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CAT(κ)-SPACES: CONSTRUCTION AND CONCENTRATION

To Victor Abramovich Zalgaller

We expose spaces X with negative curvature having in mind applications to *fractally hyperbolic groups*, such as *random groups* and *infinite Burnside* groups. Originally these spaces were introduced by Alexandrov in the axiomatization spirit and a similar class (of *convex* spaces) was later isolated by Busemann.

Till relatively recently the major thrust of geometric research was laid on suppressing singularities, emphasizing the properties equally shared by smooth and singular spaces and proving *regularization theorems* claiming, under certain assumptions, that X can be approximated by *smooth* manifolds with curvature $K \leqslant 0$. This was accomplished for surfaces in a famous treatise by Alexandrov and Zalgaller.

But the bulk of spaces with $K \leqslant 0$ is badly singular, starting from trees and most abundant among 2-polyhedra. Furthermore, almost all "natural" spaces with $K \leqslant 0$, such as the *Bruhat–Tits buildings*, are non-smooth and (unlike trees) cannot be usually approximated by smooth spaces. But geometers remained unaware of this for a stretch of time.

From another angle, the idea of negative curvature was injected into the group theory by Dehn and grew up into the *small cancellation theory*. In the course of the development, the geometric roots were forgotten and the role of curvature was reduced to a metaphor. (Algebraists do not trust geometry.)

It eventually turned out that the geometric language of Dehn and Alexandrov (sometimes slightly modified and/or generalized) accomplishes many needs of combinatorial group theory more efficiently than the combinatorial language.

Summing up, geometry furnishes a proper language, while the combinatorial group theory (especially random groups) provides a pool of objects for a meaningful usage of this language.

In this paper we present basic constructions of spaces X with $K \leqslant 0$ relevant for applications in group theory (see [8]) as well as basic *isoperimetric concentration* properties of maps of *metric measure* spaces (see [14] and [11]) into X. We observe, for example, that conical singularities based on *expanders* (with $K \leqslant 1$) cannot be smoothed, not even with the most generous notion of smoothing. (This will be brought into the group theoretic framework in [8].)

We furnish all necessary definitions and illustrate them by examples but refer to the textbooks for the details of standard arguments (see [2] and references therein).

§1. Metrics and geodesics

Given a metric space $X = (X, \text{dist})$ we often abbreviate and write

$$|x - y| = |x - y|_X = \text{dist}(x, y).$$

We call X a *geodesic* space if every two points x and y in X can be joined, albeit non-uniquely, by a *shortest* (*geodesic*) *segment* denoted $[x, y] \subset X$, that is an isometric embedding of a real segment of length $= \text{dist}_X(x, y)$ into X.

Actually, the existence of such a shortest, or *minimizing*, segment is not so crucial: it is enough for most purposes to have $\text{dist}(x, y)$ equal to the infimum of the length of paths in X joining x and y, where this infimum does not have to be achieved.

Also, one could use the *middle point condition*: the existence of $z \in X$ such that

$$\text{dist}(x, z) = \text{dist}(z, y) = \frac{1}{2}\text{dist}(x, y).$$

For *complete* metric spaces the last condition is equivalent to existence of a minimizing segment.

Sometimes one could require even less, the existence of $z = z_\varepsilon$ for each $\varepsilon > 0$, such that both distances $\text{dist}(x, z)$ and $\text{dist}(z, y)$ are $\leqslant \frac{1}{2}\text{dist}(x, y) + \varepsilon$.

From now on we assume the existence of our segments $[x, y] \subset X$ when we deal with geodesic spaces.

§2. Basic examples

(a) Every metrically complete connected Riemannian manifold X, possibly with a boundary, is path metric in an obvious way (where the minimizing segments may touch the boundary). In particular, every smooth connected domain $X \subset \mathbb{R}^n$ carries the *induced path metric* dist_X which is greater than the *restricted* metric $\mathrm{dist}_{\mathbb{R}^n}\big|_X$ and where the equality $\mathrm{dist}_X = \mathrm{dist}_{\mathbb{R}^n}\big|_X$ holds if and only if X is convex.

(b) Let X be a simplicial polyhedron. If we identify each simplex in X with a unit Euclidean simplex, we can speak of the length of a curve in X using the Euclidean geometry in all $\Delta \subset X$. Then we define dist_X by taking infima of length of paths between points x and y in X. This is a true geodesic metric for locally finite polyhedra where the infimum is achieved by some $[x, y] \in X$, while more general polyhedra sometimes need a completion in order to become geodesic in the strict sense.

The simplest polyhedra X are the 1-dimensional ones, i.e., *graphs*, where the above metric amounts to assigning unit length to all edges. Of course, one could live with edges of variable lengths, but when the dimension goes up, one should be careful if one assigns variable sizes and shapes to simplices in X as these must agree across common k-faces with $k \geqslant 1$.

(b$'$) It is often necessary to assign non-Euclidean geometries to simplices in X, e.g., by identifying each $\Delta \subset X$ with a regular *spherical* or *hyperbolic* simplex of a certain size. The resulting, e.g., piecewise spherical and piecewise hyperbolic, geodesic metric in X may reveal some combinatorial properties of X invisible in the (piece-wise) Euclidean light.

§3. Model-spaces

The *standard* or *model* spaces of *constant curvature* are

(i) The round 2-sphere of radius R, denoted $S^2(R)$. This has (by definition, if you wish) curvature $K(S^2(R)) = R^{-2}$.

(ii) The Euclidean plane $\mathbb{R}^2$, where $K(\mathbb{R}^2) = 0$.

(iii) The hyperbolic plane H_κ with curvature $-\kappa^2$. This H_κ can be represented as the plane with coordinates (t, y) and the Riemannian metric $dt^2 + \varepsilon^{2\sqrt{-\kappa}t} dy^2$. The *$t$-lines* here are *geodesic*, i.e., the embeddings $\mathbb{R} \to (\mathbb{R}, y) \subset H_\kappa$ are isometric for all $y \in \mathbb{R}$. On the other hand the *y-lines* $(t, \mathbb{R})$ are curved in H_κ and they shrink exponentially fast as t increases.

(iv) If $\kappa \to \infty$, then H_κ converges in a natural way (see [4]) to an infinite metric tree branching at all points. This serves as the model space for $\kappa = -\infty$.

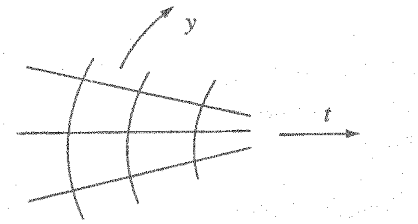

Fig. 1.

Similarly to (i)–(iii) we have n-dimensional spaces with curvature κ, that are $S^n(R)$, $\mathbb{R}^n$, and H^n_κ (with the metric

$$dt^2 + e^{2\sqrt{-\kappa}t} \sum_{i=2}^{n-1} (dy_i)^2),$$

denoted in the unified manner by $X^n_{\mathrm{mod}}(\kappa)$ for all κ (including $\kappa = \pm\infty$), where $X^n_{\mathrm{mod}}(+\infty)$ is the single point space and $X^n_{\mathrm{mod}}(-\infty)$ is the above tree for all $n = 2, 3, \dots, \infty$.

§4. Comparison relation between the model spaces

Let $x_1, \dots, x_k$, $x_{k+1} = x_1$ be a cyclically ordered k-tuple of points in $X = X^n_{\mathrm{mod}}(\kappa)$ for some $\kappa \leqslant 0$. *Then for every κ' in the interval $[0, \kappa]$ there exist points $x'_i \in X' = X^2_{\mathrm{mod}}(\kappa')$, $i = 1, \dots, k$, $k + 1 = 1$, such that*

$$\left|x'_i - x'_j\right|_X \geqslant |x_i - x_j|_X \quad \textit{for all} \;\; i,\; j = 1, \dots, k$$

and

$$\left|x'_i - x'_{i+1}\right|_{X'} = |x_i - x_{i+1}|_X \quad \textit{for} \;\; i = 1, \dots, k.$$

This is standard and elementary, where one chooses x'_i making a convex k-gon in the plane $H^2_{\kappa'}$ (which equals $\mathbb{R}^2$ for $\kappa = 0$). Notice that the above extends to $\kappa \geqslant 0$ if the points x_i are contained in a sufficiently small ball in X.

§5. Positivity relations

Recall that a symmetric matrix d_{ij}, $i, j = 1, \dots, h$, can be realized by the distances between k points x_i in $\mathbb{R}^{n \geqslant k}$ if and only if the quadratic form $\Phi\{t_i\} = -\sum\limits_{i,j=1}^{k} d_{ij}^2\, t_i t_j$ is positive *definite* on the hyperplane $H_0 = \left\{\sum\limits_{i=1}^{k} t_i = 0\right\} \subset \mathbb{R}^k$.

Observe that this imposes infinitely many *linear* inequalities on the numbers d_{ij}^2.

If we interpret Φ as the integral of $d_{ij} = \|x_i - x_j\|^2$ over $\mathbb{R}^n \times \mathbb{R}^n$ with the weights $t_i t_j$, then the positivity of Φ generalizes as follows.

Let μ and ν be probability measures on $\mathbb{R}^n$, then

$$\Phi_0(\mu,\nu) \stackrel{\text{def}}{=} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2\, d\mu d\nu -$$

$$\frac{1}{2}\left(\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2\, d\mu d\mu + \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2\, d\nu d\nu\right) \geqslant 0.$$

In fact, $\Phi(\mu,\nu)$ obviously equals the squared distance between the centers of mass of the measures,

$$\Phi(\mu,\nu) = \left(\int\limits_{\mathbb{R}^n} x d\mu - \int\limits_{\mathbb{R}^n} y d\nu\right)^2,$$

as a straightforward computation shows.

Example. Given $x_1, \ldots, x_\ell$, $y_1, \ldots, y_\ell$ in $\mathbb{R}^N$, then the average of the squared distances $\|x_i - x_j\|^2$ and $\|y_i - y_j\|^2$ is bounded by the average of $\|x_i - y_j\|^2$ as follows

$$\frac{1}{\ell(\ell-1)}\left(\sum_{i<j}\|x_i - x_j\|^2 + \sum_{i<j}\|y_i - y_j\|^2\right) \leqslant \frac{\lambda}{\ell^2}\sum_{i,j=1}^{\ell}\|x_i - y_j\|^2, \quad (\boxtimes_\ell)$$

for $\lambda = \frac{\ell}{\ell-1}$.

The form $\Phi_0(\mu,\nu)$ makes sense for an arbitrary metric space X with $\|x-y\|^2$ replaced by $|x-y|_X^2$. Clearly, positivity of Φ_0 for all probability measures on X is necessary and sufficient for the existence of an isometric embedding of X into a Hilbert space.

Similarly, one can characterize the spaces X embeddable into spaces of radius $R = 1/\sqrt{\kappa}$, $\kappa > 0$, by looking at the *R-cone* $Y_\kappa = \mathrm{Con}_R X \supset X$, where

$$|x-y|_\kappa \overset{\text{def}}{=} \begin{cases} 2R\sin\dfrac{|x-y|_X}{2R} & \text{for } |x-y|_X \leqslant \pi R \\ 2R & \text{for } |x-y|_X \geqslant \pi R \end{cases}$$

for all $x, y \in X \subset Y_\kappa$. Here the relevant form Φ_κ equals

$$\Phi_\kappa(t_1, \dots, t_k) = \sum_{i,j=1,\dots,k} \left(R^2 - \frac{1}{2}|x_i - x_j|_\kappa^2\right) t_i t_j$$

for $x_i \in X \subset Y_\kappa$ and it is positive if X is embeddable into the Hilbertian R-sphere.

If $\kappa < 0$, then

$$|x-y|_\kappa \overset{\text{def}}{=} 2R\sinh\frac{|x-y|_X}{2R}$$

for $R = 1/\sqrt{-\kappa}$ and

$$\Phi_\kappa(t_1, \dots, t_k) = \sum_{i,j=1}^{k} -\left(R^2 + \frac{1}{2}|x_i - x_j|_\kappa^2\right) t_i t_j.$$

The space X embeds into a hyperbolic space of curvature $\kappa < 0$ iff the form Φ_κ has at most *one* negative square (in the diagonalizing basis).

If $\kappa = -\infty$ and $k = 4$, one considers three numbers

$$m_1 = d_{1,2} + d_{3,4}, \quad m_2 = d_{2,3} + d_{1,4}, \quad \text{and} \quad m_3 = d_{3,1} + d_{2,4},$$

and sets $m_+ = \max_{i=1,2,3} m_i$ and $m_- = \min_{i=1,2,3} m_i$. Then $d_{i,j}$ are resizable by distances in $X_{\text{mod}}(-\infty)$ iff

$$3m_+ - \sum_{i=1}^{3} m_i = m_+ - m_-,$$

i.e., iff the second maximal among m_i equals m_+. Furthermore, if every quadruple of points in a (finite) metric space $\{x_i\}_{i=1,\dots,k}$ has this property, then $\{x_i\}$ isometrically embeds into the tree $X_{\text{mod}}(-\infty)$.

All of the above is well known and pretty obvious. But there are amusing corollaries.

§6. Wirtinger inequalities

Consider cyclically ordered points

$$x_i \in \mathbb{R}^n, \ i = 1, \dots, k, \ k+1 = 1,$$

let $W_j\{x_i\} = \sum\limits_{i=1}^{k} |x_i - x_{i+j}|^2$, and set $W_j(k)$ to be the value of W_j on the regular k-gon in $\mathbb{R}^2$ inscribed into the unit circle. *Then*

$$W_1\{x_i\}/W_j\{x_i\} \geqslant W_1(k)/W_j(k) \qquad (*_j^k)$$

for all $\{x_1, \dots, x_k\} \subset \mathbb{R}^n$.

Proof. According to Fourier (on the group $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$) one needs to check $(*)_j^k$ only for $\mathbb{Z}_k$-equivariant maps $\{x_i\} = \mathbb{Z}_k \to \mathbb{R}^2$, where this is obvious. Q.E.D.

Remarks and corollaries. (a) The Wirtinger inequality for four points is equivalent to the above $(\boxtimes_\ell)$ for $\ell = 2$. Furthermore, each $(*)_j^k$ can be algebraically derived from $(\boxtimes_\ell)$ for some $\ell = \ell(k)$, but a direct derivation is rather messy starting from $k = 5$. In fact, the negative definiteness of the distance matrices $\{d_{ij}^2\}$ in $\mathbb{R}^n$ (see §5) harbors infinitely many linear inequalities non-reducible to anything like Wir_1 and their linear combinations. One exhibits particular inequalities by looking at specific arrangements of points in $\mathbb{R}^n$, e.g., coming from *expanding graphs* and related combinatorial structures.

(b) There are further relations between W_j's. In fact, in order for an inequality

$$\sum_j c_j W_j\{x_i\} \geqslant 0$$

with given $c_i \in \mathbb{R}$, $i = 1, \dots, k-1$, to hold true for *all* k-tuples in $\mathbb{R}^n$, one only needs this for *equivariant* tuples in $\mathbb{R}^2$. For example, if $k = 6$, then $c_1W_1 + c_2W_2 + c_3W_3 \geqslant 0$ provided this holds for the following three 3-tuples of numbers: $(1,3,2)$, $(1,1,0)$, and $(1,0,1)$.

(c) Since the spheres $S^n(R) = X^n_{\mathrm{mod}}(\kappa)$, $\kappa = R^{-2}$, embed into $\mathbb{R}^{n+1}$ with the spherical distance d going to $2R \sin\frac{d}{2R}$ in $\mathbb{R}^{n+1}$, the Wirtinger inequality holds for $\{x_i\} \subset S^n(R)$ with W_j made of

$$a_{ij} = a_{ij}(\kappa) = \sin\frac{d_{ij}}{2R}, \quad i, j = 1, \dots, k,$$

instead of d_{ij}. The extremal configurations all lie on circles in $S^n(R)$ and they make regular k-gons if k is not divisible by 2 and 3.

(c′) The hyperbolic space H^n_κ is realized by the (halves of the) sphere of radius $R = -1/\sqrt{\kappa}$ in the Lorentz space $(\mathbb{R}^{n+1}, \sum\limits_{i=1}^{n} y_i^2 - y_0^2)$ where the

Lorentz "distance" equals $2R\sinh\frac{d}{2R}$ for the hyperbolic distance d. One cannot apply the Euclidean Wirtinger as our $\mathbb{R}^{n+1}$ is not Euclidean, but one can project points $x_i \in H^n_c \subset \mathbb{R}^{n+1}$ to the hyperplane H normal to $\sum\limits_i x_i = 0$ and apply the Wirtinger inequality to the projections $\overline{x}_i \in H$. The distances $\overline{d}_{ij} = |\overline{x}_i - \overline{x}_j|$ are expressible in terms of the hyperbolic $d_{ij} = |x_i - x_j|$. This gives us Wirtinger-type inequalities between d_{ij} which are sharp for regular k-gons in H^n_κ but not especially elegant.

(d) One can combine the quadratic (i.e., Euclidean) Wirtinger inequalities with the comparison inequality for $0 = \kappa' \geqslant \kappa < 0$ and conclude to the quadratic inequalities for H^n_κ. Unfortunately these are not sharp, except for some k-tuples of points lying on geodesic lines in H^n_κ.

(e) The Wirtinger inequalities extend to an arbitrary finite group G in place of $\mathbb{Z}_k$. Here, for a real function c on G and a map $f\colon G \to X$, we set

$$W_c(f) = \sum_{h\in G} c(h) \sum_{g\in G} |f(g) - f(gh)|^2_X$$

and we ask for which c and X *every* map f has $W_c(f) \geqslant Q$ (where one may try more sophisticated non-quadratic expressions). Here again we easily see for $X = \mathbb{R}^n$ that $W_c(f) \geqslant 0$ for *all* f iff it is $\geqslant 0$ for all *equivariant* maps $G \to \mathbb{R}^N$, i.e., orbits of the (irreducible, if you wish) orthogonal representations of G. The same equally applies to all compact groups G and Borel functions c (better measures) and maps f with the sums replaced by the corresponding integrals. In particular, we may take $G = S^1$, where all this can be derived from the case of $\mathbb{Z}_k$ for the obvious approximation $\mathbb{Z}_k \underset{k\to\infty}{\longrightarrow} S^1$, which allows $X = H^n_\kappa$ in the picture. Also observe that averaging Wirtingers inequalities $(*)_j$ over j and then sending $k \to \infty$ gives us the traditional Wirtinger inequality: *every smooth map* $f\colon S^1 \to X = H^n_\kappa$ *satisfies*

$$\frac{1}{2}\iint |f(s_1) - f(s_2)|^2\, ds_1 ds_2 \leqslant 2\pi \int \|df\|^2 ds.$$

This for $X = \mathbb{R}$ amounts to the evaluation of the first eigenvalue (of the Laplace operator) of S^1,

$$\lambda_1(S^1) = 1.$$

(f) One can generalize further and take a measure space H with a measure preserving action of G, where one studies weighted integrals of $|f(h) - f(gh)|^2_X$ for maps $f\colon H \to X$ and where Wirtinger inequalities for H can be derived from those for G. Here one can allow non-compact groups G, especially Kazhdan's T-groups. Also, one may look at particular H, such as the unit tangent bundle of Riemannian manifold V with the action of the geodesic flow in this case this flow is periodic. Thus the Wirtinger inequality for all *compact symmetric spaces* of *rank one* mapped into general CAT(0) and more general spaces X follows from the classical Wirtinger for $S^1 \to \mathbb{R}$. Another example of such inequality is (ℓ) in §5, where $H = \{x_i, y_i\}$ and G is generated by the permutations of x_i's of y_i's and the involution $x_i \leftrightarrow y_i$.

§7. Cycl$_k(\kappa)$ and Wir$_k$-spaces

A metric space is called Cycl$_k(\kappa)$ if, for each cyclically ordered k-tuple of points $x_i \in X$, $i = 1, \dots, k$, there exist comparison points $x_i' \in X' = X^2_{\mathrm{mod}}(\kappa')$ for some $\kappa' \leqslant \kappa$, such that

$$\left|x_i' - x_{i+1}'\right|_{X'} \leqslant |x_i - x_{i+1}|_X \,, \ i = 1, \dots, k,$$

and

$$\left|x_i' - x_j'\right|_{X'} \geqslant |x_i - x_j|_X \ \text{ for all and } j \neq i+1 \ .$$

(This definition is well suited for $\kappa \leqslant 0$, while the case $\kappa > 0$ needs a modification where the existence of comparison points is required only for "small" k-tuples $\{x_i\} \subset X$.)

The most important case is that of $k = 4$, where the existence of a comparison quadruple $\{x_i'\}$ implies (at least for $\kappa' \leqslant 0$) the existence of $\{x_i''\} \subset X'' = X^3_{\mathrm{mod}}(\kappa'')$ with $\kappa'' \leqslant \kappa' \leqslant \kappa$, such that

$$\left|x_i'' - x_j''\right|_{X''} = |x_i - x_j|_X \ \text{ for all } i, \ j = 1, \dots, 4,$$

as an elementary argument shows.

Remark. The Cycl$_4$(0)-property can be expressed by a family of linear inequalities between $|x_i - x_j|^2$. Namely, X is Cycl$_4$(0) iff the squared distance function $|x - y|^2$ satisfies $(\boxtimes_\ell)$ from §5 for every pair of *two-point* probability measures μ and ν on X. (A measure is called *two-point* if its support contains at most two points.) This can be checked by a direct computation and will be proven later on without computation.

Next we introduce Wir$_k$-spaces where, by definition, every k-tuple $\{x_i\} \subset X$, $i = 1, \dots, k$, satisfies the Wirtinger inequalities $(*)_j$, $j = 1, \dots, k-1$, i.e.,

$$\sum_i |x_i - x_{i+1}|^2_X \Big/ \sum_i |x_i - x_{i+j}|^2_X \geqslant W_1^\circ / W_j^\circ ,$$

where W_k° denote the corresponding sums for the regular k-gons in $\mathbb{R}^2$ (see 1.2.C). According to our discussion in 6

$$\mathrm{Cycl}_k(0) \Rightarrow \mathrm{Wir}_k \quad \text{for all } k = 4, 5, \dots .$$

§8. Geometric application

Let V be a compact symmetric space of rank 1 (e.g., the n-sphere) and let $T_\varepsilon \subset V \times V$ denote the subset of points (v_1, v_2) with $|v_1 - v_2|_V = \varepsilon$. Since V is two point homogeneous, the isometry group of V is transitive on T_ε and we give the *normalized*, with the total mass one, *invariant (Haar) measure* μ_ε to T_ε. Let the diameter D of V satisfy $D = \frac{1}{2}k\varepsilon$ for some *integer* $k = 2, 3, \dots$, and let $\rho \leqslant \mathrm{diam}V$ be of the form $\rho = j\varepsilon$ for some $j = 1, 2, \dots$.

If X is Wir_k*, then every (say bounded) Borel map $f : V \to X$ satisfies*

$$E_\varepsilon \overset{\text{def}}{=} \iint\limits_{T_\varepsilon} |f(v_1) - f(v_2)|_X^2 \, d\mu_\varepsilon \geqslant \lambda_{\varepsilon\rho} \iint\limits_{T_\rho} |f(v_1) - f(v_2)|_X^2 \, d\mu_\rho,$$

where $\lambda_{\varepsilon\rho} = W_1(k)/W_j(k)$.

This is seen by integrating the Wir_k-inequality over the orbits of $\mathbb{Z}_k \subset S^1$ in the unit tangent bundle $S(V)$ for $S^1 = \mathbb{R}/\mathbb{Z}$ acting on $S(V)$ by the geodesic flow.

Observe that $E_\varepsilon/\varepsilon^2$ converges, for $\varepsilon \to 0$, to the average squared partial derivation of f for smooth f,

$$E_\varepsilon/\varepsilon \to E(f) \overset{\text{def}}{=} \int\limits_S \|\partial_s f\|^2 \, ds,$$

where the measures in the spherical fibers of the unit tangent bundle $S = S(V)$ are normalized to mass one as well as the Riemannian measure on V. On the other hand, a suitably weighted sum of $E_{j\varepsilon}$, i.e., $\sum\limits_{j=1}^{k} p_j E_{j\varepsilon}$ where kp_j equals the reciprocal of the Jacobian of the exponential map $\mathbb{R}^{\dim V} = T_v(V) \to V$, on the sphere of radius $j\varepsilon$ in $\mathbb{R}^{\dim V}$, converges to the mean (average) $A(f)$ of the *squared oscillation* $|f(v_1) - f(v_2)|_X^2$ over $V \times V$. Thus, if X is Wir_∞, i.e., Wir_k for all k, then $E(f)$ bounds $A(f)$ by

$$2E(f) \geqslant \lambda^{-1} A(f)$$

where $\lambda = \lambda(V) = \frac{1}{2}A(f_0)/E(f_0)$ for the (isometric!) *Veronese* embedding of V to $\mathbb{R}^N$, $N = N(V)$, (where $N(S^n) = n + 1$, $N(P^n) =$

$\frac{(n+1)(n+2)}{2} - 1$, etc.). If $X = \mathbb{R}$, the above (+) boils down to the familiar evaluation of the first eigenvalue λ_1 of (the Laplace operator on) V,

$$\lambda_1(V) = (\dim V)\lambda(V). \qquad (+)$$

The $\dim V$-factor is due to the fact that the average of the square of a linear function $\partial\colon \mathbb{R}^n \to \mathbb{R}$ over $S^{n-1} \subset \mathbb{R}^n$ equals $n^{-1}\|\partial\|^2$ and, consequently,

$$\int_V \|\mathrm{grad} f\|^2\, dv = E(f)\,\mathrm{Vol}\, V/\dim V$$

for all Riemannian manifolds V and functions $f\colon V \to \mathbb{R}$. This yields (+) since

$$\iint_{V\times V} \|f(v_1) - f(v_2)\|^2\, dv_1 dv_2 = 2\,\mathrm{Vol}\, V \int_V \|f(v)\|^2\, dv$$

for all $\mathbb{R}^n$-valued f with zero mean, $\int_V f(v)dv = 0$. (This explains why we brought up this "2" earlier.)

§9. Remarks

(a) Observe that

$$\lambda(V) \geqslant (\mathrm{Diam}\,(\mathrm{Veronese}(V)))^{-2} \geqslant \frac{\pi^2}{4}(\mathrm{Diam}\, V)^{-2}.$$

This makes $\lambda_1(V) \approx \dim V$ and implies "high concentration" of functions f on V for large $\dim V$. Such concentration persists for maps $f\colon V \to X$ where X is a Riemannian Wir_∞-space of relatively small dimension $\dim X = \delta \dim V$. Namely,

$$\mathrm{Vol}\,(V) \int_V \|df\|^2\, dv \geqslant \delta^{-1}\lambda(V) \iint_{V\times V} |f(v_1) - f(v_2)|_X^2\, dv_1 dv_2 \qquad (\star)$$

(compare Ch. $3\frac{1}{2}$ in [11]).

(b) The role of the Wir_∞-property is minor in the above discussion: it is needed only for identifying the *explicit* value of $\lambda(V)$. For example, the inequality ($\star$) holds true for *all* (non-Wirtinger) Riemannian manifolds X with the constant $\lambda(V)$ replaced by a slightly smaller number

$$\lambda_{-(V)} = (\,\mathrm{Vol}\, V)^2 \Big/ \iint_{V\times V} |v_1 - v_2|_V^2\, dv_1 dv_2,$$

where the proof is identical to (actually easier than) that of $(*)$ and where it applies to all *piecewise* smooth Riemannian X. In particular, every 1-*Lipschitz* (i.e., distance decreasing) map $f\colon V \to X$ satisfies

$$\iint\limits_{V\times V} |f(v_1) - f(v_2)|_X^2 \, dv_1 dv_2 \leqslant \delta \iint\limits_{V\times V} |v_1 - v_2|_V^2 \, dv_1 dv_2 \qquad (\star\star)$$

for $\delta = \dim X / \dim V$, that signifies "high concentration" of f for small δ.

One can generalize further by allowing (arbitrarily singular) non-Riemannian metric spaces X, where $(\star\star)$ can be sharpened for Banach spaces X that are far from being Euclidean. One can identify extremal X and $f\colon V \to X$. One can bring in symmetric spaces X of $\mathrm{rank}_{\mathbb{R}} X \geqslant 2$ and more general Riemannian (and non-Riemannian) V with distinguished families $W_\sigma \subset V$, $\sigma \in \Sigma$, of subvarieties, such as flat tori in symmetric V and minimal geodesic segments in V with Ricci $V \geqslant \rho$ (where one may allow singular V and sometimes use the Brownian orbits for W_σ). Eventually many results on concentration of functions, e.g., various Sobolev and isoperimetric inequalities, extend to maps into rather general spaces X. For example, Levi's concentration generalizes to the following

Theorem. *Let $f\colon S^n \to \mathbb{R}^{n-m}$, $m \geqslant 0$, be an arbitrary continuous map. Then there exists a point $x \in \mathbb{R}^{n-m}$ such that the pull-back $S_x = f^{-1}(x) \subset S^n$ is larger than an equatorial sphere $S^m \subset S^n$ in the following sense: the volumes of the ε-neighborhoods of the two subsets satisfy*

$$\mathrm{Vol}\, U_\varepsilon(S_x) \geqslant \mathrm{Vol}\, U_\varepsilon(S^m) \qquad (*)$$

for all $\varepsilon \geqslant 0$.

(More generally, one considers pairs of maps, $f\colon \Sigma \to X$ and $\varphi\colon \Sigma \to V$, and seeks $x \in X$ such that the image $S_x = \varphi(f^{-1}(x)) \subset V$ has a large ε-neighborhood. For example, if Σ is a closed manifold of dimension n, X is a manifold of dimension $n-m$, the map f is contractible, φ has non-zero degree mod2, then, in the case $V = S^n$, there is an $x \in X$ such that $\mathrm{Vol}\, U_\varepsilon(S_x) \geqslant \mathrm{Vol}\, U_\varepsilon(S^m)$ for all $\varepsilon \geqslant 0$.) We shall prove this in [9] by constructing a suitable convex partition of S^n "transversal" to the fibers of $f\colon S^n \to \mathbb{R}^{n-m}$ (compare [12] and [11]).

(c) The role of negative curvature of X in the concentration of maps $f\colon V \to X$ becomes more pronounced if we look at maps f with large Lipschitz constants (or, alternatively, scale X with small $\varepsilon > 0$). For example, if $K(X) \leqslant -\kappa < 0$ (or, hyperbolic, in general), then it is approximately *one-dimensional* at infinity with a logarithmic error; thus maps f to X concentrate as much as maps to trees, up to a logarithmic error. Similarly, maps to non-compact symmetric spaces of rank k (and to buildings) concentrate, up to certain error, as do maps to k-dimensional spaces.

§10

It is well known that $\lambda_1(V \times V') = \min(\lambda_1(V),\ \lambda_1(V'))$ for products of Riemannian manifolds. This extends to maps to arbitrary Wir_4-spaces X assuming there is almost everywhere defined $\int\limits_{V \times V'} |\mathrm{grad} f|^2\, dv$ for our maps f and the squared gradient of an f on $V \times V'$ equals the sum of the two squared fiberwise gradients along the V- and V'-fibers (as obviously holds true, for instance, for smooth V's, X's, and f's). On the other hand, if X is Wir_4, then every map

$$f\colon W = V \times V' \to X$$

satisfies

$$\begin{aligned} &\|f(v_1, v_1') - f(v_2, v_2')\|^2 + \|f(v_1, v_2') - f(v_2, v_1')\|^2 \leqslant \\ &\|f(v_1, v_1') - f(v_1, v_2')\| + \|f(v_1, v_1') - f(v_2, v_1')\|^2 + \\ &\|f(v_2, v_2') - f(v_2, v_1)\|^2 + \|f(v_2, v_2') - f(v_1, v_2)\|^2 \end{aligned}$$

(see Figure below),

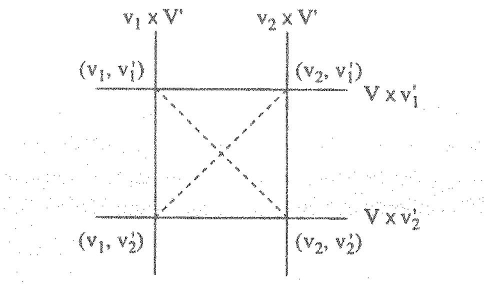

Fig. 2.

which integrates to the bound of $\iint\limits_{W \times W} \|f(v_1, v_1') - f(v_2, v_2')\|^2$ by the corresponding fiberwise integrals, since our measure on the product

$V \times V' \times V \times V'$ is symmetric under the permutations of the components.

Hence the bounds

$$\int \|\mathrm{grad} f\|^2 \geqslant \lambda_1 \frac{1}{2} \iint |f(v_1) - f(v_2)|_X^2$$

for all maps f of both V and V' to X imply such a bound with the same λ_1 for the maps $V \times V' \to X$.

Remarks. (a) The above argument is similar to one used by S. Bobkov in his thesis for functions on metric probability spaces.

(b) One may use two different metrics on X, one for evaluation of $\|\mathrm{grad}\|^2$ and the other for $|f(v_1) - f(v_2)|^2$.

(c) The measures used in the double integrals on each V and V' do not have to be product measures. Furthermore, one can use L_p norms for $p \neq 2$, which may be useful for products with L_p-product metrics. Notice that the ℓ_1-metric is implicit in the inequality Wir_4, where 4-tuples of points in X may be thought of as maps of the *Hamming square* $\{0,1\}^2 \to X$. Then one looks at the Hamming cube $\{0,1\}^n$ (where the Hamming metric is induced from the ℓ_1-metric on $\mathbb{R}^n \supset \{0,1\}^n$) put to X by some map $\{0,1\}^n \to X$. If X is Wir_4, then the standard (and obvious) computation (similar to our evaluation of $\lambda_1(V_1 \times V_2)$) shows that the mean of the squared great diagonals of $\{0,1\}^n \to X$ is bounded by n times squared edge length. (One should keep in mind that the sharpness of the L_2-estimate depends on the *global* Wir_4 for X confronted with the *inverse* Wir_4 for "infinitesimal parallelograms" in X.)

§11

Combining Remark 9 (a) and §10, we obtain concentration for maps of products V of rank 1 symmetric spaces into (mildly non-singular) Wir_∞-spaces X, including polyhedral CAT(0)-spaces, for instance. Namely, such maps for these V concentrate as much as maps to $\mathbb{R}^k$ with $k = \dim X$.

§12

Many standard tricks of the concentration theory for real-valued maps extend to general Wirtinger (and especially CAT(0))-spaces as targets. For example, concentration for a V implies that for suitable (e.g., Riemannian) quotients of V with low dimensional fibers. Also, mildly distorted and sufficiently spread subvarieties $W \subset V$ of low codimension concentrate (when mapped to X) almost as strongly as V itself (where one is additionally aided by the Lipschitz extension theorem of [13] for CAT-spaces). Thus one sees concentration of maps of Grassmann manifolds to Wir_∞-spaces.

§13. Concentration in smooth X

If X is a complete simply connected manifold with non-positive curvature (or a more general *smooth* CAT(0)-space), then the L_2-concentration of maps $f\colon V \to X$ is almost as good as that for maps $V \to \mathbb{R}^N$ of *all* manifolds V, due to the following simple

Observation. *For every $f: V \to X$ there exists a map $f_0\colon V \to \mathbb{R}^N$ for $N = \dim X$ (where this dimension is allowed to be $+\infty$), such that*

(i) $$\|df_0(v)\| \leqslant \|df(v)\| \quad \text{for all } v \in V,$$

(ii) $$\iint\limits_{V\times V} \|f_0(v_1) - f_0(v_2)\|^2 \, dv_1 dv_2 \geqslant \frac{1}{2} \iint\limits_{V\times V} |f(v_1) - f(v_2)|_X^2 \, dv_1 dv_2.$$

Proof. Take the *Riemannian center of mass* $x_0 \in X$ of the f-push-forward measure from V to X and observe that the map $f_0 = \exp_{x_0}^{-1} f$ has $\int\limits_V f_0(v)dv = 0$. Q.E.D.

The inequality (i) now follows from the contracting property of the inverse exponential map $\exp^{-1}\colon X \to T_{x_0}(X) = \mathbb{R}^N$ (since $K \leqslant 0$), while (ii) depends upon the non-decreasing (in fact isometric) feature of $\exp^{-1}$ on the rays issuing from x_0 and on the triangle inequality (where the latter is responsible for the unfortunate coefficient 1/2).

Remarks. (a) The above remains valid for *graphs* in the place of manifolds V. For example, if V is the complete bipartite graph on vertices $x_1, \dots, x_\ell$ and $y_1, \dots, y_\ell$ in X, then the above argument combined with ($\boxtimes_\ell$) in 5 shows that the averaged squared distances $|x_i - x_j|^2$ and $|y_i - y_j|^2$ are bounded by the averaged (over the edges) $|x_i - y_j|^2$ as follows

$$\frac{1}{\ell(\ell-1)} \left(\sum_{i<j} |x_i - x_j|^2 + \sum_{i<j} |y_i - y_j|^2 \right) \leqslant \frac{\lambda}{\ell^2} \sum_{i,j=1}^{\ell} \|x_i - y_j\|^2 \quad (2\boxtimes_\ell)$$

for $\lambda = 2\frac{\ell}{\ell - 1}$.

(a′) This inequality is sharp as is seen in the Cartesian product of two hyperbolic spaces, $X = H_\kappa^{\ell-1} \times H_\kappa^{\ell-1}$ with $\kappa < 0$, where the ratio of the left hand side and the right hand side of ($2\boxtimes_\ell$) converges to 1 for $\{x_i\}$ and $\{y_i\}$ converging to the vertices of regular ideal ℓ-simplices on the ideal boundaries of $H_\kappa^{\ell-1} \times \kappa_0$ and $x_0 \times H_\kappa^{\ell-1}$ respectively. (Bipartite graphs can be approximated by surfaces, which shows that the extra factor 2 is unavoidable in the Riemannian category as well.)

(b) There is *no* Euclidean reduction of the concentration property for *singular* CAT(0)-spaces (defined later on). Counter-examples are provided by cones over expanders. These play an essential role in the study of random groups (see [8]). However, most elementary (local) bounds on λ_1 (e.g., for Ricci $\leqslant -\kappa$) are likely to extend to maps into singular CAT(0)-spaces (possibly without the 2 factor).

(c) The essential property of X in the above observation is not so much $K \leqslant 0$, but rather the existence of "many sufficiently contracting" and "sufficiently proper" maps to $\mathbb{R}^N$, that is known as *parametric hyper-Euclidean property* involved in most proofs of the strong Novikov conjecture. (This property is violated by singular X with cones over arbitrary large expanders.)

§14. Diffusion, codiffusion, and harmonic maps

A *prediffusion* on V is a map from V to the space of $\mathbb{R}_+$-paths of probability measures on V, denoted

$$V \mapsto \mu_\varepsilon(v, v')dv', \varepsilon \in (0, \infty),$$

such that $\mu_\varepsilon(v, v')dv'$ converges to the δ-measure $\delta(v)dv$ for all $v \in V$ and $\varepsilon \to 0$. A prediffusion is called *diffusion* if the family $\{\mu_\varepsilon\}$ makes a *semigroup* under the composition (convolution) of measures:

$$\mu_{\varepsilon_1} * \mu_{\varepsilon_2} = \mu_{\varepsilon_1+\varepsilon_2}$$

(compare Chap. 7 in [3]).

A *codiffusion* on X is a retraction c of the space $\mathcal{P}(X)$ of probability measures on X back to $X \subset \mathcal{P}(X)$, where X is embedded to $\mathcal{P}(X)$ by $x \mapsto \delta(x)dx$. (To be consistent, one should deal with *homotopy* retractions, e.g., given by contractive semigroups of maps $\mathcal{P}(X) \hookleftarrow$ but these do not enter the present framework.)

If V and X are endowed with prediffusion and codiffusion respectively, one defines ε-*harmonic* maps $f: V \to X$ as those where $f_*(\mu_\varepsilon(v)) \in \mathcal{P}(X)$

retracts to $f(v) \in X$ for all $v \in V$. Traditionally, one defines *harmonic* maps by the equality $c(f_*(\mu_\varepsilon(v))) = f(v)$ for an *infinitesimal* ε, i.e., in the limit for $\varepsilon \to 0$. Another attractive possibility is passing to the limit $\varepsilon \to \infty$ and augmenting the spaces $\mathcal{P}(V)$ and $\mathcal{P}(X)$ by the measures on suitable ideal boundaries of the spaces V and X. In any case, the harmonicity amounts to $f\colon V \to X$ being a *fixed point* of the diffusion-codiffusion flow map

$$f(v) \mapsto c\,(f_*(\mu_\varepsilon(v)))$$

at some $\varepsilon \in [0, \infty]$. (This brings harmonic maps on equal footing with classifying maps into spaces supporting expanding maps.)

If X has negative curvature, one defines codiffusion as the *center of mass*: first, a measure $\nu \in \mathcal{P}(X)$ is mapped to the function on X that equals the ν-average of the squared distance functions on X,

$$\nu \mapsto d_\nu(x') \overset{\text{def}}{=} \int\limits_X |x - x'|_X^2 \, d\nu(x).$$

For $K(X) \leqslant 0$, this function d_ν is strictly convex on X and, hence, has a *unique* minimum point $x_{\min} = x(\nu) \in X$: this is taken for $c(\nu) \in X$. The essential feature of this c is the contraction property for the L_2-*transportation metric* (see [11]). This contraction property, when confronted with the smoothing properties of the diffusion in V (characteristic to curvature $\geqslant -\kappa > -\infty$, compare [1]), allows "good" (e.g., Lipschitz regular) harmonic maps $V \to X$ (where in interesting cases these maps commute with a given symmetry group operating on V and on X).

§15. Geodesic triangles and CAT(κ)-spaces

A metric space X is called CAT(κ) if it is geodesic and $\mathrm{Cycl}_4(\kappa)$. The geodesic property, essentially equivalent to the existence of a *middle point* x between arbitrary x_1 and x_2, i.e., satisfying

$$|x_1 - x| + |x - x_2| = |x_1 - x_2|,$$

enhances the power of distance inequalities. For example, if X is geodesic, then the general $\mathrm{Cycl}_4(\kappa)$-inequality follows (by an easy argument) from that for the special quadruples $\{x_i\} \subset X$ where x_3 lies *between* x_2 and x_4, i.e., on a (shortest geodesic) segment $[x_2, x_4]$, which amounts to the equality

$$|x_2 - x_3| + |x_3 - x_4| = |x_2 - x_4|.$$

If one thinks of $|x_1 - x_i|$, $i = 2, 3, 4$, as the values of the distance function $X \mapsto d(x) = \operatorname{dist}(x_1, x)$, then one can interpret this *inequality* as a *convexity* property of $d(x)$ saying that it is "more convex" than the corresponding distance function on the model space of curvature κ.

Another, apparently stronger but, in fact, equivalent characterization of CAT(κ)-spaces X expresses the idea of *geodesic triangles* in X being narrower than the *comparison* triangles. Here a geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space X is defined as the union of the three edges $[x_i, x_j] \subset X$, $1 \leqslant i < j \leqslant 3$, where one allows every edge between two points if there are several of them. A *comparison triangle* Δ' in a space X' (which will be taken of constant curvature later on) is, by definition, a geodesic triangle $\Delta' = \Delta(x'_i) \subset X'$, $x'_i \in X'$, $i = 1, \dots, 3$ such that

$$|x_i - x_j|_X = \left|x'_i - x'_j\right|_{X'}.$$

This $\Delta' \subset X'$ does not necessarily exist. If it does, it comes along with a canonical map $c\colon \Delta' \to \Delta$, where $x'_i \mapsto x_i$ and each segment $[x'_i, x'_j]$ *isometrically* goes to $[x_i, x_j]$.

A space X is CAT(κ) iff each D in X admits a comparison triangle Δ' in a model space X' with constant curvature $\kappa' \leqslant \kappa$ such that the *comparison map* $c\colon \Delta' \to \Delta$ is (non-strictly) distance decreasing with respect to the (non-path) metrics $\operatorname{dist}_{X'}|_{\Delta'}$ and $\operatorname{dist}_X|_\Delta$,

$$|c(x') - c(y')|_X \leqslant |x' - y'|_{X'} \tag{Δ_κ}$$

for all x', $y' \in \Delta'$.

This is a fundamental, albeit easy to prove, result by Alexandrov.

If $\kappa = 0$, then a comparison triangle always exists in X'_κ and is unique up to isometry; the same is true for $\kappa > 0$ if

$$|x_i - x_j| < \frac{\pi}{\sqrt{\kappa}} \quad \text{and} \quad \sum_{1 \leqslant i < j \leqslant 3} |x_i - x_j| \leqslant \frac{2\pi}{\sqrt{\kappa}}.$$

In general, the existence of Δ' can be dropped from the definition as we allow $\kappa' \leqslant \kappa$.

Basic examples. (a) The standard space X'_κ is CAT(κ') for all $\kappa' \geqslant \kappa$. This is the essential feature of these model spaces allowing a meaningful definition of general CAT-spaces.

(a$'$) A complete simply connected Riemannian manifold withconstant (as well as variable) sectional curvature $\leqslant \kappa$ is CAT(κ): the n-spheres of radii $\geqslant \kappa^{-1/2}$ are CAT(κ); $\mathbb{R}^n$ is CAT(0); the hyperbolic spaces

$$H^n_\kappa = \left(\mathbb{R} \times \mathbb{R}^{n-1},\ dt^2 + e^{-2t\sqrt{\kappa}} \sum_{i=1}^{n-1} dy_i^2\right), \quad \kappa \geqslant 0$$

are CAT($-\kappa$) for all $n = 2, 3, \dots, \infty$.

(b) Every (simplicial or non-simplicial) tree is CAT($-\infty$), i.e., CAT(κ) for *all* $\kappa \in \mathbb{R}$.

(c) Every smooth domain X in $\mathbb{R}^2$ is CAT(0) for the induced path metric and such domains in H^2_κ are CAT($-\kappa$). (This is an easy but useful property which does not directly extend to higher dimensions.)

Remarks. (a) Cycl_4 as well as Cycl_k for all k are instances of *concentration* (of isoperimetric kind) inequalities which can be defined with an arbitrary (measuring rod) graph with the vertex set V and edges $E \subset V \times V$ by requiring a certain bound on distances $|f(v_1) - f(v_2)|_X$ for maps $f\colon V \to X$ in terms of distances $|f(v_1) - f(v_2)|_X$ for $(v_1, v_2) \in E$. It is convenient to allow infinitesimal graphs V where E consists of pairs of infinitesimally close points. We have met such a bound for maps of Riemannian manifolds V into X, where E was represented by unit tangent vectors in V and the relevant bound(s) was/were of the form

$$\iint\limits_{V\times V} |f(v_1) - f(v_2)|^\alpha \, dv_1 dv_2 \leqslant F\left(\int\limits_E \|df\|^\delta \, de\right).$$

We have also seen that *smooth* X with $K \leqslant 0$ satisfy additional inequalities of this type but one does not know what is the full set of such inequalities characterizing a given class (e.g., of smooth X) of spaces with $K \leqslant 0$.

(b) The geodesic property is one logical level up from concentration inequalities as it involves the *existential quantifier*. It is unclear if there is a simple $\exists$-free description of (non-geodesic!) *subspaces* in CAT(κ)-spaces. (We shall see later on that $\mathrm{Cycl}_4 \Rightarrow \mathrm{Cycl}_k$ for all $k \geqslant 5$ in the *geodesic* case but this is apparently not so in general.)

(c) Consider two probability measures μ and ν in X, let $c(\mu), c(\nu) \in X$ be their centers of mass, and let

$$\Phi_X(\mu, \nu) = \iint\limits_{X\times X} |x - y|^2 \, d\mu d\nu$$

$$-\frac{1}{2}\left(\iint\limits_{X\times X}|x-y|^2\,d\mu d\mu+\iint\limits_{X\times X}|x-y|^2\,d\nu d\nu\right).$$

If $X=\mathbb{R}^n$, then

$$\Phi_X(\mu,\nu)=|c(\mu)-c(\nu)|^2,$$

(see §5), and if X is CAT(0), then

$$|c(\mu)-c(\nu)|^2\leqslant\Phi_X(\mu,\nu)\qquad(\boxtimes\,(0))$$

for *two-point* measures μ and ν. This follows from the fact that the squared distance functions d on X, and hence the convex combinations of d^2's, are "more convex" on geodesic lines than the function x^2 on $\mathbb{R}$. This means the second derivatives of d^2's are $\geqslant 2$, or equivalently, the difference of two squared distance functions, $d_y^2-d_x^2$, is convex on each geodesic in X passing through x (and concave on geodesics through y).

(c′) The inequality $\boxtimes\,(0)$ fails for general (non-two-point) measures in CAT(0)-spaces but does hold true if the supports of μ and ν are contained in (possibly different) *flat* convex subspaces in X.

(c″) The general CAT(κ) (i.e., Cycl_4) property can be brought to the $\boxtimes\,(\kappa)$-form. Yet, this does not (?) appear sufficiently illuminating.

§16. Geodesic convexity and convex gluing of spaces

A subset $Y\subset X$ is called (geodesically) *convex* if it contains every segment with the ends in Y.

Examples. (a) Every geodesic segment in the CAT(0)-space is convex, and every segment strictly shorter than π/κ is convex in a CAT($\kappa>0$)-space.

(b) Every subtree in a tree is convex.

(c) Every ball B in a CAT(0)-space is convex, where

$$B=B_{x_0}(R)\overset{\text{def}}{=}\{x\in X\mid |x-x_0|\leqslant R\}.$$

Let $Y_i\subset X_i$, $i=1,2$, be non-empty convex spaces, $\varphi\colon Y_1\to Y_2$ a bijective isometry, and denote by $X_1\vee_\varphi X_2$ the disjoint union of X_1 and X_2 where Y_1 is identified with Y_2 via φ.

Gluing theorem. *If X_1 and X_2 are* CAT(κ), *then so is $X_1 \vee_\varphi X_2$.*

The proof is straightforward, modulo elementary geometry of the model spaces (see [2]).

Examples. (a) To apply the theorem one needs X_i with mutually isometric convex $Y_i \subset X_i$. One can use, for instance, segments in X_i of equal lengths (which are always convex for $\kappa \leqslant 0$). Or, if X_1 happens to be isometric to X_2, one can use the restriction of the implied isometry $X_1 \to X_2$ to some convex subset $Y_1 \subset X_1$, e.g., to a ball $B \subset X_1$ (which is always convex for $\kappa \leqslant 0$).

(b) *Tree-like polyhedra.* A connected simplicial polyhedron P is called tree-like if

$$P = \bigcup_i P_i$$

for i ranging over a well-ordered set I, such that $P_{i+1} = P_i \cup \Delta_i$, $i \in I$, where the simplex Δ_i meets P_i over a *single face* $\Delta_i' \subset \Delta_i$. (Clearly, trees *are* tree-like.) If we give to such P the metric where each simplex $\Delta \subset P$ is isometric to a regular simplex of a fixed size in a simply connected space of constant curvature κ (i.e., spherical, Euclidean, or hyperbolic), then P becomes a CAT(κ)-space by the Gluing theorem.

(b$'$) *Nerves of subtrees.* Let Q be a tree, Q_j, $j = 1, \dots, k$, a finite collection of subtrees, and P be the *nerve* of this family $\{Q_j\}$.

If P is connected, then it is tree-like.

Proof. Assume there is a point $q \in Q \setminus \bigcap_j Q_j$ and let $Q_{j^\bullet}$ be the *farthest* subtree from q. Then, clearly, P is obtained from the nerve $P^\bullet$ of

$$(Q_1, \dots, Q_{j^\bullet - 1}, Q_{j^\bullet + 1}, \dots, Q_k)$$

by attaching a simplex to $P^\bullet$ across a single face, and an obvious induction concludes the proof. Q.E.D.

Remark. Another significant property of this P (shared by all tree-like polyhedra and possibly characterizing them) is the following *sharp* combinatorial isoperimetric inequality: *every cyclic path of k-edges bounds a* (possibly degenerate) *disk made of at most $k-2$ triangles.*

§17. Convexity and CAT$_\kappa$-convexity

Take a geodesic line ℓ' in the model space X' of curvature κ (this line is a topological circle for $\kappa > 0$) and consider the distance function to ℓ',

$$d_\kappa(x') = \inf_{y' \in \ell'} |x' - y'|_{X'} .$$

The restriction of this function to a segment $[a', b']$ in a connected component (half-plane) of the complement $X'\backslash\ell'$ is uniquely determined by the values $d_\kappa(a')$, $d_\kappa(b')$ and the length $|a'-b'|$ of $[a',b']$. So we can regard d_κ as a real function, called *κ-function*. Notice that the κ-functions are positive and 1-*Lipschitz*, i.e.,

$$|d_\kappa(t_1) - d_\kappa(t_2)| \leqslant |t_1 - t_2|.$$

Next, a positive 1-Lipschitz function d defined on some segment in $\mathbb{R}$ is called *κ-convex*, if, for every two points a and b in this segment, there is a κ'-function $d_{\kappa'}$ on $[a,b]$ with $\kappa' \leqslant \kappa$, such that

$$d_{\kappa'}(a) = d(a), \ \ d_{\kappa'}(b) = d(b)$$

and

$$d_{\kappa'}(t) \geqslant d(t) \text{ for } t \in [a,b].$$

In other words, d must be *more convex* than d_κ. For example, if $\kappa = 0$, then the κ-convexity amounts to ordinary convexity for positive 1-Lipschitz functions.

One checks elementarily that the κ-convexity is a *local property*: if d is κ-convex in a small subinterval around each point, then it is κ-convex.

About $\kappa = -\infty$. This convexity means the κ-convexity, for all $\kappa \in \mathbb{R}$, which corresponds to the behavior of the distance to a geodesic line in a tree. Clearly, every $(-\infty)$-convex function $d(t)$ on $[a,b]$ equals

$$\max\left(d(a) - |a-t|,\ 0,\ d(b) - |b-t|\right).$$

A (positive, 1-Lipschitz) function on a geodesic space is called CAT_κ-*convex* if its restriction to every geodesic segment is κ-convex. Then a space X is called CAT_κ-*convex* if the distance function to each segment $Y \subset X$, i.e., $d(x) = \inf_{y\in Y} |x-y|$, is κ-convex on X. If X is CAT_κ-convex, then, clearly,

(i) *the distance function to every convex subset $Y \subset X$ is κ-convex;*

(ii) *the R-balls in X are convex for all R if $\kappa \leqslant 0$ and for $R \leqslant \pi/\sqrt{\kappa}$ for $\kappa > 0$;*

(iii) *the ρ-neighborhood $Y + \rho$ of each convex subset $Y \subset X$ is convex for $\kappa \leqslant 0$*, where, recall

$$Y + \rho \overset{\text{def}}{=} \{x \in X \mid \operatorname{dist}(x, Y) \leqslant \rho\}.$$

Convexity theorem. *Every* CAT(κ)*-space is* CAT$_\kappa$*-convex.*

This is well-known and the proof is straightforward (see [2]). Notice that the converse is true (and rather obvious) for *Riemannian* manifolds but not for general X. For example, *Banach spaces* are 0-convex; yet these are *not* CAT(0) unless they are Hilbertian. But for $\kappa \to -\infty$ the distinctions between the classes of spaces disappear:

$$\mathrm{CAT}(-\infty) = (-\infty)\text{-convexity}.$$

On the topology of CAT(κ). If $\kappa \leqslant 0$, then every two points in a CAT(κ)-convex space X are joined by a *unique* geodesic segment and so CAT(0)-convex spaces are contractible. Moreover, the balls in these spaces are convex and contractible. (If $\kappa > 0$, then convexity of the R-balls is ensured only for $R \leqslant \pi/(2\sqrt{\kappa})$ and contractibility for $R < \pi/\sqrt{\kappa}$.)

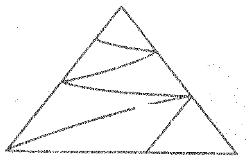

Fig. 3.

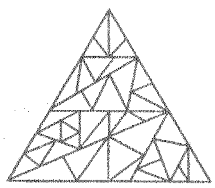

Fig. 4.

§18. CAT-(κ) AND CURVATURE

If a geodesic triangle Δ in X is subdivided into (smaller) triangles Δ_i with all vertices on Δ then the CAT(κ)-comparison inequalities for Δ_i imply (by an easy and well-known argument) that for Δ itself. Then such subdivision can be applied to all Δ_i etc., thus reducing verification of CAT(κ)-property to arbitrary small triangles.

There is a catch in this however: such subdivisions are rather special as every vertex must lie inside a geodesic edge and there is no guarantee that the new triangles will be smaller than the original Δ. Yet, everything works if X contains no *almost minimal* closed curves (geodesics), which amounts to requiring that the extrinsic distance $\mathrm{dist}_X\Delta$ is "significantly" smaller than the induced path metric on Δ. This means, by definition, that every geodesic Δ contains a pair of points x and y such that

$$\mathrm{dist}_X(x,y) \leqslant (1-\varepsilon)\mathrm{dist}_\Delta(x,y)$$

for the path metric dist_Δ on Δ, and

$$\mathrm{dist}_\Delta(x,y) \geqslant \varepsilon \operatorname{diam}\Delta$$

for some $\varepsilon = \varepsilon(X) > 0$ independent of Δ.

These considerations suggest the following

Definition. *We say that X has curvature $K \leqslant \kappa$ at $x \in X$ if there is a neighborhood $U \subset X$ of x such that every Δ contained in U is more narrow than the model triangle Δ', i.e., the comparison map $c\colon \Delta' \to \Delta$ is distance decreasing. (Equivalently, one could say that a small ε-ball around x is* CAT(κ)*.) Next we define spaces X with $K(X) \leqslant \kappa$, i.e., with curvatures $\leqslant \kappa$, by requiring this property at every point $x \in X$.*

§19. Examples

(a) Riemannian manifolds X (of finite or infinite dimension) with sectional curvature $\leqslant \kappa$ have $K(X) \leqslant \kappa$ in our (i.e., Alexandrov's) sense.

(b) Let X be a polyhedron built of (convex) simplices of constant curvature κ (i.e., simplices from a complete simply connected space with constant curvature κ). The link L_x of every vertex $x \in X$ is again a space of this kind, built of spherical simplices, i.e., those with $\kappa = 1$. Then $K(X) \leqslant \kappa$ if and only if every such link is CAT(1).

In particular, if $\dim X = 2$ and thus every L_x is a 1-polyhedron, i.e., a graph with the length of the edges measured by the angles of the corresponding triangles. Here the CAT(1) condition for L_x says that every cycle in L_x has length $\geqslant 2\pi$.

More generally, CAT(1) needs, besides $K \leqslant 1$, the uniqueness property for geodesic segments between the pairs of points with distance $< \pi$ between them. For instance, if X is CAT(κ) for $\kappa \leqslant 1$ and $Y = X/\Gamma$ for an isometry group Γ with $|x - \gamma(x)| \geqslant 2\pi$ for all $x \in X$ and $\mathrm{id} \neq \gamma \in \Gamma$, then Y is CAT(1).

§20. Ramified covers

Let X be a 2-polyhedron and let $f\colon \tilde{X} \to X$ be a *ramified covering*, i.e., the pull-backs $f^{-1}(x) \subset X$ are discrete for all $x \in X$ and, furthermore, there are discrete subsets $X_0 \subset X$ and $\tilde{X}_0 \subset \tilde{X}$ such that f maps the complement $\tilde{X}\backslash\tilde{X}_0$ to $X\backslash X_0$ with $\tilde{X}\backslash\tilde{X}_0 \to X\backslash X_0$ being a *covering* map.

Every path-metric in X (obviously) induces such a metric in $\tilde{X}$, and if the former had $K \leqslant 0$, so, obviously, does the latter. Moreover, if the metric in X is flat (Euclidean) on all 2-simplices in X, then "most" ramified coverings $\tilde{X} \to X$ have $K(\tilde{X}) \leqslant 0$, regardless of the curvature of X.

For example, let X be built of plane equilateral triangles and $\tilde{X} \to X$ *ramified* at each vertex in X with order $\geqslant 2$. This means X_0 contains all vertices in X and for every pair of points $\tilde{x} \in \tilde{X}_0$ and $x \in X_0$ the induced *covering* map $f_{\tilde{x}}$ of the link $\tilde{L}_{\tilde{x}} \subset \tilde{X}_0$ to $L_x \subset X$ *non-trivially* covers each *simple* cycle $C \subset L_x$, i.e., there is *no* cycle $\tilde{C} \subset \tilde{L}_{\tilde{x}}$ *injectively* sent by $f_{\tilde{x}}$ to C. Then, clearly, each cycle in $\tilde{L}_{\tilde{x}}$ has at most 6 edges and thus $K(\tilde{X}) \leqslant 0$.

Notice that whenever X_0 contains all vertices in X, there are plenty of ramified covers $\tilde{X} \to X$ with the above property. In fact the fundamental group π' of the complement $X' = X\backslash X_0$ is *free* and therefore it contains lots of subgroups $\tilde{\pi} \subset \pi'$ such that the classes of the simple cycles $C \subset L_x$, $x \in X_0$, are *not* contained in $\tilde{\pi}$. Then the completions of the $\tilde{\pi}$-coverings of X' are our $\tilde{X}$ with $K(\tilde{X}) \leqslant 0$.

If we are concerned with *finite* polyhedra X and $\tilde{X}$, we need subgroups $\tilde{\pi} \subset \pi'$ of finite index in order to have *finitely sheeted* ramified covers $\tilde{X} \to X$. Since free groups are *residually finite*, we do have plenty of such $\tilde{\pi} \subset \pi'$ and, consequently, we have as many *finite* 2-polyhedra $\tilde{X}$ with $K(\tilde{X}) \leqslant 0$.

To be specific, let X be the 2-skeleton of the $(n-1)$-simplex. This X is simply connected and is built of $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$ triangles. If we remove the set $X_0 \subset X$ of the vertices of X, the complement $X\backslash X_0$ contracts to the graph $X' \subset X\backslash X_0$ spanned by the baricenters of the triangles and edges in X. This X' has $3\binom{n}{3}$ edges and $\binom{n}{3} + \binom{n}{2}$ vertices, where $\binom{n}{2} = \frac{n(n-1)}{2}$ is the number of edges in X. Thus the fundamental group of $X\backslash X_0$ is free with $m = 2\binom{n}{3} - \binom{n}{2} + 1$ generators. The most obvious subgroup $\tilde{\pi} \subset \pi' = F_{m'}$ which makes $K(\tilde{X}) \leqslant 0$ is the kernel of the canonical homomorphism

$$F_m \to (\mathbb{Z}/2\mathbb{Z})^m$$

(while every $m > 2$ makes $K(\tilde{X}) < 0$). The multiplicity of the corresponding ramified cover $\tilde{X} \to X$ equals 2^m away from X_0 and $2^{m-\ell}$ at X_0, where

$$\ell = \ell_n = \frac{(n-1)(n-2)}{2} - (n-1) + 1$$

is the rank of the fundamental group of the 1-skeleton of the $(n-1)$-simplex.

(There are smaller ramified covers of this X with $K \leqslant 0$, and one, probably, can enlist the minimal ones. Similarly, one can ramify other symmetric 2-polyhedra, such as the 2-skeletons of the cubes and octahedra.)

§21. On construction of polyhedra X with $K \leqslant 0$ for $\dim X \geqslant 3$

As dimension grows, there seem to appear fewer and fewer new spaces with $K \leqslant 0$ and getting them with $K < 0$ is especially difficult. (In fact, all known high-dimensional hyperbolic groups are built out of "arithmetic blocks" but we are far from stating and proving any definite result in this direction.)

For example, if we sufficiently ramify 3-polyhedra X over 1-dimensional loci $X_1 \subset X$, the resulting $\tilde{X}$ will have negative curvature everywhere except the vertices $\tilde{x} \in \tilde{X}$, where we can ensure curvature ≤ 1 of the (2-dimensional) links $L_{\tilde{x}}$, but not the CAT(1)-property.

(The latter could be achieved if these links had sufficiently many coherent finite coverings, i.e., if the fundamental groups of $\tilde{X} \setminus \{\text{vertices}\}$ were residually finite. In fact, a suitable residual finiteness of n-dimensional groups with $K \leqslant 0$ (or $K < 0$) would lead to many examples of $(n+1)$-dimensional groups with $K \leqslant 0$ (or $K < 0$); this indicates, in my view, that typical groups with $K \leqslant 0$ (or $K < 0$) have no non-trivial finite quotients.)

The spaces like $\tilde{X}$, where the curvature is negative away from the vertices, can be modified to have $K \leqslant 0$ (or $K < 0$) everywhere in two ways.

(1) Remove the vertices and replace all simplices by the ideal hyperbolic simplices. Then the resulting space $\tilde{X}'$ becomes a complete space of *finite volume* and $K \leqslant 0$ (or $K < 0$).

(2) Suitably truncate the hyperbolic simplices and double the resulting space. This results in a *compact* $\tilde{X}''$ with $K \leqslant 0$ (or $K < 0$), homeomorphic to the double of

$$\tilde{X}\backslash\{\text{small balls around the vertices}\},$$

where the "double" gluing takes place over the boundaries of these balls. In particular, one obtains in this way some (not especially exciting) 3-polyhedra with $K \leq 0$.

Finally, if we depart from a 3-dimensional *pseudomanifold* X, then we always can arrange ramified coverings with $K < 0$. In fact, such an X can be obtained from a compact 3-manifold $X_\bullet$ by attaching cones to the boundary components of $X_\bullet$ (followed by some irrelevant identifications). If $X_\bullet$ happens to have constant negative curvature with mildly curved boundary, then, after passing to finite covering $\tilde{X}_\bullet$ and coning the boundary of $\tilde{X}_\bullet$, we get a compact pseudomanifold with $K < 0$.

Of course, not every X gives us such an $X_\bullet$ but the desired property is satisfied by a suitable preliminary ramified cover of X as can be easily derived from Thurston's theory. So, with Thurston, we have a huge pool of compact 3-dimensional pseudomanifolds with $K < 0$.

§22. Assembling ($K \leqslant 0$)-spaces over geodesic graphs

It is hard to construct high dimensional spaces X with $K \leqslant 0$ (and, especially, with $K < 0$) from scratch, but given such an X one can construct many others as follows.

Let $X_1 \subset X$ be a *geodesic subgraph* in X, i.e., a union of geodesic segments e_i, $i \in I$, where every two segments meet, if at all, at one of their end points. Take several copies of X (where, more generally, one may take various numbers of different connected components of X, in case X was disconnected) and then glue together some among edges of equal length in the corresponding union of the copies of X_1, where we do not exclude gluing edges in the same connected component in (the union of copies of) X. (There are exactly two ways to glue together two equilong edges, where a particular gluing can be specified if we orient our graph.)

It is easy to figure out when the resulting space, say Y, has $K \leqslant 0$ (or $K < 0$). Namely, if $K(X) \leqslant 0$ (or < 0), the same inequality holds at all points in Y except, possibly, the (points coming from the) vertices of our graph X_1. Now, at every vertex points $y \in Y$ consider all edges e_j, $j \in J_x$, from the union of copies of X_1 adjacent to some point in (a copy of) X

and call two such edges *x-neighbors* if they come from edges adjacent to the same point x in X. In this case there is a well (and obviously) defined *angle* measured in X between these edges, denoted $\sphericalangle_x(e_{j_1}, e_{j_2})$. Observe that the same pair of adjacent edges in Y may come from different x's in X and the resulting angle depends on which x is used. Just look at the pair of triangles glued over two pairs of edges.

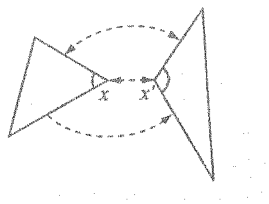

Fig. 5.

Next consider cyclic chains of edges at y, say $e_1, e_2, \ldots, e_{k+1} = e_1$, where e_{i+1} is x_i-adjacent to e_i for all $i = 1, \ldots, k$, and where $x_i \neq x_{i+1}$ for all i.

Clearly, Y has $K \leqslant 0$ (or $K < 0$) iff the total sum of angles,

$$\sphericalangle_{x_1}(e_1, e_2) + \ldots + \sphericalangle_{x_k}(e_k, e_1),$$

is $\geqslant 2\pi$ (or $> 2\pi$ if we want $K < 0$) for all such chains of edges. This (trivially) generalizes the case of 2-polyhedra, where X equals the union of Euclidean triangles with $X_1 \subset X$ being the union of the edges of these triangles (and where something new enters the picture if we take, for instance, 3-simplices with their edges instead of the triangles).

All of the above would be rather pointless if we had no simple way to arrange gluings satisfying the ($\geqslant 2\pi$)-condition. Fortunately, there are, roughly, as many such gluings for general (X, X_1) as for triangles $(\Delta, \partial\Delta)$; in particular the ramified covering trick works for all (X, X_1) as follows.

Start with an arbitrary Y, e.g., obtained by doubling X across X_1. This Y has $K \leqslant 0$ everywhere except the vertices of X_1 and then we take a ramified cover $\tilde{Y} \to Y$ which ramifies at these vertices. Technically speaking, we remove $X_0 = \{\text{vertices of } X_1\}$ from Y, take a (finite if you wish) covering of the complement $Y \backslash X_0$ which is trivial over each of the two copies of X in Y, and then metrically complete this covering by adding back the vertices.

The triviality condition says, in effect, that our covering comes from an auxiliary 2-polyhedron where each copy of X is replaced by the cone over

$X_1 \subset X$. After removing the vertices in X_1 the resulting 2-polyhedron contracts to a graph, and so coverings are determined by subgroups of a *free* group so that we have the same freedom of choosing them as for 2-polyhedra. In particular, we can construct finite ramified covers of Y with $K \leqslant 0$ (or $K < 0$), provided all angles between the edges of X_1 at the vertices are *strictly positive*. (This is a rather mild condition; actually it takes a special effort to make up examples where it is violated.)

Remark. One can look at gluing across k-dimensional subpolyhedra $X_k \subset X$ with totally geodesic simplices but making specific examples becomes rather difficult for $k \geqslant 3$.

§23

Let us isolate a purely combinatorial aspect of the above construction. Say that a graph $(\tilde{V}, \tilde{E})$ is *tessilated* by (copies of) a graph (V, E) (where V and $\tilde{V}$ stand for the sets of vertices and E's for the edges) if we are given embeddings $\varphi_i : V \to \tilde{V}$, $i \in I$, such that

(a) $\bigcup_{i\in I} \varphi_i(V) = \tilde{V}$;

(b) $\operatorname{card}(\varphi_i(V) \cap \varphi_j(V)) \leqslant 1$ for all $i \neq j \in I$;

(c) edges go to edges, i.e., the Cartesian squares

$$\varphi_i^2 : V \times V \to \tilde{V} \times \tilde{V}$$

map $E \subset V \times V$ to $\tilde{E} \subset \tilde{V} \times \tilde{V}$; furthermore, the images of $\varphi_i^2(E) \subset \tilde{E}$ are mutually disjoint and

$$\bigcup_{i\in I} \varphi_i^2(E) = \tilde{E}.$$

Proposition. *Given finitely many finite graphs,* $(V_1, E_1), \dots, (V_k, E_k)$, *there exists a finite graph* $(\tilde{V}, \tilde{E})$ *tessilated by each of* $(V_1, E_1), \dots, (V_k, E_k)$.

In fact, such a $(\tilde{V}, \tilde{E})$ is obtained by factoring some universal infinite graph $(\check{V}, \check{E})$ by a suitable cofinite subgroup of the free group operating on $(\check{V}, \check{E})$. To make it clear, we shall state a more precise form of the above proposition, where we assume, for simplicity's sake that there are only two graphs, (V_1, E_1) and (V_2, E_2), with $\operatorname{card} E_1 = \operatorname{card} E_2$. We assume, moreover, that there is given a bijection $E_1 \leftrightarrow E_2$, where the edges $E_1 \ni e_1 \leftrightarrow e_2 \in E_2$ are regarded as *equivalent*. We also fix directions on all edges and require the above correspondence to preserve the directions.

Furthermore, we assign positive lengths to the edges, thus turning V_1 and V_2 into metric spaces (presuming the graphs are connected) and assume that the corresponding (equivalent) edges have equal length. Then we shall speak of *marked directed isometric* tessilations of $(\tilde{V}, \tilde{E})$ by (V_1, E_1) and (V_2, E_2) meaning that

"*isometric*": the implied embeddings φ_i of V_1 and V_2 into V are isometries;

"*directed*": the graph $(\tilde{V}, \tilde{E})$ is directed and the maps φ_i (both for E_1 and E_2) preserve the direction of edges;

"*marked*": if an edge $\tilde{e} \in \tilde{E}$ comes from some $e_1 \in E_1$ and $e_2 \in E_2$, then edges are equivalent, i.e., $e_1 \leftrightarrow e_2$. (In other words, the tessilations agree with a marking of $\tilde{E}$ by equivalence classes of edges.)

Proposition$^+$. *There exists a finite graph $(\tilde{V}, \tilde{E})$ with marked directed isometric tessilations by (V_1, E_1) and by (V_2, E_2).*

Proof. Attach a copy of (V_2, E_2) to (V_1, E_1) at each edge in E_1 according to "$\leftrightarrow$". Then attach copies of (V_1, E_1) to all newly created E_2-edges and continue ad infinitum. Thus we get a tree-like graph $(\check{V}, \check{E})$ suitably tessilated by (V_1, E_1) and (V_2, E_2) with an obvious cocompact action of the free group F_ℓ with $\ell = \frac{c(c-1)}{2}$ for $c = \mathrm{card} E_1 = \mathrm{card} E_2$. This is the automorphism group of the tree with 2-colored vertices and c-colored edges as sketched for $c = 3$ below.

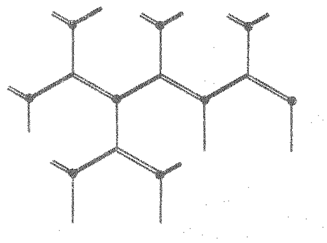

Fig. 6.

Then a quotient of $(\check{V}, \check{E})$ by a sufficiently small co-finite subgroup in F_ℓ is our $(\tilde{V}, \tilde{E})$. (This can be used for construction of Enflo type expanders departing from bipartite graphs of Remark 13(a).)

§24. Effective universal coverings of spaces with $K \leqslant 0$

If X has $K(X, x_0) \leqslant 0$, then small balls $B(x_0, \varepsilon) \subset X$ are convex and if $K(X) \leqslant 0$ everywhere, then such a ball remains locally convex in-so-far as it does not meet itself somewhere.

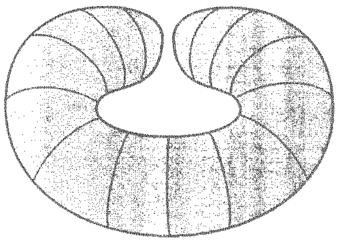

Fig. 7.

If this happens, we ignore the meeting points and continue to enlarge the ball, but not as a subset in X but rather as an abstract metric space along with a locally isometric map to X. These are called *overballs* $\tilde{B}(x_0, R) \to X$, $R > 0$, which all have locally convex boundaries since $K(X) \leqslant 0$ and so one can go from $\tilde{B}(x_0, R)$ to $B(x_0, R + \varepsilon)$ for small ε (where a little extra care is needed if X is not locally compact). Thus we obtain a space $\tilde{X} = \breve{B}(x_0,\, R = \infty)$ along with a locally isometric map $p\colon \tilde{X} \to X$. This $\tilde{X}$, being locally isometric to X, has $K(\tilde{X}) \leqslant 0$ and it is exhausted by locally convex balls. It follows (by the above considerations) that, in fact, $\tilde{X}$ is CAT(0) and it is easy to see that $p\colon \tilde{X} \to X$ is a covering map. In particular, *if a simply connected space X has $K(X) \leqslant 0$, then it is a* CAT(0)*-space.* This is the classical Cartan–Hadamard theorem (usually stated for non-singular spaces). Here are additional remarks clarifying the picture.

(a) If a space X with $K(X) \leqslant \kappa \leqslant 0$ admits a filtration by locally convex subsets $X_t \subset X$, $t \in \mathbb{R}_+$, where $X_t \subset X_{t'}$ for $t \leqslant t'$ and $X_{t+\varepsilon}$ is contained in the ε-neighborhood $X_t + \varepsilon$ of X_t for all $t \geqslant 0$ and some $\varepsilon = \varepsilon(t) > 0$, then X $(= \bigcup\limits_{t \in \mathbb{R}_+} X_t)$ is CAT(κ) provided X_0 is CAT(κ). (Recall that "locally convex" signifies convexity of some neighborhood of each point of the subset in question.)

(b) If X is CAT(0)-convex, then every connected locally convex subset in X is convex. Moreover, if Y is an abstract connected metric space and

$p\colon Y \to X$ is a locally isometric map sending a small neighborhood of each $y \in Y$ onto a convex subset in X, then p is one-to-one and the image $p(Y) \subset X$ is convex. (This is easy but not totally trivial even for $X = \mathbb{R}^n$.)

(c) The Cartan–Hadamard theorem remains valid for certain *orbispaces* with $K \leqslant 0$ (see [5]) and a version of this underlies the small cancellation theory (see below and [2]).

§25. Filling closed curves by disks in CAT-spaces

There is an alternative, in effect, more functorial, definition of CAT-spaces, at least for $\kappa \leqslant 0$ (due to Reshetnyak (?)), which says that X is CAT(κ) if every closed curve in X bounds a disk with curvature $\leqslant \kappa$. Actually one only needs Riemannian disks D with metrics of constant curvature κ (every such D appears as a multi-domain over $H^2(\kappa)$). Namely,

X is CAT(κ) *if and only if for every $\ell > 0$ and every distance (non-strictly) decreasing map α of the circle $S = S_\ell$ of length ℓ to X there exists a metric μ of constant curvature κ on D with* length $(\partial D) = \ell$ *and a distance decreasing map $\beta\colon (D,\mu) \to X$ extending α, where the boundary ∂D is naturally identified with S_ℓ.*

Sketch of the proof. If a geodesic triangle, viewed as a (mapped) circle in X, can be filled by (D,μ), then, in the case $\kappa \leqslant 0$, it is κ-narrow since (D,μ) is CAT(κ) for every metric with curvature $\leqslant \kappa$ as an elementary argument shows. Conversely, every closed curve S in a CAT(κ)-space X can be "subdivided" into "infinitesimally small" geodesic triangles as in Fig. 3 giving in the limit a filling disk D' with curvature $\leqslant \kappa$, which can then be "enlarged" to (D,μ) with curvature $\equiv \kappa$.

Then one can *define* spaces with $K(X) \leqslant 0$ by requiring the existence of (D,μ) and β for all *contractible* closed curves in X. It is not hard to show (by using, for instance, suitable minimal disks filling in curves) that the existence of (D,μ)-fillings for *short* curves in X (with shortness $\ell = \ell(x) > 0$ depending on $x \in X$ for curves contained in a ball of radius 2ℓ around x) implies that for *all* closed curves, and so this definition is essentially local.

Reshetnyak theorem and application. *The above disk D of constant negative curvature κ, a priori, only immerses into the model κ-plane H_κ. But, according to Reshetnyak, one can find a convex domain $D'_x \subset X_x$*

with

$$\operatorname{length}(\partial D'_X) = \operatorname{length}(\partial D) = \ell$$

and a distance non-decreasing homeomorphism $D' \to D$. Thus our $S = S_\ell \subset X$ can be filled in by a convex disk $D' \subset H_\kappa$.

Consequently CAT(κ)-*spaces are* $\mathrm{Cycl}_i(\kappa)$ *for all $i = 4, 5, \ldots,$ and thus they all are* Wir_∞ *for $\kappa = 0$,* and our bounds on λ_1 for various maps $V \to X$ from §8–12 apply to CAT(0)-spaces.

Question. Does the Cycl_4-inequality imply all Wir_k, $k = 5, 6\ldots,$ *without* assuming the space in question is geodesic?

§26. CAT-families of groups

Consider a closed subset $P \subset X$ and isometry groups Γ_p of X assigned to all $p \in P$ (where one could suppress P by defining $\Gamma_x = \{\mathrm{id}\}$ for $x \in X \backslash P$). Call $\{\Gamma_p\}$ a *rotation family* if the following two conditions are satisfied:

(i) Γ_p fixes p for all $p \in P$;

(ii) each Γ_{p_0} maps $P \to P$ and acts on the family $\{\Gamma_p\}$ by conjugation:

$$\gamma \Gamma_p \gamma^{-1} = \Gamma_{p'} \text{ for } p' = \gamma(p) \text{ and all } \gamma \in \Gamma_{p_0}.$$

Examples. (a) Take a finite subset $\underline{P} \in \mathbb{C}$ and let X be the universal cover of $\mathbb{C}$ ramified at $\underline{P}$. Then the (cyclic) monodromy groups around the lifts $p \in X$ of the points $\underline{p} \in \underline{P}$ make a rotation family generating the full Galois group acting on X.

(b) Take a union $\underline{P}$ of finitely many lines in $\mathbb{C}P^2$. Here again the Galois group of the universal cover ramified at $\underline{P}$ is generated by "rotations" about intersections of lines. (The Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$ is also generated by "rotations" corresponding to the Frobenius automorphisms.)

Denote by $\Gamma_P \subset \mathrm{Iso}X$ the *rotation group* generated by all Γ_p and let us reduce the CAT(0)-property of the quotient space X/Γ_P to that of the spaces X/Γ_p, $p \in P$, under the following *disjointness assumption*:

(iii) The set P is discrete and Γ_p acts freely and discretely on the complement $X \backslash \{p\}$ for all $p \in P$. (This is the case for the above (a) but not for (b).)

§27. Proposition

Let $\{\Gamma_p\}$ *satisfy* (i)–(iii), the space X be CAT(κ) for some $\kappa \leqslant 0$, and $K(X/\Gamma_p) \leqslant \kappa$ for all $p \in P$. Then the group Γ_P is discrete and the quotient space X/Γ_P is CAT(κ). Furthermore,

(a) *Γ_P acts freely on the complement $X \backslash P$ and the isotropy subgroup of each $p \in P$ equals (exactly!) Γ_p. Moreover, if a ball $B_p \subset X$ around a point $p \in P$ contains no $p' \neq p$ in P, then the obvious map $B_p/\Gamma_p \to X/\Gamma_P$ is one-to-one.*

(b) *If P is separated* on bounded subsets, i.e., $|p_1 - p_2| \geqslant r$ *for some positive monotone decreasing function $r = r_{x_0}(R) > 0$ depending on the distance $R = |p_1 - x_0|$ from a chosen point $x_0 \in X$, and for all $p_1 \neq p_2$ in P, then there is a subset $Q \subset P$ such that Γ_P is freely generated by the groups Γ_q, $q \in Q$. That is, the natural homomorphism from the free product $*_Q \Gamma_q$ to Γ_P is an isomorphism.*

Proof. Take a convex subset $Y \subset X$ and see how it behaves under the projection to X/Γ_p for some $p \in P$.

If p does not lie in the closure of Y, our map $Y \to X/\Gamma_p$ is locally isometric with locally convex boundary, and since X/Γ_p is CAT(0), our Y *isometrically* maps onto a *convex* subset in X/Γ_p. In other words, the γ-translates $\gamma(Y)$ do not meet Y for all $\mathrm{id} \neq \gamma \in \Gamma_p$.

Next, let us assume p lies in the boundary ∂Y and suppose Y is *strictly* convex at p, i.e., there is *no* geodesic segment in the closure of Y containing p as an *interior* point of this segment. Then every such segment, apart from one of its ends, is locally convex in X/Γ_p and, hence, convex. Consequently, Y injects to X/Γ_p away from p, and thus the Γ_p-orbit $\Gamma_p(Y) \subset X$ consists of the translates $\gamma(Y)$ meeting at p and *nowhere else*. This applies, in particular, to ε-neighborhoods of convex subsets in $X \backslash \{p\}$ for $\varepsilon > 0$ as these are *strictly* convex at *all* their boundary points in CAT(0)-spaces.

We denote the ε-neighborhood of Y by $Y + \varepsilon$, observe that $\Gamma_p(Y+\varepsilon) = \Gamma_p(Y) + \varepsilon$, and see that $\Gamma_p(Y+\varepsilon)$ consists of *disjoint* translates of $Y + \varepsilon$ for $\varepsilon < \mathrm{dist}(p, Y)$ which meet together at p for $\varepsilon = \mathrm{dist}(p, Y)$ so that $\Gamma_p(Y+\varepsilon)$ becomes convex as well as Γ_p-invariant for $\varepsilon > \mathrm{dist}(p, Y)$.

Now see what happens when such a Γ_p-invariant growing ε-neighborhood hits another point $p' \in P$. More generally, let $Y \subset X$ be a strictly convex subset invariant under the group Γ_{Y° generated by all p in P contained in the interior of Y and let Γ_Y be generated by Γ_{Y° and all $p \in P$ contained in the boundary of Y. Two γ-translates of Y for $\gamma \in \Gamma_Y$

may be in three kinds of mutual positions:

(1) $\gamma_1(Y) = \gamma_2(Y)$ for $\gamma_1\gamma_2^{-1} \in \Gamma_{Y^\circ}$;

(2) $\gamma_1(Y)$ meets $\gamma_2(Y)$ at a single point, e.g., $\gamma(Y)$ meets Y at p for every $\gamma \in \Gamma_p$ and $p \in P \cap \partial Y$;

(3) $\gamma_1(Y)$ is disjoint from $\gamma_2(Y)$.

Since the local convexity is preserved at the meeting points p, the orbit $\Gamma_Y(Y)$ is convex and the map $Y/\Gamma_{Y^\circ} \to X/\Gamma_Y$ is injective. Furthermore, the group Γ_Y is *freely* generated by Γ_{Y° and the groups Γ_p for $p \in R$, where $R \subset P \cap \partial Y$ intersects each Γ_{Y°-orbit of $p \cap \partial Y$ at a single point. This is sufficient to prove (a).

Indeed, take R-balls $B(R) \subset X$ around some point $x_\bullet \in X$ (e.g., some $p_0 \in P$) and let

$$\tilde{B}(R) = \Gamma_{B^\circ(R)}(B(R))$$

(where "balls" are assumed closed,

$$B(R) = \{x \in X : |x - x_\bullet| \leqslant R\},$$

and $B^\circ(R)$ denotes the interior where $|x - x_\bullet| < R$). These $\tilde{B}(R)$ are convex as well as $\Gamma_{B^\circ(R)}$ invariant and their projections to $X/\Gamma_{B(R)}$ are injective and convex. Therefore, these remain locally convex as we pass to X/Γ_P and so $\tilde{B}(R)$ *injectively* project to balls in X/Γ_P. This ensures the inequality $K(X/\Gamma_P) \leqslant 0$ at the (suspicious) points coming from $p \in P$ and proves the local, and hence global, convexity of balls in X/Γ_P as these are isometric to $\tilde{B}(R)/\Gamma_{B(R)}$.

Finally, we turn to (b) and notice that the above suffices to show that every *finite* subset $P' \subset P$ contains a subset $Q' \subset P'$ such that the group Γ' generated by Γ_p, $p \in P'$, is *freely* generated by Γ_q, $q' \in Q'$. However, if, for example, points in $P \backslash B(R)$ accumulate to the boundary of B, then we cannot (?) claim the freedom property. (Yet the injectivity of the map $B(R)/\Gamma_{B^\circ(R)} \to X/\Gamma_P$ follows from what happens to finite subset $P' \subset P$.) We need at this stage the separation property of P along with the *uniform* convexity of the balls that yields (this is all we need) a universal upper bound on the diameter of the intersection $(\tilde{B}(R)+\varepsilon) \cap \gamma(\tilde{B}(R)+\varepsilon)$ in terms of R and ε, where $\tilde{B}(R)$ is supposed to be disjoint from $\gamma\tilde{B}(R)$.

It follows that for every $R < \infty$ and $\delta > 0$ there exists $\varepsilon > 0$ such that there is no non-trivial triple intersection between γ-translates of $\tilde{B}(R)$ for $\gamma \in \Gamma_{P \cap (\tilde{B}(R)+\varepsilon)}$: that is, if

$$\gamma_1\tilde{B}(R) \cap \gamma_2\tilde{B}(R) \cap \gamma_3\tilde{B}(R) \neq \emptyset,$$

then $\gamma_i \tilde{B}(R) = \gamma_j \tilde{B}(R)$ for some $i \neq j = 1, 2, 3$, provided the subset $P \cap (\tilde{B}(R) + 1) \subset X$ is δ-separated or, equivalently, $P \cap B(R+1)$ is δ-separated. Hence, the group $\Gamma_{\tilde{B}(R)+\varepsilon}$ is *freely* generated by $\Gamma_{\tilde{B}(R)}$ and Γ_p, $p \in R$, where $R \subset P \cap (\tilde{B}(R) + \varepsilon)$ intersects each $\Gamma_{\tilde{B}(R)}$ orbit of $P \cap (\tilde{B}(R) + \varepsilon) \backslash \tilde{B}(R)$ at a single point and (e) follows as in the case of finite sets $P \cap B(R)$. Q.E.D.

§28. Remarks

(a) The above argument shows that *strict convex independence* of points $p \in P' \subset P$ implies free independence of subgroups $\Gamma_{p'}$, $p' \in P'$, where "strict convex independence" refers to the existence of a strictly convex subset $Y \subset X$ with $P' \subset \partial Y$.

(b) Let us indicate a generalization of Proposition 27 (in the spirit of the Cartan–Hadamard theorem for *non-rigid* orbispaces, compare [5] and [2], where the relevant subsets (e.g., $\tilde{B}(R)$) may be non-convex in X but project to (locally) convex subsets in X/Γ_p).

A rotation family $\{\Gamma_p\}$ is called *regular* if the subset P is closed and the function $p \mapsto \Gamma_p$ is semicontinuous, i.e., each $p \in P$ admits a neighborhood $U_p \subset P$ such that $\Gamma_{p'} \subset \Gamma_p$ for all $p' \in U_p$. (There often exists a rather regular stratification of P such that Γ_p is constant on each stratum.)

Generalization of Proposition 27. *If $\{\Gamma_p\}$ is regular, free away from P and the quotient spaces X/Γ_p are* CAT(κ) *for some $\kappa \leqslant 0$ and all $p \in P$, then the rotation group Γ_P is discrete and the quotient space X/Γ_P is* CAT(κ).

(c) Proposition 27 in its present form has rather limited applications (see below) but it gains in significance when generalized to spaces X with "approximately negative curvature" (see [6]).

§29. Coning CAT-spaces and their subspaces

Let us look at the disk of radius r in the standard space of constant curvature κ as the cone over its boundary, $D_{\kappa,r} = C_{\kappa,r}(\partial D_{\kappa,r})$, where we are mainly interested in $\kappa \leqslant 0$ (and where one should restrict to $r < \pi/\sqrt{\kappa}$ for $\kappa > 0$). Then, for an arbitrary geodesic (path) metric space X, one defines the (path) metric cone $C_{\kappa,r}(X)$ as $X \times [0, r]$ with the base $X \times 0$ shrunk to a single point: the apex, also called the *center* of the cone, where the metric is given by the same rule as in $D_{\kappa,r}$. Namely, every short geodesic segment S in $C_{\kappa,r}(X)$ away from the apex projects to a *geodesic* segment in X, say $\underline{S} \subset X$, and the cone $C_{\kappa,r}(\underline{S}) \subset C_{\kappa,r}(X)$ is isometric to the sector in $D_{\kappa,r}$ over the arc $\underline{S}' \subset \partial D_{\kappa,r}$ with length equal that of $\underline{S}$.

The curvature K of $C_{\kappa,r}(X)$ away from the apex can be evaluated in terms of $K(X)$: this curvature K is $\leqslant \kappa$ if (and only if) the curvature $K(X)$ is bounded by the curvature κ_r of the 2-sphere of radius r in the standard 3-space with curvature κ. In particular, $K < 0$ if $K(X) \leqslant 0$. Furthermore, if X is CAT(κ_r), then $K \leqslant \kappa$ also at the apex of the cone. (All this is well known and rather obvious.) In particular, the *unit Euclidean cone* $C_{0,1}(X)$ has $K \leqslant 0$ for all CAT(0)-spaces X. (One may think of $C_{0,1}(X)$ as the ordinary Euclidean cone over X, where X is isometrically immersed into the unit sphere in some $\mathbb{R}^n$.)

Next, let $U_i \subset X$, $i \in I$, be a collection of subsets in X and

$$X^\bullet \overset{\text{def}}{=} C_{\kappa,r}(X, \{U_i\})$$

be obtained by attaching the cones $U_i^\bullet = C_{\kappa,r}(U_i)$ to X across $U_i = U_i \times r$, for all $i \in I$. Notice that every two cones $U_i^\bullet$ and $U_j^\bullet$ in $X^\bullet$ intersect across $U_i \cap U_j \subset X \subset X^\bullet$. One can artificially enlarge these intersections by gluing pairs $U_i^\bullet$ and $U_j^\bullet$ across larger subsets in $C_{\kappa,r}(U_i \cap U_j)$. For example, given a positive function $\varphi(d)$, we define functions φ_{ij} on $U_i \cap U_j$, as φ of the distance $d = d(x)$, $x \in U_i \cap U_j$, to the boundary of $U_i \cap U_j$ in $U_i \cup U_j$, i.e.,

$$d(x) = \operatorname{dist}(x, (U_i \cup U_j) \backslash (U_i \cap U_j)).$$

Then we glue $U_i^\bullet$ to $U_j^\bullet$ across the subset of pairs (x, ρ) where $\rho \leqslant \varphi_{ij}(x)$ and observe that this "gluing" defines an *equivalence relation* on the disjoint union of the cones $U_i^\bullet$, and hence on $X^\bullet$, provided the function $\varphi(d)$ is monotone increasing.

§30. Useful example

Let X be a tree and U_i be double infinite geodesic lines in X, where all intersections are segments of lengths $\ell_{ij} \leqslant \ell_0 < \infty$. Take $\varphi = \varphi_{\kappa,r,\ell_0}$ such that

$$W_{ij} \subset C_{\kappa,r}(U_i \cap U_j) \subset C_{\kappa,r}(U_j) = C_{\kappa,r}(U_i) = C_{\kappa,r}(\mathbb{R})$$

looks as in the picture below.

Consider the space X^φ obtained from $X^\bullet = C_{\kappa,r}(X, \{U_i\})$ by gluing every $U_i^\bullet$ to $U_j^\bullet$ across the above W_{ij}.

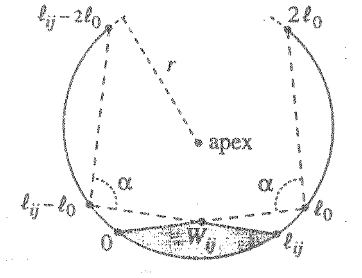

Fig. 8.

If the angle $\alpha = \alpha_\varphi = \alpha_{\kappa,r,\ell_0}$ *in Figure* 8 *is* $\leqslant 2\pi/3$, *then the space* X^φ *is* CAT(κ). *In particular, if* $\ell_0 \leqslant 2\pi r/\sigma$, *then the space* X^φ *for* $\varphi = \varphi_{0,r,\ell_0}$ *is* CAT(0).

Proof. It is clear that X remains CAT(κ) if we attach our cones to *disjoint* lines U_i or, more generally, if there are no triple meeting points between U_i, since the intersections W_{ij} are *convex* in $U_i^\bullet$. The problem may appear when three (or more) lines come together as three lines joining the pairs of ends in the infinite tripod do. But the condition $\alpha \geqslant 2\pi/3$ makes the cycles in the links of such meeting point longer than 2π, which implies $K(X^\ell) \leqslant \kappa$ at all points. This yields CAT(κ)-property since X^φ is (obviously) simply connected. Q.E.D.

§31

Corollary. *Let* Γ_i, $i \in I$, *be isometry groups acting on* X, *where each* Γ_i *is generated by a single isometry* $\gamma_i : X \to X$ *mapping the line* $U_i = \mathbb{R}$ *into itself via a translation* $x \mapsto x + R_i$, *and let* $\Gamma_\bullet$ *be generated by the groups* Γ_i, $i \in I$. *If* $R \geqslant 6\ell_0$, *then the space* X^φ/Γ *is* CAT(0) *for a suitable* φ *and if* $R > 6\ell_0$, *one can achieve* CAT($\kappa > 0$) *for* X^φ/Γ. *Consequently,* Γ *is freely generated by some subgroups* Γ_j *among* Γ_i.

Proof. Take $r = 3\ell_0/\pi$, apply the coning construction $C_{0,r}$. Then the corresponding X^φ is CAT(0) for $R \geqslant 6\ell_0$. And if $R > 6\ell_0$, we use $C_{\kappa,r}$ with $\kappa < 0$ with $|\kappa|$ being small compared to $R - 6\ell_0$, and r slightly larger than $3\ell_0/\pi$, thus getting $K(X^\varphi/\Gamma_\bullet) \leqslant k < 0$. Q.E.D.

§32. Remarks

(a) This Corollary shows, in particular, that the *small cancellation groups* Γ with the *metric* $^1/_6$-condition are CAT(0) serving as fundamental groups of 2-polyhedra with $K \leqslant 0$, while the ${}^1/_{6+\varepsilon}$-condition ensures CAT($\kappa < 0$)-property.

Recall, that such a Γ is given by finitely many *relations* that are just some elements, say $\gamma_1, \dots, \gamma_k$ in the free group F on some generators. This F acts on the standard tree X, and Γ_i are generated by the (infinitely many) F-conjugates of $\gamma_1, \dots, \gamma_k$. Then $\Gamma = F/\Gamma_\bullet$ freely and isometrically acts on our space $X^\varphi/\Gamma_\bullet$ that is CAT(0) in the ${}^1/_6$-case by the above discussion, where the quotient $(X/\Gamma_\bullet)/\Gamma$ is obtained from the standard 2-polyhedron P representing Γ by little geometric tinkering (corresponding to $X^\bullet \rightsquigarrow X^\varphi$) making $K \leqslant 0$ while keeping the homotopy type (and the dimension) of P intact.

(b) The above approach to small cancellation groups (which is essentially well known) will be extended in another paper to spaces X with approximately negative curvature and general "convex" groups Γ_i.

(c) There is another way of turning $X/\Gamma_\bullet$ into a CAT(0)-space, consisting in taking the nerve Y of the covering of X by U_i and then dividing Y by $\Gamma_\bullet$ (compare §16). Unfortunately, the upper curvature bound at the fixed vertices of Γ_i depends, besides R, on the dimension of Y, i.e., the maximal multiplicity of intersection of U_i, which makes the nerve construction unsuitable for most interesting Γ. I do not exclude, however, an improvement of this making all *combinatorially* ${}^1/_6$-groups CAT(0), but this seems hard to achieve for general small cancellation groups (see [7]), where the traditional approach via Dehn's diagrams remains indispensable.

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