

Some numerical observations
on number-theoretical properties
of the theta functions
of Dirichlet characters

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Abstract. The classical functional equation for a Dirichlet L -function, $L_\chi(s)$, is equivalent to the functional equation for the corresponding theta function $\theta_\chi(\tau)$. Motivated by a certain form of the latter functional equation, we define special non-degenerate matrices. The inverses of these matrices are of interest due to their relationship with the corresponding Dirichlet L -function; however, this relationship will be explored in a future paper. In this paper, we examine these inverse matrices *per se* because they demonstrate interesting number-theoretic properties. The numerical data allow us to make a number of conjectures.

Key words: Dirichlet characters, theta functions

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1 Main definitions

Let $q > 1$ and $\chi(n)$ be a *primitive Dirichlet character modulo q* . For τ with positive real part let

$$\theta_\chi(\tau) = \sum_{n=1}^{\infty} \chi(n) n^\delta e^{-\frac{\pi n^2}{q} \tau}, \quad (1.1)$$

where

$$\delta = \delta(\chi) = \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1. \end{cases} \quad (1.2)$$

Function $\theta_\chi(\tau)$ satisfies (see, for example, [5, Ch. 1, Sect. 4, Lemma 2]) the identity (named the *functional equation*)

$$\theta_\chi(\tau^{-1}) = \omega \tau^{\delta + \frac{1}{2}} \theta_{\bar{\chi}}(\tau), \quad (1.3)$$

where

$$\omega = \omega(\chi) = \frac{\sum_{k=1}^q \chi(k) e^{\frac{2\pi i k}{q}}}{i^\delta \sqrt{q}} \quad (1.4)$$

and $\bar{\chi}(n)$ is the character conjugate to $\chi(n)$,

$$\bar{\chi}(n) = \overline{\chi(n)}. \quad (1.5)$$

After the foundational work of B. Riemann [4], the identity (1.3) is traditionally used to establish the functional equation for *Dirichlet L -function* $L_\chi(s)$,

$$\xi_\chi(s) = \omega \xi_{\bar{\chi}}(1-s), \quad (1.6)$$

where

$$\xi_\chi(s) = \left(\frac{\pi}{q}\right)^{-\frac{s+\delta}{2}} \Gamma\left(\frac{s+\delta}{2}\right) L_\chi(s). \quad (1.7)$$

A. F. Lavrik [6] showed that, *vice versa*, (1.3) can be deduced from (1.6). Thus the identity (1.3) can be seen as a faithful form of the functional equation for $L_\chi(s)$.

The identity (1.3) can be restated in a slightly different form. Specifically, this *functional* equality implies an infinite series of *numerical* equalities, namely,

$$\left. \frac{d^m}{d\tau^m} \theta_\chi(\tau^{-1}) \right|_{\tau=1} = \left. \frac{d^m}{d\tau^m} \omega \tau^{\delta + \frac{1}{2}} \theta_{\bar{\chi}}(\tau) \right|_{\tau=1}, \quad m = 0, 1, \dots \quad (1.8)$$

These equalities tell us that the left- and right-hand sides of (1.3) have the same derivatives of all orders at $t = 1$. This means that the infinite series of numerical equalities (1.8) implies the identity (1.3). Thus, (1.8) is yet another form of the functional equation for the corresponding L -function $L_\chi(s)$.

Let us introduce some notation.

For the left-hand side of (1.8), we have:

$$\left. \frac{d^m}{d\tau^m} \theta_\chi(\tau^{-1}) \right|_{\tau=1} = \left. \frac{d^m}{d\tau^m} \sum_{n=1}^{\infty} \chi(n) n^\delta e^{-\frac{\pi n^2}{q} \tau^{-1}} \right|_{\tau=1} \quad (1.9)$$

$$= \sum_{n=1}^{\infty} \chi(n) n^\delta \left. \frac{d^m}{d\tau^m} e^{-\frac{\pi n^2}{q} \tau^{-1}} \right|_{\tau=1} \quad (1.10)$$

$$= \sum_{n=1}^{\infty} \chi(n) \lambda_{n,m}(\delta, q), \quad (1.11)$$

where

$$\lambda_{n,m}(d, q) = n^d \frac{d^m}{d\tau^m} e^{-\frac{\pi n^2}{q}\tau} \Big|_{\tau=1} \quad (1.12)$$

$$= n^d E_m\left(\frac{\pi n^2}{q}\right) e^{-\frac{\pi n^2}{q}}, \quad (1.13)$$

$$E_m(x) = \begin{cases} 1, & \text{if } m = 0, \\ \sum_{k=1}^m (-1)^{m+k} \frac{m!}{k!} \binom{m-1}{k-1} x^k, & \text{otherwise.} \end{cases} \quad (1.14)$$

Similar, for the right-hand side of (1.8), we have:

$$\frac{d^m}{d\tau^m} \left(\tau^{\delta+\frac{1}{2}} \theta_{\bar{\chi}}(\tau) \right) \Big|_{\tau=1} = \frac{d^m}{d\tau^m} \left(\tau^{\delta+\frac{1}{2}} \sum_{n=1}^{\infty} \bar{\chi}(n) n^{\delta} e^{-\frac{\pi n^2}{q}\tau} \right) \Big|_{\tau=1} \quad (1.15)$$

$$= \sum_{n=1}^{\infty} \bar{\chi}(n) n^{\delta} \frac{d^m}{d\tau^m} \left(\tau^{\delta+\frac{1}{2}} e^{-\frac{\pi n^2}{q}\tau} \right) \Big|_{\tau=1} \quad (1.16)$$

$$= \sum_{n=1}^{\infty} \bar{\chi}(n) \mu_{n,m}(\delta, q), \quad (1.17)$$

where

$$\mu_{n,m}(d, q) = n^d \frac{d^m}{d\tau^m} \left(\tau^{\delta+\frac{1}{2}} e^{-\frac{\pi n^2}{q}\tau} \right) \Big|_{\tau=1} \quad (1.18)$$

$$= n^d F_{d,m}\left(\frac{\pi n^2}{q}\right) e^{-\frac{\pi n^2}{q}}, \quad (1.19)$$

$$F_{d,m}(x) = \sum_{k=0}^m (-1)^k \binom{m}{k} \left(d + \frac{1}{2}\right)^{m-k} x^k; \quad (1.20)$$

here n^l is the *falling factorial power*,

$$n^l = n(n-1)\dots(n-l+1) \quad (1.21)$$

(we use notation from [1]).

In the above notation, equalities (1.8) can be rewritten as

$$\sum_{n=1}^{\infty} \chi(n) \lambda_{n,m}(\delta, q) = \omega \sum_{n=1}^{\infty} \bar{\chi}(n) \mu_{n,m}(\delta, q). \quad (1.22)$$

As explained above, this infinite series (for $m = 0, 1, \dots$) of numerical equalities is also equivalent to the functional equation for $L_{\chi}(s)$.

Now, we rearrange the numbers $\lambda_{n,m}(d, q)$ and $\mu_{n,m}(d, q)$ into an infinite matrix. More formally, let

$$B_N(d, q) = \begin{bmatrix} \lambda_{N,0}(d, q) & \dots & \lambda_{N,2N-1}(d, q) \\ \vdots & \ddots & \vdots \\ \lambda_{1,0}(d, q) & \dots & \lambda_{1,2N-1}(d, q) \\ \mu_{1,0}(d, q) & \dots & \mu_{1,2N-1}(d, q) \\ \vdots & \ddots & \vdots \\ \mu_{N,0}(d, q) & \dots & \mu_{N,2N-1}(d, q) \end{bmatrix}. \quad (1.23)$$

Matrix $B_N(d, q)$ is a submatrix of matrix $B_{N+1}(d, q)$, and thus can be viewed as a submatrix of the infinite matrix $B_\infty(d, q)$.

Matrices $B_N(d, q)$ are not degenerate (see the Appendix). Therefore, we can consider their inverse matrices. Let

$$B_N^{-1}(d, q) = \begin{bmatrix} \alpha_{N,0,N}(d, q) & \dots & \alpha_{N,0,1}(d, q) & \beta_{N,0,1}(d, q) & \dots & \beta_{N,0,N}(d, q) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{N,2N-1,N}(d, q) & \dots & \alpha_{N,2N-1,1}(d, q) & \beta_{N,2N-1,1}(d, q) & \dots & \beta_{N,2N-1,N}(d, q) \end{bmatrix}. \quad (1.24)$$

The inverse matrices $B_N^{-1}(d, q)$ are of interest due to their relationship with the corresponding Dirichlet L -function; this we shall discuss in a future paper. In this paper we examine the matrices $B_N^{-1}(d, q)$ *per se* because they exhibit interesting number-theoretical properties.

2 Numerical observations and conjectures

The behaviour of $B_N^{-1}(d, q)$ for even characters ($d = 0$) and odd characters ($d = 1$) is rather different. In this paper, we consider only the case of $d = 0$.

We begin by examining the entries in the top rows of the matrices $B_N^{-1}(0, q)$, that is, the numbers $\alpha_{N,0,j}(0, q)$ and $\beta_{N,0,j}(0, q)$.

2.1 Case $q = 2$

Table 1 presents the values of the α -entries from the top rows of some matrices $B_N^{-1}(0, 2)$. These and other similar data suggest the following

Conjecture 1. *For every fixed j and $N \rightarrow \infty$,*

$$\alpha_{N,0,j}(0, 2) \rightarrow \begin{cases} -2, & \text{if } j \equiv 0 \pmod{2}, \\ 2, & \text{otherwise.} \end{cases} \quad (2.1.1)$$

j	$N = 20$	$N = 25$	$N = 40$
1	$2 + 2.21 \dots 10^{-272}$	$2 - 1.57 \dots 10^{-424}$	$2 + 4.88 \dots 10^{-1087}$
2	$-2 - 6.46 \dots 10^{-271}$	$-2 + 4.68 \dots 10^{-423}$	$-2 - 1.44 \dots 10^{-1085}$
3	$2 + 1.37 \dots 10^{-268}$	$2 - 1.04 \dots 10^{-420}$	$2 + 3.17 \dots 10^{-1083}$
4	$-2 - 4.87 \dots 10^{-265}$	$-2 + 4.07 \dots 10^{-417}$	$-2 - 1.20 \dots 10^{-1079}$
5	$2 + 3.38 \dots 10^{-260}$	$2 - 3.26 \dots 10^{-412}$	$2 + 9.29 \dots 10^{-1075}$
6	$-2 - 4.74 \dots 10^{-254}$	$-2 + 5.65 \dots 10^{-406}$	$-2 - 1.54 \dots 10^{-1068}$
7	$2 + 1.31 \dots 10^{-246}$	$2 - 2.17 \dots 10^{-398}$	$2 + 5.62 \dots 10^{-1061}$
8	$-2 - 6.34 \dots 10^{-238}$	$-2 + 1.86 \dots 10^{-389}$	$-2 - 4.55 \dots 10^{-1052}$
9	$2 + 2.22 \dots 10^{-228}$	$2 - 3.60 \dots 10^{-379}$	$2 + 8.23 \dots 10^{-1042}$
10	$-2 + 3.37 \dots 10^{-216}$	$-2 + 1.56 \dots 10^{-367}$	$-2 - 3.32 \dots 10^{-1030}$

Table 1: Values of $\alpha_{N,0,j}(0, 2)$

j	$N = 20$	$N = 25$	$N = 40$
1	2.828427124746...	2.828427124746...	2.828427124746...
2	$6.74995 \dots \cdot 10^{-271}$	$-4.66247 \dots \cdot 10^{-423}$	$1.45626 \dots \cdot 10^{-1085}$
3	2.828427124746...	2.828427124746...	2.828427124746...
4	$6.45301 \dots \cdot 10^{-265}$	$-3.95149 \dots \cdot 10^{-417}$	$1.28869 \dots \cdot 10^{-1079}$
5	2.828427124746...	2.828427124746...	2.828427124746...
6	$1.04346 \dots \cdot 10^{-253}$	$-5.21244 \dots \cdot 10^{-406}$	$1.86079 \dots \cdot 10^{-1068}$
7	2.828427124746...	2.828427124746...	2.828427124746...
8	$4.02948 \dots \cdot 10^{-237}$	$-1.59125 \dots \cdot 10^{-389}$	$6.57582 \dots \cdot 10^{-1052}$
9	2.828427124746...	2.828427124746...	2.828427124746...
10	$3.77003 \dots \cdot 10^{-215}$	$-1.19819 \dots \cdot 10^{-367}$	$6.15094 \dots \cdot 10^{-1030}$

Table 2: Values of $\beta_{N,0,j}(0, 2)$

Table 2 presents the values of the β -entries from the top row of some matrices $B_N^{-1}(0, 2)$. It is not difficult to recognize the number 2.828427124746... as an approximation to $2\sqrt{2}$. In order to see how well $\beta_{N,0,j}(0, 2)$ approximates $2\sqrt{2}$, here and in the sequel we scale $\beta_{N,0,j}(0, q)$ by $\sqrt{q}$. Table 3 presents such scaled values. These and other similar data suggest the following

Conjecture 2. *For every fixed j and $N \rightarrow \infty$,*

$$\sqrt{2}\beta_{N,0,j}(0, 2) \rightarrow \begin{cases} 0, & \text{if } j \equiv 0 \pmod{2}, \\ 4, & \text{otherwise.} \end{cases} \quad (2.1.2)$$

j	$N = 20$	$N = 25$	$N = 40$
1	$4 - 3.1664 \dots \cdot 10^{-272}$	$4 + 2.2289 \dots \cdot 10^{-424}$	$4 - 6.9186 \dots \cdot 10^{-1087}$
2	$9.54587402 \dots \cdot 10^{-271}$	$-6.59373217 \dots \cdot 10^{-423}$	$2.05947263 \dots \cdot 10^{-1085}$
3	$4 - 2.2158 \dots \cdot 10^{-268}$	$4 + 1.4605 \dots \cdot 10^{-420}$	$4 - 4.6354 \dots \cdot 10^{-1083}$
4	$9.12593582 \dots \cdot 10^{-265}$	$-5.58825992 \dots \cdot 10^{-417}$	$1.82249087 \dots \cdot 10^{-1079}$
5	$4 - 7.8516 \dots \cdot 10^{-260}$	$4 + 4.3739 \dots \cdot 10^{-412}$	$4 - 1.4830 \dots \cdot 10^{-1074}$
6	$1.47568587 \dots \cdot 10^{-253}$	$-7.37150902 \dots \cdot 10^{-406}$	$2.63155866 \dots \cdot 10^{-1068}$
7	$4 - 6.1485 \dots \cdot 10^{-246}$	$4 + 2.7327 \dots \cdot 10^{-398}$	$4 - 1.0415 \dots \cdot 10^{-1060}$
8	$5.69855483 \dots \cdot 10^{-237}$	$-2.25036837 \dots \cdot 10^{-389}$	$9.29962008 \dots \cdot 10^{-1052}$
9	$4 - 1.1726 \dots \cdot 10^{-226}$	$4 + 4.1331 \dots \cdot 10^{-379}$	$4 - 1.8847 \dots \cdot 10^{-1041}$
10	$5.33163008 \dots \cdot 10^{-215}$	$-1.69450458 \dots \cdot 10^{-367}$	$8.69874368 \dots \cdot 10^{-1030}$

Table 3: Values of $\sqrt{2}\beta_{N,0,j}(0, 2)$.

j	$N = 20$	$N = 30$	$N = 40$
1	$1 + 9.67 \dots \cdot 10^{-171}$	$1 + 6.61 \dots \cdot 10^{-392}$	$1 + 1.44 \dots \cdot 10^{-703}$
2	$1 - 5.10 \dots \cdot 10^{-170}$	$1 - 3.47 \dots \cdot 10^{-391}$	$1 - 7.61 \dots \cdot 10^{-703}$
3	$-2 + 4.95 \dots \cdot 10^{-169}$	$-2 + 3.33 \dots \cdot 10^{-390}$	$-2 + 7.31 \dots \cdot 10^{-702}$
4	$1 - 3.29 \dots \cdot 10^{-167}$	$1 - 2.15 \dots \cdot 10^{-388}$	$1 - 4.77 \dots \cdot 10^{-700}$
5	$1 + 1.78 \dots \cdot 10^{-164}$	$1 + 1.12 \dots \cdot 10^{-385}$	$1 + 2.51 \dots \cdot 10^{-697}$
6	$-2 - 7.82 \dots \cdot 10^{-161}$	$-2 - 4.68 \dots \cdot 10^{-382}$	$-2 - 1.07 \dots \cdot 10^{-693}$
7	$1 + 2.77 \dots \cdot 10^{-156}$	$1 + 1.56 \dots \cdot 10^{-377}$	$1 + 3.67 \dots \cdot 10^{-689}$
8	$1 - 7.87 \dots \cdot 10^{-151}$	$1 - 4.17 \dots \cdot 10^{-372}$	$1 - 1.01 \dots \cdot 10^{-683}$
9	$-2 + 1.78 \dots \cdot 10^{-144}$	$-2 + 8.84 \dots \cdot 10^{-366}$	$-2 + 2.26 \dots \cdot 10^{-677}$
10	$1 - 3.20 \dots \cdot 10^{-137}$	$1 - 1.47 \dots \cdot 10^{-358}$	$1 - 4.04 \dots \cdot 10^{-670}$

Table 4: Values of $\alpha_{N,0,j}(0,3)$

2.2 Case $q = 3$

Table 4 presents the values of the α -entries from the top rows of some matrices $B_N^{-1}(0,3)$. These and other similar data suggest the following

Conjecture 3. *For every fixed j and $N \rightarrow \infty$,*

$$\alpha_{N,0,j}(0,3) \rightarrow \begin{cases} -2, & \text{if } j \equiv 0 \pmod{3}, \\ 1, & \text{otherwise.} \end{cases} \quad (2.2.1)$$

j	$N = 20$	$N = 30$	$N = 40$
1	$3 - 1.67 \dots \cdot 10^{-170}$	$3 - 1.14 \dots \cdot 10^{-391}$	$3 - 2.50 \dots \cdot 10^{-703}$
2	$3 + 8.81 \dots \cdot 10^{-170}$	$3 + 6.05 \dots \cdot 10^{-391}$	$3 + 1.32 \dots \cdot 10^{-702}$
3	$0 - 8.46 \dots \cdot 10^{-169}$	$0 - 5.89 \dots \cdot 10^{-390}$	$0 - 1.28 \dots \cdot 10^{-701}$
4	$3 + 5.51 \dots \cdot 10^{-167}$	$3 + 3.95 \dots \cdot 10^{-388}$	$3 + 8.53 \dots \cdot 10^{-700}$
5	$3 - 2.88 \dots \cdot 10^{-164}$	$3 - 2.16 \dots \cdot 10^{-385}$	$3 - 4.63 \dots \cdot 10^{-697}$
6	$0 + 1.21 \dots \cdot 10^{-160}$	$0 + 9.70 \dots \cdot 10^{-382}$	$0 + 2.05 \dots \cdot 10^{-693}$
7	$3 - 4.07 \dots \cdot 10^{-156}$	$3 - 3.54 \dots \cdot 10^{-377}$	$3 - 7.38 \dots \cdot 10^{-689}$
8	$3 + 1.08 \dots \cdot 10^{-150}$	$3 + 1.05 \dots \cdot 10^{-371}$	$3 + 2.16 \dots \cdot 10^{-683}$
9	$0 - 2.28 \dots \cdot 10^{-144}$	$0 - 2.52 \dots \cdot 10^{-365}$	$0 - 5.15 \dots \cdot 10^{-677}$
10	$3 + 3.74 \dots \cdot 10^{-137}$	$3 + 4.91 \dots \cdot 10^{-358}$	$3 + 9.97 \dots \cdot 10^{-670}$

Table 5: Values of $\sqrt{3}\beta_{N,0,j}(0,3)$

Table 5 presents the values of the β -entries (scaled by $\sqrt{3}$) from the top row of some matrices $B_N^{-1}(0,3)$. These and other similar data suggest the following

Conjecture 4. *For every fixed j and $N \rightarrow \infty$,*

$$\sqrt{3}\beta_{N,0,j}(0,3) \rightarrow \begin{cases} 0, & \text{if } j \equiv 0 \pmod{3}, \\ 3, & \text{otherwise.} \end{cases} \quad (2.2.2)$$

j	$N = 20$	$N = 40$	$N = 80$
1	$1 + 3.80 \dots 10^{-13}$	$1 + 1.80 \dots 10^{-26}$	$1 + 1.90 \dots 10^{-53}$
2	$0 - 7.61 \dots 10^{-13}$	$0 - 3.61 \dots 10^{-26}$	$0 - 3.80 \dots 10^{-53}$
3	$1 + 3.80 \dots 10^{-13}$	$1 + 1.80 \dots 10^{-26}$	$1 + 1.90 \dots 10^{-53}$
4	$-2 + 5.89 \dots 10^{-120}$	$-2 + 1.12 \dots 10^{-511}$	$-2 + 2.88 \dots 10^{-2113}$
5	$1 + 3.80 \dots 10^{-13}$	$1 + 1.80 \dots 10^{-26}$	$1 + 1.90 \dots 10^{-53}$
6	$0 - 7.61 \dots 10^{-13}$	$0 - 3.61 \dots 10^{-26}$	$0 - 3.80 \dots 10^{-53}$
7	$1 + 3.80 \dots 10^{-13}$	$1 + 1.80 \dots 10^{-26}$	$1 + 1.90 \dots 10^{-53}$
8	$-2 + 1.36 \dots 10^{-108}$	$-2 + 2.74 \dots 10^{-500}$	$-2 + 7.11 \dots 10^{-2102}$
9	$1 + 3.80 \dots 10^{-13}$	$1 + 1.80 \dots 10^{-26}$	$1 + 1.90 \dots 10^{-53}$
10	$0 - 7.61 \dots 10^{-13}$	$0 - 3.61 \dots 10^{-26}$	$0 - 3.80 \dots 10^{-53}$

Table 6: Values of $\alpha_{N,0,j}(0,4)$

2.3 Case $q = 4$

Table 6 presents the values of the α -entries from the top rows of some matrices $B_N^{-1}(0,4)$. As in the cases $q = 2$ and $q = 3$, one is tempted to state

Conjecture 5. *For every fixed j and $N \rightarrow \infty$,*

$$\alpha_{N,0,j}(0,4) \rightarrow \begin{cases} -2, & \text{if } j \equiv 0 \pmod{4}, \\ 0, & \text{if } j \equiv 2 \pmod{4}, \\ 1, & \text{otherwise.} \end{cases} \quad (2.3.1)$$

However, this conjecture is wrong. Table 6 is misleading because all values of N are divisible by 4. Tables 6 and 7 demonstrate a new phenomenon (compared to the cases $q = 2$ and $q = 3$): the limiting values depend on the value of $N \bmod q$.

j	$N = 21$	$N = 41$	$N = 81$
1	$1/2 - 2.08 \dots 10^{-14}$	$1/2 - 9.62 \dots 10^{-28}$	$1/2 - 1.00 \dots 10^{-54}$
2	$1 + 4.17 \dots 10^{-14}$	$1 + 1.92 \dots 10^{-27}$	$1 + 2.00 \dots 10^{-54}$
3	$1/2 - 2.08 \dots 10^{-14}$	$1/2 - 9.62 \dots 10^{-28}$	$1/2 - 1.00 \dots 10^{-54}$
4	$-2 - 4.51 \dots 10^{-133}$	$-2 - 2.02 \dots 10^{-538}$	$-2 - 2.74 \dots 10^{-2167}$
5	$1/2 - 2.08 \dots 10^{-14}$	$1/2 - 9.62 \dots 10^{-28}$	$1/2 - 1.00 \dots 10^{-54}$
6	$1 + 4.17 \dots 10^{-14}$	$1 + 1.92 \dots 10^{-27}$	$1 + 2.00 \dots 10^{-54}$
7	$1/2 - 2.08 \dots 10^{-14}$	$1/2 - 9.62 \dots 10^{-28}$	$1/2 - 1.00 \dots 10^{-54}$
8	$-2 - 1.05 \dots 10^{-121}$	$-2 - 4.96 \dots 10^{-527}$	$-2 - 6.77 \dots 10^{-2156}$
9	$1/2 - 2.08 \dots 10^{-14}$	$1/2 - 9.62 \dots 10^{-28}$	$1/2 - 1.00 \dots 10^{-54}$
10	$1 + 4.17 \dots 10^{-14}$	$1 + 1.92 \dots 10^{-27}$	$1 + 2.00 \dots 10^{-54}$

Table 7: Values of $\alpha_{N,0,j}(0,4)$

j	$N = 23$	$N = 43$	$N = 83$
1	$2.512950 \dots 10^{14}$	$5.719972 \dots 10^{27}$	$5.639178 \dots 10^{54}$
2	$-5.025901 \dots 10^{14}$	$-1.143994 \dots 10^{28}$	$-1.127835 \dots 10^{55}$
3	$2.512950 \dots 10^{14}$	$5.719972 \dots 10^{27}$	$5.639178 \dots 10^{54}$
4	$-1.999999 \dots 10^0$	$-1.999999 \dots 10^0$	$-1.999999 \dots 10^0$
5	$2.512950 \dots 10^{14}$	$5.719972 \dots 10^{27}$	$5.639178 \dots 10^{54}$
6	$-5.025901 \dots 10^{14}$	$-1.143994 \dots 10^{28}$	$-1.127835 \dots 10^{55}$
7	$2.512950 \dots 10^{14}$	$5.719972 \dots 10^{27}$	$5.639178 \dots 10^{54}$
8	$-1.999999 \dots 10^0$	$-1.999999 \dots 10^0$	$-1.999999 \dots 10^0$
9	$2.512950 \dots 10^{14}$	$5.719972 \dots 10^{27}$	$5.639178 \dots 10^{54}$
10	$-5.025901 \dots 10^{14}$	$-1.143994 \dots 10^{28}$	$-1.127835 \dots 10^{55}$

Table 8: Values of $\alpha_{N,0,j}(0, 4)$

Table 8 demonstrates another new phenomenon: the limiting values can be infinite, both $+\infty$ and $-\infty$.

2.4 Limiting values and their pereodical structure

The above tables and other similar numerical data suggest the following

Conjecture 6. *For every q , M , and j , there are (finite or infinite) limits*

$$\alpha_{M,j}^\infty(q) = \lim_{N \rightarrow \infty} \alpha_{qN+M,0,j}(0, q) \quad (2.4.1)$$

and

$$\beta_{M,j}^\infty(q) = \lim_{N \rightarrow \infty} \beta_{qN+M,0,j}(0, q). \quad (2.4.2)$$

Clearly, if $M_1 \equiv M_2 \pmod{q}$, then

$$\alpha_{M_1,j}^\infty(q) = \alpha_{M_2,j}^\infty(q) \quad \text{and} \quad \beta_{M_1,j}^\infty(q) = \beta_{M_2,j}^\infty(q). \quad (2.4.3)$$

Conjectures 1–4 and Tables 6–8 demonstrate similar periodicity modulo q for j when $q = 2, 3$, or 4. Numerical data for larger q suggest the following generalization.

Conjecture 7. *For every q , M , j_1 and j_2 , if $j_1 \equiv j_2 \pmod{q}$ then*

$$\alpha_{M,j_1}^\infty(q) = \alpha_{M,j_2}^\infty(q) \quad \text{and} \quad \beta_{M,j_1}^\infty(q) = \beta_{M,j_2}^\infty(q). \quad (2.4.4)$$

Conjecture 7 supports the following extension of definitions (2.4.1) and (2.4.2):

$$\alpha_{M,0}^\infty(q) = \alpha_{M,q}^\infty(q) \quad \text{and} \quad \beta_{M,0}^\infty(q) = \beta_{M,q}^\infty(q). \quad (2.4.5)$$

According to the periodicity given in (2.4.3) and expected the periodicity in (2.4.4), it is sufficient to provide the values of $\alpha_{M,j}^\infty(q)$ and $\beta_{M,j}^\infty(q)$ for $M = 0, \dots, q-1$, $j = 0, \dots, q-1$ only.

With the new notation, we can restate the previous conjectures more concisely as follows.

Conjecture (reformulation of Conjecture 1). *For $q = 2$, the values of $\alpha_{M,j}^\infty(0, q)$ are as follows:*

j	$M = 0$	$M = 1$
0	-2	-2
1	2	2

Conjecture (reformulation of Conjecture 2). *For $q = 2$, the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are as follows:*

j	$M = 0$	$M = 1$
0	0	0
1	4	4

Conjecture (reformulation of Conjecture 3). *For $q = 3$, the values of $\alpha_{M,j}^\infty(0, q)$ are as follows:*

j	$M = 0$	$M = 1$	$M = 2$
0	-2	-2	-2
1	1	1	1
2	1	1	1

Conjecture (reformulation of Conjecture 4). *For $q = 3$, the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are as follows:*

j	$M = 0$	$M = 1$	$M = 2$
0	0	0	0
1	3	3	3
2	3	3	3

Conjecture 8. *For $q = 4$, the values of $\alpha_{M,j}^\infty(0, q)$ are as follows:*

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$
0	-2	-2	-2	-2
1	1	1/2	1	$+\infty$
2	0	1	0	$-\infty$
3	1	1/2	1	$+\infty$

Conjecture 9. *For $q = 4$, the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are as follows:*

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$
0	0	0	0	0
1	2	3	2	$-\infty$
2	4	2	4	$+\infty$
3	2	3	2	$-\infty$

j	$N = 25$	$N = 31$	$N = 37$
1	0.8090169943...	0.1909830056...	0.1909830056...
2	0.1909830056...	0.8090169943...	0.8090169943...
3	0.1909830056...	0.8090169943...	0.8090169943...
4	0.8090169943...	0.1909830056...	0.1909830056...
5	$-2 + 2.47 \dots \cdot 10^{-145}$	$-2 + 7.86 \dots \cdot 10^{-231}$	$-2 + 5.75 \dots \cdot 10^{-336}$
6	0.8090169943...	0.1909830056...	0.1909830056...
7	0.1909830056...	0.8090169943...	0.8090169943...
8	0.1909830056...	0.8090169943...	0.8090169943...
9	0.8090169943...	0.1909830056...	0.1909830056...
10	$-2 - 1.00 \dots \cdot 10^{-131}$	$-2 - 3.32 \dots \cdot 10^{-217}$	$-2 - 2.47 \dots \cdot 10^{-322}$
11	0.8090169943...	0.1909830056...	0.1909830056...
12	0.1909830056...	0.8090169943...	0.8090169943...

Table 9: Values of $\alpha_{N,0,j}(0, 5)$ for $1 \leq j \leq N$

2.5 Case $q = 5$

Table 9 presents some values of the $\alpha_{N,0,j}(0, q)$ for $q = 5$. Note that these data support Conjecture 7.

Using continued fractions, one can recognize that

$$0.8090169943 \dots \approx (1 + \sqrt{5})/4, \quad (2.5.1)$$

$$0.1909830056 \dots \approx (3 - \sqrt{5})/4. \quad (2.5.2)$$

Table 9 and other similar numerical data suggest

Conjecture 10. For $q = 5$, the values of $\alpha_{M,j}^\infty(0, q)$ are as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$
0	-2	-2	-2	-2	-2
1	$\frac{1+\sqrt{5}}{4}$	$\frac{3-\sqrt{5}}{4}$	$\frac{3-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$+\infty$
2	$\frac{3-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{3-\sqrt{5}}{4}$	$-\infty$
3	$\frac{3-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$\frac{3-\sqrt{5}}{4}$	$-\infty$
4	$\frac{1+\sqrt{5}}{4}$	$\frac{3-\sqrt{5}}{4}$	$\frac{3-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	$+\infty$

Similar, for $\beta_{M,j}^\infty(0, q)$ we state

Conjecture 11. For $q = 5$, the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$
0	0	0	0	0	0
1	$\frac{5+\sqrt{5}}{4}$	$\frac{15-\sqrt{5}}{4}$	$\frac{15-\sqrt{5}}{4}$	$\frac{5+\sqrt{5}}{4}$	$+\infty$
2	$\frac{15-\sqrt{5}}{4}$	$\frac{5+\sqrt{5}}{4}$	$\frac{5+\sqrt{5}}{4}$	$\frac{15-\sqrt{5}}{4}$	$-\infty$
3	$\frac{15-\sqrt{5}}{4}$	$\frac{5+\sqrt{5}}{4}$	$\frac{5+\sqrt{5}}{4}$	$\frac{15-\sqrt{5}}{4}$	$-\infty$
4	$\frac{5+\sqrt{5}}{4}$	$\frac{15-\sqrt{5}}{4}$	$\frac{15-\sqrt{5}}{4}$	$\frac{5+\sqrt{5}}{4}$	$+\infty$

2.6 Case $q = 6$

For $q = 6$ corresponding numerical data suggest the following conjectures.

Conjecture 12. For $q = 6$, the values of $\alpha_{M,j}^\infty(0, q)$ are as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$
0	-2	-2	-2	-2	-2	-2
1	0	0	0	0	0	$+\infty$
2	4	0	1	0	4	$-\infty$
3	-6	2	0	2	-6	$+\infty$
4	4	0	1	0	4	$-\infty$
5	0	0	0	0	0	$+\infty$

Conjecture 13. For $q = 6$, the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$
0	0	0	0	0	0	0
1	0	4	3	4	0	$+\infty$
2	12	0	3	0	12	$+\infty$
3	-12	4	0	4	-12	$-\infty$
4	12	0	3	0	12	$+\infty$
5	0	4	3	4	0	$+\infty$

2.7 Case $q = 7$

Table 10 presents $\alpha_{N,0,j}(0, q)$ for $q = 7$. These data also support Conjecture 7.

In the case $q = 7$, continued fractions do not help to discover symbolic expressions for the numbers $\alpha_{M,j}^\infty(q)$. We suspect that they are algebraic numbers with a higher degree. In particular, $2.80193773580483\dots$ is close to $1 + 2\cos(\pi/7)$, which is a root of the polynomial $1 + 3x - 4x^2 + x^3$.

j	$N = 75$	$N = 105$
1	$2.15235907\dots \cdot 10^{-27}$	$-8.45220924\dots \cdot 10^{-39}$
2	$2.80193773580483\dots$	$2.80193773580483\dots$
3	$-1.80193773580483\dots$	$-1.80193773580483\dots$
4	$-1.80193773580483\dots$	$-1.80193773580483\dots$
5	$2.80193773580483\dots$	$2.80193773580483\dots$
6	$2.15235907\dots \cdot 10^{-27}$	$-8.45220924\dots \cdot 10^{-39}$
7	$-2 + 5.4143\dots \cdot 10^{-1036}$	$-2 + 7.1587\dots \cdot 10^{-2065}$
8	$2.15235907\dots \cdot 10^{-27}$	$-8.45220924\dots \cdot 10^{-39}$
9	$2.80193773580483\dots$	$2.80193773580483\dots$
10	$-1.80193773580483\dots$	$-1.80193773580483\dots$
11	$-1.80193773580483\dots$	$-1.80193773580483\dots$
12	$2.80193773580483\dots$	$2.80193773580483\dots$
13	$2.15235907\dots \cdot 10^{-27}$	$-8.45220924\dots \cdot 10^{-39}$
14	$-2 - 6.8899\dots \cdot 10^{-1017}$	$-2 - 8.2595\dots \cdot 10^{-2046}$

Table 10: Values of $\alpha_{N,0,j}(0, 7)$.

In general, our guesses for the values of $\alpha_{M,j}^\infty(q)$ and $\beta_{M,j}^\infty(q)$ for $q = 7$ are as follows.

Conjecture 14. *Let $q = 7$ and $c_q = \cos(\pi/q)$; the values of $\alpha_{M,j}^\infty(0, q)$ are expected to be as follows:*

j	$M \in \{0, 5\}$	$M \in \{1, 4\}$	$M \in \{2, 3\}$	$M = 6$
0	-2	-2	-2	-2
1	0	$-4c_q^2 + 2c_q + 1$	$-4c_q^2 + 3$	$+\infty$
2	$2c_q + 1$	0	$4c_q^2 - 2$	$-\infty$
3	$-2c_q$	$4c_q^2 - 2c_q$	0	$+\infty$
4	$-2c_q$	$4c_q^2 - 2c_q$	0	$+\infty$
5	$2c_q + 1$	0	$4c_q^2 - 2$	$-\infty$
6	0	$-4c_q^2 + 2c_q + 1$	$-4c_q^2 + 3$	$+\infty$

Conjecture 15. *Let $q = 7$ and $c_q = \cos(\pi/q)$; the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are expected to be as follows:*

j	$M \in \{0, 5\}$	$M \in \{1, 4\}$	$M \in \{2, 3\}$	$M = 6$
0	0	0	0	0
1	0	$-4c_q^2 + 6c_q + 3$	$-12c_q^2 + 4c_q + 9$	$+\infty$
2	$8c_q^2 + 2c_q + 1$	0	$12c_q^2 - 4c_q - 2$	$-\infty$
3	$-8c_q^2 - 2c_q + 6$	$4c_q^2 - 6c_q + 4$	0	$+\infty$
4	$-8c_q^2 - 2c_q + 6$	$4c_q^2 - 6c_q + 4$	0	$+\infty$
5	$8c_q^2 + 2c_q + 1$	0	$12c_q^2 - 4c_q - 2$	$-\infty$
6	0	$-4c_q^2 + 6c_q + 3$	$-12c_q^2 + 4c_q + 9$	$+\infty$

2.8 Case $q = 8$

Conjecture 16. *Let $q = 8$ and $c_q = \cos(\pi/q)$; the values of $\alpha_{M,j}^\infty(0, q)$ are expected to be as follows:*

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$	$M = 7$
0	-2	-2	-2	-2	-2	-2	-2	-2
1	0	$-2c_q^2 + 1$	$-2c_q^2 + 1$	$-c_q^2 + \frac{1}{2}$	$-4c_q^2 + 2$	$2c_q^2 - 1$	0	$+\infty$
2	$4c_q^2 - 1$	0	$2c_q^2 + \frac{1}{2}$	1	$4c_q^2 - 1$	0	$+\infty$	$-\infty$
3	$-4c_q^2 + 2$	$2c_q^2 - 1$	0	$c_q^2 - \frac{1}{2}$	0	$-2c_q^2 + 1$	$-\infty$	$+\infty$
4	0	2	-1	0	0	2	$+\infty$	$-\infty$
5	$-4c_q^2 + 2$	$2c_q^2 - 1$	0	$c_q^2 - \frac{1}{2}$	0	$-2c_q^2 + 1$	$-\infty$	$+\infty$
6	$4c_q^2 - 1$	0	$2c_q^2 + \frac{1}{2}$	1	$4c_q^2 - 1$	0	$+\infty$	$-\infty$
7	0	$-2c_q^2 + 1$	$-2c_q^2 + 1$	$-c_q^2 + \frac{1}{2}$	$-4c_q^2 + 2$	$2c_q^2 - 1$	0	$+\infty$

Conjecture 17. Let $q = 8$ and $c_q = \cos(\pi/q)$; the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$	$M = 7$
0	0	0	0	0	0	0	0	0
1	0	6	2	3	4	2	0	$-\infty$
2	$8c_q^2$	0	$4c_q^2 + 4$	4	$8c_q^2$	0	$-\infty$	$+\infty$
3	4	2	0	1	0	6	$+\infty$	$-\infty$
4	$-16c_q^2 + 8$	0	$-8c_q^2 + 4$	0	$-16c_q^2 + 8$	0	$-\infty$	$+\infty$
5	4	2	0	1	0	6	$+\infty$	$-\infty$
6	$8c_q^2$	0	$4c_q^2 + 4$	4	$8c_q^2$	0	$-\infty$	$+\infty$
7	0	6	2	3	4	2	0	$-\infty$

2.9 Case $q = 9$

Conjecture 18. Let $q = 9$ and $c_q = \cos(\pi/q)$; the values of $\alpha_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$
0	-2	-2	-2	-2
1	0	$-\frac{10c_q^2}{3} + \frac{c_q}{3} + \frac{7}{6}$	$-\infty$	$-\frac{2c_q^2}{3} - \frac{4c_q}{3} + \frac{4}{3}$
2	$c_q + 1$	0	$+\infty$	$\frac{2c_q^2}{3} + \frac{4c_q}{3} - \frac{5}{6}$
3	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
4	$-c_q + \frac{1}{2}$	$\frac{10c_q^2}{3} - \frac{c_q}{3} - \frac{2}{3}$	$-\infty$	0
5	$-c_q + \frac{1}{2}$	$\frac{10c_q^2}{3} - \frac{c_q}{3} - \frac{2}{3}$	$-\infty$	0
6	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
7	$c_q + 1$	0	$+\infty$	$\frac{2c_q^2}{3} + \frac{4c_q}{3} - \frac{5}{6}$
8	0	$-\frac{10c_q^2}{3} + \frac{c_q}{3} + \frac{7}{6}$	$-\infty$	$-\frac{2c_q^2}{3} - \frac{4c_q}{3} + \frac{4}{3}$

j	$M = 4$	$M = 5$	$M = 6$	$M = 7$	$M = 8$
0	-2	-2	-2	-2	-2
1	$-\frac{2c_q^2}{3} - \frac{4c_q}{3} + \frac{4}{3}$	$+\infty$	$-2c_q^2 + c_q + \frac{3}{2}$	0	$+\infty$
2	$\frac{2c_q^2}{3} + \frac{4c_q}{3} - \frac{5}{6}$	$-\infty$	0	$+\infty$	$-\infty$
3	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\infty$	$+\infty$
4	0	$+\infty$	$2c_q^2 - c_q$	$+\infty$	$-\infty$
5	0	$+\infty$	$2c_q^2 - c_q$	$+\infty$	$-\infty$
6	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\infty$	$+\infty$
7	$\frac{2c_q^2}{3} + \frac{4c_q}{3} - \frac{5}{6}$	$-\infty$	0	$+\infty$	$-\infty$
8	$-\frac{2c_q^2}{3} - \frac{4c_q}{3} + \frac{4}{3}$	$+\infty$	$-2c_q^2 + c_q + \frac{3}{2}$	0	$+\infty$

Conjecture 19. Let $q = 9$ and $c_q = \cos(\pi/q)$; the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$
0	0	0	0	0
1	0	$-2c_q^2 + 5c_q + \frac{11}{2}$	$-\infty$	$-10c_q^2 + 4c_q + 8$
2	$3c_q + 3$	0	$+\infty$	$10c_q^2 - 4c_q - \frac{1}{2}$
3	$\frac{9}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
4	$-3c_q + \frac{3}{2}$	$2c_q^2 - 5c_q + 2$	$-\infty$	0
5	$-3c_q + \frac{3}{2}$	$2c_q^2 - 5c_q + 2$	$-\infty$	0
6	$\frac{9}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
7	$3c_q + 3$	0	$+\infty$	$10c_q^2 - 4c_q - \frac{1}{2}$
8	0	$-2c_q^2 + 5c_q + \frac{11}{2}$	$-\infty$	$-10c_q^2 + 4c_q + 8$

j	$M = 4$	$M = 5$	$M = 6$	$M = 7$	$M = 8$
0	0	0	0	0	0
1	$-10c_q^2 + 4c_q + 8$	$+\infty$	$-6c_q^2 + 3c_q + \frac{9}{2}$	0	$-\infty$
2	$10c_q^2 - 4c_q - \frac{1}{2}$	$-\infty$	0	$-\infty$	$+\infty$
3	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	$+\infty$	$-\infty$
4	0	$+\infty$	$6c_q^2 - 3c_q$	$-\infty$	$+\infty$
5	0	$+\infty$	$6c_q^2 - 3c_q$	$-\infty$	$+\infty$
6	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	$+\infty$	$-\infty$
7	$10c_q^2 - 4c_q - \frac{1}{2}$	$-\infty$	0	$-\infty$	$+\infty$
8	$-10c_q^2 + 4c_q + 8$	$+\infty$	$-6c_q^2 + 3c_q + \frac{9}{2}$	0	$-\infty$

2.10 Case $q = 10$

Conjecture 20. Let $q = 10$ and $c_q = \cos(\pi/q)$; the values of $\alpha_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M \in \{0, 7\}$	$M \in \{1, 6\}$	$M \in \{2, 5\}$	$M \in \{3, 4\}$	$M = 8$	$M = 9$
0	-2	-2	-2	-2	-2	-2
1	0	$-8c_q^2 + 4$	$16c_q^2 - 4$	$-4c_q^2 + 3$	0	$+\infty$
2	0	0	$-48c_q^2 + 16$	1	$+\infty$	$-\infty$
3	$16c_q^2 - 4$	0	0	$4c_q^2 - 3$	$-\infty$	$+\infty$
4	$-48c_q^2 + 16$	$8c_q^2 - 4$	0	0	$+\infty$	$-\infty$
5	$64c_q^2 - 22$	2	$64c_q^2 - 22$	0	$-\infty$	$+\infty$
6	$-48c_q^2 + 16$	$8c_q^2 - 4$	0	0	$+\infty$	$-\infty$
7	$16c_q^2 - 4$	0	0	$4c_q^2 - 3$	$-\infty$	$+\infty$
8	0	0	$-48c_q^2 + 16$	1	$+\infty$	$-\infty$
9	0	$-8c_q^2 + 4$	$16c_q^2 - 4$	$-4c_q^2 + 3$	0	$+\infty$

Conjecture 21. Let $q = 10$ and $c_q = \cos(\pi/q)$; the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M \in \{0, 7\}$	$M \in \{1, 6\}$	$M \in \{2, 5\}$	$M \in \{3, 4\}$	$M = 8$	$M = 9$
0	0	0	0	0	0	0
1	0	$16c_q^2$	$64c_q^2 - 20$	$-8c_q^2 + 10$	0	$+\infty$
2	0	0	$-160c_q^2 + 60$	5	$+\infty$	$-\infty$
3	$64c_q^2 - 20$	0	0	$8c_q^2 - 5$	$-\infty$	$+\infty$
4	$-160c_q^2 + 60$	0	0	0	$+\infty$	$-\infty$
5	$192c_q^2 - 60$	$-32c_q^2 + 20$	$192c_q^2 - 60$	0	$-\infty$	$+\infty$
6	$-160c_q^2 + 60$	0	0	0	$+\infty$	$-\infty$
7	$64c_q^2 - 20$	0	0	$8c_q^2 - 5$	$-\infty$	$+\infty$
8	0	0	$-160c_q^2 + 60$	5	$+\infty$	$-\infty$
9	0	$16c_q^2$	$64c_q^2 - 20$	$-8c_q^2 + 10$	0	$+\infty$

2.11 Case $q = 11$

Conjecture 22. Let $q = 11$ and $c_q = \cos(\pi/q)$; the values of $\alpha_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M \in \{0, 8\}$	$M \in \{1, 7\}$	$M \in \{2, 6\}$
0	-2	-2	-2
1	0	$-16c_q^4 - 8c_q^3 + 4c_q^2 + 2c_q + 1$	$-8c_q^3 + 8c_q^2 + 2c_q - 1$
2	0	0	$-8c_q^3 + 4c_q^2$
3	$16c_q^4 + 8c_q^3 - 12c_q^2 - 4c_q + 2$	0	0
4	$-16c_q^4 - 16c_q^3 + 4c_q^2 + 8c_q + 2$	$8c_q^3 + 4c_q^2 - 4c_q - 1$	0
5	$8c_q^3 + 8c_q^2 - 4c_q - 3$	$16c_q^4 - 8c_q^3 + 2c_q + 1$	$16c_q^3 - 12c_q^2 - 2c_q + 2$
6	$8c_q^3 + 8c_q^2 - 4c_q - 3$	$16c_q^4 - 8c_q^3 + 2c_q + 1$	$16c_q^3 - 12c_q^2 - 2c_q + 2$
7	$-16c_q^4 - 16c_q^3 + 4c_q^2 + 8c_q + 2$	$8c_q^3 + 4c_q^2 - 4c_q - 1$	0
8	$16c_q^4 + 8c_q^3 - 12c_q^2 - 4c_q + 2$	0	0
9	0	0	$-8c_q^3 + 4c_q^2$
10	0	$-16c_q^4 - 8c_q^3 + 4c_q^2 + 2c_q + 1$	$-8c_q^3 + 8c_q^2 + 2c_q - 1$

j	$M \in \{3, 4, 5\}$	$M = 9$
0	-2	-2
1	$-32c_q^4 - 8c_q^3 + 28c_q^2 + 8c_q$	0
2	$48c_q^4 + 16c_q^3 - 44c_q^2 - 16c_q + 2$	$+\infty$
3	$-16c_q^4 - 8c_q^3 + 16c_q^2 + 8c_q - 1$	$-\infty$
4	0	$+\infty$
5	0	$-\infty$
6	0	$-\infty$
7	0	$+\infty$
8	$-16c_q^4 - 8c_q^3 + 16c_q^2 + 8c_q - 1$	$-\infty$
9	$48c_q^4 + 16c_q^3 - 44c_q^2 - 16c_q + 2$	$+\infty$
10	$-32c_q^4 - 8c_q^3 + 28c_q^2 + 8c_q$	0

Conjecture 23. Let $q = 11$ and $c_q = \cos(\pi/q)$; the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M \in \{0, 8\}$	$M \in \{1, 7\}$	$M \in \{2, 6\}$
0	0	0	0
1	0	$80c_q^4 + 40c_q^3 - 52c_q^2 - 14c_q + 9$	$32c_q^4 - 24c_q^3 + 2c_q + 1$
2	0	0	$-40c_q^3 + 12c_q^2 + 12c_q + 4$
3	$80c_q^4 + 8c_q^3 - 60c_q^2 + 10$	0	0
4	$-48c_q^4 - 48c_q^3 + 12c_q^2 + 20c_q + 8$	$-64c_q^4 - 8c_q^3 + 52c_q^2 + 4c_q - 3$	0
5	$-32c_q^4 + 40c_q^3 + 48c_q^2 - 20c_q - 7$	$-16c_q^4 - 32c_q^3 + 10c_q + 5$	$-32c_q^4 + 64c_q^3 - 12c_q^2 - 14c_q + 6$
6	$-32c_q^4 + 40c_q^3 + 48c_q^2 - 20c_q - 7$	$-16c_q^4 - 32c_q^3 + 10c_q + 5$	$-32c_q^4 + 64c_q^3 - 12c_q^2 - 14c_q + 6$
7	$-48c_q^4 - 48c_q^3 + 12c_q^2 + 20c_q + 8$	$-64c_q^4 - 8c_q^3 + 52c_q^2 + 4c_q - 3$	0
8	$80c_q^4 + 8c_q^3 - 60c_q^2 + 10$	0	0
9	0	0	$-40c_q^3 + 12c_q^2 + 12c_q + 4$
10	0	$80c_q^4 + 40c_q^3 - 52c_q^2 - 14c_q + 9$	$32c_q^4 - 24c_q^3 + 2c_q + 1$

j	$M \in \{3, 4, 5\}$	$M = 9$
0	0	0
1	$128c_q^4 + 24c_q^3 - 124c_q^2 - 28c_q + 14$	0
2	$-208c_q^4 - 16c_q^3 + 188c_q^2 + 32c_q - 8$	$+\infty$
3	$80c_q^4 - 8c_q^3 - 64c_q^2 - 4c_q + 5$	$-\infty$
4	0	$+\infty$
5	0	$-\infty$
6	0	$-\infty$
7	0	$+\infty$
8	$80c_q^4 - 8c_q^3 - 64c_q^2 - 4c_q + 5$	$-\infty$
9	$-208c_q^4 - 16c_q^3 + 188c_q^2 + 32c_q - 8$	$+\infty$
10	$128c_q^4 + 24c_q^3 - 124c_q^2 - 28c_q + 14$	0

2.12 Case $q = 12$

Conjecture 24. Let $q = 12$ and $c_q = \cos(\pi/q)$; the values of $\alpha_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$
0	-2	-2	-2	-2	-2	-2
1	0	$-\infty$	$-2c_q^2 + 5/2$	$-2c_q^2/3 - 1/6$	$-2c_q^2/3 - 1/6$	$-2c_q^2/3 - 1/6$
2	0	0	-2	$2c_q^2/3 + 1/6$	$-c_q^2 + 7/4$	$-c_q^2 + 7/4$
3	$8c_q^2 - 1$	0	0	$-4c_q^2/3 + 13/6$	$2c_q^2/3 + 1/6$	$2c_q^2/3 + 1/6$
4	$-12c_q^2 + 1$	-2	0	0	$c_q^2 - 3/4$	$c_q^2 - 3/4$
5	$4c_q^2 + 1$	$+\infty$	$2c_q^2 - 1/2$	0	0	0
6	0	-6	2	$8c_q^2/3 - 7/3$	0	0
7	$4c_q^2 + 1$	$+\infty$	$2c_q^2 - 1/2$	0	0	0
8	$-12c_q^2 + 1$	-2	0	0	$c_q^2 - 3/4$	$c_q^2 - 3/4$
9	$8c_q^2 - 1$	0	0	$-4c_q^2/3 + 13/6$	$2c_q^2/3 + 1/6$	$2c_q^2/3 + 1/6$
10	0	0	-2	$2c_q^2/3 + 1/6$	$-c_q^2 + 7/4$	$-c_q^2 + 7/4$
11	0	$-\infty$	$-2c_q^2 + 5/2$	$-2c_q^2/3 - 1/6$	$-2c_q^2/3 - 1/6$	$-2c_q^2/3 - 1/6$

j	$M = 6$	$M = 7$	$M = 8$	$M = 9$	$M = 10$	$M = 11$
0	-2	-2	-2	-2	-2	-2
1	$-4c_q^2 + 3$	$+\infty$	$-2c_q^2 + \frac{5}{2}$	0	0	$+\infty$
2	$4c_q^2 - 3$	-2	0	0	$+\infty$	$-\infty$
3	1	0	0	$+\infty$	$-\infty$	$+\infty$
4	0	0	-2	$-\infty$	$+\infty$	$-\infty$
5	0	$-\infty$	$2c_q^2 + \frac{7}{2}$	$+\infty$	$-\infty$	$+\infty$
6	0	2	-6	$-\infty$	$+\infty$	$-\infty$
7	0	$-\infty$	$2c_q^2 + \frac{7}{2}$	$+\infty$	$-\infty$	$+\infty$
8	0	0	-2	$-\infty$	$+\infty$	$-\infty$
9	1	0	0	$+\infty$	$-\infty$	$+\infty$
10	$4c_q^2 - 3$	-2	0	0	$+\infty$	$-\infty$
11	$-4c_q^2 + 3$	$+\infty$	$-2c_q^2 + \frac{5}{2}$	0	0	$+\infty$

Conjecture 25. Let $q = 12$ and $c_q = \cos(\pi/q)$; the values of $\sqrt{q}\beta_{M,j}^\infty(0, q)$ are expected to be as follows:

j	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$
0	0	0	0	0	0	0
1	0	$+\infty$	$-4c_q^2 + 11$	$4c_q^2 - 1$	$4c_q^2 - 1$	$4c_q^2 - 1$
2	0	0	-4	$-4c_q^2 + 9$	$2c_q^2 + \frac{7}{2}$	$2c_q^2 + \frac{7}{2}$
3	$24c_q^2$	0	0	$4c_q^2$	$-4c_q^2 + 7$	$-4c_q^2 + 7$
4	$-24c_q^2 + 6$	12	0	0	$-2c_q^2 + \frac{5}{2}$	$-2c_q^2 + \frac{5}{2}$
5	$-24c_q^2 + 6$	$-\infty$	$4c_q^2 + 1$	0	0	0
6	$48c_q^2$	24	8	$-8c_q^2 + 8$	0	0
7	$-24c_q^2 + 6$	$-\infty$	$4c_q^2 + 1$	0	0	0
8	$-24c_q^2 + 6$	12	0	0	$-2c_q^2 + \frac{5}{2}$	$-2c_q^2 + \frac{5}{2}$
9	$24c_q^2$	0	0	$4c_q^2$	$-4c_q^2 + 7$	$-4c_q^2 + 7$
10	0	0	-4	$-4c_q^2 + 9$	$2c_q^2 + \frac{7}{2}$	$2c_q^2 + \frac{7}{2}$
11	0	$+\infty$	$-4c_q^2 + 11$	$4c_q^2 - 1$	$4c_q^2 - 1$	$4c_q^2 - 1$

j	$M = 6$	$M = 7$	$M = 8$	$M = 9$	$M = 10$	$M = 11$
0	0	0	0	0	0	0
1	$-8c_q^2 + 10$	$-\infty$	$12c_q^2 - 9$	0	0	$-\infty$
2	$8c_q^2 - 2$	-4	0	0	$-\infty$	$+\infty$
3	$8c_q^2 - 4$	0	0	$-\infty$	$+\infty$	$-\infty$
4	0	0	12	$+\infty$	$-\infty$	$+\infty$
5	0	$+\infty$	$-12c_q^2 - 3$	$-\infty$	$+\infty$	$-\infty$
6	$-16c_q^2 + 16$	8	24	$+\infty$	$-\infty$	$+\infty$
7	0	$+\infty$	$-12c_q^2 - 3$	$-\infty$	$+\infty$	$-\infty$
8	0	0	12	$+\infty$	$-\infty$	$+\infty$
9	$8c_q^2 - 4$	0	0	$-\infty$	$+\infty$	$-\infty$
10	$8c_q^2 - 2$	-4	0	0	$-\infty$	$+\infty$
11	$-8c_q^2 + 10$	$-\infty$	$12c_q^2 - 9$	0	0	$-\infty$

2.13 Conjectures for a general q

The above stated conjectures admit the following generalizations.

Conjecture 26. *For all q , j , and M , the value $\alpha_{M,j}^\infty(q)$ is either equal to $\pm\infty$ or belongs to the field $\mathbb{Q}(\cos(\pi/q))$.*

Conjecture 27. *For all q , j , and M , the value $\sqrt{q}\beta_{M,j}^\infty(q)$ is either equal to $\pm\infty$ or belongs to the field $\mathbb{Q}(\cos(\pi/q))$.*

For $j = 0$, we can make two more specific conjectures.

Conjecture 28. *For all q and M , $\alpha_{M,0}^\infty(q) = -2$.*

Conjecture 29. *For all q and M , $\beta_{M,0}^\infty(q) = 0$.*

Within the periods (described by (2.4.4)), we observe a kind of symmetry.

Conjecture 30. *For every q , M , j_1 and j_2 , if*

$$j_1 + j_2 \equiv 0 \pmod{q} \tag{2.13.1}$$

then

$$\alpha_{M,j_1}^\infty(q) = \alpha_{M,j_2}^\infty(q) \quad \text{and} \quad \beta_{M,j_1}^\infty(q) = \beta_{M,j_2}^\infty(q).$$

This conjecture implies that under condition (2.13.1)

$$\frac{\alpha_{qN+M,0,j_2}(0,q)}{\alpha_{qN+M,0,j_1}(0,q)} \rightarrow \frac{\alpha_{M,j_2}^\infty(q)}{\alpha_{M,j_1}^\infty(q)} = 1 \tag{2.13.2}$$

as $N \rightarrow \infty$ provided that the value of $\alpha_{M,j_1}^\infty(q)$ is not equal to 0 or $\pm\infty$. It is interesting to observe that the convergence in (2.13.2) appears to be much faster than each of the two underlying convergences in

$$\alpha_{qN+M,0,j_2}(0,q) \rightarrow \alpha_{M,j_2}^\infty(q), \tag{2.13.3}$$

$$\alpha_{qN+M,0,j_1}(0,q) \rightarrow \alpha_{M,j_1}^\infty(q). \tag{2.13.4}$$

Tables 11 and 12 illustrate this phenomenon and its counterpart for $\beta_{qN+M,0,j}(0,q)$.

The above conjectures about the values of $\alpha_{M,j}^\infty$ and $\beta_{M,j}^\infty$ suggest the following two conjectures.

Conjecture 31. *For all q and M , if all the values of $\alpha_{M,0}^\infty(q), \dots, \alpha_{M,q-1}^\infty(q)$ are finite, then*

$$\sum_{j=0}^{q-1} \alpha_{M,j}^\infty(q) = 0. \tag{2.13.5}$$

Conjecture 32. *For all q and M , if all the values of $\beta_{M,0}^\infty(q), \dots, \beta_{M,q-1}^\infty(q)$ are finite, then*

$$\sum_{j=0}^{q-1} \sqrt{q}\beta_{M,j}^\infty(q) = 2q. \tag{2.13.6}$$

q	M	j_1	j_2	N	$\frac{\alpha_{qN+M,0,j_1}(q)}{\alpha_{M,j_1}^\infty(q)} - 1$	$\frac{\alpha_{qN+M,0,j_2}(q)}{\alpha_{qN+M,0,j_1}(q)} - 1$
4	2	1	3	5	$1.82489 \dots 10^{-14}$	$-3.42997 \dots 10^{-148}$
4	1	1	3	10	$-1.92505 \dots 10^{-27}$	$1.92335 \dots 10^{-539}$
5	1	1	4	8	$-2.74429 \dots 10^{-42}$	$-2.42904 \dots 10^{-417}$
5	1	1	4	10	$-5.23088 \dots 10^{-53}$	$-4.85932 \dots 10^{-658}$
6	2	2	4	8	$1.30423 \dots 10^{-21}$	$3.74313 \dots 10^{-539}$
7	3	1	6	8	$3.90028 \dots 10^{-21}$	$2.59440 \dots 10^{-632}$
7	3	1	6	12	$7.04072 \dots 10^{-32}$	$1.24261 \dots 10^{-1406}$
10	4	2	8	8	$1.19614 \dots 10^{-21}$	$-1.67492 \dots 10^{-902}$
10	4	1	9	10	$5.24047 \dots 10^{-27}$	$-4.97789 \dots 10^{-1399}$
12	2	1	11	8	$2.60688 \dots 10^{-20}$	$-6.95177 \dots 10^{-1005}$

Table 11: Comparison of the convergence rate in (2.13.2) and (2.13.3)

q	M	j_1	j_2	N	$\frac{\sqrt{q}\beta_{qN+M,0,j_1}(q)}{\beta_{M,j_1}^\infty(q)} - 1$	$\frac{\beta_{qN+M,0,j_2}(q)}{\beta_{qN+M,0,j_1}(q)} - 1$
4	2	1	3	5	$1.82489 \dots 10^{-14}$	$-3.42997 \dots 10^{-148}$
4	1	1	3	10	$-1.92505 \dots 10^{-27}$	$1.92335 \dots 10^{-539}$
5	1	1	4	8	$-2.74429 \dots 10^{-42}$	$-2.42904 \dots 10^{-417}$
5	1	1	4	10	$-5.23088 \dots 10^{-53}$	$-4.85932 \dots 10^{-658}$
6	2	2	4	8	$1.30423 \dots 10^{-21}$	$3.74313 \dots 10^{-539}$
7	3	1	6	8	$3.90028 \dots 10^{-21}$	$2.59440 \dots 10^{-632}$
7	3	1	6	12	$7.04072 \dots 10^{-32}$	$1.24261 \dots 10^{-1406}$
10	4	2	8	8	$1.19614 \dots 10^{-21}$	$-1.67492 \dots 10^{-902}$
10	4	1	9	10	$5.24047 \dots 10^{-27}$	$-4.97789 \dots 10^{-1399}$
12	2	1	11	8	$2.60688 \dots 10^{-20}$	$-6.95177 \dots 10^{-1005}$

Table 12: The ounterpart of Table 11 for $\beta_{qN+M,0,j}(0, q)$.

2.14 Cases of zero or infinite limiting values

The behavior of $\alpha_{qN+M,0,j}(0, q)$ and $\beta_{qN+M,0,j}(0, q)$ is not clear when $\alpha_{M,j}^\infty(q)$ or $\beta_{M,j}^\infty(q)$ are equal to 0 or when they are equal to $\pm\infty$. Nevertheless, some of the above conjectures can be extended to these cases as well.

For example, even if ratios $\alpha_{M,j_2}^\infty(q)/\alpha_{M,j_1}^\infty(q)$ is an indefinite expressions, such as $0/0$ or $\pm\infty/\pm\infty$, it seems that left-hand side in (2.13.2) still tends to 1.

Conjecture 33. For all M and j_1, j_2 satisfying (2.13.1),

$$\frac{\alpha_{qN+M,0,j_2}(0, q)}{\alpha_{qN+M,0,j_1}(0, q)} \rightarrow 1 \quad (2.14.1)$$

as N tends to infinity.

Similar conjecture can be made for $\beta_{qN+M,0,j_1}(0, q)$.

Conjecture 34. For all M and j_1, j_2 satisfying (2.13.1),

$$\frac{\beta_{qN+M,0,j_2}(0, q)}{\beta_{qN+M,0,j_1}(0, q)} \rightarrow 1 \quad (2.14.2)$$

as N tends to infinity.

q	M	j_1	j_2	N	$\alpha_{M,j_1}^\infty(q)$	$\frac{\alpha_{qN+M,0,j_2}(q)}{\alpha_{qN+M,0,j_1}(q)} - 1$
7	0	1	6	5	0	$8.39093 \dots \cdot 10^{-201}$
7	0	1	6	7	0	$1.68951 \dots \cdot 10^{-413}$
7	0	1	6	9	0	$1.51061 \dots \cdot 10^{-702}$
7	0	1	6	11	0	$5.72953 \dots \cdot 10^{-1068}$
7	0	1	6	13	0	$8.99531 \dots \cdot 10^{-1510}$
9	1	2	7	5	0	$-4.64608 \dots \cdot 10^{-266}$
9	1	2	7	7	0	$-2.29737 \dots \cdot 10^{-543}$
9	1	2	7	9	0	$-7.52208 \dots \cdot 10^{-919}$
9	1	2	7	11	0	$-1.54744 \dots \cdot 10^{-1392}$
8	6	2	6	5	$+\infty$	$3.91745 \dots \cdot 10^{-351}$
8	6	2	6	7	$+\infty$	$3.09265 \dots \cdot 10^{-642}$
8	6	2	6	9	$+\infty$	$1.12525 \dots \cdot 10^{-1020}$
10	8	3	7	3	$-\infty$	$8.32708 \dots \cdot 10^{-181}$
10	8	3	7	5	$-\infty$	$5.70848 \dots \cdot 10^{-433}$
10	8	3	7	7	$-\infty$	$2.90808 \dots \cdot 10^{-794}$
10	8	3	7	9	$-\infty$	$1.07854 \dots \cdot 10^{-1264}$

Table 13: Data supporting Conjecture 33.

q	M	j_1	j_2	N	$\beta_{M,j_1}^\infty(q)$	$\frac{\beta_{qN+M,0,j_2}(q)}{\beta_{qN+M,0,j_1}(q)} - 1$
6	1	2	4	5	0	$2.27340 \dots \cdot 10^{-189}$
6	1	2	4	7	0	$4.23522 \dots \cdot 10^{-378}$
6	1	2	4	9	0	$3.16022 \dots \cdot 10^{-632}$
6	1	2	4	11	0	$8.66986 \dots \cdot 10^{-952}$
6	1	2	4	13	0	$8.37832 \dots \cdot 10^{-1337}$
7	1	2	5	5	0	$1.38517 \dots \cdot 10^{-213}$
7	1	2	5	7	0	$9.53090 \dots \cdot 10^{-432}$
7	1	2	5	9	0	$2.95147 \dots \cdot 10^{-726}$
7	1	2	5	11	0	$3.89260 \dots \cdot 10^{-1097}$
7	6	1	6	5	$+\infty$	$-2.08349 \dots \cdot 10^{-294}$
7	6	1	6	7	$+\infty$	$-5.09156 \dots \cdot 10^{-545}$
7	6	1	6	9	$+\infty$	$-4.70732 \dots \cdot 10^{-872}$
7	6	1	6	11	$+\infty$	$-1.67244 \dots \cdot 10^{-1275}$
9	2	1	8	3	$-\infty$	$6.93014 \dots \cdot 10^{-100}$
9	2	1	8	5	$-\infty$	$9.19843 \dots \cdot 10^{-290}$
9	2	1	8	7	$-\infty$	$7.75569 \dots \cdot 10^{-578}$

Table 14: Data supporting Conjecture 34.

It seems that two other conjectures can be extended to the cases with infinite limiting values.

Conjecture 35 (generalization of Conjectures 31). *For all q , L , and M ,*

$$\sum_{j=0}^{q-1} \alpha_{N,0,L+j}(0, q) \rightarrow 0 \quad (2.14.3)$$

as N tends to infinity.

Conjecture 36. *For all q , L , and M ,*

$$\sum_{j=0}^{q-1} \sqrt{q} \beta_{N,0,L+j}(0, q) \rightarrow 2q \quad (2.14.4)$$

as N tends to infinity.

Tables 15 and 16 support Conjectures 2.14.3 and 2.14.4, respectively.

q	L	N	$\alpha_{N,L}^{\infty}(q)$	$\sum_{j=0}^{q-1} \alpha_{N,0,L+j}(q)$
8	2	15	$-\infty$	$-7.18248 \dots \cdot 10^{-20}$
8	2	23	$-\infty$	$-2.85712 \dots \cdot 10^{-62}$
8	2	31	$-\infty$	$-2.42848 \dots \cdot 10^{-126}$
8	2	39	$-\infty$	$-3.70631 \dots \cdot 10^{-212}$
9	2	25	$+\infty$	$-1.39268 \dots \cdot 10^{-68}$
9	2	34	$+\infty$	$2.43214 \dots \cdot 10^{-140}$
9	2	43	$+\infty$	$-1.21692 \dots \cdot 10^{-236}$
9	2	52	$+\infty$	$1.72256 \dots \cdot 10^{-357}$

Table 15: Data supporting Conjecture 2.14.3.

q	L	N	$\beta_{N,L}^{\infty}(q)$	$\sum_{j=0}^{q-1} \sqrt{q} \beta_{N,0,L+j}(q)$
7	1	13	$+\infty$	$14 + 3.08004 \dots \cdot 10^{-18}$
7	1	20	$+\infty$	$14 - 3.95278 \dots \cdot 10^{-55}$
7	1	27	$+\infty$	$14 + 4.49935 \dots \cdot 10^{-111}$
7	1	34	$+\infty$	$14 - 4.43640 \dots \cdot 10^{-186}$
9	1	20	$-\infty$	$18 - 2.00702 \dots \cdot 10^{-36}$
9	1	29	$-\infty$	$18 + 5.50037 \dots \cdot 10^{-92}$
9	1	38	$-\infty$	$18 - 4.83907 \dots \cdot 10^{-172}$
9	1	47	$-\infty$	$18 + 1.28622 \dots \cdot 10^{-276}$

Table 16: Data supporting Conjecture 2.14.4.

2.15 Bottom row

So far we have considered the entries in the top rows of matrices $B_N^{-1}(0, q)$. Now we state a conjecture regarding the bottom rows of these matrices. Specifically, numerical data suggest

Conjecture 37. *For all q, N , and $j = 1, \dots, N$*

$$\alpha_{N,2N-1,j}(0, q) = -\beta_{N,2N-1,j}(0, q). \quad (2.15.1)$$

For small values of N , this conjecture can be easily verified by any computer algebra system.

Appendix

To justify the definitions (1.24) of $\alpha_{N,i,j}(d, q)$ and $\beta_{N,i,j}(d, q)$, we need to prove that the matrix $B_N(d, q)$ (defined by (1.23)) is not degenerate.

Let us consider matrix

$$\hat{B}_N(d, x) = \begin{bmatrix} \hat{\lambda}_{N,0}(d, x) & \dots & \hat{\lambda}_{N,2N-1}(d, x) \\ \vdots & \ddots & \vdots \\ \hat{\lambda}_{1,0}(d, x) & \dots & \hat{\lambda}_{1,2N-1}(d, x) \\ \hat{\mu}_{1,0}(d, x) & \dots & \hat{\mu}_{1,2N-1}(d, x) \\ \vdots & \ddots & \vdots \\ \hat{\mu}_{N,0}(d, x) & \dots & \hat{\mu}_{N,2N-1}(d, x) \end{bmatrix} \quad (3.1)$$

where

$$\hat{\lambda}_{n,m}(d, x) = E_m(n^2 x), \quad (3.2)$$

and

$$\hat{\mu}_{n,m}(d, x) = F_{d,m}(n^2 x). \quad (3.3)$$

We have:

$$\lambda_{n,m}(d, q) = n^d e^{-\frac{\pi n^2}{q}} \hat{\lambda}_{n,m}(d, \pi/q), \quad (3.4)$$

$$\mu_{n,m}(d, q) = n^d e^{-\frac{\pi n^2}{q}} \hat{\mu}_{n,m}(d, \pi/q). \quad (3.5)$$

This implies that

$$\det(B_N(d, q)) = N!^{2d} e^{-\frac{\pi N(N+1)(2N+1)}{3q}} \det(\hat{B}_N(d, \pi/q)) \quad (3.6)$$

and hence it is sufficient to show that the matrix $\hat{B}_N(d, \pi/q)$ is not degenerate.

The determinant $\det(\hat{B}_N(d, x))$ is a polynomial in x . Both $E_m(n^2 x)$ and $F_{d,m}(n^2 x)$ are polynomials of degree m in x , so the degree of $\det(\hat{B}_N(d, x))$ is at most $\sum_{m=0}^{2N-1} m = N(2N-1)$. Therefore,

$$\det(\hat{B}_N(d, x)) = \sum_{k=0}^M r_k x^k \quad (3.7)$$

where $M = N(2N-1)$ and $r_0, \dots, r_M$ are rational numbers.

Let us check that $r_M \neq 0$. Calculating this number, we can replace polynomials $\hat{\lambda}_{n,m}(d, x)$ and $\hat{\mu}_{n,m}(d, x)$ in (3.1) with their leading coefficients:

$$r_M = \det(\hat{B}_N^{\text{lc}}(d)) \quad (3.8)$$

where

$$\hat{B}_N^{\text{lc}}(d) = \begin{bmatrix} \text{lc}(\hat{\lambda}_{N,0}(d, x)) & \dots & \text{lc}(\hat{\lambda}_{N,2N-1}(d, x)) \\ \vdots & \ddots & \vdots \\ \text{lc}(\hat{\lambda}_{1,0}(d, x)) & \dots & \text{lc}(\hat{\lambda}_{1,2N-1}(d, x)) \\ \text{lc}(\hat{\mu}_{1,0}(d, x)) & \dots & \text{lc}(\hat{\mu}_{1,2N-1}(d, x)) \\ \vdots & \ddots & \vdots \\ \text{lc}(\hat{\mu}_{N,0}(d, x)) & \dots & \text{lc}(\hat{\mu}_{N,2N-1}(d, x)) \end{bmatrix}, \quad (3.9)$$

and $\text{lc}(P)$ is the leading coefficient of the polynomial P . According to (1.14) and (1.20),

$$\text{lc}(\hat{\lambda}_{n,m}(d, x)) = (n^2)^m, \quad (3.10)$$

$$\text{lc}(\hat{\mu}_{n,m}(d, x)) = (-n^2)^m. \quad (3.11)$$

Thus, $\hat{B}_N^{\text{lc}}(d)$ is a Vandermonde matrix, and hence its determinant, which is equal to r_M , is non-zero.

Since π is a transcendental number, the polynomial (3.7) does not vanish when x is equal to any non-zero rational multiple of π . Therefore, according to (3.6), matrix $B_n(d, q)$ is non-singular for $q = 1, 2, \dots$.

Remark. While polynomials (3.7) do not vanish for x equal to a non-zero rational multiple of π , these polynomials seem to have zeros close to π/q for $q = 4, 5, \dots$. To observe this phenomenon, one can calculate the zeros of the polynomial

$$q^M \det\left(\hat{B}_N\left(0, \frac{\pi}{q}\right)\right) = \sum_{k=0}^M r_k \pi^k q^{M-k}. \quad (3.12)$$

Table 17 presents 40 least positive zeros of (3.12) for $N = 12, 15$, and 20 .

4 Summary

We have introduced certain matrices $B_N(d, q)$ of size $2N \times 2N$; the entries to these matrices were suggested by a special form of the functional equation for the Dirichlet $L_\chi(s)$ -function for a Dirichlet character modulo q .

We have proven that these matrices are not singular. Numerical calculations indicate that matrices the $B_N(0, q)$ and their inverses $B_N^{-1}(0, q)$ exhibits the following unexpected properties.

The formal definition of $B_N(d, q)$ does not imply that integer values of q could play any special role. However, the determinants of $B_N(0, q)$ vanish when q is close to $4, 5, \dots$.

The entries in bottom row of matrices $B_N^{-1}(0, q)$ are symmetrical, this implies the equality of the corresponding minors of the matrices $B_N(0, q)$.

In addition to these *exact* equalities between the entries of matrices $B_N^{-1}(0, q)$, there are also many *approximate* equalities. In the left-hand half of the top rows of these matrices, the entries at distance q are very close to each other; the same is true for the right-hand half of the same top rows. Also there are similar approximate equalities between entries (at similar positions) to

$N = 12$	$N = 15$	$N = 20$
$4 + 1.43487 \dots 10^{-39}$	$4 - 1.40426 \dots 10^{-73}$	$4 + 1.53607 \dots 10^{-119}$
$5 + 3.19394 \dots 10^{-27}$	$5 + 4.91675 \dots 10^{-46}$	$5 - 2.23138 \dots 10^{-88}$
$6 + 5.51749 \dots 10^{-22}$	$6 + 3.25396 \dots 10^{-36}$	$6 - 1.34398 \dots 10^{-67}$
$6 + 4.73456 \dots 10^{-19}$	$6 + 4.89556 \dots 10^{-34}$	$6 + 4.55472 \dots 10^{-70}$
$7 - 2.65631 \dots 10^{-13}$	$7 - 1.39365 \dots 10^{-25}$	$7 - 9.60329 \dots 10^{-59}$
$7 - 1.08677 \dots 10^{-15}$	$7 - 4.45839 \dots 10^{-27}$	$7 - 9.64307 \dots 10^{-62}$
$8 - 2.25893 \dots 10^{-9}$	$8 - 5.06780 \dots 10^{-22}$	$8 - 9.70568 \dots 10^{-42}$
$8 + 8.04754 \dots 10^{-16}$	$8 - 2.23904 \dots 10^{-24}$	$8 + 4.08603 \dots 10^{-52}$
$8 + 3.95665 \dots 10^{-11}$	$8 - 2.18157 \dots 10^{-31}$	$8 + 4.29936 \dots 10^{-44}$
$9 - 1.92047 \dots 10^{-7}$	$9 - 3.48374 \dots 10^{-15}$	$9 - 3.76566 \dots 10^{-39}$
$9 + 1.94667 \dots 10^{-13}$	$9 - 9.75265 \dots 10^{-16}$	$9 - 1.71319 \dots 10^{-41}$
$9 + 2.77801 \dots 10^{-6}$	$9 + 1.16528 \dots 10^{-18}$	$9 + 3.50304 \dots 10^{-33}$
$10 - 3.53263 \dots 10^{-4}$	$10 - 1.66989 \dots 10^{-11}$	$10 + 1.82065 \dots 10^{-37}$
$10 - 1.74943 \dots 10^{-5}$	$10 - 6.70672 \dots 10^{-19}$	$10 + 4.21558 \dots 10^{-34}$
$10 + 8.52601 \dots 10^{-11}$	$10 + 1.65192 \dots 10^{-15}$	$10 + 4.34898 \dots 10^{-28}$
$10 + 8.12171 \dots 10^{-8}$	$10 + 3.96915 \dots 10^{-10}$	$10 + 5.83012 \dots 10^{-26}$
$11 - 5.07370 \dots 10^{-3}$	$11 - 3.58106 \dots 10^{-7}$	$11 - 2.04383 \dots 10^{-20}$
$11 - 3.85253 \dots 10^{-6}$	$11 - 5.39248 \dots 10^{-14}$	$11 - 2.38453 \dots 10^{-22}$
$11 + 4.01883 \dots 10^{-7}$	$11 + 1.43976 \dots 10^{-15}$	$11 - 9.40356 \dots 10^{-35}$
$11 + 6.08232 \dots 10^{-3}$	$11 + 3.10096 \dots 10^{-8}$	$11 - 1.06456 \dots 10^{-37}$
$12 + 5.77332 \dots 10^{-9}$	$12 - 2.84041 \dots 10^{-6}$	$12 - 1.67378 \dots 10^{-16}$
$12 + 1.78233 \dots 10^{-5}$	$12 - 1.02577 \dots 10^{-8}$	$12 - 1.18415 \dots 10^{-17}$
$12 + 9.82536 \dots 10^{-4}$	$12 + 6.41326 \dots 10^{-14}$	$12 - 6.85463 \dots 10^{-23}$
$12 + 4.44090 \dots 10^{-2}$	$12 + 1.16164 \dots 10^{-9}$	$12 - 2.29403 \dots 10^{-29}$
$13 - 6.36804 \dots 10^{-2}$	$12 + 6.51164 \dots 10^{-5}$	$12 + 1.06921 \dots 10^{-23}$
$13 - 1.01354 \dots 10^{-3}$	$13 - 2.08510 \dots 10^{-3}$	$13 - 7.06669 \dots 10^{-14}$
$13 - 3.03801 \dots 10^{-5}$	$13 - 7.52582 \dots 10^{-4}$	$13 - 2.18236 \dots 10^{-19}$
$13 - 2.11913 \dots 10^{-8}$	$13 + 1.53274 \dots 10^{-10}$	$13 - 4.28641 \dots 10^{-24}$
$14 + 2.03601 \dots 10^{-10}$	$13 + 1.58782 \dots 10^{-7}$	$13 + 8.83051 \dots 10^{-18}$
$14 + 1.09077 \dots 10^{-7}$	$13 + 2.55442 \dots 10^{-6}$	$13 + 1.40217 \dots 10^{-12}$
$14 + 1.46130 \dots 10^{-4}$	$14 - 4.09509 \dots 10^{-2}$	$14 - 1.48535 \dots 10^{-9}$
$14 + 3.24088 \dots 10^{-3}$	$14 + 6.79009 \dots 10^{-11}$	$14 - 1.18676 \dots 10^{-14}$
$15 - 1.19707 \dots 10^{-6}$	$14 + 2.49762 \dots 10^{-8}$	$14 - 1.44754 \dots 10^{-19}$
$15 - 6.55257 \dots 10^{-9}$	$14 + 3.03301 \dots 10^{-5}$	$14 + 3.10819 \dots 10^{-21}$
$16 + 5.46932 \dots 10^{-12}$	$14 + 1.04236 \dots 10^{-4}$	$14 + 7.81810 \dots 10^{-16}$
$16 + 6.34726 \dots 10^{-4}$	$14 + 3.40819 \dots 10^{-2}$	$14 + 1.47459 \dots 10^{-10}$
$17 + 5.58344 \dots 10^{-4}$	$15 + 8.00550 \dots 10^{-9}$	$15 - 5.47130 \dots 10^{-8}$
$18 + 2.82566 \dots 10^{-4}$	$15 + 1.77435 \dots 10^{-6}$	$15 - 2.22946 \dots 10^{-11}$
$20 - 2.91020 \dots 10^{-1}$	$15 + 5.24417 \dots 10^{-4}$	$15 - 5.70439 \dots 10^{-22}$
$20 - 4.37113 \dots 10^{-2}$	$15 + 9.02722 \dots 10^{-3}$	$15 + 2.70141 \dots 10^{-20}$

Table 17: Least positive zeros of (3.12).

the top rows of matrices $B_{N_1}^{-1}(0, q)$ and $B_{N_2}^{-1}(0, q)$ provided that $N_1 \equiv N_2 \pmod{q}$. Moreover, there are (finite or infinite) limits of the values of such entries as N tends to ∞ , running along an arithmetic progression modulo q . If the limits are finite, they belong to the field $\mathbb{Q}(\cos(\pi/q))$ (up to a scaling factor of $\sqrt{q}$).

The appearance of this field and above-described periodicities modulo q is rather surprising as this number occurs in the definition of matrices $B_N(0, q)$ in the denominators of certain real-valued expressions only. Hypothetically, the periodicities could be connected to the periodicity of the characters, but they are not used in the definition $B_N(0, q)$ at all.

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