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On a stability of the time-optimal version of the Boundary Control method

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Abstract.

Let Ω be a Riemannian manifold, let Γ be its boundary. The time-optimal version of the BC-method determines the parameters in the T-neighborhood Ω^T of Γ from the boundary observations (response operator) R^{2T} on the time segment $[0, 2T]$. It visualizes the invisible waves supported in Ω^T, by reconstructing the operator W^T that creates these waves. The visualization is based on a triangular factorization of the operator $C^T := W^{T*}W^T$ in the form $C^T := F^{T*}F^T$ with a triangular factor $F^T = U^T W^T$, where U^T is a unitary operator.

The factorization $C^T \mapsto F^T$ has certain continuity properties, due to which the time-optimal reconstruction $R^{2T} \mapsto C^T \mapsto F^T \mapsto W^T$ turns out to be continuous (stable) in the sense of relevant operator topologies (convergences).

As an example, determination of the potential q in the wave equation $u_{tt} - \Delta u + qu = 0$ on the known Ω from R^{2T} is considered. We show that, under certain assumptions on the convergence $R^{2T}_j \to R^{2T}$, it implies $q_j \to q$ in $H^{-2}(\Omega^T)$. However, the question of quantitative estimates of stability (the rate of convergence) remains open.

Keywords: BC-method, time-optimal determination of parameters and manifolds, triangular factorization, wave visualization, stability.

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1 Introduction

• A principal feature and advantage of the boundary control method (BCm, [1]) is a time-optimality of the procedure that determines manifolds and their parameters from the time-domain boundary inverse data (observations). The optimality means that for determination in the T-neighborhood of the boundary, the observations R^{2T} on the time segment $[0, 2T]$ are sufficient, while the observations on $[0, 2T']$ with $T' < T$ are insufficient and the observations on $[0, 2T']$ with $T' > T$ are redundant. This feature of the *time-optimal reconstruction* (TOR) reflects the fundamental property of the wave motion, which is the finiteness of the wave propagation velocity. Meanwhile, there are other versions of the BCm that use $R^{2T'}$, $T' > T$: see, e.g., [4, 25, 31, 32, 34, 35].

• The stability of BCm has been and continues to be studied in a series of papers by [16, 19, 29, 30, 35]. However, to our knowledge, all known quantitative results (estimates) are obtained for the versions, which work with the data $[0, 2T']$ with $T' > T$. Moreover, some experts doubt the existence of similar estimates for TOR [30]. Meanwhile, the physically natural character of TOR and examples of its successful numerical implementation [7, 8, 23, 24] inspire hope for the existence of at least a qualitative stability.

The latter is the subject of our paper. Let (Ω, g) be a Riemann manifold with the boundary Γ, $\Omega^T \subset \Omega$ the T-neighborhood of Γ. We show that a certain convergence $R^{2T}_j \to R^{2T}$ at $\Gamma \times [0, 2T]$ leads to a convergence $V^T_j \to V^T$ of the operators, which visualize on $\Gamma \times [0, T]$ invisible waves supported in Ω^T. The latter convergence is equivalent to a convergence $W^T_j \to W^T$ of the operators which create waves in Ω^T. So, if Ω *is known*, which is the case in the given paper, then the convergence $R^{2T}_j \to R^{2T}$ leads to $W^T_j \to W^T$. This is our main result. Its application, in which the wave evolution is governed by the equation $u_{tt} - \Delta_g u + qu = 0$, is considered and, eventually, we arrive at convergence of the potentials $q_j \to q$ in $H^{-2}(\Omega^T)$. However, the important, difficult and challenging question on the quantitative estimates (the rate of convergence) in TOR remains open. Interesting results in this direction are recently obtained in [20].

• The plan of the paper is as follows.

Section 2 is a portion of the operator theory, which concerns the triangular factorization (TF) of positive operators. The concept of convergence regular on a nest of subspaces is introduced. A known result (Theorem 1 from [15]) is presented in a form convenient for subsequent application.

Section 3 contains a brief description of the TOR for a dynamical system with boundary control governed by the wave equation $u_{tt} - \Delta u + qu = 0$.

In Section 4 we show that Theorem 1 is applicable to TOR and provides a result on the weak operator convergence $W_j^T \to W^T$. Its consequence is a convergence $q_j \to q$.

Section 5 contains a technical Lemma.

• In the paper, we deal with a (small enough) times $T < T_0$ such that the subdomain Ω^{T_0} is covered by the regular semigeodesic coordinates with the base on Γ. This is done purely for technical simplicity: some TOR operators are unitary (and not just isometric, as in the general case), Geometrical Optics is simpler, and so on. All results in the paper are valid for arbitrary $T < T_*$, where $T_* := \inf\{T > 0 \,|\, \Omega^T = \Omega\} \leqslant \infty$ is the filling time.

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2 Operator theory preliminaries

Unless otherwise stated, all operators are considered bounded.

An operator C in a Hilbert space is *nonnegative*, if $(Cf, f) \geqslant 0$ holds for all f; *positive* if $(Cf, f) > 0$ holds for all $f \neq 0$; and *positive definite* if $(Cf, f) \geqslant c\|f\|^2$ with a constant $c > 0$ is valid for all f. We write $C \geqslant \mathbb{O}$, $C > \mathbb{O}$, and $C \geqslant c\,\mathbb{I}$ respectively.

2.1 Convergences

• Let $\mathscr{F}$ and $\mathscr{H}$ be the Hilbert spaces, $A, A_j : \mathscr{F} \to \mathscr{H}$ $(j = 1, 2, \dots \to \infty)$ the operators. Defining the uniform (norm), strong and weak convergence, we write $A_j \overset{\rm u}{\to} A$, $A_j \overset{\rm s}{\to} A$ and $A_j \overset{\rm w}{\to} A$ if $\|A_j - A\| \to 0$, $\|(A_j - A)f\| \to 0$ and $((A_j - A)f, h) \to 0$ hold respectively for all $f \in \mathscr{F}$ and $h \in \mathscr{H}$.

We denote by $|A| := \sqrt{A^*A} \geqslant \mathbb{O}$, $|A| : \mathscr{F} \to \mathscr{F}$ the operator module and use the polar decomposition $A = \Phi|A|$ with the phase operator $\Phi : \mathscr{F} \to \mathscr{H}$, which is an isometry mapping $\overline{{\rm Ran}\,|A|} \subset \mathscr{F}$ onto $\overline{{\rm Ran}\,A} \subset \mathscr{H}$. The operators $\Phi^*\Phi$ and $\Phi\Phi^*$ are the projections in $\mathscr{F}$ onto $\overline{{\rm Ran}\,|A|}$ and onto

$\overline{\mathrm{Ran}\, A}$ respectively. If $\overline{\mathrm{Ran}\,|A|} = \mathscr{F}$ and $\overline{\mathrm{Ran}\, A} = \mathscr{H}$ then Φ is a unitary operator, and $\Phi^*\Phi = \mathbb{I}_{\mathscr{F}}$, $\Phi\Phi^* = \mathbb{I}_{\mathscr{H}}$ holds.

Lemma 1. *Let $A = \Phi|A|$ and $A_j = \Phi_j|A_j|$ be the polar decompositions with the unitary Φ and Φ_j. If $A_j \xrightarrow{\mathrm{u}} A$ holds then the convergences $|A_j| \xrightarrow{\mathrm{u}} |A|$, $\Phi_j \xrightarrow{\mathrm{s}} \Phi$ and $\Phi_j^* \xrightarrow{\mathrm{s}} \Phi^*$ occur.*

Proof. Representing

$$\||A_j|^2 - |A|^2\| = \|A_j^*(A_j - A) + (A_j^* - A^*)A\| \leqslant \|A_j^*\|\|A_j - A\| + \|A_j^* - A^*\|\|A\|$$

and using the known $A_j^* \xrightarrow{\mathrm{u}} A^*$, we get $\||A_j|^2 - |A|^2\| \to 0$. The latter implies $\||A_j| - |A|\| \to 0$ (see, e.g., [18]), so that $|A_j| \xrightarrow{\mathrm{u}} |A|$ holds.

Representing

$$(\Phi_j - \Phi)|A| = A_j - A - \Phi_j(|A_j| - |A|)$$

and taking $f \in \mathscr{F}$ arbitrarily, we have

$$\|(\Phi_j - \Phi)|A|f\| \leqslant \|(A_j - A)f\| + \|(|A_j| - |A|)f\| \to 0.$$

Since $\mathrm{Ran}\,|A|$ is dense in $\mathscr{F}$ and $\|\Phi_j\|$ are uniformly bounded, we get $\Phi_j \xrightarrow{\mathrm{s}} \Phi$.

For $h \in \mathscr{H}$ we have

$$\|(\Phi_j^* - \Phi^*)h\|^2 = 2\|h\|^2 - 2\Re(\Phi_j^*h, \Phi^*h) = 2\|h\|^2 - 2\Re(h, \Phi_j\Phi^*h) \to 0$$

in view of $\Phi_j \xrightarrow{\mathrm{s}} \Phi$ proved above. $\square$

Note and recall that the strong convergence of the unitary operator sequence, in general, does not provide the limit to be unitary. Also, the conjugation $A \mapsto A^*$ is not continuous in the strong operator topology.

- For a lineal set $\mathscr{G}$, by $P_{\mathscr{G}}$ we denote the projection on $\overline{\mathscr{G}}$. If U is a unitary operator then $P_{U\mathscr{G}} = UP_{\mathscr{G}}U^*$ holds.

Let $A_j \xrightarrow{\cdot} A$ be one of the operator convergences. We say that it is *regular on a subspace* $\mathscr{F}' \subset \mathscr{F}$ (and write $A_j \xrightarrow{\cdot,\mathrm{r}} A$) if $P_{A_j\mathscr{F}'} \xrightarrow{\mathrm{s}} P_{A\mathscr{F}'}$ holds [15]. The case $\mathscr{F}' = \mathscr{F} = \mathscr{H}$, $A_j = j^{-1}\mathbb{I} \xrightarrow{\mathrm{u}} A = \mathbb{O}$, $P_{A_j\mathscr{F}} = \mathbb{I} \not\to \mathbb{O} = P_A$ is a trivial example of the irregular convergence. For more interesting examples see [15].

If A_j regularly converges on a family $\mathfrak{f} = \{\mathscr{F}^\alpha\}$ we say that the convergence is regular on $\mathfrak{f}$.

Lemma 2. *Assume that* $A_j = \Phi_j|A_j|$, $A = \Phi|A|$, *all* Φ_j *and* Φ *are unitary, and* $A_j \overset{\text{u,r}}{\to} A$ *holds on a subspace* $\mathscr{F}' \subset \mathscr{F}$. *Then the convergence* $|A_j| \overset{\text{u,r}}{\to} |A|$ *occurs on* $\mathscr{F}'$.

Proof. The regularity of $A_j \overset{\text{u}}{\to} A$ on $\mathscr{F}'$ means that

$$P_{A_j\mathscr{F}'} = P_{\Phi_j|A_j|\mathscr{F}'} = \Phi_j P_{|A_j|\mathscr{F}'}\Phi_j^* \overset{\text{s}}{\to} P_{A'} = P_{\Phi|A|\mathscr{F}'} = \Phi P_{|A|\mathscr{F}'}\Phi^*$$

holds. Therefore, we have

$$\Psi_j P_{|A_j|\mathscr{F}'}\Psi_j^* \overset{\text{s}}{\to} P_{|A|\mathscr{F}'}$$

with a unitary $\Psi_j := \Phi_j^*\Phi$, whereas $\Psi_j \overset{\text{s}}{\to} \mathbb{I}_{\mathscr{H}}$ holds by Lemma 1. For $h \in \mathscr{H}$, one has

$$\begin{aligned} 0 \leftarrow \|\left(\Psi_j P_{|A_j|\mathscr{F}'}\Psi_j^* - P_{|A|\mathscr{F}'}\right)h\| \,=\, \|\Psi_j\left(P_{|A_j|\mathscr{F}'}\Psi_j^* - \Psi_j^* P_{|A|\mathscr{F}'}\right)h\| = \\ = \|\left(P_{|A_j|\mathscr{F}'}\Psi_j^* - \Psi_j^* P_{|A|\mathscr{F}'}\right)h\| \,=\, \|P_{|A_j|\mathscr{F}'}\Psi_j^* h - \Psi_j^* P_{|A|\mathscr{F}'}h\|. \end{aligned}$$

In view of $\Psi_j^* \overset{\text{s}}{\to} \mathbb{I}_{\mathscr{H}}$, the terms in the last $\|\cdot\|$ are of the form

$$P_{|A_j|\mathscr{F}'}\Psi_j^* h = P_{|A_j|\mathscr{F}'}h + o_j(1), \qquad \Psi_j^* P_{|A|\mathscr{F}'}h = P_{|A|\mathscr{F}'}h + \tilde{o}_j(1)$$

with $\|o_j(1)\|$ and $\|\tilde{o}_j(1)\|$ tending to 0. As a result, we arrive at $\|P_{|A_j|\mathscr{F}'}h - P_{|A|\mathscr{F}'}h\| \to 0$ that proves the Lemma. $\square$

2.2 Triangular factorization

• Let $\mathfrak{f} = \{\mathscr{F}^s\}_{s\in[0,T]}$ be a *nest* in a Hilbert space $\mathscr{F}$, i.e., a family of extending subspaces [17, 21]. Let X^s be the projection in $\mathscr{F}$ onto $\mathscr{F}^s$. We assume $\mathfrak{f}$ to be continuous:

$$\mathscr{F}^s \subset \mathscr{F}^{s'},\ 0 \leqslant s < s' \leqslant T; \quad X^\sigma \underset{\sigma\to s}{\to} X^s; \quad \mathscr{F}^0 = \{0\},\ \mathscr{F}^T = \mathscr{F}.$$

(the projectors converge strongly).

An operator F is *triangular* w.r.t. a nest $\mathfrak{f}$ if $F\mathscr{F}^s \subset \mathscr{F}^s$ (equivalently, $FX^s = X^sFX^s$) holds for all $s \in [0,T]$. An operator $C \geqslant \mathbb{O}$ admits the *triangular factorization* (TF) w.r.t. the nest $\mathfrak{f}$ if the representation $C = F^*F$ holds with a triangular F. Such a factorization is not unique: if U is unitary and triangular then UF satisfies $(UF)^*(UF) = C$ and also provides a TF of C. One way to ensure uniqueness is to fix the *diagonal* of the factor. For further convenience, we introduce this notion in more general situation: see [12, 13, 15] for detail and motivation.

• Let $\mathscr{H}$ be one more Hilbert space. For an operator $W\,:\,\mathscr{F}\to\mathscr{H}$, the subspaces $\mathscr{H}^s:=\overline{W\mathscr{F}^s},\ s\in[0,T]$ form a nest. By P^s we denote the projections in $\mathscr{H}$ onto $\mathscr{H}^s$. Fix a $T<\infty$ and choose a partition $\Xi=\{s_k\}_{k=0}^n\,:\quad 0=s_0<s_1<\cdots<s_n=T$ of $[0,T]$ of the range $r_W^\Xi:=\max\limits_{k=1,\dots,K}(s_k-s_{k-1})$. Denote $\Delta X^{s_k}:=X^{s_k}-X^{s_{k-1}}$, $\Delta P^{s_k}:=P^{s_k}-P^{s_{k-1}}$, and put

$$D_W^\Xi\;:=\;\sum_{k=1}^n\Delta P^{s_k}\,W\,\Delta X^{s_k}\,. \tag{1}$$

The operator

$$D_W\;=\;\text{w-}\lim_{r_W^\Xi\to 0}\,D_W^\Xi=:\int_{[0,T]}dP^s\,W\,dX^s\;:\;\mathscr{F}\to\mathscr{H} \tag{2}$$

is called a *diagonal* of W w.r.t. the nest $\mathfrak{f}$. For its adjoint one has $D_W^*=\int_{[0,T]}dX^sW^*dP^s\,:\,\mathscr{H}\to\mathscr{F}$, where the integral converges (or diverges) in the same sense as for D_W. The relation $\|D_W\|\leqslant\|W\|$ is valid [12].

The diagonal intertwines the projections: the relations

$$D_WX^s\;=\;P^sD_W,\quad D_W^*P^s\;=\;X^sD_W^*\qquad s\in[0,T] \tag{3}$$

hold.

• Now, let $\mathscr{H}=\mathscr{F}$ and let C be a positive operator in $\mathscr{F}$. Assume that its positive square root $\sqrt{C}$ has the diagonal $D_{\sqrt{C}}=\int_{[0,T]}d\tilde{P}^s\sqrt{C}\,dX^s$, where $\tilde{P}^s$ projects in $\mathscr{F}$ onto $\overline{\sqrt{C}\mathscr{F}^s}$. As is easily seen from (3), the operator $F:=D^*_{\sqrt{C}}\sqrt{C}$ is triangular: $F\mathscr{F}^s\subset\mathscr{F}^s$ holds. If, in addition, the diagonal satisfies the conditions

$$\operatorname{Ran}D_{\sqrt{C}}=\mathscr{F},\qquad D_{\sqrt{C}}\,D^*_{\sqrt{C}}=\mathbb{I}_{\mathscr{F}},$$

then the TF

$$C=F^*F,\qquad F=D^*_{\sqrt{C}}\sqrt{C} \tag{4}$$

holds and is referred to as *canonical* [12]. Its peculiarity and advantage is that the triangular factor F is determined constructively via the factorizable operator C. Also, we have $|F| = \sqrt{F^*F} = \sqrt{C}$, so that a possible look at (4) is that by $F = D^*_{|W|}|W|$ one recovers a triangular operator from its module.

• The triangular factorization $C = F^*F$ is *stable* if $C_j \to C$ and $C_j = F_j^* F_j$, $C = F^*F$ imply $F_j \to F$ (with a relevant understanding of convergences). The following result is established in [15].

Theorem 1. *Let $C_j \overset{\mathrm{s}}{\to} C$ hold, operators C_j and C admit the canonical TF (4) on a nest $\mathfrak{f}$, and $\sqrt{C_j} \overset{\mathrm{s,r}}{\to} \sqrt{C}$ hold on $\mathfrak{f}$. Let the integral sums, which define $D_{\sqrt{C}}$ and $D_{\sqrt{C_j}}$ by (1), converge to their limits (2) uniformly w.r.t. to j. Then the convergence of the triangular factors $F_j \overset{\mathrm{w,r}}{\to} F$ occurs.*

2.3 Towards application

Below, in an inverse problem, we deal with the operators $W_j, W : \mathscr{F} \to \mathscr{H}$, for which the following holds.

(i) All operators are injective, so that the operators $C_j := W_j^* W_j$ and $C := W^*W$, are positive. The relations $\overline{W_j \mathscr{F}^s} = \overline{W \mathscr{F}^s} = \mathscr{H}^s$, $s \in [0, T]$ hold.

(ii) All W_j and W have the diagonals D_{W_j} and D_W, whereas the integrals (2) converge to their limits uniformly w.r.t. j. The diagonals satisfy $D_{W_j} = D_W =: D$, where $D : \mathscr{F} \to \mathscr{H}$ is a unitary operator.

(iii) Operators C_j and C admit the canonical factorization (4)

$$C_j = F_j^* F_j, \;\; F_j = D^*_{\sqrt{C_j}} \sqrt{C}_j = D^*_{|W_j|}|W_j|;$$
$$C = F^*F, \;\; F = D^*_{\sqrt{C}} \sqrt{C} = D^*_{|W|}|W|.$$

(iv) The equalities $F_j = UW_j$ and $F = UW$ are valid with the same unitary operator $U : \mathscr{H} \to \mathscr{F}$.

Then, summarizing the previous results, we arrive at the following.

Proposition 1. **1.** *If the convergence $W_j \overset{\mathrm{u,r}}{\to} W$ holds on a nest $\mathfrak{f} \subset \mathscr{F}$, then, by Lemmas 1 and 2, the convergence $|W_j| \overset{\mathrm{s,r}}{\to} |W|$ (and, moreover, $|W_j| \overset{\mathrm{u,r}}{\to} |W|$) also holds.*

2. *If the convergence $|W_j| \overset{\mathrm{s,r}}{\to} |W|$ on $\mathfrak{f}$ holds, then we have $F_j \overset{\mathrm{w,r}}{\to} F$ by Theorem 1. The latter, due to* **(iv)**, *follows to $W_j \overset{\mathrm{w,r}}{\to} W$.*

3 Dynamical system with boundary control

3.1 Images

• Let (Ω, g) be a C^∞-smooth compact Riemannian manifold of dimension $n \geqslant 2$ with a C^∞-smooth boundary Γ. By τ we denote the distance function $\tau(x) := \operatorname{dist}(x, \Gamma)$. We denote $\Omega^T := \{x \in \Omega \,|\, \tau(x) < T\}$ and put $T_* := \inf\{T > 0 \,|\, \Omega^T = \Omega\}$.

The set $\gamma(x) := \{\gamma \in \Gamma \mid \operatorname{dist}(x, \gamma) = \tau(x)\}$ is a geodesic projection of x on Γ. We put $T_0 := \sup\{T > 0 \,|\, \sharp\gamma(x) = 1,\ x \in \Omega^T\} \leqslant T_*$ and in what follows, unless otherwise is specified, deal with $T < T_0$, so that the projection of any $x \in \Omega^T$ consists of a unique point $\gamma(x) \in \Gamma$ (see the comments in Introduction). In this case, the pair $(\gamma(x), \tau(x))$ constitutes the semi-geodesic coordinates (s.g.c.) of x. We also write $x = x(\gamma, \tau)$ for $(\gamma, \tau) \in \Sigma^T := \Gamma \times [0, T]$ and call Σ^T a *screen*.

The length and volume elements in s.g.c. are $dl^2 = d\tau^2 + \tilde{g}_{ij}(\gamma, \tau) d\gamma^i\, d\gamma^j$, where γ_i are the local coordinates on Γ, and $dv = \beta(\gamma, \tau) d\Gamma d\tau$ holds, where $\beta := \sqrt{\det\{\tilde{g}_{ij}\}}$ is a smooth function satisfying $\beta|_{\tau=0} = 1$, $\beta > 0$ in $\overline{\Omega^T}$, and $d\Gamma$ is a surface element at the boundary.

• Put $\mathscr{H} := L_2(\Omega)$, $\mathscr{H}^T := \{y \in \mathscr{H} \mid \operatorname{supp} y \subset \overline{\Omega^T}\} \subset \mathscr{H}$, $\mathscr{F}^T := L_2(\Sigma^T)$. Recall that $T < T_0$ is accepted and introduce the *image operator* $I^T : \mathscr{H}^T \to \mathscr{F}^T$,

$$\tilde{y} = (I^T y)(\gamma, \tau) := \beta^{\frac{1}{2}}(\gamma, \tau)\, y(x(\gamma, \tau)), \quad (\gamma, \tau) \in \Sigma^T.$$

We call $\tilde{y}$ the image of y and say that I^T visualizes y on the screen Σ^T. As is easy to check, I^T is a unitary operator and

$$I^{T\,*} : \mathscr{F}^T \to \mathscr{H}^T; \quad (I^{T\,*} f)(x) \;=\; \beta^{-\frac{1}{2}} f(\gamma(x), \tau(x)), \quad x \in \Omega^T$$

holds. The adjoint operator transfers the nest $\mathfrak{f}^T := \{\mathscr{F}^s\}_{s \in [0,T]}$, $\mathscr{F}^s := \{f \in \mathscr{F}^T \mid \operatorname{supp} f \subset \Gamma \times [0, s]\}$ to the nest $\mathfrak{h}^T := \{\mathscr{H}^s\}_{s \in [0,T]}$, $\mathscr{H}^s := \{y \in \mathscr{H}^T \mid \operatorname{supp} f \subset \overline{\Omega^s}\}$ by $I^{T\,*} \mathscr{H}^s = \mathscr{F}^s$.

Remark 1. The image operator I^T can be introduced not only for $T < T_0$ but for any $T > 0$ as a map, which transfers functions on Ω^T to functions on $\Theta^T := \{(\gamma(x), \tau(x)) \,|\, x \in \Omega \setminus \mathrm{c}\} \subset \Sigma^T$, where c is the separation set (*cut locus*) of Ω w.r.t. Γ: see [2,3,6]. For $T < T_0$, one has $\Theta^T = \Sigma^T$.

3.2 System and operators

In this section, $T > 0$ is arbitrarily fixed; we will return to $T < T_0$ later.

• A dynamical system α^T with boundary control (DSBC) is the system of the form

$$u_{tt} - \Delta u + qu = 0 \qquad \text{in } \mathscr{H},\ \ t \in (0,T); \tag{5}$$
$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in } \mathscr{H}; \tag{6}$$
$$u = f \qquad \text{on } \Sigma^T, \tag{7}$$

where Δ is the Beltrami-Laplace operator on (Ω, g), f is a *boundary control*, $u = u^f(x,t)$ is a solution (*wave*). A *potential* q is a real function of the class $C^N(\Omega)$ with a large enough N specified later as necessary.

There is an integer $N_{\rm cl}$ depending on $\dim \Omega$, such that, under the assumption $q \in C^N(\Omega)$, $N \geqslant N_{\rm cl}$, the solution u^f corresponding to the smooth controls $f \in \mathscr{M}^T := \{f \in C^\infty(\Sigma^T) \,|\, f \text{ vanishes near } t = 0\}$, is classical. Note that $\mathscr{M}^T$ is dense in $L_{2,\,d\Gamma d\tau}(\Sigma^T)$.

For $q \in L_\infty(\Omega)$, the map $\mathscr{M}^T \ni f \mapsto u^f$ is continuous from $L_{2,\,d\Gamma d\tau}(\Sigma^T)$ to $C([0,T]; L_2(\Omega))$: see [27,28] and Lemma 4 below. The latter allows to introduce the (generalized) L_2-solutions u^f for $f \in L_{2,\,d\Gamma d\tau}(\Sigma^T)$ via the extension of the map by continuity, and in what follows we deal with these solutions.

There are system theory attributes of α^T.

1. The *outer space* of controls is $\mathscr{F}^T = L_{2,\,d\Gamma d\tau}(\Sigma^T)$. It contains the continuous nest $\mathfrak{f}^T$ of the delayed control subspaces

$$\mathscr{F}^{T,s} := \{f \in \mathscr{F}^T \mid f\,|_{0 \leqslant t \leqslant T-s} = 0\}, \qquad s \in [0,T];$$

here s is the action time and $T - s$ is the delay, so that $\mathscr{F}^{T,0} = \{0\}$ and $\mathscr{F}^{T,T} = \mathscr{F}^T$ holds.

2. The *inner space* of states (waves) is $\mathscr{H}^T = L_2(\Omega^T)$. It contains the nest of the *reachable subspaces*

$$\mathscr{U}^s := \overline{\{u^f(\cdot, s) \mid f \in \mathscr{F}^T\}} \subset \mathscr{H}^s, \qquad s \in [0,T],$$

where the embedding corresponds to the finiteness of the wave propagation velocity into Ω. Meanwhile, thanks to the Holmgren-John-Tataru Theorem [33], which is crucial for the BCm, the equality $\mathscr{U}^s = \mathscr{H}^s$ holds for all $s > 0$, so the nest $\{\mathscr{U}^s\}_{s \in [0,T]}$ coincides with $\mathfrak{h}^T = \{\mathscr{H}^s\}_{s \in [0,T]}$. This fact is referred to as an (approximate) boundary controllability of the system α^T.

The controllability also holds in the classes of the smooth functions. Using the results of [5], one can show that, under assumption $q \in C^N(\Omega)$, $N \geqslant N_{\rm sm}$ with a large enough $N_{\rm sm}$, the equality

$$\overline{\{u^f(\cdot,s) \mid f \in \mathscr{M}^T\}} \,=\, \overline{\{y \in H^2(\Omega) \mid \operatorname{supp} y \subset \Omega^s\}}, \qquad s \in [0,T] \tag{8}$$

is valid (the closure in $H^2(\Omega)$-norm).

3. As it follows from the above mentioned regularity results on solutions u^f, if $q \in L_\infty(\Omega)$ then the *control operator* $W^T : \mathscr{F}^T \to \mathscr{H}^T$, $W^T f := u^f(\cdot,T)$ is bounded. If W_j, $j = 1,2,\dots$ correspond to the systems with the potentials $\|q_j\|_{L_\infty(\Omega)} \leqslant$ const, then $\|W_j\|$ are also uniformly bounded.

For times $T < T_*$, operator W^T is injective.

4. The *response operator* $R^T : \mathscr{F}^T \to \mathscr{F}^T$, $\operatorname{Dom} R^T = \{f \in H^1(\Sigma^T) \mid f|_{t=0} = 0\} =: \dot H^1(\Sigma^T)$, $R^T f := \partial_\nu u^f$ on Σ^T (∂_ν is the derivative w.r.t. the outward normal to Γ) describes the response of system α^T on the action of controls. It acts continuously from $\operatorname{Dom} R^T$ (endowed with the Sobolev H^1-norm) to $\mathscr{F}^T$ ([27], Theorem 2.2, (b)) but is unbounded as operator in $\mathscr{F}^T$.

5. The operator $C^T := W^{T*}W^T \geqslant \mathbb{O}$ is bounded in $\mathscr{F}^T$ and injective for $T < T_*$. It connects the metrics of the outer and inner spaces by

$$(C^T f, g)_{\mathscr{F}^T} = (W^T f, W^T g)_{\mathscr{H}^T} = (u^f(\cdot,T), u^g(\cdot,T))_{\mathscr{H}^T},$$

and is called the *connecting operator*. The relation

$$C^T \,=\, 2^{-1}\, S^{T*} R^{2T} J^{2T} S^T \tag{9}$$

holds, where $S^T : \mathscr{F}^T \to \mathscr{F}^{2T}$ extends controls from $[0,T]$ to $[0,2T]$ by oddness w.r.t. $t = T$, $J^{2T} : \mathscr{F}^{2T} \to J^{2T}$ acts by $(J^{2T} f)(\cdot,t) = \int_0^t f(\cdot,s)\,ds$, and R^{2T} is the response operator of the system α^{2T} [2, 3]. Thus, C^T is explicitly expressed via response operator, which plays the role of the inverse data in time-domain inverse problems.

• However, one should be careful when using the formula (9). The representation of this kind has appeared in [2] for the Neumann-to-Dirichlet response operator, which is bounded. In our case, the representation is of fairly similar form but the Dirichlet-to-Neumann operator R^{2T} is unbounded. Therefore, as such, (9) is only applicable to controls f satisfying $S^T f \in \operatorname{Dom} R^{2T} J^{2T}$, while it is unknown whether the operator $R^{2T} J^{2T}$ is bounded or unbounded. It is bounded in one-dimensional cases, but we have never encountered a generalization to $n \geqslant 2$ in the literature. Moreover, there is some reason to expect that it is unbounded, while the boundedness of C^T is provided due to the bordering by S^T and S^{T*}. An accurate handle with the right-hand side of (9) requires to define it on admissible f's and then extend it to $\overline{C^T}$ by continuity. For instance, a dense set $\{f \in \mathscr{F}^T \mid J^{2T}S^T f \in \operatorname{Dom} R^{2T}\}$ is appropriate. However, no efficient representation of $\overline{C^T}$ that allows its application to all $f \in \mathscr{F}^T$ is known. This question has remained open for over 30 years. However, this is not an obstacle for numerical realization of BCm because in calculations one uses (and has to invert) not C^T itself but its large size Gram matrix $\{(C^T f_i, f_j)\}_{i,j=1}^N$, where the controls f_i can be chosen admissible for (9) [7, 8, 31, 32].

One more remark to (9) is the following. If the norms $\|q_j\|_{L_\infty(\Omega)}$ are uniformly bounded, then the norms $\|W_j^T\|$ of the control operators of the systems with potentials q_j are also uniformly bounded. As a consequence, $\|C_j^T\|$ are uniformly bounded. If $\|R_j^{2T} - R^{2T}\|_{\dot H^1(\Sigma^{2T}) \to \mathscr{F}^{2T}} \to 0$ holds, then by (9) we have $\|(C_j^T - C^T)f\| \to 0$ for all f provided $J^{2T}S^T f \in \dot H^1(\Sigma^{2T})$. Such f's constitute a set dense in $\mathscr{F}^T$. Therefore, due to the uniform boundedness of C_j^T, the convergence $C_j^T \xrightarrow{\text{s}} C^T$ holds. Thus, we have

Proposition 2. *The condition* $\|q_j\|_{L_\infty(\Omega)} \leqslant \text{const}$ *and the convergence*

$$\|R_j^{2T} - R^{2T}\|_{\dot H^1(\Sigma^{2T}) \to \mathscr{F}^{2T}} \to 0$$

imply $C_j^T \xrightarrow{\text{s}} C^T$.

4 Stability of TOR

4.1 TOR

• Let's briefly recall how the TOR procedure works [3, 6, 9]. We start with the properties of the control and connecting operators that ensure the realizability of the procedure and allow to analyze its stability. The following holds for $0 < T < T_*$.

(i) Operator W^T is injective, so that the connecting operator $C^T = W^{T*}W^T$ is positive. Due to the boundary controllability of the system α^T, one has $\overline{W^T \mathscr{F}^s} = \mathscr{H}^s$, $s \in [0, T]$.

(ii) Operator W^T has the diagonal and the remarkable equality

$$I^T \;=\; Y^T D^*_{W^T} \tag{10}$$

holds, where $I^T : \mathscr{H}^T \to \mathscr{F}^T$ is the image operator (recall Remark 1) and $Y^T = Y^{T*} : \mathscr{F}^T \to \mathscr{F}^T$ changes the time by $t \mapsto T - t$. It represents a "purely geometrical" I^T in "dynamical terms" related with the wave propagation in Ω. This is proved by the use of the Geometric Optics relations, which describe propagation of singularities (discontinuities) of the waves u^f and evolution of their amplitudes [3, 9]. However, to be applicable to our case, the proof requires $q \in C^N(\Omega)$ with a sufficiently large $N \geqslant N_{\rm GO}$, which is specified in [3, 9] and depends on $\dim \Omega$, and in what follows we will assume that this condition is satisfied.

The proof [3, 9] ensures the weak convergence of the integral (2) for the general case $T < T_*$ and strong convergence for $T < T_0$. Also, the same proof provides that if a family of potentials obeys $\|q_j\|_{C^N(\Omega)} \leqslant {\rm const}$, then the integrals (2) for $D_{W_j^T}$, converge uniformly w.r.t. j (uniformly strongly if $T < T_0$). By (10), we have $D_{W_j^T} = D_{W^T} = I^{T*} Y^T$.

(iii) The connecting operators C_j and C corresponding the potentials q_j and q admit the canonical factorization (4)

$$C_j^T = F_j^{T*} F_j^T,\ F_j^T = D^*_{\sqrt{C_j^T}} \sqrt{C_j^T};\quad C^T = F^{T*} F^T,\ F^T = D^*_{\sqrt{C}^T} \sqrt{C}^T. \tag{11}$$

(iv) The equalities $F_j^T = U^T W_j^T$ and $F^T = U^T W^T$ are valid with the same operator $U^T = D^*_{W^T} \overset{(10)}{=} Y^T I^T : \mathscr{H}^T \to \mathscr{F}^T$, which is isometric for $T < T_*$ and unitary for $T < T_0$.

Applying Proposition 1, we get

Proposition 3. *Assume that* $\|q_j\|_{C^N(\Omega)}, \|q\|_{C^N(\Omega)} \leqslant {\rm const}$ *is valid with an* $N \geqslant N_{\rm GO}$. *Then,*
1. *if the convergence* $W_j^T \overset{\rm u,r}{\to} W^T$ *holds on a nest* $\mathfrak{f}^T \subset \mathscr{F}^T$, *then the convergence* $\sqrt{C_j^T} \overset{\rm s,r}{\to} \sqrt{C^T}$ *(and, moreover,* $\sqrt{C_j^T} \overset{\rm u,r}{\to} \sqrt{C^T}$*) also holds;*
2. *if the convergence* $\sqrt{C_j^T} \overset{\rm s,r}{\to} \sqrt{C^T}$ *holds on* $\mathfrak{f}^T$, *then we have* $F_j^T \overset{\rm w,r}{\to} F^T$. *The latter, due to* **(iv)**, *leads to* $W_j^T \overset{\rm w,r}{\to} W^T$.

(v) Since the operator $L = -\Delta + q$ which governs the evolution of the system α^T, does not depend on the time, the evolution is invariant w.r.t. the time shift $t \mapsto t - \sigma$, which implies $u^{f_t} = u^f_t$ and leads to

$$u^{f_{tt}} = u^f_{tt} \overset{(5)}{=} -L u^f. \tag{12}$$

By (8), one has $u^f(\cdot,t)\in H^2(\Omega)$ for $f\in\mathscr{M}^T$. As is easy to see, the graph

$$\begin{aligned}\operatorname{graph} L=\{\langle u^f(\cdot,T),Lu^f(\cdot,t)\rangle \mid f\in\mathscr{M}^T\}&\overset{(12)}{=}\\ =\{\langle W^Tf,-W^Tf_{tt}\rangle \mid f\in\mathscr{M}^T\}\subset\mathscr{H}^T\times\mathscr{H}^T&\end{aligned}\tag{13}$$

determines the metric g together with potential q. Respectively, the graph of the operator $\tilde L:=I^TLI^{T\,*}$ is

$$\operatorname{graph}\tilde L\;=\;\{\langle V^Tf,-V^Tf_{tt}\rangle \mid f\in\mathscr{M}^T\}\subset\mathscr{F}^T\times\mathscr{F}^T,\tag{14}$$

where the operator $V^T:\mathscr{F}^T\to\mathscr{F}^T$,

$$V^T:=I^TW^T=I^TI^{T\,*}Y^TF^T\;=\;Y^TF^T\tag{15}$$

visualizes the waves on the screen Σ^T. More precisely, the images of waves $\tilde u^f\;=\;V^Tf$ turn out to be supported on the set $\Theta^T\subset\Sigma^T$ (see Remark 1). The graph (14) determines metric $\tilde g$ (that is g in s.g.c.) and potential $\tilde q$ (q in s.g.c.) on Θ^T [3].

• Let a positive $T<T_*$ be fixed. The TOR procedure works as follows: given the operator R^{2T}, the external observer acts according to the scheme

$$R^{2T}\overset{(9)}{\mapsto}C^T\;\mapsto\;\sqrt{C^T}\overset{(11)}{\mapsto}F^T\overset{(15)}{\mapsto}V^T\,.$$

If (Ω^T,g) is *known* (so that the image operator I^T is given) and the only problem is to determine the potential q on it, the observer recovers the control operator $W^T\overset{(15)}{=}I^{T\,*}V^T$, and finds q from the graph (13).

If the manifold is *unknown*, the observer finds the operator V^T, recovers the set $\Theta^T\;=\;\overline{\{\operatorname{supp}V^Tf \mid f\in\mathscr{F}^T\}}$ along with the metric $\tilde g$ and potential $\tilde q$ on it (in s.g.c.) from the graph (14), endows Θ^T with the metric $\tilde g$ that turns it to an isometric copy $(\Theta^T,\tilde g)$ of (Ω^T,g), and claims that the inverse problem is solved [3].

4.2 Stability

Fix a positive $T<T_0$ and recall that we assume $q_j,q\in C^N(\Omega)$ with $N\geqslant\max\{N_{\rm cl},N_{\rm sm},N_{\rm GO}\}$.

• A stability of the TOR is analyzed by the use of Proposition 3 as follows. Assume that a convergence $R_j^{2T}\to R^{2T}$ yields $C_j^T\overset{\rm s}{\to}C^T$ and, hence, $\sqrt{C_j^T}\overset{\rm s}{\to}$

$\sqrt{C^T}$ holds. Then the observer checks whether the latter convergence is regular on the nest $\mathfrak{f}^T = \{\mathscr{F}^{T,s}\}_{s\in[0,T]}$.

$\star$ If it is not regular, the observer reports that the uniform convergence $W_j^T \overset{\rm u}{\to} W^T$ is absent. However, a weaker convergence or divergence are not excluded, but nothing more can be said. For the potentials this means that the convergence $q_j \to q$, if exists, is weaker than convergence in $L_\infty(\Omega)$, because the latter leads to $W_j^T \overset{\rm u}{\to} W^T$.

$\star\star$ If the convergence is regular, i.e., $\sqrt{C_j^T} \overset{\rm s,r}{\to} \sqrt{C^T}$ is valid, then the convergences $V_j^T \overset{\rm w,r}{\to} V^T$ and $W_j^T \overset{\rm w,r}{\to} W^T$ hold.

• In the latter (affirmative) case, the convergence $W_j^T \overset{\rm w,r}{\to} W^T$ follows to a convergence of the potentials.

Lemma 3. *The convergence $W_j^T \overset{\rm w,r}{\to} W^T$ leads to $\|q_j - q\|_{H^{-2}(\Omega^T)} \to 0$.*

Proof. Fix an $f \in \mathscr{M}^T$, put $g := (J^T)^2 f = (\int_0^t)^2 f \in \mathscr{M}^T$ and note that $u^g_{tt} = u^f$ holds by (12). Subscribing the wave equations, we have

$$(u_j^f - u^f) - \Delta(u_j^g - u^g) + q_j(u_j^g - u^g) = (q_j - q)u^g.$$

Taking $t = T$, multiplying by a test function $\omega \in H_0^2(\Omega^T)$, integrating by parts, and tending $j \to \infty$, we get

$$\begin{aligned} 0 \leftarrow ((W_j^T - W^T)f, \omega)_{\mathscr{H}^T} - ((W_j^T - W^T)g, \Delta\omega)_{\mathscr{H}^T} = \\ = ((q_j - q)\, u^g(\cdot, T), \omega))_{\mathscr{H}^T} := \langle q_j - q, \eta\rangle \end{aligned}$$

with $\eta = u^g(\cdot, T)\,\omega \in H_0^2(\Omega^T)$. Using (8), one can show that such η's form a set dense in $H_0^2(\Omega)$. Meanwhile, understanding $\langle q_j - q, \eta\rangle$ as the coupling in the triple $H_0^{-2}(\Omega^T) \subset \mathscr{H}^T = L_2(\Omega^T) \subset H^2(\Omega^T)$ (i.e., $q_j - q$ as an H^{-2}-functionals), we see that $\langle q_j - q, \eta\rangle \to 0$ holds on a dense set, while the norms $\|q_j - q\|_{H^{-2}(\Omega^T)}$ are obviously uniformly bounded. As a result, we arrive at $q_j - q \to 0$ in $H^{-2}(\Omega^T)$. □

4.3 Comments

• According to traditional views in the IP-community, an inverse problem is considered to be completely solved if

1) *Uniqueness* is established, i.e., the injectivity of the correspondence "objects to be determined $\mapsto$ inverse data" is proven;

2) A *procedure* that recovers the objects is developed;
3) A *stability* of the determination procedure is established;
4) Necessary and sufficient conditions for solvability of the problem (*inverse data characterization*) are found.

For time-domain problems, the BC-method provides the TOR procedure that settles **1)** and **2)** [3,6]. It also offers some insight into **4)** [14]. Our paper gives at least some result on the item **3)**. However, to claim that the time-optimal reconstruction problem is completely solved would be overstatement. We guess and hope that these results can be substantially improved. First of all, this concerns the quantitative stability estimates for TOR that is very important, difficult and challenging problem.

• The original version of TOR has appeared in [2]. In the same paper, representation (9) based on the A.S.Blagoveshchenskii relations, and the concept of the amplitude integral (later interpreted as an operator diagonal) have been proposed. It generalizes M,S.Brodskii's operator construction known as a triangular truncation integral [21]. The approach [2], with no substantial modifications, is extended in [11] to the reconstruction of Riemannian manifolds. In [10], a variant of TOR uses visualization of Gaussian beams for reconstruction.

5 Appendix

We denote $Q^T := \Omega \times (0,T)$. The following result is well known "in folklore", but we did not find it in the literature in the required form.

Lemma 4. *If $q \in L_\infty(\Omega)$, then the control operator W^T of the system (5)–(7) is bounded.*

Proof. Denoting u_0^f the solution and W_0^T the control operator for $q = 0$, we have

$$(u-u_0)_{tt} - \Delta(u-u_0) + q(u-u_0) = -qu_0 \qquad \text{in } Q^T; \tag{16}$$

$$(u-u_0)|_{t=0} = (u-u_0)_t|_{t=0} = 0 \qquad \text{in } L_2(\Omega); \tag{17}$$

$$(u-u_0) = 0 \qquad \text{on } \Sigma^T. \tag{18}$$

By [27], $\|u_0^f\|_{C([0,T];L_2(\Omega))} \leqslant \text{const}\|f\|_{L_2(\Sigma^T)}$ holds and shows that W_0^T is bounded. Hence, the right-hand side of (16) satisfies $\|qu_0^f\|_{L_2([0,T];L_2(\Omega))} \leqslant \text{const}\|f\|_{L_2(\Sigma^T)}$. The latter, by [22,26], implies $\|u^f - u_0^f\|_{H_0^1(Q^T)} \leqslant \text{const}\|f\|_{L_2(\Sigma^T)}$, which leads to $\|u^f(\cdot,T) - u_0^f(\cdot,T)\|_{L_2(\Omega)} \leqslant \text{const}\|f\|_{L_2(\Sigma^T)}$ by the Sobolev trace theorems. The latter means that $\|(W^T - W_0^T)f\|_{L_2((\Omega)} \leqslant \text{const}\|f\|_{L_2(\Sigma^T)}$ holds, which implies $\|W^T - W_0^T\| < \infty$ and proves the Lemma. $\square$

In the notation of the given paper, we also easily get

Corollary 1. *If $\|q_j - q\|_{L_\infty(\Omega)} \to 0$ holds, then $\|W_j^T - W^T\| \to 0$ is valid.*

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