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ON THE TOPOLOGY OF CYCLES IN PSEUDOLINEAR PROGRAMS

Abstract. By demand coming from the quasicrystal community, we present here a construction realizing an arbitrary oriented link in $\mathbb{R}^3$ by the graph of a 3d pseudolinear (or matroid) program. This gives a hint about possible "topological complexity" of rank 4 oriented matroids $\approx$ 3d pseudoplane arrangements $\approx$ 3d quasicrystal tilings.

To V. A. Zalgaller on his 80th birthday

§1. Oriented matroids, arrangements, and quasicrystals

There is a well-known but still surprising connection between the rank $n+1$ generic oriented matroids, arrangements of topological hyperplanes, and tilings of n-dimensional Euclidean space by parallelepipeds (which are Minkowski sums of n generic closed intervals = affine images of cubes). We briefly outline this connection. For a complete general reference one can use [1].

One can imagine a rank $n+1$ oriented matroid as an axiomatic ordered partial n-dimensional projective geometry, i.e., a collection of abstract k-dimensional planes $k = 0, \dots, n-1$ which are subject to all usual incidence and order axioms of ordered projective space, but we skip the requirement that any abstract k points in general position define a $(k-1)$ subspace. Everything is done so that any arrangement of projective hyperplanes in $P^n(\mathbb{R})$ produces a correct model of the oriented matroid axiomatics.

Such a voluntary "finite" shortening of the list of axioms of projective geometry gives an interesting result. It is a theorem saying that we cannot restrict projective geometry axiomatics so that in the restricted theory the arrangements of projective hyperplanes will be the only models. We will always get some virtual, nonrealizable models. It turns out

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that for the case of "oriented matroids restriction" such a virtual models has a very nice description, which is subject to rather topology then geometry. This description is the following. Let us pass from $P^n(\mathbb{R})$ to its canonical two-fold covering by the standard sphere S^n. Then an arrangement of projective hyperplanes in $P^n(\mathbb{R})$ is covered by an arrangement of $(n-1)$-dimensional "equatorial" spheres embedded into S^n. So an arrangement of equatorial $(n-1)$-spheres in S^n is the same thing as an arrangement of projective hyperplanes in $P^n(\mathbb{R})$. Call by *arrangement of pseudospheres* a collection of arbitrary topologically embedded $(n-1)$-spheres $\mathcal{H} = \{H_0, \ldots, H_m\}$ such that two axioms holds:

1: The common intersection of spheres in an arbitrary subset of $\mathcal{H}$ is a sphere of some dimension.
2: For any pair $H_i, H_j, i \neq j$, there is a homeomorphism of S^n sending pair (H_i, H_j) into a pair of equatorial spheres.

It is the content of Folkman–Lawrence theorem that all oriented matroids can be modeled by arrangements of pseudospheres, and, vice versa, any arrangement of pseudospheres is a model of some oriented matroid, in particular, chamber complexes on S^n generated by arrangements form a nice, tame class of regular cell decompositions of the sphere with a finite axiomatic description. Starting from that point, those who prefer the language of classical combinatorics can play with matroids axiomatically, all the others can forget axiomatics and draw pictures. Arrangements of pseudospheres have affine and projective models. For an arrangement $\mathcal{H}$, one can find an isotopic model consisting of centrally-symmetric pseudospheres, which can be identified with projective pseudohyperplanes – embeddings of $P^{n-1}(\mathbb{R})$ into $P^n(\mathbb{R})$ isotopic to linear embeddings. For "affine" model, one can select a pseudosphere H_0 of $\mathcal{H}$ as the "infinity", choose H_0^+ to be one of the two closed semispheres in S^n with boundary H_0, and form the arrangement of closed $(n-1)$-dimensional topological balls

$$\{D_1 = H_0^+ \cap H_1, \ldots, D_m = H_0^+ \cap H_m\}.$$

We can send H_0^+ by a homeomorphism to the standard unit ball D^n and get an object which can be call an "arrangement of affine pseudohyperplanes": a collection $\mathcal{A} = \{D_1, \ldots, D_m\}$ of embeddings of the standard disk D^{n-1} into the standard disk D^n such that the intersection of any subcollection is a disk of some dimension and any pair of embeddings is isotopic to the standard coordinate pair.

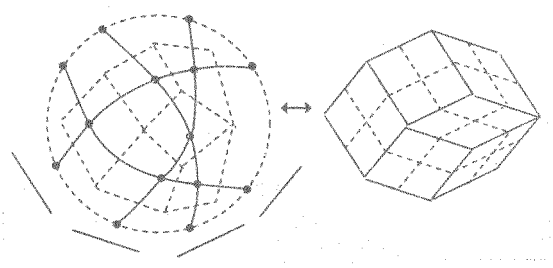
Suppose we have an arrangement $\mathcal{L} = \{L_0, \ldots, L_m\}$ of n-dimensional linear subspaces in $\mathbb{R}^{n+1}$. Let $\bar{\mathcal{L}} = \{L_0 \cap S^d, \ldots, L_k \cap S^d\}$ be the induced arrangement of equatorial spheres on the unit sphere S^d. Let $C(\bar{\mathcal{L}})$ be the set of all chambers in the decomposition of S^d defined by $\bar{\mathcal{L}}$. The set $C(\bar{\mathcal{L}})$ has a natural poset structure by the adjacency of chambers. The nice fact is that the dual poset $C(\bar{\mathcal{L}})^{\text{op}}$ has a canonical presentation as the face poset of a convex polytope. One can construct this polytope as follows. For every $L_i, i = 0, \ldots, k$, pick an orthogonal closed interval e_i. Then the convex polytope (called *zonotope*)

$$P(\mathcal{L}) = \text{ Minkowski } \sum_{i=0}^{k} e_i$$

has $C(\bar{\mathcal{L}})^{\text{op}}$ as a face poset. If the arrangement $\mathcal{L}$ is generic (i.e., all of its hyperplanes are in general position), then all faces of $P(\mathcal{L})$ are parallelepipeds.

Now we return to the arrangement $\mathcal{A}$ of affine pseudohyperplanes. Suppose that the boundary $(n-1)$ arrangement of pseudospheres $\partial\mathcal{A} = \{D_1 \cap S^{n-1}, \ldots, D_k \cap S^{n-1}\}$ is linearly representable, i.e., it is isotopic to the arrangement of equatorial spheres in S^{n-1} with normal intervals $f_1, \ldots, f_k$. Consider the poset of all chambers $C(\mathcal{A})$ of the chamber decomposition of the open disk $D^n \setminus S^{n-1}$ defined by $\mathcal{A}$. The poset $C(\mathcal{A})$ contains the poset $C(\partial\mathcal{A})$ as a subposet.

We know that $C(\partial\mathcal{A})^{\text{op}}$ is representable by the face poset of the zonotope $P(\partial\mathcal{A}) = \sum_{i=1}^{k} f_i$. A simple, but surprising theorem is that $C(\mathcal{A})^{\text{op}} \supset C(\partial\mathcal{A})^{\text{op}}$ is representable by the face poset of a tiling of $P(\partial\mathcal{A})$ using the set of tiles $\{\sum_{i\in I} f_i \mid I \subset \{1, \ldots, k\}\}$, and the correspondence (extensions of $\partial\mathcal{A}$ up to an arrangement of affine pseudohyperplanes) $\longleftrightarrow$ (tilings of $P(\partial\mathcal{A})$) is one-to-one.



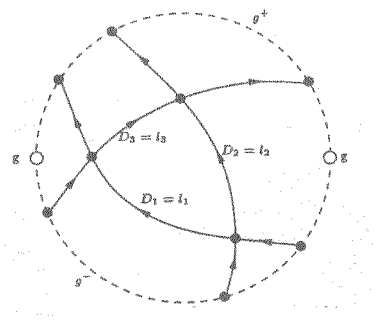
If the arrangement $\mathcal{A}$ is generic and $\partial\mathcal{A}$ is linear representable, then we obtain that $C(\mathcal{A})^{\mathrm{op}}$ is representable by a tiling of a generic zonotope by parallelepipeds.

The low-dimensional tilings of zonotopes by parallelepipeds have a special value as local models for physical quasicrystals as in [3]. (Penrose tilings are the most famous examples of 2-dimensional quasicrystals.) Here we study a special nonlinear effect which shows up in arrangements (and hence in quasicrystals) starting from dimension 3 – appearance of embedded nontrivial links, which generally can affect the dynamics of quasicrystals.

§2. Links in pseudolinear programs

We follow the terminology of Oriented Matroids book [1], where Chapter 10 is devoted to (pseudo) linear programming.

The constructions have topological nature and so, our main object is a generic arrangement of affine pseudohyperplanes. The pseudolinear n-program for us is a data $\mathcal{P} = \langle M, g\rangle$, where $M = (H_1, \ldots, H_m)$ is an arrangement of m $(n-1)$-dimensional affine pseudohyperplanes in the unit n-disk D^n. The object g is supposed to be an $(n-2)$-dimensional pseudosphere, $g \subset S^{n-1} = D^{n-1}$, which is a generic *extension* of ∂M, i.e., $(g, \partial M)$ is a correct generic $(n-1)$-dimensional arrangement of $m+1$ $(n-2)$-pseudospheres. We also suppose that on the spheres S^{n-1}, g some orientations are fixed. So, two closed (pseudo)hemispheres g^+ and g^- in the sphere S^{n-1} are defined.



Now we define an oriented topological graph $\Gamma(\mathcal{P})$, together with a canonical embedding $|\Gamma(\mathcal{P})| \hookrightarrow D^n$. For two natural numbers $a \geq b$ denote by $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ the set of multi-indices

$$\{(i_1, .., i_b) \mid 1 \leq i_1 < i_2 < .. < i_b \leq a\}.$$

Consider the set of 1-disks

$$L = \left\{ l_I = \bigcap_{i \in I} D_i \;\middle|\; I \in \begin{bmatrix} m \\ n-1 \end{bmatrix} \right\}.$$

Every $l \in L$ intersects S^{n-1} at two opposite points $p_l^{\pm} = g^{\pm} \cap l$. We orient the interval l as an arc with source p_l^- and target p_l^+.

Now consider the set of points in D^n:

$$P = \left\{ p_J = \bigcap_{i \in J} D_i \;\middle|\; J \in \begin{bmatrix} m \\ n \end{bmatrix} \right\} \cup \{p_l^-, p_l^+\}_{l \in L}$$

The points in P sit on the oriented topological intervals in L and break them into a set A of smaller arcs with induced orientation. So we obtain an oriented topological graph $\Gamma(\mathcal{P})$ in H^+ associated to the topological program $\mathcal{P}$.

Now we start to estimate the topological complexity of $\Gamma(\mathcal{P})$ when dimension of the program is $n = 3$. First define an oriented link in the 3-disk D^3. Take a free sum $\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup \cdots \sqcup \overrightarrow{S}^1_q$ of oriented topological circles. Then a *topological link* in D^3 is a topological embedding

$$F : \overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup \cdots \sqcup \overrightarrow{S}^1_q \to D^3.$$

Next, we note that the graph $\Gamma(\mathcal{P}) \hookrightarrow D^3$ of a 3-dimensional program can have cycles as an oriented graph. The cycles exist together with an embedding of the corresponding topological circles into D^3. So, if we denote by $\Omega(\mathcal{P})$ the set of cycles in $\Gamma(\mathcal{P})$, then a map is defined

$$\omega : \bigsqcup_{i \in \Omega(\mathcal{P})} \overrightarrow{S}^1_i \to D^3.$$

Usually, ω is very far from being an embedding, and, probably, always is very singular. Our claim is:

Theorem 1. *For any link*

$$F : \overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup \cdots \sqcup \overrightarrow{S}^1_q \to D^3,$$

there exists a 3-dimensional program $\mathcal{Q}_F$ and an embedding ϵ of the set of circles $\{1, \ldots, q\}$ into the set of cycles $\Omega(\mathcal{Q}_F)$ such that the composite map

$$\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup \cdots \sqcup \overrightarrow{S}^1_q \xrightarrow{\mathrm{id}_{\epsilon(1)} \sqcup \cdots \sqcup \mathrm{id}_{\epsilon(k)}} \bigsqcup_{i \in \Omega(\mathcal{Q})} \overrightarrow{S}^1_i \xrightarrow{\omega(\mathcal{Q}_F)} D^3$$

is isotopic to F. The number of the pseudospheres in P is a linear function in the complexity of the presentation of the link F by a piecewise-linear 2-dimensional diagram.

We describe a construction of $\mathcal{Q}_{\mathcal{F}}$ in §§1–3.

§3. PsL-diagrams

Here we start our constructions. The main point of the construction is a realization of *pseudolines linking diagram* (*PsL-diagram*) by a 3-program.

Consider some generic affine arrangement of *oriented* pseudolines $\mathcal{A}$, as in [1, Chapter 6]. So $\mathcal{A}$ is a collection of *oriented* topological intervals $(t_1, \dots, t_k)$ embedded into the unit 2-disk D^2. Every interval t_i transversally intersects the boundary unit sphere g at its source and target points r_j^- and r_j^+. Every two intervals $t_i, t_j, i < j$ intersect transversally at a single point $r_{i,j}$. We introduce two symbols: "bridge"$= \div$ and "pass"$= -|-$

Definition. *A PsL-diagram is pair $\mathcal{D} = \langle \mathcal{A}, \xi \rangle$, where $\mathcal{A}$ is an arrangement of k numbered oriented pseudolines and ξ is a map*

$$\xi : \left[\begin{smallmatrix} k \\ 2 \end{smallmatrix}\right] \to \{\text{bridge}, \text{pass}\}.$$

We imagine the map ξ as assigning the symbol $\xi(i,j)$ to the point $r_{i,j}$. (See [2] for the developed science on links of (projective) lines and diagrams of such links.) The points

$$\{r_{1,j}, \dots, r_{j-1,j}, r_{j,j+1}, \dots, r_{j,k}, r_j^+, r_j^-\}$$

split the oriented interval t_j into a set B_j of k oriented arcs. Thus we get a topological oriented graph $\Phi(\mathcal{D}) = (B, R)$ in D^2, where the set of arcs is $B = B_1 \cup \dots \cup B_k$ and the set of vertices is

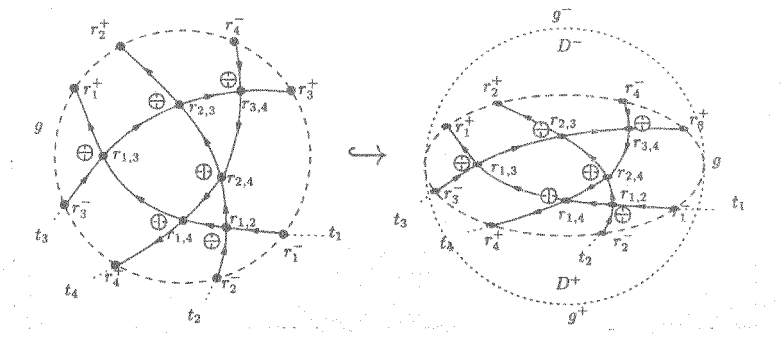
$$R = \{r_{i,j}\}_{(i,j) \in \left[\begin{smallmatrix} k \\ 2 \end{smallmatrix}\right]} \cup \{r_i^+\}_{i=1}^k \cup \{r_i^-\}_{i=1}^k.$$

Here is a picture.

Now we will use PsL-diagram $\mathcal{D}$ to construct an oriented topological graph $\Xi(\mathcal{D}) \hookrightarrow D^3$.

First, we embed by horizontal coordinate way a unit disk D^2, carrying the diagram $\mathcal{D}$, into a unit 3 ball D^3. This induces an embedding of graph $\Phi(\mathcal{P}) \hookrightarrow D^3$ as shown. Now we orient the boundary circle g in the embedding and got a positive and negative closed semispheres g^+ and

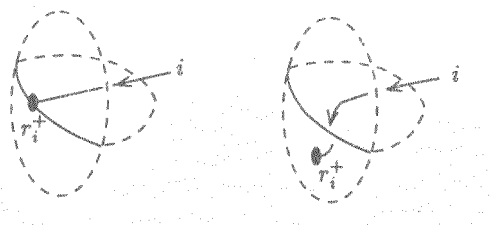
g^-,



$g^+ \cap g^- = g$ in the boundary 2-sphere of D^3. Then to the constructed embedding of $\Phi(\mathcal{P})$, we apply following rules of local surgery:

1a: Local transform around a point of type r_i^+.

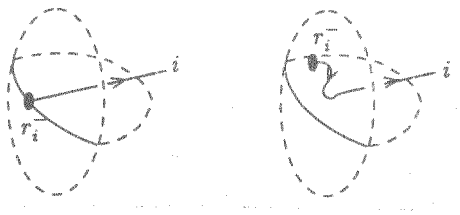
We pick a small neighborhood of r_i^+ in D^3 and slightly change in it the embedding, moving r_i^+ to the interior of the positive hemisphere g^+, as shown:



the remaining part of graph remains unchanged.

2a: Local transform around a point of type r_i^-.

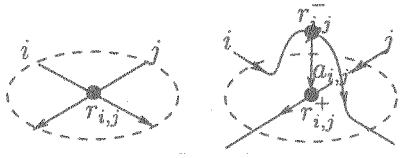
All the same, but the move sends r_j^- to the interiour of the negative hemisphere g^-



3a: Local transform around a point of type $r_{i,j}$ when $\xi(i,j) =$ "bridge" .

We pick a small neighborhood of $r_{i,j}$ in D^3. We replace the single point r_{ij} by two points $r^+_{i,j}$ and $r^-_{i,j}$, slightly moving $r^-_{i,j}$ up into open negative semiball. Then we introduce a new small oriented straight arc $a_{i,j}$ with source $r^-_{i,j}$ and target $r^-_{i,j}$. We deform the line t_i to make a "bridge" over t_j with "base" $a_{i,j}$, as shown:

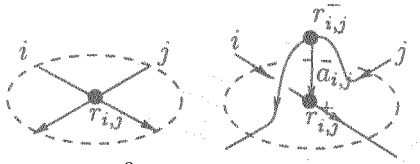
$\xi(i,j) =$ $\oplus$



4a: Local transform around a point of type $r_{i,j}$ when $\xi(i,j) =$ "pass".

All the same with i, j mutated. This creates a "pass" of t_i under t_j

$\xi(i,j) =$ $\oplus$



We define a new graph $\Xi(\mathcal{D}) \hookrightarrow D^3$ as a result of applying rules **1a–4a** to all vertices of $\Phi(\mathcal{D}) \hookrightarrow D^3$.

What remains is to do two things: to embed an arbitrary link in a graph of type $\Xi(\cdot)$ and to embed the graph $\Xi(\cdot)$ into the graph $\Gamma(\cdot)$ of some 3-program.

§4. Diagram of link $\longmapsto$ Ξ (PsL-diagram)

Let us have a link

$$F : \overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup \cdots \sqcup \overrightarrow{S}^1_q \to D^3.$$

For a PsL-diagram $\mathcal{D}$, denote by $\Omega(\mathcal{D})$ the set of cycles in the graph $\Xi(\mathcal{D})$ and we denote by $\omega_{\mathcal{D}}$ the map

$$\omega_{\mathcal{D}} : \bigsqcup_{i \in \Omega(\mathcal{D})} \overrightarrow{S}^1_i \to D^3$$

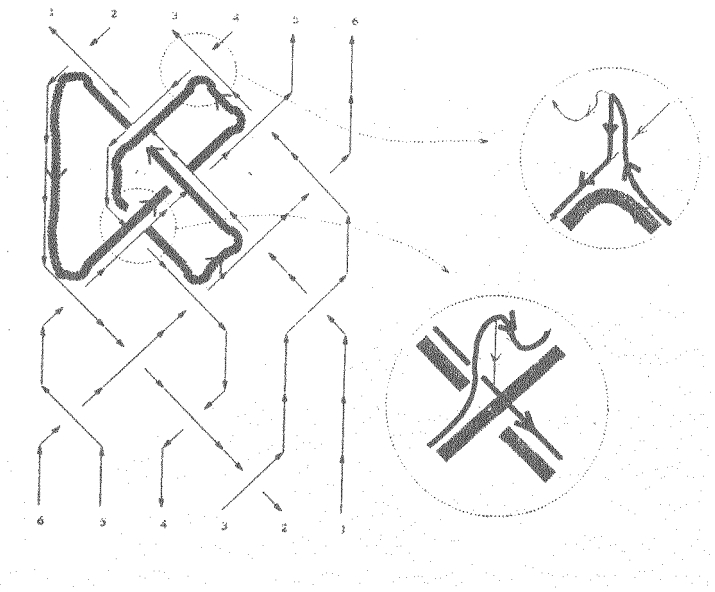
naturally defined by the embedding $\Xi(\mathcal{D}) \hookrightarrow D^3$.

Proposition 2. *For any piecewise-linear planar diagram U of the link F such that U has q linear intervals, there exists a PsL-diagram $\mathcal{D}(U)$ having q pseudolines and an embedding e of the set of circles $\{1,\dots,q\}$ into the set of cycles $\Omega(\mathcal{D}(U))$ such that composite map*

$$\overrightarrow{S}^1_1 \sqcup \overrightarrow{S}^1_2 \sqcup \cdots \sqcup \overrightarrow{S}^1_q \xrightarrow{\mathrm{id}_{e(1)} \sqcup \cdots \sqcup \mathrm{id}_{e(k)}} \bigsqcup_{i \in \Omega(\mathcal{D}(U))} \overrightarrow{S}^1_i \xrightarrow{\omega_{\mathcal{D}(U)}} \mathcal{D}^3$$

is isotopic to F.

We do not describe the construction here in detail – it is quite obvious: pick up a pl-link-diagram, generate lines by the linear intervals of diagram, orient them by orientation of the intervals induced from the orientations of components of link, and then distribute the path-bridge information as illustrated here on the example of the trefoil knot:



We present here the background arrangement of pseudolines, using "wiring diagram" as in [1, p. 260].

§5. PsL-diagram $\longmapsto$ 3-program

Now we embed $\Xi(\mathcal{D})$ into the graph $\Gamma(\mathcal{P}(\mathcal{D}))$ of some affine 3-program $\mathcal{P}(\mathcal{D})$.

Proposition 3. *For any PsL-diagram $\mathcal{D}$ having k pseudolines, there exists a 3-program $\mathcal{P}(\mathcal{D})$ with $2k$ pseudoplanes such that there is canonical embedding $\Xi(\mathcal{D}) \hookrightarrow \Gamma(\mathcal{P}(\mathcal{D}))$.*

Let us have a PsL-diagram $\mathcal{D}$. We suppose that the unit 2-disk D^2 carrying $\mathcal{D}$ is already embedded in D^3 by horizontal coordinate way. Suppose that all notation of §2 is valid. Suppose that the boundary circle $g \hookrightarrow S^2$ of $\mathcal{D}$ is oriented and this will be an "object function" of our future 3-program. Now for any pseudoline t_i of $\mathcal{D}$ we choose two 2-dimensional pseudoplanes T_i^1 and T_i^2 in D^3 such that

(1) $T_i^1 \cap T_i^2 = t_i$, and

(2) $T_i^\alpha \cap D^2 = t_i$ for $\alpha = 1, 2$ and $(T_1^1, T_2^2, \ldots, T_k^1, T_k^2)$ is a correct arrangement.

It is always possible to do so. Our program $\mathcal{P}(\mathcal{D})$ consists of the "object function" g and an arrangement of $2k$ pseudoplanes $(\tilde{T}_1^1, \tilde{T}_2^2, \ldots, \tilde{T}_k^1, \tilde{T}_k^2)$, where $\tilde{T}_i^\alpha$ is a result of a small deformation of T_i^α by the following local rules:

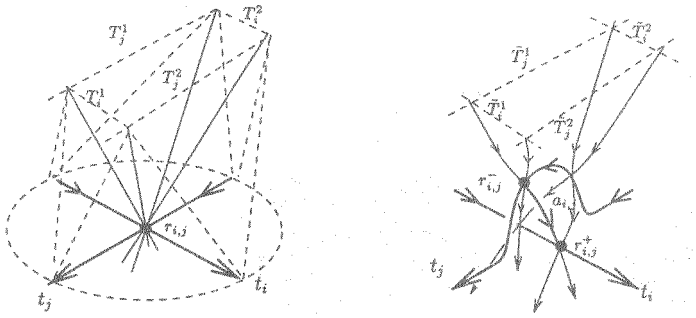
3b, 4b: Working around the interior vertices of type $r_{i,j}$, when $\xi(i,j) =$ "bridge" or "pass". Consider the case of "pass" .

Pick up a small neighborhood V of $r_{i,j}$. We have:

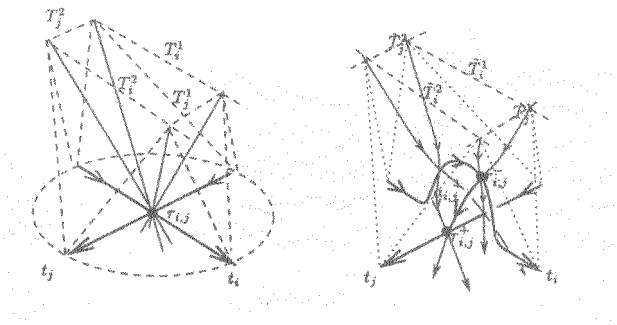
$$r_{i,j} = t_i \cap t_j = (T_i^1 \cap T_i^2) \cap (T_i^1 \cap T_j^2).$$

By continuous isotopy $f : [0,1] \times D^3 \to D^3$ such that $f|_{[0,1]\times(D^3 \setminus V)} \equiv \mathrm{id}$ we slightly shift up a pair of pseudoplanes $\{T_j^1, T_j^2\}$ into $\{\tilde{T}_j^1, \tilde{T}_j^2\}$ which is in general position with $\{T_i^1, T_i^1\}$ and such that $(\tilde{T}_j^1, \tilde{T}_j^2, T_i^1, T_i^2)$ is a correct arrangement. The condition "not to leave the class of correct arrangements" poses a constraint on the continuous deformation of the pseudoplanes of the arrangement. The important thing is that these constrains are topologically local. Hence, together with all other pseudoplanes, the deformed ones forms a correct arrangement. After such a surgery, the point r_{ij} blows up to a simplex. We have some freedom in the choice of the points $r_{i,j}^\pm$ and the arc $a_{i,j}$ from the elements of this

simplex. I hope that all is clear from the picture:



We have a symmetric picture for the transform in the case where $\xi(i,j)$ ="bridge":



1b, 2b: Working around the endpoints r_i^+ and r_i^-.

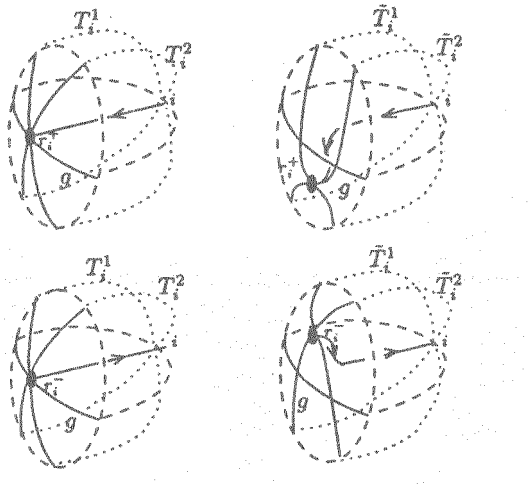
The oriented pseudoline t_i of the PsL-diagram $\mathcal{D}$ has two endpoints marked r_i^- – "source" and r_i^+ – "target". We have presented t_i as the intersection of two pseudoplanes T_i^1 and T_i^2. Also, the points r_i^- and r_i^+ belong to the boundary circle g. Let us pick a small neighborhood V of r_i^- in D^3. The arrangement of the pseudocircles

$$T_1^1 \cap S^2,\ T_1^2 \cap S^2, \ldots, T_k^1 \cap S^2,\ T_K^2 \cap S^2$$

is a correct centrally-symmetric arrangement. So, $-V$ is a neighborhood of r_i^+. By continuous centrally-symmetric isotopy

$$f\colon [0,1] \times D^3 \to D^3$$

such that $f|_{[0,1]\times(D^3\setminus(V\cup -V))} \equiv \mathrm{id}$ we can slightly deform the pair of pseudoplanes $\{T_i^1, T_i^2\}$ into $\{\tilde{T}_i^1, \tilde{T}_i^2\}$ so that r_i^- moves to the interior of g^-, r_i^+ moves to the interior of g^+, and $(\tilde{T}_i^1, \tilde{T}_i^2)$ together with the remaining pseudoplanes and g forms a correct 3-program. Here are the drawings.



For $i = \mathbf{1,2,3,4}$, each rule $i\mathbf{b}$ induces the rule $i\mathbf{a}$ of cooking the graph $\Xi(\mathcal{D})$ from the PsL-diagram $\mathcal{D}$ together with the embedding of this graph into the graph Γ of the resulting 3-program $\mathcal{P}(\mathcal{D})$. So, we are done.

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