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UNIQUENESS THEOREMS FOR ANALYTIC VECTOR–VALUED FUNCTIONS

ABSTRACT. Using the Berezin transformation, we give a multidimensional analog of a uniqueness theorem of N.Nikolski concerning distance functions and subspaces of a Hilbert space of analytic functions. Then, we establish some uniqueness properties drawing connections between two analytic X-valued functions F and G that satisfy $\|F(z)\| \equiv \|G(z)\|$, $\forall z \in \Omega$, where X is a Banach space and Ω a connected domain in $\mathbb{C}^n$. The particular case where $X = \ell_n^p$ and $\Omega = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ will lead us to the notion of flexible and inflexible functions. We give a complete description of these functions when $p = +\infty$, $n \in \mathbb{N}^*$ and when $n = 2$, $1 \leqslant p \leqslant +\infty$.

INTRODUCTION

Let Ω be a domain in $\mathbb{C}$ and X a Banach space of holomorphic functions continuously imbedded into the space $\mathrm{Hol}(\Omega)$ of all such functions in Ω . Assume that X is stable with respect to multiplication by the independent variable z:

$$f \in X \Longrightarrow zf \in X .$$

The problem (very far from being solved) is to describe all (closed) z-invariant subspaces of X:

$$E \subset X, \quad E = \overline{E}, \quad zE \subset E .$$

Solutions are known for very scarce cases related to the Hardy spaces $H^p(\mathbb{D})$, where $1 \leqslant p \leqslant +\infty$ and $\mathbb{D}$ is the unit disk:

$$H^p(\mathbb{D}) := \{ f \in \mathrm{Hol}(\mathbb{D}) \ : \ \sup_{0 \leqslant r < 1} \int_{\partial \mathbb{D}} |f(r\zeta)|^p \, dm(\zeta) \ < +\infty \};$$

this case is classical with the following Beurling description of z-invariant subspaces $E \subset H^p$:

$$E = \Theta H^p \ ,$$

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where Θ stands for an inner function in the disk $\mathbb{D}$ (see, for example, [15]).

In order to study this problem, in [16] N. Nikolski, for a closed subspace $E \subset X$, introduced the distance function defined on Ω by

$$\Theta_E(\lambda) := \sup\{\, |f(\lambda)| \,:\, f \in E\,,\ \|f\| \leqslant 1\,\}\,, \quad \lambda \in \Omega.$$

In the case of Hilbert spaces, N. Nikolski gave a uniqueness theorem establishing one-to-one correspondence between subspaces and distance functions. The key step to this result is the following lemma about analytic vector-valued functions (going back to [7]).

Lemma 0.1. *If F and G are analytic Hilbert space valued functions on a connected domain $\Omega \subset \mathbb{C}$ such that*

$$\|F(z)\| = \|G(z)\|\,, \quad \forall z \in \Omega\,,$$

then there is an isometry V from the closed linear hull $\operatorname{Span}\{\,F(z) \,:\, z \in \Omega\,\}$ *onto* $\operatorname{Span}\{\,G(z) \,:\, z \in \Omega\,\}$ *such that*

$$V\,F(z) = G(z), \quad \forall z \in \Omega\,.$$

In §1, we give our first generalization of this result: a uniqueness theorem for the Berezin transformation. Then, from this theorem we deduce a multidimensional analog of the result of [16]. We conclude that section with studying the examples of $\mathbb{C}^n$ and Sobolev spaces $W_p^l(\Omega)$: we prove that the distance function distinguishes invariant subspaces, but without this invariance assumption there exist some subspaces for which uniqueness fails.

As we have already emphasized, in the case of Hilbert spaces the description of subspaces in terms of the distance function is closely related to the theory of analytic vector-valued functions. Though this theory is well developped (see, for example, [17]), there exist interesting open problems, in particular, the following uniqueness problem, which we consider in §2.

Let Ω be a connected domain in $\mathbb{C}^n$, $n \geqslant 1$, and let X be a Banach space; let further F and G be analytic X-valued functions on Ω such that

$$\|F(z)\| = \|G(z)\|\,, \quad \forall z \in \Omega\,. \tag{$*$}$$

The problem is to draw from $(*)$ as much information as possible about the relationship between F and G. In order to understand these relationship, we try to answer the following questions:

(Q1) Is there a transformation V verifying

$$V\,F(z) = G(z), \quad \forall z \in \Omega\,?$$

(Q2) Does relation $(*)$ imply that

$$\dim(\mathrm{Span}\{\,F(z),\, z \in \Omega\,\}) = \dim(\mathrm{Span}\{\,G(z),\, z \in \Omega\,\})\,?$$

(Q3) If we suppose that F admits analytic continuation across the boundary $\partial\Omega$, does G admit analytic continuation too?

(Q4) Finally, if we only suppose that relation $(*)$ is true on a nonempty domain strictly contained in Ω, is it possible to obtain the identity in the entire Ω?

The results of §2 are organized as follows. First, we suppose that X is a Hilbert space; Theorem 2.1.1 is a multidimensional analog of Lemma 0.1, which allows us to answer three other questions in the positive. Then, with the hypothesis that $\dim(\mathrm{Span}\{\,F(z)\,:\,z \in \Omega\,\})=1$, we answer these questions when X is a smooth and strictly convex Banach space. Without this additional hypothesis, in the case of ℓ^p_n spaces, Theorem 2.2.5 gives a positive answer to question (Q4) and Theorem 2.2.7 shows that there exists a finite number of singular points for the analytic extension of G in question (Q3).

Finally, in order to answer question (Q1) in the case of ℓ^p_n spaces, we introduce the notion of flexible and inflexible functions. Then, we obtain a complete description of inflexible functions in ℓ^∞_n, $n \in \mathbb{N}^*$, and a complete description in ℓ^p_2, $p \geqslant 1$; as we shall see, $p = +\infty$, $p = 1$, and $p = 2$ are exceptional values for different reasons and the answer in other cases depends on the arithmetical nature of p.

§1. Distance functions and invariant subspaces

Let Ω be a connected domain in $\mathbb{C}^n$, $n \geqslant 1$, and let $(H, \langle\,,\,\rangle)$ be a Hilbert space continuously imbedded into the space $C(\Omega)$ of continuous functions on Ω. We denote by $L(H)$ the Banach space of all bounded linear operators on H and by $\lambda \longmapsto k_\lambda$ the reproducing kernel of H, i.e., a unique H-valued function in Ω such that

$$f(\lambda) = \langle f, k_\lambda\rangle\,, \quad f \in H,\, \lambda \in \Omega\,.$$

1.1. Berezin transformation.

Let $T \in L(H)$. The *Berezin transformation* of T is a function B_T

defined on Ω by

$$B_T(\lambda) := \langle Tk_\lambda, k_\lambda\rangle\,, \qquad \lambda \in \Omega.$$

(For an introduction to the Berezin transformation, we refer the reader to [4] or [5]).

In what follows, we assume that H is a Hilbert space of analytic functions on Ω, which is equivalent to the fact that $\lambda \longmapsto k_\lambda$ is a co-analytic function on Ω.

In the theorem below, we establish a one-to-one correspondence between $L(H)$ and the set $\{B_T,\, T \in L(H)\}$. This correspondence has been mentioned several times in a very particular case of the space $H^2(\mathbb{C}^n\,,\, d\mu)$ of entire functions that are L^2-summable with respect to the Gaussian measure $d\mu(z) := \frac{1}{(2\pi)^n} e^{-\frac{|z|^2}{2}} dm(z)$, where m is Lebesgue measure (see, for instance, [6], [8], and [9]).

Theorem 1.1.1. *Let $T_1,\, T_2 \in L(H)$ be such that*

$$B_{T_1}(z) \equiv B_{T_2}(z)\,, \quad \forall z \in \Omega.$$

Then $\quad T_1 \equiv T_2$.

Proof. Fix a point $\mu \in \Omega$ and consider the following scalar-valued functions f_1 and f_2:

$$f_1(z) := \langle T_1 k_\mu, k_z\rangle\,, \ \text{ and } \ f_2(z) := \langle T_2 k_\mu, k_z\rangle\,, \ z \in \Omega.$$

The reproducing kernel being co-analytic, it is clear that both f_1 and f_2 are holomorphic on Ω with

$$\partial^\alpha f_1(\mu) = \langle T_1 k_\mu, \overline{\partial}^\alpha k_\mu\rangle\,, \quad \forall \alpha \in \mathbb{N}^n, \tag{1}$$

and

$$\partial^\alpha f_2(\mu) = \langle T_2 k_\mu, \overline{\partial}^\alpha k_\mu\rangle, \quad \forall \alpha \in \mathbb{N}^n, \tag{2}$$

where ∂^α denotes the usual differential operator of complex analysis. Moreover, we have

$$\langle T_1 k_z, k_z\rangle \equiv \langle T_2 k_z, k_z\rangle\,, \quad \forall z \in \Omega\,.$$

By differentiation of this identity, we obtain, for all multi-indices $\alpha \in \mathbb{N}^n$

$$\langle T_1 k_z, \overline{\partial}^\alpha k_z\rangle = \langle T_2 k_z, \overline{\partial}^\alpha k_z\rangle\,, \quad \forall z \in \Omega\,.$$

Using (1) and (2), we arrive at

$$\partial^\alpha f_1(\mu) = \partial^\alpha f_2(\mu)\,, \quad \forall \alpha \in \mathbb{N}^n,$$

whence we deduce that $f_1 \equiv f_2$. Since $\mathrm{Span}\{k_z\,,\, z \in \Omega\} = H$, we get the desired conclusion $T_1 \equiv T_2$. $\square$

Corollary 1.1.2. *Let $T_1, T_2 \in L(H)$ be such that*

$$\|T_1 k_z\| \equiv \|T_2 k_z\|, \quad \forall z \in \Omega .$$

Then $T_1^* T_1 = T_2^* T_2$ and there are two partial isometries U and V such that $T_2 = VT_1$ and $T_1 = UT_2$.

Proof. The hypothesis is equivalent to the identity

$$B_{T_1^* T_1}(z) = B_{T_2^* T_2}(z), \quad \forall z \in \Omega .$$

Then, by Theorem 1.1.1, we have

$$T_1^* T_1 \equiv T_2^* T_2 .$$

It is now easy to construct two partial isometries U and V such that $T_2 = VT_1$ and $T_1 = UT_2$. □

1.2. Uniqueness theorem for subspaces of a Hilbert space of analytic functions.

H is always assumed to be a Hilbert space of analytic functions on Ω continuously imbedded into the space $C(\Omega)$.

Now we state the promiced uniqueness property for distance functions and subspaces. This generalizes the result obtained by N. Nikolski in [16] from one to several variables.

Theorem 1.2.1. *Let E_1 and E_2 be two closed subspaces of H such that*

$$\Theta_{E_1}(\lambda) \equiv \Theta_{E_2}(\lambda), \quad \forall \lambda \in \Omega .$$

Then $E_1 = E_2$.

Proof. For $i = 1, 2$ and $\lambda \in \Omega$, we have

$$\Theta_{E_i}(\lambda) = \sup\{|\langle f, P_{E_i} k_\lambda\rangle| \,:\, f \in E_i\,,\ \|f\| \leqslant 1\},$$

where P_{E_i} denotes the orthogonal projection onto the subspace E_i. It is evident that $\Theta_{E_i}(\lambda) \leqslant \|P_{E_i} k_\lambda\|$; putting $f := \dfrac{P_{E_i} k_\lambda}{\|P_{E_i} k_\lambda\|}$, we get

$$\Theta_{E_i}(\lambda) = \|P_{E_i} k_\lambda\| \quad \text{for } i = 1, 2 \text{ and } \lambda \in \Omega .$$

Therefore, by Corollary 1.1.2, we have $P_{E_1}^* P_{E_1} = P_{E_2}^* P_{E_2}$, which yields $E_1 = E_2$. □

In the next subsection, we shall show that for Hilbert spaces whose reproducing kernel is not co-analytic the above uniqueness property is no longer true in general.

1.3. Examples of uniqueness and nonuniqueness.

-Distance function and subspaces of $\mathbb{C}^n$ -

For a subspace E of $\mathbb{C}^n$, we define the distance function Θ_E by

$$\Theta_E(p) := \sup\{|x_p|\,, x = (x_1, \dots, x_n) \in E\,, \|x\| = 1\}\,, \quad p = 1, \dots, n\,.$$

Then, we consider

$$E_1 := \mathbb{C}(1, \dots, 1) := \{(\lambda, \dots, \lambda),\, \lambda \in \mathbb{C}\}$$

and

$$E_2 := \mathbb{C}(1, \dots, 1, -1) := \{(\lambda, \dots, \lambda, -\lambda),\, \lambda \in \mathbb{C}\}\,.$$

It is easy to see that

$$\Theta_{E_1}(i) = \Theta_{E_2}(i) = \frac{1}{\sqrt{n}}\,, \quad \forall i = 1, \dots, n,$$

and yet $E_1 \neq E_2$. This example shows that the distance function cannot distinguish arbitrary subspaces of $\mathbb{C}^n$ and underlines the role of analyticity in Theorem 1.2.1.

Nevertheless, if we limit ourselves to some invariant subspaces of $\mathbb{C}^n$, we have a uniqueness property. Indeed, consider the operator U defined by

$$\begin{aligned} U \,:\, \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (x_1, \dots, x_n) &\longmapsto (x_n, x_1 \dots, x_{n-1})\,, \end{aligned}$$

and let k_{λ_j} be the eigenvector associated with the eigenvalue $\lambda_j = \exp(\dfrac{2ij\pi}{n})$. Finally, for a subspace E of $\mathbb{C}^n$, we define the distance function $\widetilde{\Theta}_E$ related to the orthonormal basis $(k_{\lambda_j})_{1\leqslant k\leqslant n}$ by

$$\widetilde{\Theta}_E(p) := \mathrm{Sup}\{|\langle x, k_{\lambda_p}\rangle|\,, \, x \in E\,, \, \|x\| = 1\,\}\,, \quad p = 1, \dots, n\,.$$

Theorem 1.3.1. *Let E and F be subspaces of $\mathbb{C}^n$ invariant under U and such that*

$$\widetilde{\Theta}_E(p) \equiv \widetilde{\Theta}_F(p)\,, \quad \forall p = 1, \dots, n\,.$$

Then $\quad E = F$.

Proof. For the proof it suffices to remark that the subspaces E of $\mathbb{C}^n$ invariant under U are described by

$$E = \mathbb{C}k_{\lambda_{i_1}} \oplus \mathbb{C}k_{\lambda_{i_2}} \oplus \cdots \oplus \mathbb{C}k_{\lambda_{i_s}}\,, (i_1, \dots, i_s) \subset \{1, \dots, n\}. \quad \square$$

-Distance function and subspaces of Sobolev spaces-

Let Ω be a bounded open subset of $\mathbb{R}$, let $l \in \mathbb{N}^*$ and let $1 < p < +\infty$. $C_0^{l-1}(\Omega)$ denotes the space of functions $(l-1)$-times continuously differentiable and having compact support in Ω, and the Sobolev space $W_p^l(\Omega)$ consists of all functions f that, together with all their weak partial derivatives $D^\alpha f$ of order $|\alpha| \leqslant l$, belong to $L^p(\Omega)$.

It is classical that the space $W_p^l(\Omega)$ is an algebra with respect to pointwise multiplication, and that this algebra is continuously imbedded into the space $C_0^{l-1}(\Omega)$ (see, for instance, [1], [11], and [12]). Due to this imbedding, we can consider the distance function, and it is interesting to know if it distinguishes the subspaces of $W_p^l(\Omega)$.

In the case of ideals, the following results give a positive answer.

Theorem 1.3.2. *Let I_1, I_2 be closed ideals of the Sobolev space $W_p^l(\Omega)$. Assume that there is $C > 0$ such that*

$$\Theta_{I_2}(x) \leqslant C\Theta_{I_1}(x) \quad \text{for all } x \in \Omega\,.$$

Then $\quad I_2 \subset I_1$.

One of the main steps in the proof of this theorem is the following lemma.

Lemma 1.3.3. *Assume there is $x_0 \in \Omega$ and $k \leqslant l-1$ such that*

$$f^{(s)}(x_0) = 0 \quad \text{for all } s \leqslant k \text{ and all } f \in I_1\,.$$

Then

$$g^{(s)}(x_0) = 0 \quad \text{for all } s \leqslant k \text{ and all } g \in I_2\,.$$

Proof of Lemma 1.3.3. This Lemma is a straightforward consequence of the Taylor formula and the Hölder inequality. □

Proof of Theorem 1.3.2. Let x_0 be a point of the spectrum $\sigma(I_2)$ of I_2 and g a function in I_2. Recall that the spectrum of an ideal I is a closed set defined by

$$\sigma(I) := \bigcap_{f \in I} f^{-1}(0)\,.$$

We show that there is a function $f \in I_1$ such that

$$f^{(k)}(x_0) = g^{(k)}(x_0)\,, \forall k \leqslant l-1\,. \tag{3}$$

If $g^{(k)}(x_0) = 0$ for all $k \leqslant l-1$, then we may take $f :\equiv 0$. Otherwise, we put

$$r_2 := \mathrm{Min}\{\, k \leqslant l-1 \,:\, g^{(k)}(x_0) \neq 0 \,\}$$

and for $h \in I_1$ define

$$r_1(h) := \begin{cases} l, & \text{if } h^{(k)}(x_0) = 0,\ \forall k \leqslant l-1, \\ \mathrm{Min}\{\, k \leqslant l-1 \,:\, h^{(k)}(x_0) \neq 0 \,\}, & \text{otherwise.} \end{cases}$$

Owing to Lemma 1.3.3, we easily prove *ad absurdum* that there is a function $h \in I_1$ such that $r_1(h) \leqslant r_2$. Then, by using function h and taking into account that I_1 is an ideal of $W_p^l(\Omega)$, we easily construct by induction a function $f \in I_1$ that satisfies relation (3).

This yields

$$I_2 \subset I_1 + J_{x_0}\,, \forall x_0 \in \sigma(I_2)\,,$$

where $J_{x_0} = \{\, h \in W_p^l(\Omega) \,:\, h^{(k)}(x_0) = 0,\ k \leqslant l-1 \,\}$ is the minimal closed primary ideal at x_0. If we denote by I_x the primary component of the ideal I at the point x, we get

$$(I_2)_{x_0} \subset (I_1)_{x_0}, \quad \forall x_0 \in \sigma(I_2)\,.$$

On the other hand, the assumption " $\Theta_{I_2} \leqslant C\Theta_{I_1}$" clearly implies that

$$\sigma(I_1) \subset \sigma(I_2),$$

whence

$$\bigcap_{x_0 \in \sigma(I_2)} (I_2)_{x_0} \subset \bigcap_{x_0 \in \sigma(I_1)} (I_1)_{x_0}\,.$$

Then from the Hanin theorem concerning spectral synthesis of ideals in $W_p^l(\Omega)$ (see [14] and [12]) it follows that $I_2 \subset I_1$, which completes the proof. $\square$

Corollary 1.3.4. *Let I_1, I_2 be closed ideals of Sobolev spaces $W_p^l(\Omega)$. Assume there are $c > 0$, $C > 0$ such that*

$$c\Theta_{I_1}(x) \leqslant \Theta_{I_2}(x) \leqslant C\Theta_{I_1}(x) \text{ for all } x \in \Omega\,.$$

Then $\quad I_1 = I_2$.

Proof. This is an immediate consequence of Theorem 1.3.2. $\square$

Now we show that for arbitrary subspaces of $W_p^l(\Omega)$ the above uniqueness property is no longer true in general. Letting $\Omega :=]a,b[$, $-\infty < a < b < +\infty$, we fix a point $x_0 \in]a,b[$. Define

$$\begin{aligned} f_1 \ :]a,b[&\longrightarrow \mathbb{R}, \\ x &\longmapsto f_1(x) := (x-x_0)^l \end{aligned}$$

and

$$f_2 \ : \ x \longmapsto \begin{cases} f_2(x) = \dfrac{1}{\|f_1\|}(x-x_0)^l & \text{if } x \in]a, x_0], \\ f_2(x) = -\dfrac{1}{\|f_1\|}(x-x_0)^l & \text{if } x \in [x_0, b[\, . \end{cases}$$

Put $E_1 := \mathbb{R} f_1$ and $E_2 := \mathbb{R} f_2$. Then it is easy to check that E_1 and E_2 are closed subspaces of $W_p^l(\Omega)$ such that

$$\Theta_{E_1}(x) \equiv \Theta_{E_2}(x)\,, \quad \forall x \in]a,b[$$

and

$$E_1 \neq E_2 .$$

This (counter)example proves that the distance function cannot distinguish arbitrary subspaces of $W_p^l(\Omega)$.

If $p = 2$, $W_2^l(\Omega)$ is a Hilbert space whose reproducing kernel is not co-analytic; hence the previous example stresses the role of analyticity in Theorem 1.2.1. Contrary to the analytic case, we need an invariance assumption to get uniqueness.

§2. Uniqueness theorems for analytic Banach space valued functions

Let Ω be a connected domain in $\mathbb{C}^n$, X a Banach space and $\mathrm{Hol}(\Omega, X)$ the space of analytic X-valued functions on Ω. For $1 \leqslant p \leqslant +\infty$ and $n \in \mathbb{N}^*$, ℓ_n^p denotes the space $\mathbb{C}^n$ equipped with the norm $\|.\|_p$, where

$$\|(u_1, \ldots, u_n)\|_p = \begin{cases} \sup\{|u_i|, \, 1 \leqslant i \leqslant n\} & \text{if } p = +\infty, \\ \left(\displaystyle\sum_{i=1}^{n} |u_i|^p \right)^{\frac{1}{p}} & \text{if } 1 \leqslant p < +\infty \, . \end{cases}$$

Now we are interested in the second problem explained in the introduction. Consider two analytic X-valued functions F and G on Ω such that

$$\|F(z)\| \equiv \|G(z)\|\,, \quad \forall z \in \Omega\,. \tag{$*$}$$

The problem is to draw from $(*)$ as much information as possible about the relationship between F and G.

2.1. Uniqueness theorems for analytic Hilbert space valued functions.

In this section, $(H, \langle\, , \rangle)$ will denote a Hilbert space.

The theorem below is a multidimensional analog of Lemma 0.1. In fact, this result was proved, in a different framework, by Cowen and Douglas (see [7]).

Theorem 2.1.1. *Let F and G be analytic H-valued functions on Ω such that*

$$\|F(z)\| \equiv \|G(z)\|, \quad \forall z \in \Omega\,.$$

Then there is an isometry V from $\mathrm{Span}\{F(z), z \in \Omega\}$ *onto* $\mathrm{Span}\{G(z), z \in \Omega\}$ *such that*

$$VF(z) \equiv G(z)\,, \quad \forall z \in \Omega\,.$$

Proof. The proof is similar to that of Theorem 1.1.1. $\square$

Corollary 2.1.2. *Let F and G be analytic H-valued functions on Ω. Assume that there is a nonempty domain $\mathcal{O}$ contained in Ω and such that*

$$\|F(z)\| \equiv \|G(z)\|, \quad \forall z \in \mathcal{O}\,.$$

Then

a) $\|F(z)\| \equiv \|G(z)\|$ *for all* $z \in \Omega$.

b) $\dim(\mathrm{Span}\{\, F(z),\ z \in \Omega\,\}) = \dim(\mathrm{Span}\{\, G(z),\ z \in \Omega\,\})$.

c) *If F admits analytic continuation across the boundary $\partial\Omega$, then so does G.*

Proof. These statements follow immediately from Theorem 2.1.1. $\square$

2.2. Uniqueness theorems in the case of smooth and strictly convex Banach spaces.

For a detailed study of the notions of smoothness and strict convexity, we refer the reader to [3] and [13].

The next lemma will be used in the proof of Theorem 2.2.2. Probably, it is well known, but, having no reference, we provide it with a simple proof.

Lemma 2.2.1. *Let X be a strictly convex Banach space, and let $x, y \in X$, $f_x, f_y \in X^*$ be such that*

$$\|f_x\| = \|f_y\| = 1 \text{ and } f_x(x) = \|x\|\,,\ f_y(y) = \|y\|\,.$$

(The existence of f_x and f_y follows immediately from the Hahn–Banach Theorem.) Assume that

$$\|x\| = \|y\| \text{ and } f_x = f_y\,.$$

Then $\quad x = y$.

Proof. It is trivial to see that $f_x\left(\dfrac{x+y}{2}\right) = \|x\|$. Hence

$$f_x\left(\frac{1}{2}\Big(\frac{x}{\|x\|} + \frac{y}{\|y\|}\Big)\right) = 1\,.$$

Since $\|f_x\| = 1$, we have

$$1 \leqslant \left\|\frac{1}{2}\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right)\right\|\,.$$

Since the space X is strictly convex, this implies that $x = y$, which concludes the proof. □

Theorem 2.2.2. *Let X be a smooth and strictly convex Banach space and let $x_0 \in X$. Let f be an analytic scalar-valued function on Ω and G an analytic X-valued function on Ω. Put*

$$\begin{aligned} F\,:\Omega &\longrightarrow X,\\ z &\longmapsto F(z) := f(z)x_0\,, \end{aligned}$$

and assume that $\|G(z)\| \equiv \|F(z)\|$ for all $z \in \Omega$. Then there is a unit vector $C \in X$ such that $G(z) \equiv \|x_0\| f(z) C$, $\forall z \in \Omega$.

Proof. If $x_0 = 0$, the conclusion is trivial and there is nothing to prove. Therefore, we may assume without loss of generality that $x_0 \neq 0$.

The function H defined by

$$H(z) := \frac{G(z)}{\|x_0\| f(z)}$$

is an analytic X-valued function and we have

$$\|H(z)\| \equiv 1 \text{ for all } z \in \Omega\,.$$

The smoothness of X implies that there exists a unique continuous linear functional $f_{H(z)} \in X^*$ such that

$$\|f_{H(z)}\| = 1 \quad \text{and} \quad \langle H(z), f_{H(z)} \rangle = 1\,.$$

Fix a point $z_0 \in \Omega$ and put

$$\begin{aligned} \Phi \,:\, \Omega &\longrightarrow \mathbb{C}, \\ z &\longmapsto \Phi(z) := \langle H(z), f_{H(z_0)} \rangle\,. \end{aligned}$$

It is easy to check that Φ is an analytic function on Ω such that

$$|\Phi(z)| \leqslant \|H(z)\| \|f_{H(z_0)}\| = 1,\, \forall z \in \Omega, \text{ and } \Phi(z_0) = 1\,.$$

Hence, from the maximum principle it follows that $\Phi \equiv 1$. Then Lemma 2.2.1 implies that $H(z) \equiv H(z_0)$, $\forall z \in \Omega$, which completes the proof with $C := H(z_0)$. □

Remark 2.2.3. Therefore, under the conditions that X is a smooth and strictly convex Banach space and $\dim(\mathrm{Span}\{\, F(z) \;:\; z \in \Omega\,\}) = 1$, Theorem 2.2.2 gives a relationship between two functions F and G satisfying equation $(*)$. This result is true, in particular, in the case of the spaces ℓ_n^p and L^p with $1 < p < +\infty$.

Now, we show that the theorem remains valid if Ω is a connected domain in $\mathbb{C}$ and $X = \ell_n^1$. In a similar way, we consider H, the analytic ℓ_n^1-valued function on Ω defined by

$$H(z) := \frac{G(z)}{\|x_0\| f(z)}\,,$$

and put $H := (H_1, \ldots, H_n)$, where $H_i \in Hol(\Omega, \mathbb{C})$.

There is $1 \leqslant r \leqslant n$ such that $H_i \not\equiv 0$, $\forall i = 1, \ldots, r$ and $H_{r+1} \equiv \cdots \equiv H_n \equiv 0$, up to a permutation of the functions $(H_k)_k$. Let $\mathcal{O}$ be a connected domain in Ω such that

$$\bigcup_{i=1}^{r} Z(H_i) \bigcap \mathcal{O} = \emptyset\,,$$

where $Z(H_i)$ denotes the set of zeros of the analytic scalar valued function H_i. Then, for all $z \in \mathcal{O}$ we have

$$1 \equiv \sum_{i=1}^{r} |H_i(z)| \equiv \sum_{i=1}^{r} |H_i^{\frac{1}{2}}(z)|^2\,.$$

Therefore, Theorem 2.1.1 implies that there is an orthogonal matrix $V := (a_{ij})_{i,j}$ in $\mathbb{C}^r$ such that

$$(H_1^{\frac{1}{2}}(z), \ldots, H_r^{\frac{1}{2}}(z)) = V(1, 0, \ldots, 0), \text{ for all } z \in \mathcal{O},$$

whence

$$H_i \equiv a_{i1}^2, \quad \forall i = 1, \ldots, r,$$

which completes the proof for the case $X = \ell_n^1$. $\square$

The case of the space ℓ_n^∞ will be discussed in the next subsection without assumptions on $\dim(\mathrm{Span}\{ F(z) : z \in \Omega \})$. But on the other hand, the question for the spaces L^1 and L^∞ seems to be open. Moreover, in the previous proof the existence of the set $\mathcal{O}$ depends on the property that the zeros of an analytic function of one variable are isolated; this is no longer true for functions of several variables, so, it seems difficult to extend the result obtained for ℓ_n^1 to the multidimensional case.

Remark 2.2.4. If $\dim(\mathrm{Span}\{ F(z) : z \in \Omega \}) = 1$ and X is a smooth and strictly convex Banach space (or $X = \ell_n^1$), then it clearly follows from the above results that the answers to the questions (Q2), (Q3), and (Q4) are positive.

From now on, we shall restrict ourselves to the one variable case, and in order to keep our exposition elementary, we shall assume that $\Omega = \mathbb{D}$. In fact, all results of this and the following subsections remain valid for an arbitrary connected domain in $\mathbb{C}$. But on the other hand, the extension from one to several variables seems difficult because, as we shall see, except for the results in the ℓ_n^∞ case, the subsequent arguments depend on the above-mentioned property that the zeros of an analytic function of one variable on Ω have no accumulation points in Ω.

Two following theorems answer questions (Q3) and (Q4) in the case of the space ℓ_n^p, without any assumption on $\dim(\mathrm{Span}\{ F(z) : z \in \Omega \})$. The idea is to use the results obtained in the case of Hilbert spaces.

Theorem 2.2.5. *Let $1 \leqslant p < +\infty$, let $n \in \mathbb{N}^*$, and let $\mathcal{O}$ be a nonempty domain in $\mathbb{D}$. Let f and g be analytic ℓ_n^p-valued functions on $\mathbb{D}$ such that*

$$\|f(z)\|_p = \|g(z)\|_p, \quad \forall z \in \mathcal{O}.$$

Then

$$\|f(z)\|_p = \|g(z)\|_p, \quad \forall z \in \mathbb{D}.$$

Proof. Put $f := (f_1, \dots, f_n)$ and $g := (g_1, \dots, g_n)$, where $f_i, g_i \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$. If we have $f_i \equiv 0$ for all $i = 1 \dots, n$, then the conclusion is trivial. Therefore, we may assume without loss of generality that there are $1 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n$ such that

$$f_i \not\equiv 0 \text{ for all } i = 1, \dots, r, \quad f_{r+1} \equiv \cdots \equiv f_n \equiv 0$$

and

$$g_i \not\equiv 0 \text{ for all } i = 1, \dots, s, \quad g_{s+1} \equiv \cdots \equiv g_n \equiv 0\,,$$

up to a permutation of functions. Now it is easy to construct a dense and connected domain $\mathcal{U}$ in $\mathbb{D}$ such that $f_1, \dots, f_r$ and $g_1, \dots, g_s$ do not vanish on $\mathcal{U}$.

Define

$$F := (f_1^{\frac{p}{2}}, \dots, f_r^{\frac{p}{2}} \underbrace{, 0, \dots, 0}_{n-r}) \quad \text{and} \quad G := (g_1^{\frac{p}{1}}, \dots, g_s^{\frac{p}{2}} \underbrace{, 0, \dots, 0}_{n-s}).$$

We easily see that both F and G are analytic ℓ_n^2-valued functions on $\mathcal{U}$ such that

$$\|F(z)\|_2 = \|G(z)\|_2, \quad \forall z \in \mathcal{O} \cap \mathcal{U}\,.$$

From Corollary 2.1.2 a) it follows that $\|F(z)\|_2 = \|G(z)\|_2, \forall z \in \mathcal{U}$. Hence

$$\|f(z)\|_p = \|g(z)\|_p, \quad \forall z \in \mathcal{U}\,. \tag{4}$$

Since the set $\mathcal{U}$ is dense, we get identity (4) in the entire unit disk $\mathbb{D}$, which completes the proof. $\square$

Remark 2.2.6. Theorem 2.2.5 turns out to be false for $p = +\infty$, as the following simple counterexample shows. Put $f := (f_1, f_2)$ and $g := (g_1, g_2)$ with

$$\begin{aligned} f_1(z) &:= z, & f_2(z) &:= z + \tfrac{1}{2} \\ g_1(z) &:= i(z + \tfrac{1}{2}), & g_2(z) &:= -(z + \tfrac{1}{2})\,, \end{aligned}$$

and $\mathcal{O} := \{\, z \in \mathbb{D} : \Re e z > -\frac{1}{4}\}$.

Then it is easy to check that

$$\|f(z)\|_\infty = \|g(z)\|_\infty, \quad \forall z \in \mathcal{O}$$

$$\|f(z)\|_\infty \neq \|g(z)\|_\infty, \quad \forall z \in \mathbb{D} \backslash \overline{\mathcal{O}}\,. \qquad \square$$

Theorem 2.2.7. *Let $1 \leqslant p < +\infty$, and let $n \in \mathbb{N}^*$. Let f and g be analytic ℓ_n^p-valued functions on $\mathbb{D}$ such that*

$$\|f(z)\|_p \equiv \|g(z)\|_p\,, \quad \forall z \in \mathbb{D}.$$

Assume that f admits analytic continuation through an open arc Γ on $\mathbb{T}$, and let γ be an arbitrary arc on $\mathbb{T}$ such that $\overline{\gamma} \subset \Gamma$. Then there are a finite number of points $\xi_0, \dots, \xi_k$ on γ such that g admits analytic continuation through $\gamma \backslash \{\xi_0, \dots, \xi_k\}$.

Proof. The proof is similar to that of Theorem 2.2.5. The key point is to come down to analytic ℓ_n^2-valued functions and apply Corollary 2.1.2 c). $\square$

Remark 2.2.8. The conclusion of Theorem 2.2.7 is optimal in the sense that we cannot expect g to admit analytic continuation through the entire boundary γ, as the following simple counterexample shows. Let $f := (f_1, f_2)$ and $g := (g_1, g_2)$ with

$$\begin{aligned} f_1(z) &:= z^2, & f_2(z) &:= 1 \\ g_1(z) &:= \left(\tfrac{1}{\sqrt{2}}(z^p + 1)\right)^{\frac{2}{p}}, & g_2(z) &:= \left(\tfrac{1}{\sqrt{2}}(z^p - 1)\right)^{\frac{2}{p}}. \end{aligned}$$

Then it is easy to see that f and g are analytic ℓ_2^p-valued functions on $\mathbb{D}$ such that

$$\|f(z)\|_p = \|g(z)\|_p, \; \forall z \in \mathbb{D}.$$

The function f is holomorphic on the entire complex plane $\mathbb{C}$ but on the other hand, g cannot be analytically continued through any arc containing 1 or -1. $\square$

Therefore, contrary to the case of Hilbert spaces, there are singular points.

2.3. The problem of flexible and inflexible functions in ℓ_n^p-spaces.

Let $1 \leqslant p < +\infty$, $p \neq 2$, $n \in \mathbb{N}^*$ and $\Omega = \mathbb{D}$. Let $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_n)$ be analytic ℓ_n^p-valued functions on $\mathbb{D}$, $f \not\equiv 0$, $g \not\equiv 0$. Assume that f and g satisfy

$$\sum_{i=1}^{n} |f_i(z)|^p = \sum_{i=1}^{n} |g_i(z)|^p\,, \quad \forall z \in \mathbb{D}. \tag{$*$}$$

Now we are interested in the first question asked in the Introduction. Is there a transformation V verifying

$$V f(z) = g(z), \quad \forall z \in \Omega\,?$$

How to characterize such transformations? If $\dim(\mathrm{Span}\{ f(z),\ z \in \mathbb{D}\}) = 1$, Theorem 2.2.2 and Remark 2.2.3 give an answer to this question.

To study this problem in the case where $\dim(\mathrm{Span}\{ f(z),\ z \in \mathbb{D}\}) > 1$, we need to introduce the notion of flexible and inflexible functions. In order to motivate the definition, we give two examples.

Examples 2.3.1. First, we consider $f(z) := (z, 1)$, $z \in \mathbb{D}$, and $p \notin 2\mathbb{N}$. Let us show that if $g := (g_1, g_2) \in \mathrm{Hol}\,(\mathbb{D}, \ell_2^p)$ satisfies relation $(*)$, then there is a unitary transformation U of ℓ_2^p such that $g(z) = Uf(z)$ for all $z \in \mathbb{D}$.

By the hypothesis, we have

$$|g_1(z)|^p + |g_2(z)|^p = |z|^p + 1, \quad \forall z \in \mathbb{D}. \tag{5}$$

Theorem 2.2.2 (or Remark 2.2.3 if $p = 1$) implies that $g_1 \not\equiv 0$ and $g_2 \not\equiv 0$. Then there is a sufficiently small number $r > 0$ such that

$$\Big(Z(g_1) \bigcup Z(g_2)\Big) \bigcap D(0, r)\backslash\{0\} = \emptyset\,.$$

For all $z \in D(0, r)\backslash[0, r]$, (5) becomes

$$|g_1^{\frac{p}{2}}(z)|^2 + |g_2^{\frac{p}{2}}(z)|^2 = |z^{\frac{p}{2}}|^2 + 1\,.$$

Hence from Theorem 2.1.1 it follows that there is an orthogonal matrix

$$V := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

on $\mathbb{C}^2$ such that for all $z \in D(0, r)\backslash[0, r]$ we have

$$\begin{cases} g_1^{\frac{p}{2}}(z) = \alpha z^{\frac{p}{2}} + \beta, \\ g_2^{\frac{p}{2}}(z) = \gamma z^{\frac{p}{2}} + \delta. \end{cases} \tag{6}$$

If $\beta \neq 0$, the first relation in (6) implies that $g_1(0) \neq 0$; therefore, we may suppose that $g_1^{\frac{p}{2}} \in \mathrm{Hol}\,(D(0, r))$, up to a reduction of r. It follows that $\alpha z^{\frac{p}{2}}$ is analytic in a neighborhood of 0, which is impossible, in view of $p \notin 2\mathbb{N}$, unless $\alpha = 0$. Consequently, either $\beta = 0$ or $\alpha = 0$, which yields

$$\text{either } \Big(g_1 \equiv \alpha^{\frac{2}{p}} z \text{ and } g_2 \equiv \delta^{\frac{2}{p}}\Big) \text{ or } \Big(g_1 \equiv \beta^{\frac{2}{p}} \text{ and } g_2 \equiv \gamma^{\frac{2}{p}} z\Big)\,.$$

Thus, in both cases there is a unitary transformation U of ℓ_2^p such that

$$g(z) = Uf(z), \quad \forall z \in \mathbb{D}.$$

For the second example, we consider $f := (f_1, f_2)$ where f_1 and f_2 are both analytic in $\overline{\mathbb{D}}$, without zeros in $\mathbb{D}$, and such that

$$|f_1(z)| < |f_2(z)|, \quad \forall z \in \mathbb{D}.$$

Define

$$\begin{cases} g_1(z) := \left(\dfrac{1}{\sqrt{2}}(f_1^{\frac{p}{2}} + f_2^{\frac{p}{2}})\right)^{\frac{2}{p}} \\ \text{and} \\ g_2(z) := \left(\dfrac{1}{\sqrt{2}}(f_1^{\frac{p}{2}} - f_2^{\frac{p}{2}})\right)^{\frac{2}{p}} . \end{cases}$$

By our assumptions and the Rouché Theorem, it is clear that g_1 and g_2 are analytic in $\mathbb{D}$. Moreover, for all $z \in \mathbb{D}$ we have

$$|g_1(z)|^p + |g_2(z)|^p = |f_1(z)|^p + |f_2(z)|^p .$$

Hence, $g := (g_1, g_2)$ satisfies relation $(*)$ and yet there is no unitary transformation U of ℓ_2^p such that $g(z) = Uf(z)$, $\forall z \in \mathbb{D}$. $\square$

Therefore, if $\dim(\mathrm{Span}\{\, f(z),\ z \in \mathbb{D}\}) > 1$ and $X = \ell_n^p$, we are led to introduce a notion expressing the difference of the behavior of functions towards equation $(*)$.

Definition 2.3.2. *Let $1 \leqslant p \leqslant +\infty$, $p \neq 2$, and let $f := (f_1, \ldots, f_n)$ be an analytic ℓ_n^p-valued function on $\mathbb{D}$. The function f is said to be inflexible in ℓ_n^p if for any analytic ℓ_n^p-valued function $g = (g_1, \ldots, g_n)$ on $\mathbb{D}$ satisfying*

$$\|f(z)\|_p \equiv \|g(z)\|_p \, , \quad \forall z \in \mathbb{D}, \qquad (*)$$

there is a unitary transformation U_g of ℓ_n^p such that

$$g(z) \equiv U_g f(z)\, , \quad \forall z \in \mathbb{D}.$$

In the opposite case, we call f a flexible function in ℓ_n^p.

For the description of unitary transformations of ℓ_n^p, we refer the reader to [2].

Remark 2.3.3. Actually, almost every function is locally flexible. Indeed, let $f := (f_1, \ldots, f_n)$ be an analytic ℓ_n^p-valued function on $\mathbb{D}$ and assume there is $z_0 \in \mathbb{D}$ and $i \neq j$ such that $f_i(z_0) \neq 0$ and $f_j(z_0) \neq 0$. We may suppose that $f_1(z_0) \neq 0$ and $f_2(z_0) \neq 0$, up to a permutation of functions. We shall show that f is flexible in a neighborhood of z_0.

Let $\varepsilon > 0$ be a sufficiently small number such that $f_i^{\frac{p}{2}} \in \mathrm{Hol}(D(z_0, \varepsilon))$, $i = 1, 2$. By continuity, there is an orthogonal matrix $V : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ and $\delta > 0$ such that $\|V - I\| < \delta$, $V \neq I$, and if G is defined by

$$G := (G_1, G_2) := V(f_1^{\frac{p}{2}}, f_2^{\frac{p}{2}}),$$

then $G_i(z) \neq 0$ for all $z \in D(z_0, \varepsilon)$. Consequently, the function g defined by

$$g := (G_1^{\frac{2}{p}}, G_2^{\frac{2}{p}}, f_3, \ldots, f_n),$$

is an analytic ℓ_n^p-valued function on $D(z_0, \varepsilon)$ such that

$$\|g(z)\|_p \equiv \|f(z)\|_p, \quad \forall z \in \mathbb{D},$$

which implies that f is flexible in a neighborhood of z_0. The main problem is to extend the function g analytically to the entire unit disk $\mathbb{D}$. Therefore, the problem of flexible functions is global. $\square$

Now we give a complete description of inflexible functions in ℓ_n^∞.

-Description of inflexible functions in ℓ_n^∞-

Let $f := (f_1, \ldots, f_n)$ be an analytic ℓ_n^∞-valued function on $\mathbb{D}$, $f \not\equiv 0$. For $i = 1, \ldots, n$, we consider

$$\Omega_i := \{\, z \in \mathbb{D} : |f_i(z)| > |f_k(z)|, \forall k \neq i \,\}.$$

For $I := \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$, we write $|I| := p$; next, f_I is the function defined by

$$f_I := (f_{i_1}, \ldots, f_{i_p}).$$

Theorem 2.3.4. *(f is inflexible in ℓ_n^∞) $\Longleftrightarrow$ ($\Omega_i \neq \emptyset$ for $i = 1, \ldots, n$).*

Proof. First assume that there exists $i_0 \in \{1, \ldots, n\}$ such that $\Omega_{i_0} = \emptyset$. Choose $k_0 \in \{1, \ldots, n\}$ such that $f_{k_0} \not\equiv 0$. Then consider $g := (g_1, \ldots, g_n)$ defined by

$$\begin{cases} g_k := f_k, \quad \forall k \neq i_0 \\ g_{i_0} := \begin{cases} \frac{1}{2} f_{i_0} & \text{if } f_{i_0} \not\equiv 0 \\ \frac{1}{2} f_{k_0} & \text{if } f_{i_0} \equiv 0. \end{cases} \end{cases}$$

It is clear that

$$\|g(z)\|_\infty \equiv \|f(z)\|_\infty, \quad \forall z \in \mathbb{D},$$

and there is no permutation i of $\{1, \ldots, n\}$ such that $|g_{i_k}| \equiv |f_k|$, $\forall k = 1, \ldots, n$. This proves that f is flexible in ℓ_n^∞.

Conversely, assume $\Omega_i \neq \emptyset$, for $i = 1, \dots, n$. Let $g := (g_1, \dots, g_n)$ be an analytic ℓ_n^∞-valued function on $\mathbb{D}$ satisfying relation $(*)$. First, we suppose that there is a nonempty open subset Ω of $\mathbb{D}$ and $i \in \{1, \dots, n\}$ such that

$$|f_i(z)| = \|g(z)\|_\infty\,, \quad \forall z \in \Omega\,.$$

For $k = 1, \dots, n$, we consider the set W_k defined by

$$W_k := \{\, z \in \mathbb{D} \,:\, |g_k(z)| = \|g(z)\|_\infty \,\}\,.$$

The Baire theorem shows that there is $j \in \{1, \dots, n\}$ such that

$$\mathrm{Int} W_j \bigcap \Omega \neq \emptyset\,,$$

which yields

$$|g_j| \equiv |f_i|\,.$$

Then, applying this result to each Ω_i, we construct a permutation i of $\{1, \dots, n\}$ such that

$$|g_{i_k}| \equiv |f_k|, \quad \forall k = 1, \dots, n\,,$$

which proves that f is inflexible in ℓ_n^∞. $\square$

Corollary 2.3.5. *Let $f := (f_1, \dots, f_n)$ be an analytic ℓ_n^∞-valued function on $\mathbb{D}$, $f \not\equiv 0$. Then there is a subset I of $\{1, \dots, n\}$ such that*

$$f_I \text{ is an inflexible function in } \ell_{|I|}^\infty \tag{7}$$

and

$$\text{for every } z \in \mathbb{D} \text{ and every } k \notin I \text{ we have } |f_k(z)| \leqslant \|f_I(z)\|_\infty\,. \tag{8}$$

The set I is not unique; however, $|I|$ is an invariant for the previous properties in the sense that if J is another subset of $\{1, \dots, n\}$ satisfying conditions (7) and (8), then $|I| = |J|$ and for every $k \in J$ there exists $i_k \in I$ such that $|f_k| \equiv |f_{i_k}|$.

Proof. This result easily follows from the techniques used in the proof of Theorem 2.3.4. $\square$

Now we give a complete characterization of inflexible functions in ℓ_2^p .

-Description of inflexible functions in ℓ_2^p, $1 \leqslant p < +\infty$, $p \neq 2$-

Let $f := (f_1, f_2)$ be an analytic ℓ_2^p-valued function on $\mathbb{D}$ such that

$$\dim(\mathrm{Span}\{\, f(z),\ z \in \Omega \,\}) = 2\,. \tag{9}$$

We are searching for necessary and sufficient conditions for f to be inflexible in ℓ_2^p.

First, we clarify our notation. For $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$, $f \not\equiv 0$, k_f denotes the divisor of f, that is, the function defined in $\mathbb{D}$ by

$$k_f(\lambda) := \begin{cases} 0 & \text{if } f(\lambda) \neq 0, \\ m & \text{if } f(\lambda) = \cdots = f^{(m-1)}(\lambda) = 0 \text{ and } f^{(m)}(\lambda) \neq 0 \end{cases}$$

$\mathrm{Supp}\, k_f$ denotes the support of k_f, that is, the set of zeros of f.

The following lemma will be used to simplify the exposition of the theorems concerning inflexible functions in ℓ_2^p.

Lemma 2.3.6. *Let f_1, f_2, and h be analytic scalar-valued functions on $\mathbb{D}$. Then (f_1, f_2) is inflexible in ℓ_2^p if and only if $(f_1 h, f_2 h)$ is inflexible in ℓ_2^p .*

Proof. This Lemma is an immediate consequence of the definition of flexibility. □

Remark 2.3.7. In view of this Lemma, we may suppose without loss of generality that $\mathrm{Supp} k_{f_1} \cap \mathrm{Supp} k_{f_2} = \emptyset$. Indeed, define $k := \min(k_{f_1}, k_{f_2})$. Then, a theorem by Picard (see, for example, [19], Chap.1 §11.) implies that there is a function $g \in \mathrm{Hol}(\mathbb{D})$ such that $k_g \equiv k$, which yields

$$\frac{f_1}{g} \text{ and } \frac{f_2}{g} \text{ are analytic in } \mathbb{D} \text{ and } \mathrm{Supp} k_{\frac{f_1}{g}} \cap \mathrm{Supp} k_{\frac{f_2}{g}} = \emptyset\,.$$

Moreover, from Lemma 2.3.6 it follows that

$$(f_1, f_2) \text{ is inflexible in } \ell_2^p \iff \left(\frac{f_1}{g}, \frac{f_2}{g}\right) \text{ is inflexible in } \ell_2^p\,.$$

Consequently, from now on we suppose that

$$\mathrm{Supp} k_{f_1} \bigcap \mathrm{Supp} k_{f_2} = \emptyset\,. \quad \square \tag{10}$$

To expose the necessary and sufficient conditions for f to be inflexible in ℓ_2^p, we need to introduce some notation.

Notation 2.3.8. We put $\mathcal{O} := \mathbb{D} \backslash Z(f_2)$; let h be an analytic function on $\mathbb{D}$ defined by

$$h(z) := \frac{f_1(z)}{f_2(z)}\,,\ z \in \mathcal{O}\,.$$

For $n \in \mathbb{N}\backslash\{0,1\}$, E_n denotes the subset of $\mathcal{O}$ defined by

$$E_n := \begin{cases} \{\, \lambda \in \mathcal{O} \ : \ h(\lambda) \neq 0\,,\ k_{h'}(\lambda) \in n\mathbb{N}^* - 1\,\} & \text{if } n \text{ is odd} \\ \{\, \lambda \in \mathcal{O} \ : \ h(\lambda) \neq 0\,,\ k_{h'}(\lambda) \in \frac{n}{2}\mathbb{N}^* - 1\,\} & \text{if } n \text{ is even.} \end{cases}$$

We observe that E_n is invariant under the change of h by $\dfrac{1}{h} = \dfrac{f_2}{f_1}$. Finally, if there exist analytic branches of $f_1^{\frac{p}{2}}$ and $f_2^{\frac{p}{2}}$ in $\mathbb{D}$, we consider the following two conditions

$$C_1(p) : \quad \forall \xi_0 \in \overline{\mathbb{D}}\backslash\{0\} \ : \ Z(\xi_0 f_2^{\frac{p}{2}} - f_1^{\frac{p}{2}}) \cup Z(f_2^{\frac{p}{2}} + \overline{\xi_0} f_1^{\frac{p}{2}}) \neq \emptyset$$

and

$$C_2(p,n) : \quad \forall \lambda_0 \in E_n \ : \ \begin{cases} Z\left(f_2^{\frac{p}{2}}(\lambda_0) f_1^{\frac{p}{2}} - f_1^{\frac{p}{2}}(\lambda_0) f_2^{\frac{p}{2}}\right) \cap \complement E_n \neq \emptyset \\ \text{or} \\ Z\left(\overline{f_1^{\frac{p}{2}}(\lambda_0)} f_1^{\frac{p}{2}} + \overline{f_2^{\frac{p}{2}}(\lambda_0)} f_2^{\frac{p}{2}}\right) \cap \complement E_n \neq \emptyset\,. \end{cases}$$

If $n = p$, we put $C_2(p) := C_2(p,p)$.

With this notation, we give necessary and sufficient conditions for f to be inflexible in ℓ^p_2. As we said in the Introduction, the characterization of inflexible functions in ℓ^p_2 depends on the arithmetical nature of p; we are led to distinguish five cases in accordance with whether p is even, odd different from 1, equal to 1, rational, irrational.

We state the result when p is an odd integer, $p \neq 1$.

Theorem 2.3.9. *Let p be an odd integer, $p \neq 1$. If there exists $\lambda_0 \in \mathbb{D}$ such that $k_{f_1}(\lambda_0) - k_{f_2}(\lambda_0) \in 2\mathbb{Z}+1$, then f is inflexible in ℓ^p_2. Otherwise, one can define analytic branches of $f_1^{\frac{p}{2}}$ and $f_2^{\frac{p}{2}}$ in $\mathbb{D}$, and f is inflexible in ℓ^p_2 if and only if both conditions $C_1(p)$ and $C_2(p)$ are satisfied.*

We give the result concerning the case where p is an even integer.

Theorem 2.3.10. *Let $p = 2n$ be an even integer, $n > 1$. A function f is inflexible in ℓ^p_2 if and only if both conditions $C_1(p)$ and $C_2(p)$ are fulfiled.*

The following theorem deals with the case $p = 1$.

Theorem 2.3.11. *A function f is inflexible in ℓ^1_2 if and only if there is $\lambda_0 \in \mathbb{D}$ such that $k_{f_1}(\lambda_0) - k_{f_2}(\lambda_0) \in 2\mathbb{Z}+1$.*

Now, we expose the result for the case where p is rational:

Theorem 2.3.12. *Let $p \in \mathbb{Q}\backslash\mathbb{N}$, $p > 1$. Put $p = 2\frac{r}{s}$, $(r,s) \in \mathbb{N}^* \times \mathbb{N}^*$, $(r,s) = 1$. If there is $\lambda_0 \in \mathbb{D}$ such that $k_{f_1}(\lambda_0) - k_{f_2}(\lambda_0) \notin s\mathbb{Z}$, then f is inflexible in ℓ_2^p. Otherwise, one can define analytic branches of $f_1^{\frac{1}{s}}$ and $f_2^{\frac{1}{s}}$ in $\mathbb{D}$, and we have*

$$(f \text{ is inflexible in } \ell_2^p) \iff \left((f_1^{\frac{1}{s}}, f_2^{\frac{1}{s}}) \text{ is inflexible in } \ell_2^{2r}\right)$$
$$\iff (\text{both conditions } C_1(p) \text{ and } C_2(p,2r) \text{ are satisfied}).$$

Finally, we give the result for the case where p is irrational:

Theorem 2.3.13. *Let $p \in]1,+\infty[\backslash\mathbb{Q}$. If $Supp k_{f_1} \cup Supp k_{f_2} \neq \emptyset$, then f is inflexible in ℓ_2^p. Otherwise, one can define analytic branches of $f_1^{\frac{p}{2}}$ and $f_2^{\frac{p}{2}}$ in $\mathbb{D}$ and f is inflexible in ℓ_2^p if and only if condition $C_1(p)$ is fulfiled.*

We shall give the proof of Theorem 2.3.9. The other results follow from similar arguments.

Proof of Theorem 2.3.9. First, assume that there exists $\lambda_0 \in \mathbb{D}$ such that $k_{f_1}(\lambda_0) - k_{f_2}(\lambda_0) \in 2\mathbb{Z}+1$. Then, applying the argument of example 2.3.1 in a neighborhood of λ_0, we easily show that f is inflexible in ℓ_2^p.

Therefore, from now on we can assume that

$$k_{f_1}(\lambda) - k_{f_2}(\lambda) \in 2\mathbb{Z}, \quad \forall \lambda \in \mathbb{D}.$$

Using (10) yields

$$k_{f_1}(\lambda) \in 2\mathbb{N} \text{ and } k_{f_2}(\lambda) \in 2\mathbb{N}, \quad \forall \lambda \in \mathbb{D}. \tag{11}$$

Suppose that both conditions $C_1(p)$ and $C_2(p)$ are satisfied, and let g_1, g_2 be two analytic functions on $\mathbb{D}$ such that, for all $z \in \mathbb{D}$,

$$|g_1(z)|^p + |g_2(z)|^p \equiv |f_1(z)|^p + |f_2(z)|^p. \tag{12}$$

From (9) and Theorem 2.2.2 it follows that $g_1 \not\equiv 0$ and $g_2 \not\equiv 0$. Hence, there is a sufficiently small number $\varepsilon > 0$ such that

$$(Z(g_1) \cup Z(g_2) \cup Z(f_1) \cup Z(f_2)) \cap D(0,\varepsilon)\backslash\{0\} = \emptyset.$$

Then, for all $z \in \Gamma_\varepsilon := D(0,\varepsilon)\backslash[0,\varepsilon]$, (12) becomes

$$|g_1^{\frac{p}{2}}(z)|^2 + |g_2^{\frac{p}{2}}(z)|^2 = |f_1^{\frac{p}{2}}(z)|^2 + |f_2^{\frac{p}{2}}(z)|^2, \quad \forall z \in \Gamma_\varepsilon.$$

Theorem 2.1.1 implies that there is an orthogonal matrix

$$V := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

such that for all $z \in \Gamma_\varepsilon$ we have

$$\begin{cases} g_1^{\frac{p}{2}}(z) = \alpha f_1^{\frac{p}{2}}(z) + \beta f_2^{\frac{p}{2}}(z) \\ g_2^{\frac{p}{2}}(z) = \gamma f_1^{\frac{p}{2}}(z) + \delta f_2^{\frac{p}{2}}(z)\,. \end{cases}$$

By analytic continuation, we get

$$\begin{cases} g_1^p \equiv \left(\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}}\right)^2 \\ g_2^p \equiv \left(\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}}\right)^2\,. \end{cases} \tag{13}$$

Assume that $\alpha\beta \neq 0$. We show that this leads to a contradiction. If $|\beta| \leqslant |\alpha|$ then, by applying $C_1(p)$ to $\xi_0 := -\dfrac{\beta}{\alpha}$, it is easy to see that there exists

$$z_0 \in Z\left(\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}}\right) \bigcup Z\left(\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}}\right)\,.$$

If $|\alpha| < |\beta|$ then, since $1 = |\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2$, we have $|\delta| \leqslant |\gamma|$ and arguing with $\xi_0 := -\dfrac{\delta}{\gamma}$, we get the same conclusion.

On the other hand, from the system (13) and simple arguments involving on divisors of analytic functions, we deduce that

$$Z\left(\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}}\right) \bigcup Z\left(\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}}\right) \subset E_p\,. \tag{14}$$

This yields that $z_0 \in E_p$, whence, since $C_2(p)$ is fulfilled, we see that there is $\lambda_0 \in \complement E_p$ such that

$$\lambda_0 \in Z\left(f_2^{\frac{p}{2}}(z_0) f_1^{\frac{p}{2}} - f_1^{\frac{p}{2}}(z_0) f_2^{\frac{p}{2}}\right) \bigcup Z\left(\overline{f_1^{\frac{p}{2}}(z_0)} f_1^{\frac{p}{2}} + \overline{f_2^{\frac{p}{2}}(z_0)} f_2^{\frac{p}{2}}\right)\,.$$

If $\lambda_0 \in Z\left(f_2^{\frac{p}{2}}(z_0) f_1^{\frac{p}{2}} - f_1^{\frac{p}{2}}(z_0) f_2^{\frac{p}{2}}\right)$, we easily deduce that

$$\alpha f_1^{\frac{p}{2}}(\lambda_0) + \beta f_2^{\frac{p}{2}}(\lambda_0) = 0\,.$$

If $\lambda_0 \in Z\left(\overline{f_1^{\frac{p}{2}}(z_0)} f_1^{\frac{p}{2}} + \overline{f_2^{\frac{p}{2}}(z_0)} f_2^{\frac{p}{2}}\right)$, then

$$\gamma f_1^{\frac{p}{2}}(\lambda_0) + \delta f_2^{\frac{p}{2}}(\lambda_0) = 0\,.$$

Consequently,

$$\lambda_0 \in Z\left(\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}}\right) \bigcup Z\left(\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}}\right),$$

which is in contradiction with (14) and the fact that $\lambda_0 \notin E_p$. Thus $\alpha\beta = 0$. Then, arguing as in the example 2.3.1, we show that f is inflexible in ℓ_2^p.

Conversely, if condition $C_1(p)$ is not fulfilled, there exists $\xi_0 \in \overline{\mathbb{D}}\backslash\{0\}$ such that

$$Z(\xi_0 f_2^{\frac{p}{2}} - f_1^{\frac{p}{2}}) \cup Z(f_2^{\frac{p}{2}} + \overline{\xi_0} f_1^{\frac{p}{2}}) = \emptyset\,. \tag{15}$$

Put

$$\begin{aligned} \alpha &:= \frac{1}{\sqrt{1+|\xi_0|^2}}\,, \quad \beta := -\alpha\xi_0, \\ \gamma &:= \sqrt{1-|\alpha|^2}\,, \qquad \delta := \frac{-\alpha\beta}{\gamma}. \end{aligned}$$

Then it easy to check that

$$V := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is an orthogonal matrix. Now consider the functions g_1 and g_2 defined by

$$\begin{cases} g_1 := (\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}})^{\frac{2}{p}} \\ g_2 := (\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}})^{\frac{2}{p}} \end{cases}$$

Due to (11) and (15) we show that $\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}}$ and $\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}}$ are analytic and do not vanish in $\mathbb{D}$. Therefore, g_1 and g_2 are analytic in $\mathbb{D}$; since V is orthogonal, we clearly have

$$|g_1(z)|^p + |g_2(z)|^p \equiv |f_1(z)|^p + |f_2(z)|^p\,, \quad \forall z \in \mathbb{D},$$

which proves that f is flexible in ℓ_2^p.

Now, if the condition $C_2(p)$ is not fulfilled, there exists $\lambda_0 \in E_p$ such that

$$\begin{cases} Z\left(f_2^{\frac{p}{2}}(\lambda_0) f_1^{\frac{p}{2}} - f_1^{\frac{p}{2}}(\lambda_0) f_2^{\frac{p}{2}}\right) \cap \complement E_p = \emptyset \\ \text{and} \\ Z\left(\overline{f_1^{\frac{p}{2}}(\lambda_0)} f_1^{\frac{p}{2}} + \overline{f_2^{\frac{p}{2}}(\lambda_0)} f_2^{\frac{p}{2}}\right) \cap \complement E_p = \emptyset\,. \end{cases} \tag{16}$$

Put $\xi_0 := h(\lambda_0)$ ($\neq 0$ because $\lambda_0 \in E_p$) and

$$\alpha := \frac{1}{\sqrt{1+|\xi_0|^p}}, \quad \beta := -\alpha \xi_0^{\frac{p}{2}}$$
$$\gamma := \sqrt{1-|\alpha|^2}, \quad \delta := \frac{-\alpha\beta}{\gamma}.$$

Then it is easy to check that

$$V := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is an orthogonal matrix. Now, consider the functions g_1 and g_2 defined by

$$\begin{cases} g_1 := (\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}})^{\frac{2}{p}} \\ g_2 := (\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}})^{\frac{2}{p}} \end{cases}$$

From (11) it follows that $\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}}$ and $\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}}$ are analytic in $\mathbb{D}$. Moreover, thanks to (16), we show that

$$k_{\alpha f_1^{\frac{p}{2}} + \beta f_2^{\frac{p}{2}}}(\lambda) \text{ and } k_{\gamma f_1^{\frac{p}{2}} + \delta f_2^{\frac{p}{2}}}(\lambda) \in p\mathbb{N}, \quad \forall \lambda \in \mathbb{D}.$$

This implies that g_1 and g_2 are analytic in $\mathbb{D}$, and since V is orthogonal, we have

$$|g_1(z)|^p + |g_2(z)|^p \equiv |f_1(z)|^p + |f_2(z)|^p, \quad \forall z \in \mathbb{D},$$

which proves that f is flexible in ℓ_2^p and concludes the proof. □

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