $0 \leqslant k \leqslant l$, then put $P_{(k,l);a} \equiv -P_{(l,k);a}$. Then we have*

$$\begin{aligned} 4uv(u+v) \sum_{k,l \leqslant 0} \frac{P_{(k,l);a}(x_1, x_2, \dots)}{(u|a)^{k+1} (v|a)^{l+1}} &= u \left(\prod_{j=1}^{\infty} \frac{u + x_j}{u - x_j} + 1 \right) \left(- \prod_{j=1}^{\infty} \frac{v + x_j}{v - x_j} + 1 \right) \\ &\quad - v \left(- \prod_{j=1}^{\infty} \frac{u + x_j}{u - x_j} + 1 \right) \left(\prod_{j=1}^{\infty} \frac{v + x_j}{v - x_j} + 1 \right). \end{aligned} \tag{8.1}$$

Proof. The equality (8.1) is equivalent to the following two equalities:

$$P_{(k+1,l);a} + P_{(k,l+1);a} + (a_{k+1} + a_{l+1})P_{(k,l);a} = \\ = P_{(k);a}P_{(l+1);a} - P_{(k+1);a}P_{(l);a} + (a_{l+1} - a_{k+1})P_{(k);a}P_{(l);a}; \quad (8.2)$$

$$P_{(k+1);a} + P_{(k,1);a} + a_{k+1}P_{(k);a} = P_{(k);a}P_{(1);a}. \quad (8.3)$$

The equality (8.3) is a particular case of Theorem 6.2. Let us prove (8.2). From [17, Ch. III, §5, (5.7)] we get the following identity for the ordinary Schur P-functions:

$$P_{(k+1)}P_{(l)} = P_{(k+1,l)} + P_{(k+1+l)} + 2\sum_{j=1}^{l-1} P_{(k+1+j,l-j)}.$$

Consequently we have

$$P_{(k+1,l)} + P_{(k,l+1)} = P_{(k)}P_{(l+1)} - P_{(k+1)}P_{(l)}. \quad (8.4)$$

First we suppose that the numbers a_j are pairwise distinct; then from (8.4) and Theorem 5.5 we get that

$$P_{(k+1,l);a} + P_{(k,l+1);a} + \alpha P_{(k,l);a} = \\ = P_{(k);a}P_{(l+1);a} - P_{(k+1);a}P_{(l);a} + \\ + (a_{l+1} - a_{k+1})P_{(k);a}P_{(l);a} \quad (8.5)$$

for some α. Evaluating both sides of (8.5) at the point (a_{k+1}, a_{l+1}) we obtain that $\alpha = a_{k+1} + a_{l+1}$. Thus (8.2) is proved for sequences (a_j) without repetitions. Using the continuity argument we have that (8.2) holds for all sequences (a_j). This concludes the proof. □

One may compare Theorem 8.4 with its analogue for the multiparameter supersymmetric Schur functions ([28, Proposition 7.1]).

9. Giambelli–Schur-type formula

Theorem 9.1. *As in Theorem 8.4, we put $Q_{(k,l);a} \equiv -Q_{(l,k);a}$ for arbitrary integers $l \geqslant k \geqslant 0$. Also we will use the notation $\lambda_{\ell(\lambda)+1}$; certainly, $\lambda_{\ell(\lambda)+1} = 0$. If λ is a strict partition, then we have*

$$Q_{\lambda;a} = \mathrm{Pf}\left((Q_{(\lambda_i,\lambda_j);a})_{1\leqslant i,j\leqslant 2[\frac{\ell(\lambda)+1}{2}]} \right),$$

where $[x]$ stands for the integral part of x.

Proof. We use Theorem 4.3 and the method of Stembridge [34] with simple modifications. ☐

I. Schur ([31]) used a particular case of this formula ($a_j \equiv 0$) as the definition of the Q-functions. This formula may be viewed as a projective analogue of the Giambelli formula for the ordinary S-functions [17, Ch. I, §3, Examples]. We may compare Theorem 9.1 with its determinantal analogue for the multiparameter supersymmetric Schur functions [28].

10. Transition coefficients

In this section we use the method of [28, §2, §7]. From [28, Lemma 2.5] we get the following identity for arbitrary sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$:

$$\frac{1}{(u-b_1)\dots(u-b_{r'+1})} = \\ = \sum_{r=r'}^{\infty} \frac{h_{r-r'}(b_1,\dots,b_{r'+1};-a_1,\dots,-a_r)}{(u-a_1)\dots(u-a_{r+1})}, \qquad (10.1)$$

where $h_0 = 1, h_1, h_2, \dots$ denote the conventional complete homogeneous functions in the super realization of the algebra Λ of symmetric functions ([17, Ch. I, §5, Examples]). From now in this section we suppose that the sequences $(a_n)_{n\geqslant 1}$ and $(b_n)n \geqslant 1$ are fixed and $a_1 = b_1 = 0$. Then

$$\begin{aligned} &h_k(b_1,b_2,\dots,b_{r'+1};-a_1,-a_2,\dots,-a_r) \\ &\quad = h_k(b_2,\dots,b_{r'+1};-a_2,\dots,-a_r). \end{aligned} \qquad (10.2)$$

Denote by $d_{rr'}$ the value $h_k(b_2,\dots,b_{r'+1};-a_2,\dots,-a_r)$, where $r \geqslant r'$. If $r < r'$, then put $d_{rr'} = 0$.

Proposition 10.1. *Let $r, s \geqslant 1$. Then*

$$P_{(r,s);a} = \sum_{r'=1}^{r}\sum_{s'=1}^{s} d_{rr'}(a,b) d_{ss'}(a,b) P_{(r',s');b}.$$

Proof. The proof is based on (10.1) and Theorem 8.4. ☐

Theorem 10.2. *Suppose μ is a strict partition; then*

$$Q_{\mu;a} = \sum_{\nu\in DP,\ell(\nu)=\ell(\mu),\nu\subset\mu} d_{\mu\nu} Q_{\nu;b},$$

where

$$d_{\mu\nu} = \det\left((d_{\mu_i\nu_j})_{1\leqslant i,j\leqslant \ell(\mu)}\right).$$

Proof. We use Theorem 9.1 and Proposition 10.1. □

Theorem 10.2 is a projective analogue of [28, Theorem 7.3].

References

1. L. Biedenharn, J. Louck, *A new class of symmetric polynomials defined in terms of tableaux.* — Adv. in Appl. Math. **10** (1989), 396–438.
2. S. Billey, M. Haiman, *Schubert polynomials for the classical groups.* — J. Amer. Math. Soc. **8**, No. 2 (1995), 443–482.
3. A. Borodin, G. Olshanski, *Harmonic functions on multiplicative graphs and interpolation polynomials.* — Electron. J. Combin. **7** (2000), #R28; arXiv: math.CO/9912124.
4. A. M. Borodin, *Multiplicative central measures on Schur graph.* — Zap. Nauchn. Semin. POMI **240** (1997), 44–52. English transl. in J. Math. Sci. **96**, No. 5 (1999), 3472–3477.
5. S. Fomin, A. N. Kirillov, *Combinatorial B_n-analogues of Schubert polynomials.* — Trans. Amer. Math. Soc. **348** (1996), 3591–3620; arXiv:hep-th/9306093.
6. P. N. Hoffman, J. F. Humphreys, *Projective Representations of the Symmetric Groups.* Oxford University Press, New York, 1992.
7. V. N. Ivanov, *Dimension of a skew shifted Young diagram and projective representations of the infinite symmetric group.* — Zap. Nauchn. Semin. POMI **240** (1997), 115–135. English transl. in J. Math. Sci. **96**, No. 5 (1999), 3517–3530; arXiv:math.CO/0303169.
8 V. N. Ivanov, *Combinatorial formula for factorial Schur Q-functions.* — Zap. Nauchn. Semin. POMI **256** (1999), 73–94. English transl. in J. Math. Sci. **107**, No. 5 (2001), 4195–4211.
9. V. N. Ivanov, *Gaussian limit for projective characters of large symmetric groups.* — Zap. Nauchn. Semin. POMI **283** (2001), 73–97. English transl. to appear in J. Math. Sci.
10. T. Józefiak, *Characters of projective representations of symmetric groups.* — Expo. Math. **7** (1989), 193–247.
11. S. Kerov, A. Okounkov, G. Olshanski, *The boundary of Young graph with Jack edge multiplicities.* — Internat. Math. Res. Notices, No. 4 (1998), 173–199; arXiv: q-alg/9703037.
12. F. Knop, *Symmetric and non-symmetric quantum Capelli polynomials.* — Comment. Math. Helv. **72** (1997), 84–100; arXiv:q-alg/9603028.
13. F. Knop, S. Sahi, *Difference equations and symmetric polynomials defined by their zeros.* — Internat. Math. Res. Notices, No. 10 (1996), 473–486; arXiv: q-alg/9610017.
14. A. Lascoux, *Puissances extérieurs, déterminants et cycles de Schubert.* — Bull. Soc. Math. France **102** (1974), 161–179.
15. A. Lascoux, *Notes on interpolation in one and several variables.* Preprint; http://phalanstere.univ-mlv.fr/∽al/.

16. A. Lascoux, P. Pragacz, *Operator calculus for Q-polynomials and Schubert polynomials*. — Adv. Math. **140** (1998), 1–43.
17. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition (with contributions by A. Zelevinsky). The Clarendon Press, Oxford University Press, New York, 1995.
18. A. Molev, *Factorial supersymmetric Schur functions and super Capelli identities*. — In: Kirillov's Seminar on Representation Theory, G. I. Olshanski (ed.), Amer. Math. Soc. Transl., Ser. 2, vol. 181. Amer. Math. Soc., Providence, R.I., 1998, pp. 109–137; arXiv:q-alg/9606008.
19. M. L. Nazarov, *Projective representations of the infinite symmetric group*. — In: Representation Theory and Dynamical Systems, A. M. Vershik (ed.), Adv. Soviet Math. **9**, Amer. Math. Soc., Providence, R.I., 1992, pp. 115–130.
20. M. Nazarov, *Capelli identities for Lie superalgebras*. — Ann. Sci. École. Norm. Sup. **30** (1997), 847–872; arXiv:q-alg/9610032.
21. J. J. C. Nimmo, *Hall–Littlewood symmetric functions and the BKP equation*. — J. Phys. A **23** (1990), 751–760.
22. A. Yu. Okounkov, *Quantum immanants and higher Capelli identities*. — Transformation Groups **1** (1996), 99–126; arXiv:q-alg/9602028.
23. A. Okounkov, *(Shifted) Macdonald polynomials: q-integral representation and combinatorial formula*. — Compositio Math. **112**, No. 2 (1998), 147–182; arXiv: q-alg/9605013.
24. A. Okounkov, *A characterization of interpolation Macdonald polynomials*. — Adv. in Appl. Math. **20**, No. 4 (1998), 395–428; arXiv:q-alg/9712052.
25. A. Okounkov, G. Olshanski, *Shifted Schur functions*. — Algebra i Analiz **9**, No. 2 (1997), 73–146. English transl. in St. Petersburg Math. J. **9** (1998), 239–300; arXiv:q-alg/9605042.
26. A. Okounkov, G. Olshanski, *Shifted Schur functions*. II: *The binomial formula for characters of classical groups and its applications*. — In: Kirillov's Seminar on Representation Theory, G. I. Olshanski (ed.), Amer. Math. Soc. Transl., Ser. 2, vol. 181, Amer. Math. Soc., Providence, R.I., 1998, pp. 245–271; arXiv:q-alg/9612025.
27. A. Okounkov, G. Olshanski, *Shifted Jack polynomials, binomial formula, and applications*. — Math. Res. Lett. **4**, No. 1 (1997), 69–78; arXiv:q-alg/9608020.
28. G. Olshanski, A. Regev, A. Vershik, *Frobenius–Schur functions*. — In: Studies in memory of Issai Schur, A. Joseph, A. Melnikov, R. Rentschler (eds.), Progr. Math., vol. 210, Birkhäuser, 2003, pp. 251–299; arXiv:math.CO/0110077.
29. P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials*. — Lect. Notes Math. **1478** (1991), 130–191.
30. S. Sahi, *Interpolation, integrality, and a generalization of Macdonald's polynomials*. — Internat. Math. Res. Notices, No. 10 (1996), 457–471.
31. I. Schur, *Uber die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrocheme lineare Substitionen*. — J. Reine Angew. Math. **139** (1911), 155–250.
32. A. N. Sergeev, *Tensor algebra of the identity representation as a module over Lie superalgebras GL(n,m) and Q(n)*. — Mat. Sb. **123** (1984), 422–430. English transl. in Math. USSR Sb. **51**, No. 2 (1985), 419–425.
33. J. R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*. — Adv. Math. **74** (1989), 87–134.

34. J. R. Stembridge, *Nonintersecting paths, Pfaffians, and plane partitions.* — Adv. Math. **83** (1990), 96–131.
35. M. Yamaguchi, *A duality of the twisted group algebra of the symmetric group and a Lie superalgebra.* — J. Algebra **222**, No. 1 (1999), 301–327; arXiv: math.RT/9811090.
36. H. Weyl, *The Classical Groups. Their Invariants and Representations.* Princeton Univ. Press, Princeton, NJ, 1939.

Independent University of Moscow
E-mail: vivanov@mccme.ru

Поступило 5 марта 2004 г.