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N-transform and factorization of the DN-map

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Abstract.

Let (Ω, g) be a smooth compact 3D Riemannian manifold with the smooth boundary Γ, $\tau(x) := \operatorname{dist}(x, \Gamma)$, $x \in \Omega$; $\Omega^\tau := \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) < \tau\}$, $\Gamma^\tau := \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) = \tau\}$, $\tau \geqslant 0$. For the sake of technical simplicity, we deal with Ω diffeomorphic to a ball in $\mathbb{R}^3$.

Let $\mathscr{P} := \{\nabla p \mid p \in H^1(\Omega)\}$ be the space of the potential vector fields, and let $\mathscr{L}_\lambda := \{\varkappa \nabla\tau \mid \varkappa \in L_2(\Omega)\}$ be the space of the vector fields parallel to $\nabla\tau$. The N-transform is a map from $\mathscr{P}$ to $\mathscr{L}_\lambda$ defined layer-wise (in accordance with $\Omega = \cup_{\tau \geqslant 0} \Gamma^\tau$) by

$$Nh\big|_{\Gamma^\tau} := (P^\tau h)\big|_{\Gamma^{\tau-0}}, \qquad \tau > 0,$$

where P^τ are the projections in $\mathscr{P}$ onto the subspaces $\mathscr{P}^\tau := \{h \in \mathscr{P} \mid \operatorname{supp} h \subset \overline{\Omega^\tau}\}$. We show that N is a unitary operator.

Let $p = p^f(x)$ be a solution to the Dirichlet problem: $\Delta_g p = 0$ in $\Omega \setminus \Gamma$, $p = f$ on Γ. The DN-map Λ is defined by $\Lambda f := -\langle \nabla p^f, \nabla\tau \rangle$ on Γ. We show that the N-transform provides a certain factorization $\Lambda^{-1} = V^*V$ and discuss its possible usefulness for determination of (Ω, g) from Λ.

Keywords: Helmholtz-Weyl decomposition of 3D vector fields, subspace of potential fields, harmonic fields, N-transform, DN-map and its factorization.

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About the paper

• The N-transform was introduced in [1] in parallel and by analogy with the M-transform, which plays a key role in the 3D dynamical (time-domain) inverse problem for the Maxwell system [3, 2]. These transforms correspond to the Helmholtz-Weyl decomposition of the 3D vector fields into gradients and curls. However, unlike M-transform, the N-transform has not yet found application in inverse problems. In our paper, we study its fundamental properties: isometry and completeness, and discuss such a possible application in the form of a conjecture related to a factorization of the elliptic DN-map (the Calderon operator) of Ω.

• Let (Ω, g) be a smooth compact 3D Riemannian manifold with the smooth boundary Γ, $\tau(x) := \operatorname{dist}(x, \Gamma)$, $x \in \Omega$. We denote $\Omega^\tau := \{x \in \Omega \,|\, \tau(x) < \tau\}$, $\Gamma^\tau := \{x \in \Omega \,|\, \tau(x) = \tau\}$, $\tau \geqslant 0$, and represent $\Omega = \cup_{\tau \geqslant 0} \Gamma^\tau$.

Let $\mathscr{P} := \{h = \nabla p \,|\, p \in H^1(\Omega)\}$ be the space of the potential fields; let $\mathscr{L}_\lambda := \{v = \varkappa \nabla \tau \,|\, \varkappa \in L_2(\Omega)\}$ be the space of the vector fields longitudinal w.r.t. $\nabla\tau$. The N-transform is an isometry from $\mathscr{P}$ to $\mathscr{L}_\lambda$. It is constructed layer-wise (in accordance with $\Omega = \cup_{\tau \geqslant 0} \Gamma^\tau$) via the breaks $P^\tau h\big|_{\Gamma^{\tau}-0}$ of the projections on the subspaces $\mathscr{P}^\tau := \{h \in \mathscr{P} \,|\, \operatorname{supp} h \subset \overline{\Omega^\tau}\}$:

$$N h \,:=\, (P^\tau h)\big|_{\Gamma^\tau - 0} \qquad \text{on } \Gamma^\tau, \;\; \tau > 0$$

(see [1]). For the sake of technical simplicity, we deal with Ω diffeomorphic to a ball in $\mathbb{R}^3$ and show that N is a unitary operator.

• Let $\mathscr{H} := \{h \in \mathscr{P} \,|\, h = \nabla p, \;\Delta p = 0 \text{ in } \Omega\}$ (Δ the Beltrami-Laplace operator) be the subspace of the harmonic potential fields. We show that the fields $N\mathscr{H} \subset \mathscr{L}_\lambda$ are of the form $v = \varkappa \nabla \tau$, where $\varkappa$ satisfies a first-order evolutionary differential equation in $\Gamma \times (0, T)$ (in the semi-geodesic coordinates (γ, τ) with the base on Γ) and the Cauchy data $\varkappa\big|_\Gamma = \varkappa_0$. By $V \,:\, \varkappa_0 \mapsto \varkappa\big|_{\Gamma \times [0,T]}$ we denote the operator (propagator) that solves this Cauchy problem.

• The DN-map Λ is associated with the elliptic Dirichlet problem

$$\begin{cases} \Delta p = 0 & \text{in } \Omega \setminus \Gamma; \\ p = f & \text{on } \Gamma, \end{cases}$$

with a solution $p = p^f(x)$, and is defined by

$$\Lambda f \,:=\, \partial_\nu p^f = -\langle \nabla p^f, \nabla \tau \rangle \qquad \text{on } \Gamma,$$

where ν is the outward normal at the boundary.

We show that the N-transform provides a factorization $\Lambda^{-1} = V^*V$ and discuss its possible usefulness for the electric impedance tomography problem, which is determination of (Ω, g) from Λ.

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Manifold and coordinates

• Let (M, g) be a smooth[1] *three-dimensional* Riemannian manifold. By $\langle a, b\rangle$ we denote the (point-wise) inner product of vector fields a, b (sections of $T\Omega$); Δ is the Beltrami - Laplace operator on M. Let $\Omega \Subset M$ be a ball of radius T centered at a point $c \in \Omega$, $\Gamma := \partial\Omega$. We assume that $T < r_c^{\rm inj}$ (injectivity radius), so that $\exp_c$ is smooth in Ω.

Let $\tau(x) := \operatorname{dist}(x, \Gamma)$, $\Omega^\tau := \{x \in \Omega \mid \tau(x) < \tau\}$, $\Gamma^\tau := \{x \in \Omega \mid \tau(x) = \tau\}$, $0 \leqslant \tau \leqslant T$. The semi-geodesic coordinates (s.g.c.) $x \mapsto (\gamma, \tau) : \ \tau = \tau(x), \gamma = \gamma(x)$, where $\gamma(x) \in \Gamma$ satisfies $\operatorname{dist}(x, \gamma(x)) = \tau(x)$, are defined and regular in $\dot\Omega := \Omega \setminus \{c\}$. By $x(\gamma, \tau) \in \dot\Omega$ we denote the point with the s.g.c. $(\gamma, \tau) \in \Gamma \times [0, T)$. Such an Ω is diffeomorphic to a ball in $\mathbb{R}^3$.

Let γ^1, γ^2 be the local coordinates on Γ near $\gamma(x)$, and $\partial_\tau, \partial_{\gamma^1}, \partial_{\gamma^2}$ be the coordinate fields in $\dot\Omega$. The length and volume elements are

$$ds^2 = d\tau + g_{ij}(\gamma, \tau) d\gamma^i d\gamma^j; \qquad dx = g^{\frac{1}{2}}(\gamma, \tau)\, d\tau\, d\gamma^1 d\gamma^2, \quad g := \det\{g_{ij}\},$$

In $\dot\Omega$, each vector field h is represented as

$$h = h_\lambda + h_\theta; \quad h_\lambda := \langle h, \partial_\tau\rangle \partial_\tau, \quad h_\theta := h - h_\lambda \tag{1}$$

with the components h_λ and h_θ, which are called longitudinal (parallel to ∂_τ) and transversal parts of h, whereas $\langle h_\lambda(x), h_\theta(x)\rangle = 0$, $x \in \dot\Omega$ holds. Note that $\nu := -\partial_\tau\big|_\Gamma$ is the field of the outward normals at the boundary.

• The vector analysis operations (see [6]) in s.g.c. are

$$\nabla p = (\partial_\tau p)\, \partial_\tau \ + \ \left(g^{ij} \partial_{\gamma^j} p\right)\, \partial_{\gamma^i}; \ \operatorname{div} h = g^{-\frac{1}{2}} \partial_\tau (g^{\frac{1}{2}} h^0) + g^{-\frac{1}{2}} \partial_{\gamma^i} (g^{\frac{1}{2}} h^i) \tag{2}$$

[1] Everywhere *smooth* means C^∞-smooth.

in $\dot{\Omega}$, where $\{g^{ij}\} := \{g_{ij}\}^{-1}$ and $h = h^0\partial_\tau + h^i\partial_{\gamma^i}$ is a smooth vector field. We say that $\nabla_\lambda := (\partial_\tau\cdot)\,\partial_\tau$ and $\nabla_\theta := (g^{ij}\partial_{\gamma^j}\cdot)\,\partial_{\gamma^i}$ are the longitudinal and transversal parts of the gradient. The operations $\mathrm{div}_\lambda := g^{-\frac{1}{2}}\partial_\tau(g^{\frac{1}{2}}\langle\cdot,\partial_\tau\rangle)$ and $\mathrm{div}_\theta := g^{-\frac{1}{2}}\partial_{\gamma^i}(g^{\frac{1}{2}}\langle\cdot,\partial_{\gamma^i}\rangle$ are the longitudinal and transversal parts of the divergence; so, one represents

$$\nabla p \,=\, \nabla_\lambda p + \nabla_\theta p, \quad \mathrm{div}\, h = \mathrm{div}_\lambda h_\lambda + \mathrm{div}_\theta h_\theta \qquad \text{in } \dot{\Omega} \tag{3}$$

in accordance with (1).

Spaces, subspaces and projections

- Let $\mathscr{L}$ be the space of the square integrable vector fields,

$$\begin{aligned}(a,b)_{\mathscr{L}} := \int_\Omega \langle a,b\rangle\, dx = \int_{\Gamma\times[0,T)} \left[a^0b^0 + g_{ij}a^ib^j\right]\, g^{\frac{1}{2}}\, d\gamma^1 d\gamma^2\, d\tau \,= \\ = \int_0^T d\tau \int_{\Gamma^\tau} d\Gamma^\tau\, \langle a,b\rangle\big|_{\Gamma^\tau},\end{aligned} \tag{4}$$

where $a \,=\, a^0\partial_\tau + a^i\partial_{\gamma^i},\ b \,=\, b^0\partial_\tau + b^i\partial_{\gamma^i}$, and $d\Gamma^\tau \,=\, g^{\frac{1}{2}}(\gamma,\tau)\, d\gamma^1 d\gamma^2 \,=\, \left[\frac{g(\gamma,\tau)}{g(\gamma,0)}\right]^{\frac{1}{2}} d\Gamma$ is the surface element on Γ^τ, $d\Gamma$ is the surface element on Γ. The latter equality in (4) corresponds to the representations

$$\Omega = \bigcup_{0\leqslant\tau\leqslant T} \Gamma^\tau, \qquad L_2(\Omega) \,=\, \oplus \int_{[0,T]} L_2(\Gamma^\tau)\, d\tau. \tag{5}$$

Recall that our Ω is a Riemannian ball diffeomorphic to a ball in $\mathbb{R}^3$. In such a case, the Helmholtz - Weyl decomposition on the potential and solenoidal fields is

$$\mathscr{L} = \mathscr{P} \oplus \mathscr{S}_0, \tag{6}$$

where

$$\begin{aligned}\mathscr{P} &:= \{\nabla p \mid p \in H^1(\Omega)\}, \quad \mathscr{S}_0 := \overline{\{h \in \mathscr{L} \mid \mathrm{div}\, h = 0 \text{ in } \Omega,\ \langle h,\nu\rangle = 0 \text{ on } \Gamma\}} \\ &= \overline{\{h \in \mathscr{L} \mid \mathrm{div}\, h = 0 \text{ in } \Omega,\ \mathrm{supp}\, h \subset \Omega\setminus\Gamma\}}.\end{aligned}$$

The first summand is specified as follows:

$$\mathscr{P} = \mathscr{P}_0 \oplus \mathscr{H}, \tag{7}$$

where

$$\mathscr{P}_0 := \{\nabla p \,|\, p \in H^1_0(\Omega)\} = \overline{\{\nabla p \,|\, \operatorname{supp} p \subset \Omega \setminus \Gamma\}}$$

and $\mathscr{H} := \{\nabla p \,|\, \Delta p = 0 \text{ in } \Omega\}$ (see, e.g., [6]). We say $\mathscr{H}$ to be the subspace of harmonic potential fields.

- One more decomposition, which corresponds to (1), is

$$\mathscr{L} = \mathscr{L}_\lambda \oplus \mathscr{L}_\theta,$$

where $\mathscr{L}_\lambda := \{h \in \mathscr{L} \,|\, h_\theta = 0\}$ and $\mathscr{L}_\theta := \{h \in \mathscr{L} \,|\, h_\lambda = 0\}$ are the subspaces of the longitudinal and transversal fields respectively.

- Denote

$$\mathscr{P}^\tau := \{h \in \mathscr{P} \,|\, \operatorname{supp} h \subset \overline{\Omega^\tau}\}, \quad 0 < \tau \leqslant T; \quad \mathscr{P}^0 := \{0\}, \ \mathscr{P}^T = \mathscr{P},$$

and $\mathscr{P}^\tau_\perp := \mathscr{P} \ominus \mathscr{P}^\tau$. For a field $h = \nabla p \in \mathscr{P}^\tau$ we have $\nabla p = 0$ in $\Omega \setminus \Omega^\tau$, so that $p = \text{const}$ outside Ω^τ holds, and we put $p\big|_{\Omega\setminus\Omega^\tau} = 0$. By P^τ and $P^\tau_\perp$ we denote the projections in $\mathscr{P}$ onto the first and second summands in the decomposition $\mathscr{P} = \mathscr{P}^\tau \oplus \mathscr{P}^\tau_\perp$.

Lemma 1. *For a smooth $h = \nabla p \in \mathscr{P}$, the representations $P^\tau h = \nabla p^\tau$ and $P^\tau_\perp h = \nabla p^\tau_\perp$ hold with the potentials satisfying*

$$\begin{cases} \Delta p^\tau = \operatorname{div} h & \text{in } \operatorname{int}\Omega^\tau; \\ p^\tau = 0 & \text{on } \Gamma^\tau; \\ \partial_\tau p^\tau = \partial_\tau p & \text{on } \Gamma; \end{cases} \quad , \quad \begin{cases} \Delta p^\tau_\perp = 0 & \text{in } \operatorname{int}\Omega^\tau; \\ p^\tau_\perp = p & \text{on } \Gamma^\tau; \\ \partial_\tau p^\tau_\perp = 0 & \text{on } \Gamma; \end{cases} \tag{8}$$

and $p^\tau\big|_{\Omega\setminus\Omega^\tau} = 0$, $p^\tau_\perp\big|_{\Omega\setminus\Omega^\tau} = p$.

Proof. As is easy to check, the relations $h = \nabla p = \nabla p^\tau + \nabla p^\tau_\perp$, $\nabla p^\tau \in \mathscr{P}^\tau$ and $(\nabla p^\tau, \nabla p^\tau_\perp)_{\mathscr{L}} = 0$ hold, which is equivalent to the statement of the Lemma. □

So, the decomposition $h = P^\tau h + P^\tau_\perp h$ takes the form $\nabla p = \nabla p^\tau + \nabla p^\tau_\perp$ with the potentials obeying (8).

N-transform

• Fix $\tau \in (0, T)$ and a smooth $h = \nabla p \in \mathscr{P}$. The field $P^\tau h\big|_{\Gamma^\tau} := P^\tau h\big|_{\Gamma^{\tau-0}}$ is supported on Γ^τ and is orthogonal to Γ^τ. Indeed, in view of $p = p^\tau_\perp$ on Γ^τ, the equality $\nabla_\theta p = \nabla_\theta\, p^\tau_\perp$ holds on Γ^τ, and for $P^\tau h = h - P^\tau_\perp h$ we have

$$\begin{aligned} P^\tau h\big|_{\Gamma^\tau} &= (\nabla p - \nabla p^\tau_\perp)\,\big|_{\Gamma^\tau} = (\partial_\tau p\,\partial_\tau + \nabla_\theta p - \partial_\tau p^\tau_\perp\,\partial_\tau - \nabla_\theta p^\tau_\perp)\,\big|_{\Gamma^\tau} \\ &= (\partial_\tau p\,\partial_\tau - \partial_\tau p^\tau_\perp\,\partial_\tau)\,\big|_{\Gamma^\tau} = (\partial_\tau p - \partial_\tau p^\tau_\perp)\;\partial_\tau\big|_{\Gamma^\tau}. \end{aligned} \tag{9}$$

Define $N : \mathscr{P} \to \mathscr{L}_\lambda$ layer-wise, i.e., in accordance with (5), by

$$Nh\big|_{\Gamma^\tau} \;:=\; P^\tau h\big|_{\Gamma^\tau}, \qquad 0 \leqslant \tau \leqslant T \tag{10}$$

(recall that $P^0 = \mathbb{O}$ and $P^T = \mathbb{I}$) on smooth fields h. As is shown in [1], the transform N is an isometry, which is extended by continuity from smooth h's onto $\mathscr{P}$ to a unitary operator from $\mathscr{P}$ onto $\mathscr{L}_\lambda$. In Appendix, we provide a proof of these properties, which in fact basically repeats the proof in [1].

• For $0 < \tau < T$, we define the operator $\Pi^\tau : L_2(\Gamma^\tau) \to L_2(\Gamma^\tau)$ on $\operatorname{Dom}\Pi^\tau = H^1(\Gamma^\tau)$ by $\Pi^\tau f := \partial_\tau u^f$, where u^f is a solution to

$$\begin{cases} \Delta u = 0 & \text{in } \operatorname{int}\Omega^\tau; \\ u = f & \text{on } \Gamma^\tau; \\ \partial_\tau u = 0 & \text{on } \Gamma. \end{cases} \tag{11}$$

The following facts are well known.

Proposition 1. *Operator Π^τ is a 1-st order PDO with the pricipal symbol $\sigma^\tau(k) = [g^{ij}\big|_{\Gamma^\tau} k_i k_j]^{\frac{1}{2}}$. The relations $\Pi^\tau = \Pi^{\tau\,*} \geqslant \mathbb{O}$, $\operatorname{Ker}\Pi^\tau = \{\mathrm{const}\}$ hold.*

In the space $L_2(\Omega)$ (see (5)) we define the layer-wise operator Π on $\operatorname{Dom}\Pi = \oplus\int_{[0,T]} H^1(\Gamma^\tau)\,d\tau$ by

$$(\Pi\varkappa)\big|_{\Gamma^\tau} \;:=\; \Pi^\tau(\varkappa\big|_{\Gamma^\tau}), \qquad 0 < \tau < T.$$

One can easily verify the following facts.

Proposition 2. *The relations $\Pi = \Pi^* \geqslant \mathbb{O}$, $\operatorname{Ker}\Pi = \mathscr{C}$, $\operatorname{Ran}\Pi \perp \mathscr{C}$ hold, where $\mathscr{C}$ is the class of the square-summable leyer-wise constant functions.*

Taking into account (9) and (10), we represent

$$Nh = N\nabla p = (\partial_\tau p - \Pi p)\, \partial_\tau. \tag{12}$$

• For a fixed $\tau \in [0,T)$, Γ^τ is a 2-dimensional Riemannian manifold (surface) with the metric $g^\tau = g\big|_{\Gamma^\tau}$ induced by the metric g in Ω. The elements of the vector field space $\mathscr{L}^\tau := \vec{L}_2(\Gamma^\tau)$ can be identified with the traces of transversal fields in Ω: $\mathscr{L}^\tau = \overline{\{a\big|_{\Gamma^\tau} \mid a \text{ is smooth, } a \in \mathscr{L}_\theta\}}$. It contains the subspace $\mathscr{P}_{\Gamma^\tau} := \{\nabla^\tau \phi \mid \phi \in H^1(\Gamma^\tau)\}$ of the potential fields, where ∇^τ is the intrinsic gradient on Γ^τ.

Put

$$\dot{L}_2(\Gamma^\tau) := \{\phi \in L_2(\Gamma^\tau) \mid (\phi, 1) = \int_{\Gamma^\tau} \phi\, d\Gamma^\tau = 0\}, \qquad 0 \leqslant \tau < T. \tag{13}$$

The vector analysis operations $\nabla^\tau : \dot{L}_2(\Gamma^\tau) \to \mathscr{P}_{\Gamma^\tau}$ and $\operatorname{div}^\tau = -\nabla^{\tau\,*} : \mathscr{P}_{\Gamma^\tau} \to \dot{L}_2(\Gamma^\tau)$ are injective.

Introduce the spaces

$$\dot{L}_2(\Omega) \ :=\ \oplus \int_{[0,T]} \dot{L}_2(\Gamma^\tau)\, d\tau = L_2(\Omega) \ominus \mathscr{C} \ =\ \overline{\operatorname{Ran} \Pi}$$

and $\mathscr{Q}_\theta \ :=\ \oplus \int_{[0,T]} \mathscr{P}_{\Gamma^\tau}\, d\tau \subset \mathscr{L}_\theta$. The layer-wise operations

$$\nabla_\theta\big|_{\dot{L}_2(\Omega)} = \oplus \int_{[0,T]} \nabla^\tau\, d\tau : \dot{L}_2(\Omega) \to \mathscr{Q}_\theta$$

and

$$\operatorname{div}_\theta\big|_{\mathscr{Q}_\theta} = \oplus \int_{[0,T]} \operatorname{div}^\tau\, d\tau : \mathscr{Q}_\theta \to \dot{L}_2(\Omega)$$

are injective. Therefore, the operations $\nabla_\theta^{-1} : \mathscr{Q}_\theta \to \dot{L}_2(\Omega)$ and $\operatorname{div}_\theta^{-1} : \dot{L}_2(\Omega) \to \mathscr{Q}_\theta$ are well defined, and the relations

$$\nabla_\theta^{-1}\nabla_\theta = \mathrm{id}, \qquad (\nabla_\theta^{-1})^* = -\operatorname{div}_\theta^{-1} \tag{14}$$

are valid on $\dot{L}_2(\Omega)$ and $\mathscr{Q}_\theta$ respectively.

• The latter relations are used to derive the following representation.

Lemma 2. *Let $\varkappa$ be smooth in $\dot{\Omega}$, $v = \varkappa\,\partial_\tau \in \mathscr{L}_\lambda$. For $N^* : \mathscr{L}_\lambda \to \mathscr{P}$ the representation*

$$N^* v \,=\, P_{\mathscr{P}}\left[\varkappa\,\partial_\tau \,+\, \operatorname{div}{}_\theta^{-1}\Pi\varkappa\right] \tag{15}$$

is valid, where $P_{\mathscr{P}}$ is the projection onto $\mathscr{P}$ in (6). The relation

$$\operatorname{div} N^* v \,=\, \operatorname{div} \varkappa\,\partial_\tau + \Pi\varkappa \tag{16}$$

holds.

Proof. Let p and $\varkappa$ be smooth, $h = \nabla p \in \mathscr{P}$ and $v = \varkappa\partial_\tau \in \mathscr{L}_\lambda$. Let $\dot{p}$ be the projection in $L_2(\Omega)$ onto $\dot{L}_2(\Omega) = \overline{\operatorname{Ran}\Pi}$, so that $\nabla_\theta p = \nabla_\theta \dot{p}$ holds. Then we have

$$\begin{aligned}
&(Nh, v)_{\mathscr{L}} \overset{\text{see (12)}}{=} (\partial_\tau p\,\partial_\tau, \varkappa\,\partial_\tau)_{\mathscr{L}} - (\Pi p\,\partial_\tau, \varkappa\,\partial_\tau)_{\mathscr{L}} \\
&= (\partial_\tau p, \varkappa)_{L_2(\Omega)} - (\Pi p, \varkappa)_{L_2(\Omega)} = (\partial_\tau p\,\partial_\tau, \varkappa\partial_\tau)_{\mathscr{L}_\lambda} - (\dot{p}, \Pi\varkappa)_{L_2(\Omega)} \\
&\overset{(14)}{=} (\partial_\tau p\partial_\tau, \varkappa\partial_\tau)_{\mathscr{L}_\lambda} - (\nabla_\theta^{-1}\nabla_\theta p, \Pi\varkappa)_{L_2(\Omega)} \overset{(14)}{=} (\partial_\tau p\partial_\tau, \varkappa\partial_\tau)_{\mathscr{L}_\lambda} + (\nabla_\theta p, \operatorname{div}{}_\theta^{-1}\Pi\varkappa)_{\mathscr{L}_\theta} \\
&= (\partial_\tau p\,\partial_\tau + \nabla_\theta p, \varkappa\partial_\tau + \operatorname{div}{}_\theta^{-1}\Pi\varkappa)_{\mathscr{L}} = (\nabla p, \varkappa\partial_\tau + \operatorname{div}{}_\theta^{-1}\Pi\varkappa)_{\mathscr{L}} \\
&= (h, \varkappa\partial_\tau + \operatorname{div}{}_\theta^{-1}\Pi\varkappa)_{\mathscr{L}} = (P_{\mathscr{P}} h, \varkappa\partial_\tau + \operatorname{div}{}_\theta^{-1}\Pi\varkappa)_{\mathscr{L}} \\
&= (h, P_{\mathscr{P}}[\varkappa\partial_\tau + \operatorname{div}{}_\theta^{-1}\Pi\varkappa])_{\mathscr{L}} = (h, N^* v)_{\mathscr{L}}.
\end{aligned}$$

So, by the density of smooth h's in $\mathscr{P}$, for smooth $\varkappa$'s the representation (15) does hold. By the simply verified boubdedness of the composition $\operatorname{div}{}_\theta^{-1}\Pi\varkappa]$ it can be extended to $\mathscr{L}_\lambda$ by continuity.

Denoting $a := \varkappa\partial_\tau + \operatorname{div}{}_\theta^{-1}\Pi\varkappa$, we have $a = P_{\mathscr{P}} a + P_{\mathscr{S}} a$ (see (6)) and $P_{\mathscr{P}} a = a - P_{\mathscr{S}} a$, div $P_{\mathscr{S}} a = 0$, which implies

$$\begin{aligned}
\operatorname{div} N^* v &= \operatorname{div} P_{\mathscr{P}} a = \operatorname{div} a - \operatorname{div} P_{\mathscr{S}} a = \operatorname{div} a = \operatorname{div}\, \varkappa\partial_\tau + \operatorname{div}{}_\theta^{-1}\Pi\varkappa \\
&\overset{(3)}{=} \operatorname{div}\, \varkappa\partial_\tau + \operatorname{div}{}_\theta \operatorname{div}{}_\theta^{-1}\Pi\varkappa = \operatorname{div}\, \varkappa\partial_\tau + \Pi\varkappa. \qquad \square
\end{aligned}$$

• Let $h = \nabla p$ and div $h = 0$ hold, so that $\Delta p = 0$ holds, i.e., the potential p is harmonic in Ω. Let $v = \varkappa\,\partial_\tau = Nh$. Then $\operatorname{div} N^* v = \operatorname{div} N^* N h = \operatorname{div} h = \Delta p = 0$ holds and, by (16), implies

$$\operatorname{div} \varkappa\,\partial_\tau + \Pi\varkappa \,=\, 0 \qquad \text{in } \operatorname{int}\Omega. \tag{17}$$

Factorization of DN-map

- Consider the Dirichlet problem

$$\begin{cases} \Delta p = 0 & \text{in } \operatorname{int}\Omega; \\ p = f & \text{on } \Gamma; \end{cases}$$

let $p = p^f(x)$ be a solution, $\operatorname{div}\nabla p^f = \Delta p^f = 0$ holds in Ω. The operator $W : L_2(\Gamma) \to \mathscr{H} \subset \mathscr{P}$ (see (7)), $Wf := \nabla p^f$ resolves the problem.

The DN map is $\Lambda : L_2(\Gamma) \to \mathrm{L}_2(\Gamma)$, $\operatorname{Dom}\Lambda = H^1(\Gamma)$,

$$\Lambda f := \nu p^f \qquad \text{on } \Gamma,$$

where $\nu = -\partial_\tau\big|_\Gamma$ is the outward normal at the boundary. Since

$$\begin{aligned}(\nabla p^f, \nabla p^g)_{\mathscr{H}} = (Wf, Wg)_{\mathscr{H}} &= \int_\Omega \langle \nabla p^f, \nabla p^g\rangle\, dx \\ &= \int_\Gamma \nu p^f\, p^g\, d\Gamma = \int_\Gamma \Lambda f\, g\, d\Gamma,\end{aligned} \tag{18}$$

we have $\Lambda = \Lambda^* = W^*W \geqslant \mathbb{O}$, $\operatorname{Ran}\Lambda = \dot{L}_2(\Gamma)$ (see (13)) and $\operatorname{Ker}\Lambda = \{\text{const}\}$.

- Let $N\nabla p^f = \varkappa^f\,\partial_\tau \in \mathscr{L}_\lambda$. Then $\operatorname{div}\varkappa^f\partial_\tau + \Pi\varkappa^f \overset{(17)}{=} 0$ holds and takes the form $g^{-\frac{1}{2}}\partial_\tau(g^{\frac{1}{2}}\varkappa^f) + \Pi\varkappa^f = 0$ in s.g.c. (see (2) and (3)). Also, in view of $\partial_\tau p^\tau_\perp\big|_\Gamma \overset{(8)}{=} 0$, we have

$$N\nabla p^f\big|_\Gamma \overset{(9)}{=} \partial_\tau p^f \partial_\tau\big|_\Gamma = -\nu p^f \partial_\tau\big|_\Gamma = -\Lambda f\,\partial_\tau\big|_\Gamma = (\varkappa^f\,\partial_\tau)\big|_\Gamma,$$

which implies $\varkappa^f = -\Lambda f$ on Γ. As a result, $\varkappa = \varkappa^f(x(\gamma,\tau))$ satisfies

$$\begin{cases} g^{-\frac{1}{2}}\partial_\tau(g^{\frac{1}{2}}\varkappa) + \Pi\varkappa = 0 & \text{in } \Gamma\times(0,T); \\ \varkappa\big|_{\tau=0} = -\Lambda f; \end{cases}. \tag{19}$$

By V we denote the operator (propagator) $V : \varkappa\big|_\Gamma \mapsto \varkappa\big|_{\Gamma\times(0,T)}$ that resolves the Cauchy problem (19) [7, 5, 8]. Since $\operatorname{Ran}\Lambda = \dot{L}_2(\Gamma)$ holds, V acts from $\dot{L}_2(\Gamma)$ to $N\mathscr{H} \subset \mathscr{L}_\lambda$.

- By isometry of the N-transform and (18) we have

$$\begin{aligned}&(\Lambda f\, g)_{L_2(\Gamma)} = (\nabla p^f, \nabla p^g)_{\mathscr{H}} = (N\nabla p^f, N\nabla p^g)_{\mathscr{L}_\lambda} = (\varkappa^f\partial_\tau, \varkappa^g\partial_\tau)_{\mathscr{L}_\lambda}\\ &= (\varkappa^f, \varkappa^g)_{L_2(\Omega)} = (V\varkappa^f\big|_{\tau=0}, V\varkappa^g\big|_{\tau=0})_{L_2(\Omega)} = (V\Lambda f, V\Lambda g)_{L_2(\Omega)}\\ &= (\Lambda V^*V\Lambda f, g)_{L_2(\Gamma)},\end{aligned}$$

which implies $\Lambda = \Lambda V^*V\Lambda$ and leads to

$$\Lambda^{-1} \;=\; V^*V. \tag{20}$$

so that $V : \dot{L}_2(\Gamma) \to N\mathscr{H}$ provides a factorization to Λ^{-1}.

Illustration: upper half-space

- Let $\Omega = \overline{\mathbb{R}^3_+} = \{(x, z) \mid x = (x^1, x^2) \in \mathbb{R}^2,\; z \geqslant 0\}$, $\Gamma = \{(x, 0) \mid x \in \mathbb{R}^2\}$.

A harmonic potential $p = p^f$ satisfies

$$\begin{cases}\Delta p = 0 & \text{in } \operatorname{int}\Omega;\\ p = f & \text{on } \Gamma;\\ p \to 0 & \text{as } |x| + z \to \infty;\end{cases} \quad .$$

The Fourier transform

$$\tilde{u}(k, z) = (Fu(\cdot, z))(k) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^2} e^{-\langle k,x\rangle} u(x, z)\, dx,$$

implies

$$\begin{cases}\tilde{p}_{zz} + |k|\,\tilde{p} = 0, & z > 0;\\ \tilde{p} = \tilde{f}, & z = 0;\\ \tilde{p} \to 0, & |k| + z \to \infty;\end{cases}$$

and provides

$$\tilde{p}\,(k, z) \;=\; e^{-|k|z}\tilde{f}(k), \qquad k \in \mathbb{R}^2_*,\; z \geqslant 0,$$

whereas $\tilde{\Lambda} := F\Lambda F^*$ just multiplies functions by $|k|$.

- The second problem in (8) takes the form

$$\begin{cases}(\tilde{p}^\tau_\perp)_{zz} + |k|^2(\tilde{p}^\tau_\perp) = 0, & z > 0;\\ (\tilde{p}^\tau_\perp) = \tilde{p}, & z = \tau;\\ (\tilde{p}^\tau_\perp)_z = 0, & z = 0;\end{cases}$$

and provides

$$\tilde{p}^{\tau}_{\perp}(k,z)=\frac{\cosh|k|z}{\cosh|k|\tau}\,\tilde{p}(k,z),\qquad 0\leqslant z\leqslant\tau.$$

Hence we have

$$(\tilde{p}^{\tau}_{\perp})_z(k,\tau)=\left(\Pi^{\tau}\tilde{p}(\cdot.\tau)\right)(k)=\left(|k|\,\tanh|k|\tau\right)\tilde{p}(k,\tau),\qquad \tau\geqslant 0.$$

- Problem (19) takes the form

$$\begin{cases}\tilde{\varkappa}_{\tau}+(|k|\,\tanh|k|\tau)\tilde{\varkappa}=0, & \tau>0;\\ \tilde{\varkappa}=-\widetilde{\Lambda f}=-\tilde{\Lambda}\tilde{f}=-|k|\tilde{f}, & z=0;\end{cases}\tag{21}$$

the solution is

$$\tilde{\varkappa}(k,\tau)=e^{-\int_0^{\tau}|k|\,\tanh|k|\eta\,d\eta}\,\tilde{\varkappa}(k,0)=\frac{\tilde{\varkappa}(k,0)}{\cosh|k|\tau}=-\frac{|k|}{\cosh|k|\tau}\,\tilde{f}(k).$$

Thus, the operator resolving (21) acts from $L_2(\{k\in\mathbb{R}^2_*\})$ to $L_2\left(\mathbb{R}^2_*\times\{z\geqslant 0\}\right)$ by

$$(\tilde{V}\tilde{\varkappa}_0)(k,z)\;=\;\frac{\tilde{\varkappa}_0(k)}{\cosh|k|z},\qquad k\in\mathbb{R}^2_*,\;z\geqslant 0.$$

As is easy to check, for $\tilde{V}^*:L_2\left(\mathbb{R}^2_*\times\{z\geqslant 0\}\right)\to L_2(\{k\in\mathbb{R}^2_*\})$ one has

$$(\tilde{V}^*\phi)(k)\;=\;\int_0^{\infty}\frac{\phi(k,z)}{\cosh|k|z}\,dz,\qquad k\in\mathbb{R}^2_*.$$

As a result, we get

$$(\tilde{V}^*\tilde{V}\tilde{\varkappa}_0)(k)\;=\;\int_0^{\infty}\frac{dz}{\cosh|k|z}\frac{\tilde{\varkappa}_0(k)}{\cosh|k|z}=\tilde{\varkappa}_0(k)\int_0^{\infty}\frac{dz}{\cosh^2|k|z}=\frac{\tilde{\varkappa}_0(k)}{|k|}.$$

Thus, we have $\tilde{V}^*\tilde{V}=\tilde{\Lambda}^{-1}$ in the accordance with (20).

Illustration: unit ball

Let $\Omega=\mathbb{B}=:\{x\in\mathbb{R}^3\mid|x|\leq 1\}$; then $\Gamma^{\tau}:=\{x\in\mathbb{R}^3\mid|x|=1-\tau\}$, $\Gamma=\Gamma^0=\mathbb{S}^2$. Introduce the spherical coordinates (r,ϑ,φ); then $\partial_{\tau}=-\partial_r$. Let

u^τ_{lm} be a solution to (11), where $f(\vartheta,\varphi)=Y^m_l(\vartheta,\varphi)$ is a spherical harmonic. As is easy to derive,

$$u^\tau_{lm}=\frac{(l+1)r^l+lr^{-(l+1)}}{(l+1)\rho^l+l\rho^{-(l+1)}}Y^m_l(\vartheta,\varphi)\qquad(\rho:=1-\tau)$$

holds. Hence, one has

$$\Pi^\tau Y^m_l=-\partial_r u^\tau_{lm}|_{r=\rho}=\frac{l(l+1)}{\rho}\frac{1-\rho^{2l+1}}{l+(l+1)\rho^{2l+1}}Y^m_l(\vartheta,\varphi).$$

In particular, if $f=Y^m_l$ on $\mathbb{S}^2$, then $p^f=r^l f$, $\Lambda f=lf$, and one obtains

$$N\nabla p^f=-l\,\lambda_l(\rho)\,f\,\partial_\tau;\qquad Vf=\lambda_l(\rho)\,f,\qquad f=Y^m_l,$$

where

$$\lambda_l(\rho):=-\frac{(2l+1)\rho^{l-1}}{l+(l+1)\rho^{2l+1}}.$$

Note that $N\nabla p^f$ $(f\in H^{1/2}(\Gamma))$ is bounded but may have a jump discontinuity at the center of the ball.

Denote by $\mathfrak{P}_l$ the projection in $L_2(\mathbb{S}^2)$ on the eqigenspace of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$ corresponding to the eigenvalue $l(l+1)$. Then the above formulas imply

$$[Vf](\cdot,\tau)=\sum_{l=1}^\infty\lambda_l(1-\tau)\,\mathfrak{P}_l f,\quad V^*\phi=\int_0^1\sum_{l=1}^\infty[\mathfrak{P}_l\phi](\cdot,1-\rho)\lambda_l(\rho)\rho^2d\rho.$$

At last, we have

$$V^*Vf=\sum_{l=1}^\infty\Big(\int_0^1\lambda_l^2(\rho)\rho^2d\rho\Big)\mathfrak{P}_l f=\sum_{l=1}^\infty l^{-1}\mathfrak{P}_l f=\Lambda^{-1}f$$

in accordance with (20).

Comments and hopes

• Note that, in problem (19), the operator Π is a layer-wise PDO, whereas its symbol determines the metric g in s.g.c. (Proposition 1). Consider the case in which $\Omega=\mathbb{R}^3_+$ is a half-space endoved with the smooth metric

$$ds^2=(dx^3)^2+\sum_{ij=1,2}g_{ij}(x^1,x^2,x^3)\,dx^i dx^j,$$

coinciding with the euclidean one outside a sufficiently large ball $|x| < N$. Then the results of general theory of parabolic PDO [7, 5] applied to the initial problem (19) imply that its evolution operator $V^\tau : \varkappa(\cdot,0) \mapsto (V\varkappa)(\cdot,\tau)$ is a negligible PDO for any $\tau > 0$.

• Perhaps the above could suggest an approach to the 3D electrical impedance tomography problem. If we characterized the factorization (20) so that it was realizable (or at least unique), then we could recover the metric g in s.g.c. by the scheme $\Lambda \mapsto V \mapsto \Pi \mapsto \Pi^\tau \mapsto g\big|_{\Gamma^\tau}$, $\tau > 0$ (see Proposition 1).

Reducing the problem to the determination $\Lambda \mapsto V$, we encounter a canonical situation: to recover an operator V via its module $|V| := \sqrt{V^*V} = \Lambda^{-\frac{1}{2}}$, which is quite common in inverse problems. In the time-domain IP's, the property of V, due to that the factorization $\Lambda^{-1} = V^*V$ is realizable, is its triangularity, which reflects the fundamental physical fact: the finiteness of the wave propagation velocity [4]. So, in EIT we need to recognize its relevant "physical" analog. Perhaps, it is related to the complex geometrical optics.

Appendix

Recall that Ω is a ball diffeomorphic to a ball in $\mathbb{R}^3$. The proof of the following basic property of the N-transform in fact repeats the analogous proof in [1].

Theorem 1. *The N-transform is a unitary operator from $\mathscr{P}$ to $\mathscr{L}_\lambda$.*

Proof. • Show that the N-transform is an isometry. Taking smooth $h = \nabla p$ and $g = \nabla q$, we have

$$\frac{d}{d\tau}\,(P^\tau h, P^\tau g)_{\mathscr{L}_\lambda}$$
$$= \frac{d}{d\tau}\int_{\Omega^\tau} dx\,\langle P^\tau h, P^\tau g\rangle = \frac{d}{d\tau}\int_0^\tau ds\int_{\Gamma^s} d\Gamma^s\,\langle h - \nabla p^\tau_\perp, g - \nabla q^\tau_\perp\rangle \;=\; I^\tau \;+ II^\tau,$$

where $I^\tau := \int_{\Gamma^\tau} d\Gamma^\tau\, \langle h - \nabla p^\tau_\perp, g - \nabla q^\tau_\perp\rangle = \int_{\Gamma^\tau} d\Gamma^\tau\, \langle Nh, Ng\rangle$, and

$$
\begin{aligned}
I\!I^\tau &:= \int_0^\tau ds \int_{\Gamma^s} d\Gamma^s\, \left[\langle -\nabla\frac{\partial p^\tau_\perp}{\partial\tau},\, g - \nabla q^\tau_\perp\rangle + \langle h - \nabla p^\tau_\perp,\, -\nabla\frac{\partial q^\tau_\perp}{\partial\tau}\rangle\right] \\
&= \int_0^\tau ds \int_{\Gamma^s} d\Gamma^s\, \left[\langle -\nabla\frac{\partial p^\tau_\perp}{\partial\tau},\, \nabla(q - q^\tau_\perp)\rangle + \langle \nabla(p - p^\tau_\perp),\, -\nabla\frac{\partial q^\tau_\perp}{\partial\tau}\rangle\right] \\
&= \int_{\Omega^\tau} dx\, \left[\Delta\frac{\partial p^\tau_\perp}{\partial\tau}\,(q - q^\tau_\perp) + (p - p^\tau_\perp)\,\Delta\frac{\partial q^\tau_\perp}{\partial\tau}\right] \\
&- \int_\Gamma d\Gamma\, \left[\frac{\partial(\partial_\nu p^\tau_\perp)}{\partial\tau}\,(q - q^\tau_\perp) + (p - p^\tau_\perp)\,\frac{\partial(\partial_\nu q^\tau_\perp)}{\partial\tau}\right] \\
&- \int_{\Gamma^\tau} d\Gamma^\tau\, \left[\frac{\partial(\partial_\nu p^\tau_\perp)}{\partial\tau}\,(q - q^\tau_\perp) + (p - p^\tau_\perp)\,\frac{\partial(\partial_\nu q^\tau_\perp)}{\partial\tau}\right] =: \int_{\Omega^\tau} - \int_{\Gamma^\tau} - \int_\Gamma,
\end{aligned}
$$

where ν is the outward normal to $\partial\Omega^\tau = \Gamma \cup \Gamma^\tau$. The first equation of the second system in (8) easily provides $\Delta\frac{\partial p^\tau_\perp}{\partial\tau} = \frac{\partial\Delta p^\tau_\perp}{\partial\tau} = \Delta\frac{\partial q^\tau_\perp}{\partial\tau} = \frac{\partial\Delta q^\tau_\perp}{\partial\tau} = 0$ in $\operatorname{int}\Omega^\tau$, so that we have $\int_{\Omega^\tau} = 0$. The second equation implies $\int_{\Gamma^\tau} = 0$, the third one leads to $\int_\Gamma = 0$. As a result, we get $I\!I^\tau = 0$ and

$$
\frac{d}{d\tau}\,(P^\tau h, P^\tau g)_{\mathscr{L}_\lambda} = \int_{\Gamma^\tau} d\Gamma^\tau\, \langle Nh, Ng\rangle, \qquad 0 < \tau < T.
$$

Integrating over τ with regard to $P^0 = \mathbb{O}$ and $P^T = \mathbb{I}$, we arrive at

$$
(h, g)_{\mathscr{P}} = \int_0^T d\tau \int_{\Gamma^\tau} d\Gamma^\tau\, \langle Nh, Ng\rangle = \int_\Omega dx\, \langle Nh, Ng\rangle = (Nh, Ng)_{\mathscr{L}_\lambda},
$$

so that N is an isometry on the smooth fields. Hence, its extension by continuity (denoted by the same N) is an isometry from $\mathscr{P}$ to $\mathscr{L}_\lambda$.

- The following simple facts are used later.

1. Fix $0 < \tau < T$. Any $h = \nabla p \in \mathscr{P}^\tau$ is longitudinal on Γ^τ in view of $p\big|_{\Gamma^\tau} = \text{const} = 0$. By (8), the latter implies $p^\tau_\perp = 0$ in Ω^τ and, by (9), follows to

$$
Nh = h \qquad \text{on } \Gamma^\tau. \tag{22}
$$

2. For any smooth ψ given on Γ^τ $(0 < \tau < T)$, there is a field $h \in \mathscr{P}^\tau$ such that $h\big|_{\Gamma^\tau - 0} = \psi\partial_\tau$ holds. Indeed, choosing in Ω^τ a smooth function u provided $u = 0$ and $\partial_\tau u = \psi$ on Γ^τ, and putting $h\big|_{\Omega^\tau} = \nabla u$, $h\big|_{\Omega\setminus\Omega^\tau} = 0$, we get a required field.

• Show that $\mathrm{Ran}\, N = \mathscr{L}_\lambda$ holds.

The definition of the N-transform (10) easily implies

$$NP^\tau \;=\; X^\tau N, \qquad 0<\tau<T,$$

where X^τ cuts off fields on Ω^τ:

$$X^\tau v \;=\; \begin{cases} v & \text{in } \Omega^\tau; \\ 0, & \text{in } \Omega\setminus\Omega^\tau; \end{cases}.$$

As a consequence, we have $P^\tau N^* = N^* X^\tau$, which leads to

$$X^\tau \mathrm{Ker}\, N^* \;\subset\; \mathrm{Ker}\, N^*, \qquad 0<\tau<T. \tag{23}$$

By the latter, assuming $v=\varkappa\partial_\tau,\; v\perp \mathrm{Ran}\, N$ or, equivalently, $v\in \mathrm{Ker}\, N^*$, we have $X^\tau v\in \mathrm{Ker}\, N^*$ and

$$0 \overset{(23)}{=} (h, N^*X^\tau v)_{\mathscr{P}} = (Nh, X^\tau v)_{\mathscr{L}_\lambda} = \int_{\Omega^\tau} \langle Nh, v\rangle\, dx = \int_0^T ds \int_{\Gamma^\tau} d\Gamma^\tau \langle Nh, v\rangle$$

for $h\in\mathscr{P}$ and $0<\tau<T$. Differentiation provides

$$\int_{\Gamma^\tau} \langle Nh, v\rangle\, d\Gamma^\tau \;=\; 0, \qquad 0<\tau<T. \tag{24}$$

Fix a $\tau=\sigma$. Choose a smooth function ψ on Γ^τ and a field $h\in\mathscr{P}^\sigma$, $h\big|_{\Gamma^\sigma}=\psi\partial_\tau$ that is possible due to **2.**. For such a choice, one has

$$\int_{\Gamma^\sigma} \langle Nh, v\rangle\, d\Gamma^\sigma \overset{(22)}{=} \int_{\Gamma^\sigma} \langle h, v\rangle\, d\Gamma^\sigma = \int_{\Gamma^\sigma} \psi\,\varkappa\, d\Gamma^\sigma \overset{(24)}{=} 0. \tag{25}$$

Since ψ is arbitrary, (25) yields $\varkappa=0$ on Γ^σ. Since σ is arbitrary, we have $\varkappa=0$ and hence $v=0$ in Ω. Thus $\mathrm{Ker}\, N^* = \mathscr{L}_\lambda \ominus \mathrm{Ran}\, N = \{0\}$ holds, i.e., $\mathrm{Ran}\, N = \mathscr{L}_\lambda$ is valid.

• So, the transform $N:\mathscr{P}\to\mathscr{L}_\lambda$ is an isometry acting onto the image space, i.e., is a unitary operator. $\square$

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