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On non-homeomorphic surfaces with close DN maps

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Abstract.
Let Λ' be a DN map of a genus m' surface (M', g') with boundary Γ. Introduce the Schottky double X' of (M', g') and denote by $\mathcal{L}(X')$ the length of the shortest closed geodesics in the hyperbolic metrics on X'. We prove that $\mathcal{L}(X')$ is small if Λ' is close in $B(H^1(\Gamma;\mathbb{R}); L_2(\Gamma;\mathbb{R}))$ to the DN map Λ of some surface (M, g) of lower genus $m < m'$ with the same boundary Γ:

$$\Lambda' \to \Lambda \implies \mathcal{L}(X') \to 0.$$

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Keywords: Electrical Impedance Tomography, DN map, degeneration of surfaces.

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1 Introduction

Let (M,g) be an orientable surface (smooth two-dimensional Riemannian manifold) of genus with boundary $(\Gamma; dl)$ (we assume that Γ is diffeomorphic to a circle while the length element induced by the metric g on Γ coincides with dl). Introduce the Laplace-Beltrami operator Δ_g and denote by u^f the harmonic extension of $f \in C^\infty(\Gamma;\mathbb{R})$ into (M,g). Let ν be a unit outward normal on Γ. The *Dirichlet-to-Neumann map* (DN map) Λ of (M,g) is given by $\Lambda f := \partial_\nu u^f|_\Gamma$. It is well-known that DN map Λ is an order 1 pseudo-differentaional operator and $\Lambda \simeq |\partial_\gamma|$ modulo smoothing operator, where ∂_γ is the differentiation with respect to the length along Γ. In particular, Λ can be extended to a continuous operator acting from $H^1(\Gamma;\mathbb{R})$ to $L_2(\Gamma;\mathbb{R})$. It worth noting that the difference $\Lambda - \Lambda'$ of any two DN maps of surfaces with common boundary (Γ, dl) is a smoothing operator (see [2]); in particular, $\Lambda - \Lambda'$ acts continuously from $L_2(\Gamma;\mathbb{R})$ to any Sobolev space $H^l(\Gamma;\mathbb{R})$.

Let (M',g') be another surface with boundary (Γ', dl'), Laplace-Beltrami operator $\Delta_{g'}$, and DN-map $\Lambda' : f' \mapsto \partial_{\nu'} u'^{f'}|_{\Gamma'}$, where $u'^{f'}$ is the harmonic extension of $f' \in C^\infty(\Gamma';\mathbb{R})$ into (M',g') and ν' is the exterior normal. If there is a confromal diffeomorphism $\beta : M \to M'$, $\beta^* g' = \rho g$, then the harmonic functions and DN-maps of (M,g) and (M',g') are connected via $\beta^*[u'^{f'}] = u^{\beta^* f'}$ and

$$\Lambda' = \beta^{*-1} \frac{1}{\sqrt{\rho}} \Lambda \beta^*, \tag{1}$$

where $\beta^* u' := u' \circ \beta$ is a precomposition. In particular, $\Lambda' = \Lambda$ if $(\Gamma', dl') = (\Gamma, dl)$ and β does not move points of Γ. The well-known result of Lassas and Uhlmann [1] states that the converse is also true, i.e., the equalities $\Gamma = \Gamma'$ and $\Lambda' = \Lambda$ imply the existence of conformal diffeomorphism β between (M,g) and (M',g') that does not move the points of Γ.

In [4], it is shown that the topology of a surface is unstable under small perturbations of its DN map. Namely, by cutting small holes in the surface and then attaching small handles or Möbius strips to the hole boundaries, one can obtain a surface whose DN map is arbitrarily close (in the operator norm) to the DN map of the original surface. Note that one cannot lower the genus of the surface or make orientable surface from a non-orientable one without significant perturbation of its DN map (this fact is a immediate corollary of (3)). For some other results on the stability/instability of the surface topology under a small perturbation of the DN map we refer the reader to Propositions 5.5, 6.2 and Theorem 6.3, [5].

This note addresses the question of whether the converse is true, namely, whether the closeness of the DN map Λ' of a topologically perturbed surface (M', g') to the DN map Λ of the original surface (M, g) implies that all "extra" handles on (M', g') are effectively separated by small closed curves from the "exterior" part of the surface. However, the DN map does not change if one enlarges small handles by multiplying the surface metric g' by a conformal factor which is large on the "extra" handles and equal to one on the boundary. Thus, to make the above discussion correct, the length of closed curves on M' should be defined in some "canonical" metric h' belonging to the same conformal class $[g']$ as the original metric g' on M'.

The most natural choice is the *hyperbolic metric* on M' which is the metric $h' \in [g']$ of constant scalar curvature $K' = -1$ on M' such that the boundary $\partial M' = \Gamma$ is a geodesic curve on (M', h'). This metric can be constructed as follows. As a rule, we assume that M' is endowed with the complex structure associated with the conformal class $[g']$ and the choice of the orientation on M'. By attaching to M' its copy endowed with the opposite orientation, one obtains the *Schottky double* X' of M' which is the Riemann surface endowed with the antiholomorphic involution τ' that interchanges the points of the original and copy in such a way that $X'/\tau' \cong M'$ and $\partial M'$ coincides with the set of fixed points of τ. On such X', there is the (unique) hyperbolic metric (that is, the metric of constant scalar curvature -1 compatible with the complex structure on X'). Thus, $\tau'^{*}\mathrm{h}' = \mathrm{h}'$, i.e. the involution τ' is an isometry on (X', h'). Due to the last symmetry, a geodesic curve on (X', h') starting from any point $\partial M'$ with the same tangent vector as $\partial M'$, does not leave $\partial M'$. So, the $\partial M'$ is geodesic and $h' = \mathrm{h}'|_{\partial M}$ is the hyperbolic metric on $M' \subset X'$.

The *length spectrum* of M' (resp., X') is a collection of lengths of all closed geodesics (corresponding to the hyperbolic metric) on M' (resp., on X'). Denote by $\mathcal{L}(M')$ (resp., by $\mathcal{L}(X')$) the infimum of the length spectrum of M' (resp., X'), i.e., the length of shortest geodesics on M' (resp., X') in hyperbolic metric.

The main result of this note is the following statement.

Proposition 1. *Let* (M, g) *be a surface of genus* m *with boundary* (Γ, dl) *(diffeomorphic to a circle) and DN map* Λ*. Let* $\{(M_s, g_s)\}_s$ *be a sequence of surfaces of genera* $\mathrm{gen}(M_s) = m' > m$ *with the same boundary* (Γ, dl) *and DN maps* Λ_s*. Denote by* X *and* X_s *the doubles of* (M, g) *and* (M_s, g_s)*, endowed with the corresponding antiholomorphic involutions* τ *and* τ_s*, respectively.*

Then the convergence $\|\Lambda_s - \Lambda\|_{B(H(\Gamma;\mathbb{R});L_2(\Gamma;\mathbb{R}))} \to 0$ *implies* $\mathcal{L}(X_s) \to 0$.

In the particular case $m' = 1$, $m = 0$, Proposition 1 is proved in [7] by the use of explicit expressions for b-period matrices of X_s in terms of the DN-maps Λ_s provided by the results of [6].

The remaining part of this note is devoted to the proof of Proposition 1.

2 Proof of Proposition 1

Preliminaries

Hilbert transform and defect operator. Let ∂_γ^{-1} be an integration operator on Γ which is inverse to ∂_γ on the space $\partial_\gamma C^\infty(\Gamma;\mathbb{R})$ and vanishes on constants. Introduce the *Hilbert transform* H of (M,g) by $H := \partial_\gamma^{-1}\Lambda$. This operator admits the following interpretation in terms of the complex structure on M. Recall that the conformal class of the metric g and the choice of orientation determine the structure of Riemann surface (complex atlas) on M in such a way that the boundary Γ is smooth and the Cauchy-Riemann (CR) equations on M take the form $d\Im w = \star d\Re w$, where $\star$ is a Hodge operator. In what follows, we assume that the pair of vectors $(\partial_\nu, \partial_\gamma)$ is positively oriented; then the restriction of the CR equations on Γ is of the form $\partial_\gamma \Im\eta = \Lambda\Re\eta$, $\partial_\gamma\Re\eta = -\Lambda\Im\eta$, where $\eta = w|_\Gamma$. Integration of these equations yield $\Im\eta = H\Re\eta + \text{const}$, $(H^2+I)\Re\eta = \text{const}$. Thus, the Hilbert transform is the operator relating the real and imaginary parts of boundary traces of holomorphic functions on the surface. From (1) follows the conformal invariance of the Hilbert transform,

$$H'f = \beta^{*-1}H\beta^* f \text{ modulo constants.} \tag{2}$$

Introduce the *defect operator* $\mathfrak{D} = H^2 + I$. Note that

$$H^2 + I \simeq \partial_\gamma^{-1}|\partial_\gamma|\partial_\gamma^{-1}|\partial_\gamma| + I \simeq 0$$

is a smoothing operator; in particular, $(H^2+I)L_2(\Gamma;\mathbb{R}) \subset C^\infty(\Gamma;\mathbb{R})$. In what follows, it is more convenient to consider $\mathfrak{D}$ as a continuous operator acting on the quotient space $\dot{C}(\Gamma;\mathbb{R}) := C(\Gamma;\mathbb{R})/\mathbb{R}$ endowed with the norm

$$\|\!|f + \text{const}|\!\|_\Gamma := \max_\Gamma f - \min_\Gamma f.$$

The important fact is the following connection

$$\dim\partial_\gamma\mathfrak{D}C^\infty(\Gamma;\mathbb{R}) = \dim\mathfrak{D}\dot{C}(\Gamma;\mathbb{R}) = 2\mathrm{gen}(M). \tag{3}$$

between the topology of a surface and its defect operator established by Belishev [3].

Quasiconformal mappings. Let X and X' be Riemann surfaces and $\beta : X \to X'$ be a homeomorphism between them. Let z be a holomorphic coordinate of $x \in X$ while z' be a holomorphic coordinate of $\beta(x)$. The map β is called quasiconformal (QC) if, for any choice z and z', the (distributional) derivatives $\partial_{\overline{z}}z'$, $\partial_z z'$ are square integrable on their domain $D \subset \mathbb{C}$ and the ratio

$$\mu(z) := \frac{\partial_{\overline{z}}z'}{\partial_z z'}$$

(called the Beltrami quotient) obeys $\|\mu\|_{L_\infty(X;\mathbb{C})} \leq \mathrm{c}$ for some $\mathrm{c} < 1$. Note that $|\mu(z)| = |\mu(x)|$ is independent on the choice of coordinates z and z'. The condition $\|\mu\|_{L_\infty(X;\mathbb{C})} < 1$ means that β is orientation-preserving: if T is a small counterclockwise circle (in local coordinates z) with the center at x_0, then $\mathrm{wind}(\beta \circ T, \beta(x_0)) = +1$, see Lemma 6.1 in [8].

The ratio

$$K_\beta := \frac{1 + \|\mu\|_{L_\infty(X;\mathbb{C})}}{1 - \|\mu\|_{L_\infty(X;\mathbb{C})}}$$

is called the dilatation of β. The composition with the biholomorphism does not change the dilatation and $K_{\beta_1\circ\beta_2} \leq K_{\beta_1}K_{\beta_2}$, $K_{\beta^{-1}} = K_\beta$.

The QC map β between X and X' minimizing the dilatation in a given isotopy class is called the Teichmüller map and it is well-known that it exists and is unique [9]. The Beltrami quotient of the Teichmüller map is always of the form

$$\mu(x) = k\frac{\overline{\phi(z)dz^2}}{|\phi(z)dz^2|},$$

where $\phi(z)dz^2$ is some holomorphic quadratic differential on X and $k \in (0,1)$. Moreover, $K_\beta \to 1$ implies $k \to 0$. Note that the map β is smooth outside the singularities of μ (i.e., the zeroes of ϕ).

The Teichmüller space $\mathcal{T}_\mathfrak{g}$ consists of classes $[(X,\phi)]$ of the pairs (X,ϕ), where X is a Riemann surface of genus $\mathfrak{g}$ and $\phi : X_0 \to X$ is an orientation-preserving diffeomorphism from some reference surface X_0 onto X ("a marking"), under the following equivalence: $(X_1, \phi_1) \sim (X_2, \phi_2)$ if there is a biholomorphism $\beta : X_1 \to X_2$ isotopic to $\phi_2 \circ \phi_1^{-1}$. The Teichmüller metric on $\mathcal{T}_\mathfrak{g}$ is defined as

$$d_T([(X_1, \phi_1)], [(X_2, \phi_2)]) := \frac{1}{2} \min_\beta \log K_\beta,$$

where the minimum is taken over all QC maps $\beta : X_1 \to X_2$ isotopic to $\phi_2 \circ \phi_1^{-1}$. Endowed with such a metric, $\mathcal{T}_\mathfrak{g}$ is homeomorphic to an open unit ball in $\mathbb{C}^{3\mathfrak{g}-3}$. Introduce the moduli space $\mathcal{M}_\mathfrak{g}$ consisting of classes $[X]$ of genus $\mathfrak{g}$ Riemann surfaces X under the biholomorhic equivalence. The $\mathcal{M}_\mathfrak{g}$ is endowed with the quotient topology in which the map $\pi : \mathcal{T}_\mathfrak{g} \to \mathcal{M}_\mathfrak{g}$ that forgets the marking, $\pi([X, \phi]) := [X]$, is continuous.

If $\mathfrak{U}$ is a complex atlas on X and $\kappa : X_0 \to X$ is any diffeomorphism, then the surface X_0 equipped with the atlas $\mathfrak{U} \circ \kappa$ is biholomorphically equivalent to X. In addition, if there is the biholomorphism $\beta : X \to X'$, then $(X, \phi) \sim (X', \beta \circ \phi)$. Now, let $(X_0, \phi_0) \sim (X_1, \phi_1)$ and (X_2, ϕ_2) be marked Riemann surfaces and $\beta_1 : X_1 \to X_2$ be a QC map isotopic to $\phi_2 \circ \phi_1^{-1}$. Denote by β_0 the biholomorphism from X_0 onto X_1 isotopic to $\phi_1 \circ \phi_0^{-1}$ and by $\mathfrak{U}_2$ the complex atlas on X_2. Then $[X_2] = [(X_0, \mathfrak{U}_2 \circ \phi_2 \circ \phi_0^{-1})]$ while the map $\beta_2 := \phi_0 \circ \phi_2^{-1} \circ \beta_1 \circ \beta_0$ is isotopic to identity and has the same dilatation as β_1, $K_{\beta_2} = K_{\beta_1}$. As a corollary, if $[Y_k] \to [X]$ in $\mathcal{M}_\mathfrak{g}$ and $\pi([(X, \phi)] = [X]$, then there are $[(X_k, \phi_k)]$ such that $[(X_k, \phi_k)] \to [(X, \phi)]$ in $\mathcal{T}_\mathfrak{g}$ and $\pi([(X_k, \phi_k)]) = [Y_k]$.

So, if $[X_s] \to [X]$ in $\mathcal{M}_\mathfrak{g}$, then there are QC maps $\beta_s : X \to X_s$ whose Beltrami quotients μ_s obey $\mu_s = k(s)\overline{\phi_s}/|\phi_s|$, where ϕ_s are holomorphic quadratic differentials on X and $k(s) \to 0$ as $s \to 0$.

Step 1: Applying Mumford's compactness theorem

Now we prove Proposition 1. Suppose the contrary $\mathcal{L}(X_s) \not\to 0$. Then one can find a subsequence $\{(M_{s(l)}, g_{s(l)}\}_l$ such that $\mathcal{L}(X_{s(l)}) \geq \epsilon > 0$ for any l (for simplicity, in what follows we always assume that the subsequence $\{(M_{s(l)}, g_{s(l)}\}_l$ obeying the required property coincides with the original $\{(M_s, g_s\}_s)$. Due to the Mumford's compactness theorem [10], the set of conformal classes of surfaces admitting no closed geodesics (in the hyperbolic metrics) of lengths less than ϵ is compact in $\mathcal{M}_\mathfrak{g}$ ($\mathfrak{g} = 2m'$). Thus, passing to subsequences, one can assume that $[X'_s] \to [X']$ as $s \to \infty$ in $\mathcal{M}_{2m'}$. Then, as shown above, there are QC maps $\beta_s : X \to X_s$ whose Beltrami quotients μ_s obey

$\mu_s = k(s)\overline{\phi_s}/|\phi_s|$, where ϕ_s are nonzero holomorphic quadratic differentials on X and $k(s) \to 0$ as $s \to 0$.

Let $x^{(s,1)}, \ldots, x^{(s,4\mathfrak{g}-4)}$ be all zeroes of the quadratic differential ϕ_s counted with their multiplicities. Since X is compact, one can assume, passing to the sequences, that $x^{(s,k)} \to x^{(k)}$ in X.

Next, introduce the maps $\tilde{\tau}_s = \beta_s^{-1} \circ \tau_s \circ \beta_s$ (where τ_s are the antiholomorphic involutions on X_s). Let us show that $\tilde{\tau}_s$ do not converge to the identity Id on X. Indeed, suppose the contrary and chose x_0 and a curve L in $X \backslash \{x_0\}$ with the winding number $\mathrm{wind}(L, x_0) = 1$. If $\tilde{\tau}_s \to \mathrm{Id}$ uniformly on X, then $\mathrm{wind}(\tilde{\tau}_s \circ L, \tilde{\tau}_s(x_0)) = 1$ for large s. At the came time, each $\tilde{\tau}_s$ is orientation-reversing homeomorphism since τ_s is antiholomorphic and β_s is QC. Therefore, $\mathrm{wind}(\tilde{\tau}_s \circ L, x_0) = -1$ for any s. This contradiction shows that there are $q_s \in X$ obeying $\mathrm{dist}(q_s, \tilde{\tau}_s(q_s)) \geq \mathrm{const} > 0$ for all s (where dist is, say, the geodesic distance in the hyperbolic metric on X). Since X is compact, one can assume, passing to the subsequences, that $q_s \to q_+$ and $\tilde{\tau}_s(q_s) \to q_-$ in X. Then $q_+ \neq q_-$.

Denote by Γ_s the set $\{y \in X_s \mid \tau_s(y) = y\}$; then Γ_s is a smooth curve in X_s and cutting X_s along Γ_s yields two Riemann surfaces $X_{\pm,s}$ with boundaries conformally equivalent to (M_s, g_s).

Step 2: Local replacement of quasiconformal mappings

In what follows, we write $u_s \underset{s\to\infty}{\to} u$ in $C^\infty(\mathbb{C};\mathbb{C})$ if any partial derivative (of any order $l = 0, 1, \ldots$) of u_s converge to the corresponding partial derivative of u uniformly on each compact set in $\mathbb{C}$. If f_s, f are sections of some line bundle L on the Riemann surface X and $Q \subset X$ is a compact set, we write $f_s \underset{s\to\infty}{\to} f$ in $C^\infty(Q; L)$ if $f_s \underset{s\to\infty}{\to} f$ uniformly with any number of derivatives in any (independent of s) trivialization of L over Q.

Lemma 2. *Let X be a Riemann surface, let $z : U \to \mathbb{C}$ be a complex chart on X, and let V be a domain in X such that $\overline{V} \subset U$. Let $\{\beta_s\}_{s=1}^\infty$ be a family of quasiconformal homeomorphisms $\beta_s : X \to X_s$ from X onto Riemann surfaces X_s whose Beltrami quotients μ_s obey $|\mu_s| \underset{s\to\infty}{\to} 0$ in $L_\infty(X;\mathbb{C})$ and $\mu_s \underset{s\to\infty}{\to} 0$ in $C^\infty(X\backslash V;\mathbb{C})$. Suppose that there is $x_\infty \in V$ and a sequence $\{x_s, y_s\}_{s=1}^\infty$ such that $x_s \to x_\infty$ and $y_s = \beta_s(x_s) \in X_s$. Then there is a family $\{\beta'_s\}_{s=1}^\infty$ of smooth diffeomorphisms $\beta'_s : X \to X_s$ obeying $\beta'_s(x_\infty) = y_s$ and whose Beltrami quotients μ'_s obey $\mu'_s \underset{s\to\infty}{\to} 0$ in $C^\infty(X;\mathbb{C})$ and $\mu'_s|_{\overline{V}} = 0$.*

Proof. Let U' be a domain in X such that $\overline{U'} \subset U$ and $\overline{V} \subset U'$. Let also $\kappa \in C_c^\infty(U; [0, +\infty))$ be a cut-off function equal to one on $\overline{U'}$. Put $\tilde{\mu}_s(z(x)) := \kappa(x)\mu_s(z(x))$; then $\tilde{\mu}_s \underset{s\to\infty}{\to} 0$ in $L_\infty(\mathbb{C}; \mathbb{C})$ and in $C^\infty(\mathbb{C}\backslash z(V'))$. Introduce the solutions $\tilde{\beta}_s$ to the Beltrami equations

$$\partial_{\overline{z}}\tilde{\beta}_s = \tilde{\mu}_s \partial_z \tilde{\beta}_s \tag{4}$$

in $\mathbb{C}$ of the form $\tilde{\beta}_s(x) = z + \tilde{q}_s(z)$ where $\tilde{q}_s$ belong to the Schwartz space $\mathcal{S}(\mathbb{C}; \mathbb{C})$. The method of construction of solutions to (4) is as follows (see [11]). Introduce the Beurling transform (2-dim Hilbert transform) Π by

$$\Pi u = \mathcal{F}^{-1}_{\xi\mapsto z} \left(\frac{\xi}{\overline{\xi}}\right) \mathcal{F}_{z\mapsto\xi} u = \partial_z \partial_{\overline{z}}^{-1} u,$$

where $\mathcal{F}_{z\mapsto\xi}$ is the 2-dim Fourier transform (then Π can be extended by continuity to the operator $\Pi:\ H^l(\mathbb{C}; \mathbb{C}) \to H^l(\mathbb{C}; \mathbb{C})$ and $\Pi:\ L^\infty(\mathbb{C}; \mathbb{C}) \to L^\infty(\mathbb{C}; \mathbb{C})$). Then one can rewrite (4) in the form

$$\begin{aligned}\partial_{\overline{z}}\tilde{q}_s &= \tilde{\mu}_s(1 + \partial_z \tilde{q}_s) = \tilde{\mu}_s(1 + \Pi\partial_{\overline{z}}\tilde{q}_s) = \\ &= \tilde{\mu}_s + \tilde{\mu}_s \Pi \tilde{\mu}_s(1 + \Pi\partial_{\overline{z}}\tilde{q}_s) = \dots = \sum_{k=0}^{\infty} (\tilde{\mu}_s \Pi)^k \tilde{\mu}_s.\end{aligned}$$

Note that the series in the right-hand side converge in $L_\infty(\mathbb{C}; \mathbb{C})$ for $\|\tilde{\mu}_s\|_{L_\infty(\mathbb{C};\mathbb{C})} < 1$ and $\tilde{\mu}_s \underset{s\to\infty}{\to} 0$ in $L_\infty(\mathbb{C}; \mathbb{C})$ implies $\partial_{\overline{z}}\tilde{q}_s \underset{s\to\infty}{\to} 0$ in $L_\infty(\mathbb{C}; \mathbb{C})$. In addition, we have $\partial_z \tilde{\beta}_s = 1 + \partial_z \tilde{q}_s = 1 + \Pi\partial_{\overline{z}}\tilde{q}_s$. Integration of the last two equations provides the required solution $\tilde{\beta}_s = z + \tilde{q}_s(z)$ such that $\tilde{q}_s \underset{s\to\infty}{\to} 0$ in $L_\infty(\mathbb{C}; \mathbb{C})$.

Now, we observe that each $\tilde{\beta}_s$ admits smoothness increasing outside $\overline{V}$. Indeed, let V' be a small fixed neighborhood of $\overline{V}$ and let $\chi \in C^\infty(\mathbb{C}; [0, +\infty))$ be a cut-off function equal to zero on $z(\overline{V})$ and to one outside $z(V')$. Note that the commutator $[\partial_{\overline{z}}, \chi]$ is the multiplication operator $u \mapsto \tilde{\chi}u$, where $\tilde{\chi}$ is the smooth cut-off-function obeying $\mathrm{supp}\tilde{\chi} \subset \mathrm{supp}(\partial_{\overline{z}}\chi)$. We have

$$\partial_{\overline{z}}(\chi\tilde{q}_s) = \chi(\partial_{\overline{z}} - \tilde{\mu}_s\partial_z)\tilde{q}_s + [\partial_{\overline{z}}, \chi]\tilde{q}_s + \chi\tilde{\mu}_s\partial_z\tilde{q}_s = \chi\tilde{\mu}_s + \tilde{\chi}\tilde{q}_s + \chi\tilde{\mu}_s\partial_z\tilde{q}_s.$$

Since the system $\partial_{\overline{z}}Y = F$ is elliptic, i.e., (the representation

$$\partial_{\overline{z}}Y := \mathcal{F}^{-1}_{\xi\mapsto z}\,\overline{\xi}\,\mathcal{F}_{z\mapsto\xi}Y$$

is valid for $\partial_{\overline{z}}$), the estimate

$$\|Y\|_{H^{l+1}(\mathbb{C};\mathbb{C})} \le c(l)(\|\partial_{\overline{z}}Y\|_{H^l(\mathbb{C};\mathbb{C})} + \|Y\|_{L_2(\mathbb{C};\mathbb{C})})$$

is valid for all $l = 0, 1, \dots$ (in the Fourier representation, it becomes the obvious estimate $|\mathcal{P}(\xi, \overline{\xi})| \le c(\mathcal{P})(1 + |\overline{\xi}|^l)$ for any polynomial of degree $\deg(\mathcal{P}) \le l$). In particular, we have

$$\begin{aligned}\|\chi\tilde{q}_s\|_{H^{l+1}(\mathbb{C};\mathbb{C})} \le c(l)(\|\chi\tilde{\mu}_s\partial_z\tilde{q}_s\|_{H^l(\mathbb{C};\mathbb{C})} + \|\tilde{\chi}\tilde{q}_s\|_{H^l(\mathbb{C};\mathbb{C})} + \\ + \|\chi\tilde{\mu}_s\|_{H^l(\mathbb{C};\mathbb{C})} + \|\tilde{q}_s\|_{L_\infty(\mathbb{C};\mathbb{C})})\end{aligned}$$

Since $\mathrm{supp}\chi \cap \overline{V} = \varnothing$, we have $\chi\tilde{\mu}_s \underset{s\to\infty}{\to} 0$ in $C^\infty(\mathbb{C};\mathbb{C})$. Therefore, for sufficiently large $s \ge s_0(l)$ one can get rid (by increasing the constant $c(l)$) the first term in the right-hand side of the last estimate. Thus, iterating the last estimate and taking into account the convergences $\chi\tilde{\mu}_s \underset{s\to\infty}{\to} 0$ in $C^\infty(\mathbb{C};\mathbb{C})$ and $\tilde{q}_s \underset{s\to\infty}{\to} 0$ in $L_\infty(\mathbb{C};\mathbb{C})$ and the continuity of the embeddings $H^{l+2}(D;\mathbb{C}) \subset C^l(\overline{D};\mathbb{C})$ (where D is a domain with compact closure in $\mathbb{C}$), one arrives at

$$\chi\tilde{q}_s \underset{s\to\infty}{\to} 0 \text{ in } C^\infty(\mathbb{C};\mathbb{C}) \tag{5}$$

(note that χ and V' chosen above are arbitrary).

Recall the expression

$$\mu_{g\circ f^{-1}} = \frac{\mu_g - \mu_f}{1 - \overline{\mu_f}\mu_g}\frac{\partial f}{\overline{\partial f}}$$

for the composition of two quasiconformal maps (see, e.g., p.50, [12]). Due to this expression, we have $\mu_{\tilde{\beta}_s\circ\beta_s^{-1}} = 0$ on $\beta_s(\overline{U'})$, i.e., $\tilde{\beta}_s \circ \beta_s^{-1} : \overline{U'} \to \mathbb{C}$ is a holomorphic chart on X_s.

Introduce the function

$$\begin{aligned}\tilde{\beta}'_s(x) &:= (1-\chi)\Big[z + \tilde{\beta}_s(x_s) - z(x_\infty)\Big] + \chi\tilde{\beta}_s(x) = \\ &= z + \Big[z(x_s) - z(x_\infty) + \tilde{q}_s(x_s)\Big](1-\chi) + \chi\tilde{q}_s(x)\end{aligned}$$

and denote by $\tilde{\mu}'_s := \partial_{\overline{z}}\tilde{\beta}'_s/\partial_z\tilde{\beta}'_s$ the corresponding Beltrami quotient. Then the following properties are valid:

1. $\tilde{\beta}'_s = \tilde{\beta}_s$ outside $\mathrm{supp}(1-\chi)$;

2. $\tilde{\mu}'_s = 0$ on $z(\overline{V})$;

3. formula (5) and the convergence $x_s \to x_\infty$ in X imply $\partial_{\bar{z}}\tilde{\beta}'_s \underset{s\to\infty}{\to} 0$ and $\tilde{\mu}'_s \underset{s\to\infty}{\to} 0$ in $C^\infty(\mathbb{C};\mathbb{C})$;

4. in addition, $\tilde{\beta}'_s$ is a diffeomorphism from $\overline{U'}$ onto its image $\tilde{\beta}'_s(\overline{U'})$;

5. we have $\tilde{\beta}'_s(x_\infty) = \tilde{\beta}_s(x_s)$.

Now, introduce the family $\{\beta'_s\}_{s=1}^\infty$ of maps $\beta'_s : X \to X_s$ by the rules

$$\beta'_s = \beta_s \text{ outside } V', \qquad \beta'_s := (\beta_s \circ \tilde{\beta}_s^{-1}) \circ \tilde{\beta}'_s \text{ on } U'.$$

Denote by μ'_s the Beltrami quotient of β'_s. Then the above properties imply:

1. β'_s is well-defined on X;

2. $\mu'_s = 0$ on $\overline{V}$;

3. since the composition with holomorphic map does not change the absolute value of Beltrami quotient, we have $\mu'_s \underset{s\to\infty}{\to} 0$ in $C^\infty(U';\overline{K}K^{-1})$ and, due to 1., in $C^\infty(X;\overline{K}K^{-1})$;

4. $\beta'_s : X \to X_s$ is a diffeomorphism;

5. $\beta'_s(x_\infty) = y_s$.

Thus, the required family $\{\beta'_s\}_{s=1}^\infty$ is constructed. □

Note that Lemma 2 can be easily generalized to the case of several domains V violating the C^∞-convergence of μ_s to zero on X and several sequences $\{x_s\}_{s=1}^\infty$. Due to Lemma 2, one can replace the family Techmüller maps $\beta_s : X \mapsto X_s$ constructed above by the family of smooth diffeomorphisms $\beta'_s : X \mapsto X_s$ obeying the following properties:

- the Beltrmi quotients μ'_s of β'_s obey $\mu'_s \underset{s\to\infty}{\to} 0$ in $C^\infty(X;\overline{K}K^{-1})$;
- $\beta'_s(q_+) = \beta_s(q_s)$ and $\beta'_s(q_-) = \beta_s(\tilde{\tau}_s(q_s))$, in particular, we have $\beta'^{-1}_s \circ \tau_s \circ \beta'_s(q_+) = q_-$;
- $\mu'_s = 0$ in the (independent of s) neighborhoods of q and q^*.

Now, let $\{\varkappa_m\}_{m=1}^M$ be a smooth partition of unity on X subordinate to a finite complex atlas $\{z_m : U_m \to \mathbb{C}\}_{m=1}^M$ on X. Introduce the smooth metrics

$$h_s = \sum_{m=1}^{M} h_{s,m}, \text{ where } h_{s,m}(z_m) = \varkappa_m(x)|dz_m + \mu'_s(z_m)d\overline{z_m}|^2$$

and denote by $\mathfrak{U}_s$ the complex structures corresponding to the metrics g_s and the original orientation on X. Similarly, introduce the smooth conformal metric

$$h = \sum_{m=1}^{M} h_m, \text{ where } h_m(z_m) = \varkappa_m(x)|dz_m|^2,$$

corresponding to the complex structure on X. In view of the above properties, we have

$$h_s \underset{s\to\infty}{\to} h \text{ in } C^\infty(X, T^*X \otimes T^*X). \tag{6}$$

Let z' be a holomorphic coordinate of the point $\beta_s(x)$. Then

$$h_{s,m}(z_m) = \varkappa_m(x)|\partial z'/\partial z_m|^{-1}|dz'|^2$$

and, thus, the maps $\beta'_s : (X, \mathfrak{U}_s) \to X_s$ are biholomorphisms.

Without loss of generality, one can assume that each β_s is the identity on X, i.e., X_s coincides with $(X, \mathfrak{U}_s)$. Then, since β'_s constructed above are smooth, the Γ_s are smooth curves in X. The cutting (X, h_s) along Γ_s provides two conformal copies $(X_{\pm,s}, h_s|_{X_{\pm,s}})$ of (M_s, g_s). The antiholomorphic involutions τ_s on $X_s = (X, \mathfrak{U}_s)$ obey $\tau_s(q_+) = q_-$ for any s. There are two complex charts $z_+ : U_+ \to \mathbb{C}$, $z_- = \overline{z_+ \circ \tau}\, U_- \to \mathbb{C}$ on X such that $q_\pm \in U_\pm$, $z_\pm(q_\pm) = 0$ and $z_\pm$ belong to any complex atlas $\mathfrak{U}_s$.

Step 3: Introducing Mandelstam variables

Let $\mathcal{E}_s$ be a harmonic function on $(X\backslash\{q_+, q_-\}, h_s)$ admitting the asymptotics

$$\mathcal{E}_s(x) = \sum_{\pm} \mp \frac{\kappa(|z_\pm|)}{2\pi} \log|z_\pm| + \tilde{\mathcal{E}}_s(x),$$

where $\kappa \in C_c^\infty(\mathbb{R}; [0, +\infty))$ is the cut-off function equal to one near the origin and the remainder $\tilde{\mathcal{E}}_s$ is smooth on the whole X and obeys $\tilde{\mathcal{E}}_s(q_+) = -\tilde{\mathcal{E}}_s(q_-)$. As it easy to see, $\mathcal{E}_s(x) = G_s(x, q_+) - G_s(x, q_-)$, where G_s is the Green function for the Laplace operator $\Delta_{(s)}$ on (X, h_s). Since the involution τ_s is a conformal automorphism of (X, h_s), the function $-\mathcal{E}_s \circ \tau_s$ obeys the same conditions as $\mathcal{E}_s$, whence $u = \mathcal{E}_s + \mathcal{E}_s \circ \tau_s$ is a harmonic (thus, a constant) function on X obeying $u(q_\pm) = 0$. Hence $u = 0$ and

$$\mathcal{E}_s \circ \tau_s = -\mathcal{E}_s. \tag{7}$$

In particular, $-\mathcal{E}_s(x) = \mathcal{E}_s(\tau_s(x)) = \mathcal{E}(x) = 0$ for $x = \tau_s(x) \in \Gamma_s$, i.e., $\mathcal{E}|_{\Gamma_s} = 0$. This means that the restriction of $\mathcal{E}_s$ on $X_{\pm,s}$ coincides with the Green function $G^D_{s,\pm}(\cdot, q_\pm)$ for the Dirichlet Laplacian on $\Delta^D_{(s,\pm)}$ on $(X_{\pm,s}, h_s|_{X_{\pm,s}})$. Due to the Zaremba's normal derivative lemma, the exterior normal derivative of $\pm\mathcal{E}_s|_{X_{\pm,s}} \equiv G^D_{s,\pm}(\cdot, q_\pm)$ is everywhere positive on $\partial X_{\pm,s}$.

Similarly, let $\mathcal{E}$ be a harmonic function on $(X\backslash\{q_+, q_-\}, h)$ admitting the asymptotics

$$\mathcal{E}(x) = \sum_{\pm} \mp \frac{\kappa(|z_\pm|)}{2\pi} \log|z_\pm| + \tilde{\mathcal{E}}(x),$$

where $+\tilde{\mathcal{E}}$ is smooth on X and $\tilde{\mathcal{E}}(q_+) = -\tilde{\mathcal{E}}(q_-)$. Then $\mathcal{E}_s(x) = G(x, q_+) - G(x, q_-)$, where G is the Green function for the Laplace operator Δ on (X, h).

Since $z_\pm$ are holomorphic coordinates for X and each $X_s = (X, \mathfrak{U}_s)$, we have

$$\Delta\tilde{\mathcal{E}} = \Delta_{(s)}\tilde{\mathcal{E}}_s = f,$$

where

$$f(x) := \frac{1}{2\pi} \sum_{\pm} \pm \Big[-4\partial_{z_\pm}\partial_{\overline{z_\pm}}, \kappa(|z_\pm|)\Big] \log|z_\pm|$$

is independent of s, is smooth on X, and vanishes outside $U_\pm$. Due to (6), one can consider $\Delta_{(s)}$ as a small regular perturbation of the Laplacian Δ. Then the standard results of the perturbation theory for elliptic equations yield

$$\tilde{\mathcal{E}}_s \underset{s\to\infty}{\to} \tilde{\mathcal{E}} \text{ in } C^\infty(X; \mathbb{C})$$

and, hence,

$$\mathcal{E}_s - \mathcal{E} \underset{s\to\infty}{\to} 0 \text{ in } C^\infty(X; \mathbb{C}). \tag{8}$$

The above formula implies that the Hausdorff distance between the level sets $\mathcal{E} = C$ and $\mathcal{E}_s = C$ tends to zero while the critical points of $\mathcal{E}_s$ converge to those of $\mathcal{E}$.

Next, introduce the Abelian differentials of the third kind $\omega_s = (1 + i\star_s)d\mathcal{E}_s$ on $X_s = (X, \mathfrak{U}_s)$, where $\star_s$ is the Hodge star operator on X_s. Similarly, introduce the differential $\omega = (1+i\star)d\mathcal{E}$, where $\star$ is the Hodge operator in X. Then the surface $X_s \backslash \{q_+, q_-\}$ (resp., $X\backslash\{q_+, q_-\}$) endowed with the metric $|\omega_s|^2$ (or $|\omega|^2$) is isometric to some Mandelstam diagram which is a surface obtained by gluing together a finite number of finite or semi-infinite cylindrical surfaces; the isometry being provided by the map

$$X_s \ni x \mapsto \int_p^x \omega_s =: \zeta_s(x|p) \qquad (\text{resp.,} \quad X \ni x \mapsto \int_p^x \omega =: \zeta(x|p)\ ) \tag{9}$$

(see [13]), where p is an arbitrarily fixed point of $X\backslash\{q_+, q_-\}$. The metric $|\omega_s|^2$ (resp., $|\omega|^2$) is flat outside of $2\mathfrak{g}$ zeroes (counted with multiplicities) and the two poles $q_\pm$ of ω_s (resp., ω). Each level set of $\mathcal{E}_s$ (except those containing critical points of $\mathcal{E}_s$) is a finite collection of closed geodesic in the metric $|\omega_s|^2$; the same is true for $\mathcal{E}$, ω. In particular, the curve Γ_s is geodesic in the metric $|\omega_s|^2$ and there is a metric neighborhood of Γ_s which does not contain the zeroes of ω_s.

Finally, convergence (8) implies

$$\omega_s - \omega \underset{s\to\infty}{\to} 0 \text{ in } C^\infty(X; T^*X). \tag{10}$$

Note that (7) and the equality $\tau_s^* \star_s = - \star_s \tau_s^*$ lead to

$$\tau_s^*\omega_s = \tau_s^*(1 + i\star_s)d\mathcal{E}_s = (1 - i\star_s)d(\mathcal{E}_s \circ \tau) = -(1 - i\star_s)d\mathcal{E}_s = -\overline{\omega_s}. \tag{11}$$

Step 4: Proof of the existence of an anti-holomorphic involution on X

For sufficiently large N_0 the neighborhood

$$\Pi_\pm(N_0) = \{x \in X \mid \pm\mathcal{E}(x) > N_0\}$$

is free of zeroes of ω. Due to (8), the neighborhoods

$$\Pi_{s,\pm}(N_0) = \{x \in X \mid \pm\mathcal{E}_s(x) > N_0\}$$

are free of zeroes of ω_s for sufficiently large s. Then coordinates (9) with $p \in \Pi_\pm(N_0)$ provide the isometry from $\Pi_\pm(N_0)$ or $\Pi_{s,\pm}(N_0)$ onto semi-infinite cylindrical surface $\mathbb{T} \times (C, +\infty)$.

Let $N > N_0$. Introduce the surfaces

$$X_s^{(N)} := \{x \in X \mid |\mathcal{E}_s(x)| \leq N\}, \qquad X^{(N)} := \{x \in X \mid |\mathcal{E}(x)| \leq N\}$$

(endowed with the metrics $|\omega_s|^2$ and $|\omega|^2$, respectively). Fix $p_+ \in \partial X^{(N)}$ and denote by $p_{+,s}$ a point of $\partial X_s^{(N)}$ closest to p (say, in the metric $|\omega|^2$). Convergence (8) implies that $p_{+,s}$ is unique for large s and $p_{+,s} \to p$ as $s \to \infty$. Let $p_{-,s} = \tau_s(p_{+,s})$, then $\mathcal{E}_s(p_{-,s}) = -N$ and $\mathcal{E}(p_{-,s}) \to -N$ as $s \to \infty$ due to (8). Since the set $\{x \in X \mid \mathcal{E}(x) \in [N-\epsilon, N+\epsilon]\}$ is compact one can assume, passing to the subsequences, that $\exists \lim\limits_{s\to\infty} p_{-,s} =: p_-$ in X; then $\mathcal{E}(p_-) = \lim\limits_{s\to\infty} \mathcal{E}_s(p_{-,s}) = -N$, i.e., $p_- \in X^{(N)}$.

In the neighborhoods of boundaries $\partial X_s^{(N)}$, $\partial X^{(N)}$ of surfaces $(X_s^{(N)}, |\omega_s|^2)$, $(X^{(N)}, |\omega|^2)$, introduce the semi-geodesic coordinates $\mathfrak{X}_\pm = (r_\pm, l_\pm)$ and $\mathfrak{X}_{\pm,s} = (r_{\pm,s}, l_{\pm,s})$, where

$$\begin{aligned} r_{\pm,s} &:= \Re\zeta_s(x|p_{\pm,s}) = \pm N \mp \mathcal{E}_s(x), \quad l_{\pm,s} := \Im\zeta_s(x|p_{\pm,s})\mathrm{mod} 2\pi; \\ r_\pm &:= \Re\zeta(x|p_\pm) = \pm N \mp \mathcal{E}(x), \quad l_\pm := \Im\zeta(x|p_\pm)\mathrm{mod} 2\pi). \end{aligned} \tag{12}$$

Note that $\mathfrak{X}_\pm$ and $\mathfrak{X}_{\pm,s}$ also are semi-geodesic coordinates on $(\Pi_\pm(N), |\omega|^2)$ and $(\Pi_{s,\pm}(N), |\omega_s|^2)$, respectively. Coordinates (12) obey

$$\mathfrak{X}_{\pm,s}(\tau_s(x)) = \mathfrak{X}_{\mp,s}(x) \tag{13}$$

due to (11). We identify the boundaries $\partial X_s^{(N)}$, $\partial X^{(N)}$ with $\mathbb{T} \times \{\pm\}$ by the rules

$$\begin{aligned} \partial X_s^{(N)} \ni x \longleftrightarrow (l_{\pm,s}, \pm) \qquad (\mathcal{E}_s(x) = \pm N), \\ \partial X^{(N)} \ni x \longleftrightarrow (l_\pm, \pm) \qquad (\mathcal{E}(x) = \pm N). \end{aligned} \tag{14}$$

In view of (10), we have

$$\mathfrak{X}_{\pm,s}(x) \to \mathfrak{X}_\pm(x) \text{ in } C^\infty(\Pi_\pm(N_0) \backslash \Pi_\pm(2N); \mathbb{R} \times \mathbb{T}). \tag{15}$$

Let χ be a smooth cut-off function on X equal to one on the union of $\Pi_\pm(N-\epsilon)$ and to zero outside the union of $\Pi_\pm(N_0+\epsilon)$, where $\epsilon_0 > 0$. Introduce the map

$$\tilde{i}_s^{(N)} : X^{(N)} \to X_s^{(N)}$$

which is the identity outside $\Pi_\pm(N_0)$ and is given by the rule

$$\mathfrak{X}_{\pm,s} \circ \tilde{i}_s^{(N)}(x) = (1 - \chi(x))\mathfrak{X}_{\pm,s}(x) + \chi(x)\mathfrak{X}_{\pm,s}(x). \tag{16}$$

Then convergences (15) and (6) imply that the map $\tilde{i}_s^{(N)} : X^{(N)} \to X_s^{(N)}$ is a near-isometric diffeomorphism for large s,

$$\tilde{h}_s := (\tilde{i}_s^{(N)})^* h_s \to h \text{ in } C^\infty(X^{(N)}, T^*X^{(N)} \otimes T^*X^{(N)}). \tag{17}$$

By definition of $\tilde{i}_s^{(N)}$, one has

$$\tilde{i}_s^{(N)}(l_\pm, \pm) = (l_\pm, \pm) \qquad ((l_\pm, \pm) \in \mathbb{T} \times \{\pm\}) \tag{18}$$

(here identification rules (14) are applied).

Denote by $\tilde{\Lambda}_s^{(N)}$ and $\tilde{\Lambda}^{(N)}$ the DN maps of the surfaces $(X_s^{(N)}, h_s)$ and $(X^{(N)}, h)$, respectively, which are defined on $\mathbb{T}\times\{\pm\}$. Since $\tilde{i}_s^{(N)}$ is an isometry from $(X^{(N)}, \tilde{h}_s)$ onto $(X_s^{(N)}, h_s)$ obeying (18), the operator $\tilde{\Lambda}_s^{(N)}$ coincides with the DN map of $(X^{(N)}, \tilde{h}_s)$. Let us show that $\tilde{\Lambda}_s^{(N)}$ tends to $\tilde{\Lambda}^{(N)}$ in the operator norm. For $f \in C^\infty(\mathbb{T} \times \{\pm\}; \mathbb{R})$, let us denote by u^f and u_s^f the harmonic extensions of f onto $(X^{(N)}, h)$ and $(X^{(N)}, \tilde{h}_s)$, respectively. Introduce the (Dirichlet) Laplacians Δ (Δ_D) and Δ_s ($\Delta_{s,D}$) on $(X^{(N)}, h)$ and $(X^{(N)}, \tilde{h}_s)$. Note that both Δ_s, Δ are elliptic and, in each chart on $X^{(N)}$, the coefficients of the second order differential operator $\Delta_s - \Delta$ and all its derivatives converge uniformly to zero due to (17). Since $u_s^f - u^f = 0$ on $\mathbb{T} \times \{\pm\}$ and

$$\Delta_D(u_s^f - u^f) = \Delta_s u_s^f - \Delta u^f - (\Delta_s - \Delta)u^f = -(\Delta_s - \Delta)u^f,$$

one has the estimate

$$\begin{aligned}
&\|u_s^f - u^f\|_{H^l(X^{(N)})} \leq c\|(\Delta_s - \Delta)u^f\|_{H^{l-2}(X^{(N)})} \leq \\
&\quad \leq c\|(\Delta_s - \Delta)\|_{H^l(X^{(N)}) \to H^{l-2}(X^{(N)})} \|u^f\|_{H^l(X^{(N)})} \leq C\epsilon \|f\|_{H^{l-1/2}(\mathbb{T}\times\{\pm\})}
\end{aligned}$$

where

$$\epsilon = \|(\Delta_s - \Delta)\|_{H^l(X^{(N)}) \to H^{l-2}(X^{(N)})} \to 0 \qquad (s \to \infty).$$

Let $\nu^{(N)}$ and $\nu_s^{(N)}$ be unit outward normal vectors on $\partial X^{(N)} \equiv \mathbb{T} \times \{\pm\}$ corresponding to the metrics h and $\tilde{h}_s$), respectively. Then $\nu_s^{(N)} \to \nu^{(N)}$ in $C^\infty(\partial X^{(N)}, TX^{(N)})$ and the above estimate and the Sobolev trace theorem imply

$$\begin{aligned}
&\|(\tilde{\Lambda}_s^{(N)} - \tilde{\Lambda}^{(N)})f\|_{H^{l-3/2}(\mathbb{T}\times\{\pm\})} = \|\partial_{\nu_s^{(N)}} u_s^f - \partial_{\nu^{(N)}} u^f\|_{H^{l-3/2}(\mathbb{T}\times\{\pm\})} \leq \\
&\qquad \leq c_1 \|u_s^f - u^f\|_{H^l(X^{(N)})} + c_2 \|(\partial_{\nu_s^{(N)}} - \partial_{\nu^{(N)}})u^f\|_{H^{l-3/2}(\mathbb{T}\times\{\pm\})} \leq \\
&\qquad\qquad \leq C(\epsilon + \tilde{\epsilon}(s,l))\|f\|_{H^{l-1/2}(\mathbb{T}\times\{\pm\})}
\end{aligned}$$

where $\tilde{\epsilon}(s,l) := \|\nu_s^{(N)} - \nu^{(N)}\|_{C^l(X^{(N)},h)} \to 0$ as $s \to \infty$. Hence,

$$\tilde{\Lambda}_s^{(N)} \to \tilde{\Lambda}^{(N)} \text{ in } B(H^{l-1/2}(\mathbb{T}\times\{\pm\}); H^{l-3/2}(\mathbb{T}\times\{\pm\})), \tag{19}$$

where $B(Y;F)$ denotes the space of bounded operators acting from Y to F.

Next, let $\Lambda_s^{(N)}$ and $\Lambda^{(N)}$ be the DN maps of the surfaces $(X_s^{(N)}, |\omega_s|^2)$ and $(X^{(N)}, |\omega|^2)$, respectively, which are defined on $\mathbb{T}\times\{\pm\}$. Note that the metrics $|\omega_s|^2 = \rho_s h_s$, $|\omega|^2 = \rho h$ are conformally equivalent to h_s, h, respectively, and formulas (6), (10) imply the covergence of the conformal factors

$$\rho_s \to \rho \text{ in } C^\infty(\Pi_\pm(N_0); \mathbb{R}). \tag{20}$$

In view of (1), formulas (19) and (20) yield

$$\Lambda_s^{(N)} = \frac{1}{\sqrt{\rho_s}}\tilde{\Lambda}_s^{(N)} \to \frac{1}{\sqrt{\rho}}\tilde{\Lambda}^{(N)} = \Lambda^{(N)} \text{ in } B(H^{l-1/2}(\mathbb{T}\times\{\pm\}); H^{l-3/2}(\mathbb{T}\times\{\pm\})). \tag{21}$$

Finally, note that formulas (7) and $\tau_s^*\star_* = -\star_*\,\tau_s^*$ imply $\tau_s^* d\mathcal{E}_s = -d\mathcal{E}_s$ and

$$\tau_s^*\omega_s = -\overline{\omega_s}, \qquad \tau_s^*|\omega_s|^2 = |\omega_s|^2.$$

In addition, (7) implies that $\tau_s(X_s^{(N)}) = X_s^{(N)}$. Thus, the restriction of τ_s on $X_s^{(N)}$ is an isometric involution of $(X_s^{(N)}, |\omega_s|^2)$. As a corollary, one has

$$v_s^f \circ \tau_s = v^{f\circ T}, \qquad \Lambda_s^{(N)}(f\circ T) = (\Lambda_s^{(N)} f)\circ T, \tag{22}$$

where v_s^f is the harmonic extension of $f \in C^\infty(\mathbb{T}\times\{\pm\};\mathbb{R})$ into $(X_s^{(N)}, |\omega_s|^2)$ and

$$Tf(l,\pm) := f(l,\mp).$$

Passing to the limit as $s \to \infty$ in (22) and taking into account (21), one arrives at

$$\Lambda^{(N)}(f\circ T) = (\Lambda^{(N)} f)\circ T \qquad (f \in C^\infty(\mathbb{T}\times\{\pm\};\mathbb{R})). \tag{23}$$

Symmetry (23) of the DN map $\Lambda^{(N)}$ implies the symmetry of the surface $(X^{(N)}, |\omega|^2)$. Indeed, denote by $\acute{X}^{(N)}$ the surface with the boundary $\mathbb{T}\times\{\pm\}$ obtained from $(X^{(N)}, |\omega|^2)$ by applying the "reversed" identification of the boundary points

$$\partial\acute{X}^{(N)} \ni x \longleftrightarrow (l_\pm, \mp) \qquad (\mathcal{E}(x) = \pm N),$$

(compare with (14)). Let $\acute{\Lambda}^{(N)}$ be a DN-map of $(\acute{X}^{(N)}, |\omega|^2)$, then (23) yields $\acute{\Lambda}^{(N)} = \Lambda^{(N)}$. Due to the theorem of Lassas and Uhlmann [1], there is the conformal map τ between $(X^{(N)}, |\omega|^2)$ and $(\acute{X}^{(N)}, |\omega|^2)$ which does not move the points of $\mathbb{T} \times \{\pm\}$. This means that there is the orientation-reversing conformal automorphism (still denoted by τ) on $(X^{(N)}, |\omega|^2)$ obeying

$$\mathfrak{X}_{\mp}(\tau(x)) = \mathfrak{X}_{\pm}(x) \qquad (\mathcal{E}(x) = \pm N).$$

Now, let us continue τ on $\Pi_{\pm}(N)$ by the rule

$$\mathfrak{X}_{\mp}(\tau(x)) = \mathfrak{X}_{\pm}(x) \qquad (\mathcal{E}(x) > \pm N). \tag{24}$$

Then τ is well-defined and smooth on $X \backslash \{q_+, q_-\}$ and it is an isometry (and, hence, the conformal map) from $(\Pi_{\pm}(N), |\omega|^2)$ onto $(\Pi_{\mp}(N), |\omega|^2)$. Due to these facts and the Riemann theorem on removable singularity, τ is an anti-holomorphic automorphism of X. Since τ^2 is the holomorhic autimorphism of X obeying $\tau^2(x) = x$ for $x \in \Pi_{\pm}(N)$, one concludes that τ^2 is an identity and, thus, τ is the anti-holomorphic involution on X interchanging the points q_+ and q_-. Since $\mathcal{E}$ is a harmonic function with logarithmic singularities at $q_{\pm}$ (with opposite signs), one has $\mathcal{E} \circ \tau = -\mathcal{E}$ whence τ is an isometric automorphism of $(X, |\omega|^2)$, i.e., $\tau^* |\omega|^2 = |\omega|^2$.

2.1 Step 5. Proof that X is a double of a surface with boundary

Let $t \mapsto L_l(t)$ be the real curve

$$\omega[\dot{L}_l(t)] = 1, \tag{25}$$

of the differential ω passing through the point $x \in \partial X^{(N)}$ with $l_-(x) = l$ at $t = -N$. The only obstacle to the continuation of L_l to all values of $t \in \mathbb{R}$ is its intersection of the zeroes of ω which could be happen only for finite number of exceptional values of $l \in \mathbb{T}$. If l is not exceptional, than there is the arc $(l_1, l_2) \ni l$ of $\mathbb{T}$ which contains no exceptional values. The union of the curves $L_{l'}$ $(l' \in (l_1, l_2))$ constitute the strip $\mathfrak{S}$ in $(X, |\omega|^2)$ which is isometric to the strip $(l_1, l_2) \times \mathbb{R}$.

Now, let $t \mapsto L_{l,s}(t)$ be the real curve

$$\omega_s[\dot{L}_{l,s}(t)] = 1, \tag{26}$$

of the differential ω_s passing through the same point $x \in \partial X^{(N)}$, $l_-(x) = l$ at $t = t_0$. Due to convergence (10) of the differentials ω_s and ω, the strip $\mathfrak{S}$ is free of zeroes of ω_s for large s, while the curve $L_{l,s}$ is well defined and it is close to L_l,

$$L_{s,l}(t) \to L_l(t) \text{ in } X \text{ uniformly in } t \in [-N, N]. \tag{27}$$

Indeed, the functions $\zeta_s(\cdot, x)$ and $\zeta(\cdot, x)$ given by (9) are well-defined on $\mathfrak{S}$ and convergence (10) implies

$$\zeta_s(\cdot, x) \to \zeta(\cdot, x) \text{ in } C^\infty(\mathfrak{S}; \mathbb{R}).$$

The integration of equations (25) and (26) yields

$$\zeta_s(L_{l,s}(t), x) = \zeta(L_l(t), x) = \mathcal{E}_s(L_{l,s}(t)) = \mathcal{E}(L_l(t)) = t + N. \tag{28}$$

To prove (27), it remains to apply the following version of of the implicit function theorem (for a detailed proof, see, e.g., Lemma 4 and Appendix B, [14]) to equations (28).

Lemma 3. *Let X_0, X, Y be domains with compact closures in $\mathbb{R}^K$, $\overline{X} \subset X_0$, and $E_0 \in C^{l+2}(X_0 \times Y; \mathbb{R}^K)$. Suppose that the null set of E_0 is the graph of the function $\kappa_0 \in C^{l+1}(X_0; Y)$ and the Jacoby matrix of $y \mapsto E_0(x, y)$ is invertible for any $y = \kappa(x)$ and $x \in X_0$. Then, for sufficiently small $\varepsilon \in (0, \varepsilon_0)$ and any function $E \in C^{l+1}(X_0 \times Y; \mathbb{R}^K)$ obeying $\|E - E_0\|_{C^{l+1}(X_0 \times Y; \mathbb{R}^K)} \le \varepsilon$, the null set of E on $X \times Y$ is the graph of some function $\kappa \in C^{l+1}(\overline{X}; Y)$ obeying the estimate $\|\kappa - \kappa_0\|_{C^{l+1}(\overline{X}; Y)} \le c\varepsilon$, where the constant independent of E and ε.*

In view of (11), the curve

$$t \mapsto L'_{s,l}(t) := \tau_s \circ L_{s,l}(-t)$$

is a solution to equation (26); indeed

$$\omega_s[\dot{L}'_{l,s}(t)] = \omega_s[d\tau_s[-\dot{L}_{s,l}(-t)]] = -(\tau_s^* \omega_s)[\dot{L}_{s,l}(-t)] = \overline{\omega_s[\dot{L}_{s,l}(-t)]} = 1.$$

In addition, $L'_{s,l}(0) = \tau_s(L_{s,l}(0)) = L_{s,l}(0)$ since the curve $\Gamma_s = \{y \in X \mid \mathcal{E}_s(y) = 0\}$ coincides with the set of fixed points of the involution τ. Therefore, $L'_{s,l} = L_{s,l}$, i.e.,

$$L_{s,l}(t) = \tau_s \circ L_{s,l}(-t) \qquad (t \in \mathbb{R}).$$

Due to (13), one has

$$\mathfrak{X}_{\pm,s}(L_{s,l}(t)) = \mathfrak{X}_{\mp,s}(L_{s,l}(-t)) \qquad (|t| > N_0).$$

Due to convergences (15) and (27), the limit transition as $s \to \infty$ in the above formula yields

$$\mathfrak{X}_{\pm}(L_l(t)) = \mathfrak{X}_{\mp}(L_l(-t)) \qquad (|t| \in (N_0, 2N)).$$

Due to (24), this means that

$$\tau \circ L_l(t) = L_l(-t) \qquad (|t| \in (N_0, 2N)). \tag{29}$$

Note that formula (29) is valid for all non-exceptional $l \in \mathbb{T}$.

Next, let $x' = L_{l'}(-N)$ where $l' \in (l_1, l_2)$ and let $y = L_{l'}(t') \in \mathfrak{S}$. Integrating equation (25) and taking into account (12), one obtains

$$\zeta(y, x) = \zeta(x', x) + (t' + N) = t' + il' + N - il.$$

Due to (9), the function $y \mapsto \zeta(y, x)$ is a holomorhic coordinate on the strip $\mathfrak{S}$. Thus, the map

$$t' + il' \mapsto L_{l'}(t') =: \mathfrak{Y}(t' + il')$$

is a biholomorphism from the strip $\mathfrak{Q} := \{x \in \mathbb{C} \mid \Im z \in (l_1, l_2)\}$ onto $\mathfrak{S}$ obeying

$$-\overline{z} = \mathfrak{Y}^{-1} \circ \tau \circ \mathfrak{Y}(z) \qquad (|\Re z| \in (N_0, 2N)) \tag{30}$$

due to (29). Since the function in the right-hand side of (30) is anti-holomorphic on $\mathfrak{Q}$, equality (30) holds everywhere on $\mathfrak{Q}$. Put $z = il'$ into (30), then

$$\tau(L_{l'}(0)) = \tau \circ \mathfrak{Y}(il') = \mathfrak{Y}(il') = L_{l'}(0).$$

The last equality means that any point $L_l(0)$, where l is not exceptional, is a fixed point of the involution τ (note that $\mathcal{E}(L_l(0)) = 0$ due to (7)). Since each point of $\mathcal{E}^{-1}(0)$ except for some finite set can be embedded in some curve L_l and τ is continuous, one concludes that $\mathcal{E}^{-1}(0)$ is the set of fixed points of τ.

Finally, suppose that $y_0 \in \mathcal{E}^{-1}(0)$ is a critical point of $\mathcal{E}$; then it is a zero of ω of order $n > 0$ and there are the holomorphic coordinates ξ in a neighborhood $U \ni y_0$ in X obeying $\xi(y_0) = 0$ and

$$\omega = \xi^n d\xi.$$

Then $\mathcal{E} \circ \xi^{-1} = \Re\xi(y)^{n+1}/(n+1)$ and the set $\{\xi(y) \mid y \in \mathcal{E}^{-1}(0) \cap U\}$ coincides with the union $\mathfrak{U}$ of $n+1$ straight lines near the origin. The function $\tilde{\tau} := \xi \circ \tau \circ \xi^{-1}$ is anti-holomorphic on $\xi(U) \subset \mathbb{C}$ and does not move the points of $\xi(U) \cap \mathfrak{U}$. Thus, $\tilde{\tau}$ should coincide with the reflection with respect to each of the lines from $\mathfrak{U}$ which is impossible for $n+1 > 1$. This contradiction shows that the set $\mathcal{E}^{-1}(0)$ contains no critical points of $\mathcal{E}$ and, thus, it can be represented as a smooth curve Υ in X.

Now, cutting X along Υ provides two surfaces $(X_\pm, h_\pm := h|_{X_\pm})$ with smooth boundaries $\partial X_\pm = \Upsilon$ which are conformally equivalent via the involution τ, $\tau(X_\pm) = X_\mp$. Thus, X is a Schottky double of X_+. In particular, $\mathrm{gen}(X_\pm) = \mathrm{gen}(X)/2 = \mathrm{gen}(X) = m'$.

Completing the proof of the convergence $\mathcal{L}(X'_s) \to 0$ as $s \to \infty$. Now, we have surfaces $X_{+,s}$ and X_+, both embedded in X, and such that their boundaries $\Gamma_s = \partial X_{+,s}$ and $\Upsilon = \partial X_+$ are the null sets of $\mathcal{E}_s$ and $\mathcal{E}$, respectively. We already have proved that Υ and, thus, some its neighborhood $|\mathcal{E}| < e_0$, is free of the zeroes of $d\mathcal{E}$. Due to (8), the neighborhood $U_0 = \{y \in X \mid |\mathcal{E}_s(y)| < e_0\}$ is also free of zeroes of $d\mathcal{E}_s$ for large s. So, coordinates (9) provide isometries from the above neighborhoods onto the cylindrical surface $\mathbb{T} \times [-e_0, e_0]$.

Arguing in the same way as in paragraph 4, one constructs:

- the semi-geodesic coordinates $\mathfrak{X}_s$ and $\mathfrak{X}$ in the neighborhoods of Γ_s and Υ, corresponding to the metrics $|\omega_s|^2$ and $|\omega|^2$, respectively, and obeying
$$\mathfrak{X}_s \to \mathfrak{X} \text{ in } C^\infty(U_0; \mathbb{R} \times \mathbb{T})$$
(compare with (12) and (15));

- the identification (similar to (14)) of the boundaries Γ_s and Υ with the unit circle $\mathbb{T}$, that is the parametrizations $t \mapsto \Gamma_s(t)$ and $t \mapsto \Upsilon(t)$ of the Γ_s and Υ obeying
$$\mathfrak{X} \circ \Gamma_s \to \mathfrak{X} \circ \Upsilon \text{ in } C^\infty(\mathbb{T}; \mathbb{T} \times \mathbb{R}). \qquad (31)$$

- the diffeomorphism $\iota_s : X_+ \to X_{+,s}$ which is an identity far from Υ, obeys
$$\iota_s \circ \Upsilon = \Gamma_s,$$

converges uniformly to the identity as $s \to \infty$, and is "near-isometric" for large s, i.e.,

$$h'_s := \iota_s^* h_s \to h \text{ in } C^\infty(X_+, T^*X_+ \otimes T^*X_+).$$

(compare with (16), (17), and (18));

- the DN-maps Λ'_s and Λ' of the surfaces $(X_{+,s}, h)$ and (X_+, h_s) which are defined on the unit circle $\mathbb{T}$ (identified with the boundaries $\partial X_{+,s}$ and X_+) and are close to each other in the operator norm

$$\Lambda'_s \to \Lambda' \text{ in } B(H^{l-1/2}(\mathbb{T}); H^{l-3/2}(\mathbb{T})) \tag{32}$$

for any $l = 1, 2, \ldots$ (compare with (19)).

Let dl'_s and dl' are length elements on $\mathbb{T}$ induced by its embedding Γ_s and Υ into (X, h_s) and (X, h), respectively. Denote by J_s and J the integration operators on $\mathbb{T}$ corresponding to the length elements dl'_s and dl', vanishing on constants and such that the sub-spaces $J_s L_2(\mathbb{T}, dl'_s) \subset L_2(\mathbb{T}, dl'_s)$, $J L_2(\mathbb{T}, dl') \subset L_2(\mathbb{T}, dl')$ are orthogonal to constants. Then from (31) and (6), it follows that

$$J_s \to J \text{ in } \text{ in } B(H^{l-3/2}(\mathbb{T}); H^{l-1/2}(\mathbb{T})) \tag{33}$$

for any $l \in \mathbb{R}$. Introduce the Hilbert operators $H'_s := J_s \Lambda'_s$, $H' := J\Lambda'$ and the defect operators $\mathfrak{D}'_s = H'^2_s + I$ and $\mathfrak{D}' = H^2_s + I$ corresponding to surfaces $(X_{+,s}, h)$ and (X_+, h_s), respectively. From (32) and (33), it follows that

$$H'_s \to H', \quad \mathfrak{D}'_s \to \mathfrak{D}' \text{ in } B(H^l(\mathbb{T}); H^l(\mathbb{T})) \tag{34}$$

for any $l = 1/2, 3/2, \ldots$. Note that, since $H'_s, H', \mathfrak{D}'_s, \mathfrak{D}'$ are zero-order pseudo-differential operators and any two norms in $B(H^l(\mathbb{T}); H^l(\mathbb{T}))$ and $B(H^{l'}(\mathbb{T}); H^l(\mathbb{T}))$ $(l, l' \in \mathbb{R})$ are equivalent on the space of such operators, convergence (34) holds for any $l \in \mathbb{R}$.

Since $\mathrm{gen}(X_+) = m'$, formula (3) implies

$$\dim\big[\mathfrak{D}' C^\infty(\mathbb{T}; \mathbb{R})/\mathbb{R}\big] = 2m',$$

i.e., there are smooth functions $f'_1, \ldots, f'_{2m'}$ such that any nonzero linear combination of $\mathfrak{D}' f'_1, \ldots, \mathfrak{D}' f'_{2m'}$ is nonconstant. Introduce the (compact in any $C^l(\mathbb{T}; \mathbb{R})$) set

$$\mathfrak{S} := \left\{ \sum_{k=1}^{2m'} c_k f'_k \;\middle|\; \sum_{k=1}^{2m'} |c_k|^2 = 1 \right\},$$

then

$$\begin{aligned}\min_{f'\in\mathfrak{S}} \|\!|\mathfrak{D}'f' + \mathrm{const}|\!\|_{\mathbb{T}} \geq 2c_0 > 0,\\ \max_{f'\in\mathfrak{S}} \|\!|f' + \mathrm{const}|\!\|_{\mathbb{T}} \leq C_0 \leq \infty,\\ \max_{f'\in\mathfrak{S}} \|f'\|_{H^{l+3/2}(\mathbb{T};\mathbb{R})} \leq C(l) \leq \infty.\end{aligned} \tag{35}$$

Note that

$$\max_{f'\in\mathfrak{S}} \|\!|(\mathfrak{D}'_s - \mathfrak{D}')f' + \mathrm{const}|\!\|_{\mathbb{T}} \leq \|\mathfrak{D}'_s - \mathfrak{D}'\|_{H^{3/2}(\mathbb{T};\mathbb{R})} C(0) \underset{s\to\infty}{\to} 0$$

due to (34) and the continuity of the embedding $H^{3/2}(\mathbb{T};\mathbb{R}) \subset C(\mathbb{T};\mathbb{R})$. Therefore, for sufficiently large s we have

$$\min_{f'\in\mathfrak{S}} \|\!|\mathfrak{D}'_s f' + \mathrm{const}|\!\|_{\mathbb{T}} \geq c_0. \tag{36}$$

Now we recall that the surfaces $(X_{+,s}, h_s)$ constructed above are conformally equivalent to (M_s, g_s). Then the Hilbert and defect operators H_s, $\mathfrak{D}_s$ of (M_s, g_s) are related to those of $(X_{+,s}, h_s)$ via the change of variables

$$H_s = \kappa_s^{*-1} H'_s \kappa_s^*, \qquad \mathfrak{D}'_s = \kappa_s^{*-1}\mathfrak{D}_s\kappa_s^* \text{ in } \dot{C}(\Gamma;\mathbb{R}). \tag{37}$$

Here $\kappa_s^* : f' \mapsto f \circ \kappa_s$ is a precomposition operator corresponding to the restriction on Γ of the conformal diffeomorphism $\kappa : (X_{+,s}, h_s) \to (M_s, g_s)$. By definition, κ_s^* is an isometry,

$$\|\!|\kappa_s^* f' + \mathrm{const}|\!\|_{\mathbb{T}} = \|\!|f' + \mathrm{const}|\!\|_{\Gamma} \qquad (f \in C(\mathbb{T};\mathbb{R})). \tag{38}$$

From the one side, taking into account (37) and (38), one can rewrite (35) and (36) as follows

$$\begin{aligned} c_0 \leq \min_{f'\in\mathfrak{S}} \|\!|\kappa_s^{*-1}\mathfrak{D}_s\Big(\kappa_s^* f'\Big) + \mathrm{const}|\!\|_{\mathbb{T}} = \min_{f\in\kappa_s^*(\mathfrak{S})} \|\!|\mathfrak{D}_s f + \mathrm{const}|\!\|_{\Gamma};\\ C_0 \geq \max_{f'\in\mathfrak{S}} \|\!|f' + \mathrm{const}|\!\|_{\mathbb{T}} = \max_{f\in\kappa_s^*(\mathfrak{S})} \|\!|f + \mathrm{const}|\!\|_{\Gamma}.\end{aligned} \tag{39}$$

From the other side, since the surface (M, g) is of genus m, its defect operator $\mathfrak{D}$ obeys

$$\dim \mathfrak{D}\dot{C}(\Gamma;\mathbb{R}) = 2m < 2m'$$

due to (3). Since $\dim[\kappa_s^*(\mathfrak{S})/\mathbb{R}] > \dim \mathfrak{D}\dot{C}(\Gamma;\mathbb{R})$, there is an element

$$y \in \kappa_s^*(\mathfrak{S}), \qquad \mathfrak{D}y = \mathrm{const}. \tag{40}$$

The convergence $\|\Lambda_s - \Lambda\|_{B(L_2(\Gamma;\mathbb{R});L_2(\Gamma;\mathbb{R}))} \to 0$ implies

$$\|H_s - H\|_{B(L_2(\Gamma;\mathbb{R});H^1(\Gamma;\mathbb{R}))} \to 0, \qquad \|\mathfrak{D}_s - \mathfrak{D}\|_{B(L_2(\Gamma;\mathbb{R});H^1(\Gamma;\mathbb{R}))} \to 0,$$

where H and $\mathfrak{D}$ are the Hilbert transfrom and the defect operator of (M,g), respectively. Due to the continuity of the embeddings $C(\Gamma;\mathbb{R}) \subset L_2(\Gamma;\mathbb{R})$ and $H^1(\Gamma;\mathbb{R}) \subset C(\Gamma;\mathbb{R})$, the above convergences imply

$$\|\mathfrak{D}_s - \mathfrak{D}\|_{B(\dot{C}(\Gamma;\mathbb{R});\dot{C}(\Gamma;\mathbb{R}))} \underset{s\to\infty}{\to} 0. \tag{41}$$

Now formulas (40), (41), and (39) yield

$$\begin{aligned} 0 = \|\mathfrak{D}y + \mathrm{const}\|_\Gamma = \|(\mathfrak{D} - \mathfrak{D}_s)y + \mathrm{const}\|_\Gamma + \|\mathfrak{D}_s y + \mathrm{const}\|_\Gamma \geq \\ \geq c_0 - \|\mathfrak{D}_s - \mathfrak{D}\|_{B(\dot{C}(\Gamma;\mathbb{R});\dot{C}(\Gamma;\mathbb{R}))} C_0 \geq c_0/2 > 0 \end{aligned}$$

for large s. This contradiction disproves the assumption that $\mathcal{L}(X_s) \not\to 0$ made at the beginning of the proof. Thus, we have proved that $\mathcal{L}(X_s) \to 0$ as $s \to \infty$. Proposition 1 is proved. $\square$

References

[1] M. Lassas, G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map.* Ann. Scient. Ec. Norm. Sup., **34**(5) (2001), 771–787.

[2] J.M. Lee, G. Uhlmann, *Determining anisotropic real?analytic conductivities by boundary measurements.* Comm. Pure Appl. Math., **42** (1989), 1097–1112.

[3] M.I. Belishev, *The Calderon problem for two-dimensional manifolds by the BC-method.* SIAM Journal of Mathematical Analysis, **35**(1) (2003), 172–182.

[4] D.V. Korikov, *On the topology of surfaces with a common boundary and close DN-maps.* Zapiski Nauchnykh Seminarov POMI, **506** (2021), 57–66.

[5] D.V. Korikov, *Stability Estimates in Determination of Non-orientable Surface from Its Dirichlet-to-Neumann Map*. Complex Anal. Oper. Theory **18**, 29 (2024).

[6] D.V. Korikov, *Determination of period matrix of double of surface with boundary via its DN map*. Canadian Journal of Mathematics (2025), 24 p.

[7] D.V. Korikov, *On perturbations of the DN map of a disk causing changes of surface topology*. `arXiv:2511.18179v1 (math-ph)`.

[8] Yu. Reshetnyak, *Space mappings with bounded distortion*. Translations of Mathematical Monographs, **73**. AMS, Providence, RI, 362 p.

[9] L. Bers, *Quasiconformal mappings and Teichmüller's theorem*. Analytic Functions, Princeton Univ. Press, Princeton, 89–119 (1960).

[10] D. Mumford, *A Remark on Mahler's Compactness Theorem*. Proceedings of the American Mathematical Society, **28**(1) (1971), 289–294.

[11] I.N. Vekua, *Generalized Analytic Functions*. Pergamon Press. Oxford-London-New York-Paris (1962).

[12] E.M. Chirka. *Teichmüller spaces*. Lekts. Kursy NOC, **15**, Steklov Math. Institute of RAS, Moscow (2010), 3–150.

[13] S.B. Giddings, S.A. Wolpert, *A triangulation of moduli space from light-cone string theory*. Commun.Math. Phys. **109** (1987), 177-–190.

[14] M.I. Belishev, D.V. Korikov, *Stability of Determination of Riemann Surface from its Dirichlet-to-Neumann Map in Terms of Teichmüller Distance*. SIAM Journal on Mathematical Analysis, **55**(6) (2023) 7426–7448.