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On elastic waves in topographic waveguides*

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Аннотация

The existence of elastic waves with displacements localized along the edge of a triangular protrusion on the surface of an elastic half-space was rigorously proven by V. M. Babich in [1] for isotropic homogeneous topographic waveguides with an interior angle less than $\frac{\pi}{2}$. We extend Babich's result by proving the existence of such localized waves for interior angles exceeding $\frac{\pi}{2}$. Furthermore, our ansatz allows us to separately analyze the existence of symmetric and antisymmetric modes.

Keywords: topographic waveguide, wedge waves, Rayleigh wave, variational method, discrete spectrum.

Dedicated to the memory of our teacher, Professor V.M. Babich

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1 Introduction

The propagation of elastic waves along topographic features such as ridges and wedges (see Fig. 1) on an otherwise flat surface has been a subject of long-standing interest in seismology, nondestructive testing, and surface acoustic wave devices. These structures support guided modes — known as wedge waves or topographic modes — that are localized near the ridge tip and propagate with phase velocity strictly less than that of the Rayleigh wave c_R.

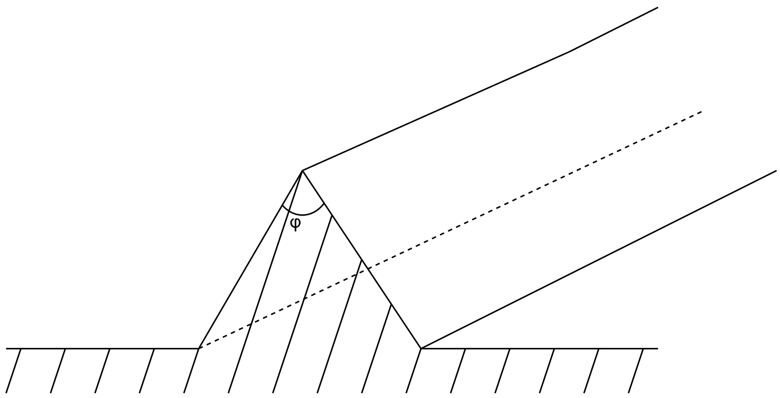

Fig. 1. Topographic waveguide.

Рис. 1: Topographic waveguide.

Numerical evidence of such modes dates back to the 1970s [3–6]. Physically rigorous asymptotic and ray methods were developed in the 1980s–1990s (see the review [8, Ch. 10]). Mathematically rigorous existence proofs appeared later: Bonnet–Duterte–Joly [9] proved existence for sufficiently small apex angles, while V. M. Babich [1] extended the result to all acute apex angles $\varphi < \pi/2$ using a variational approach in the short-wavelength limit ($k \to +\infty$).

The boundary case of a right apex angle $\varphi = \pi/2$ remained open because Babich's original trial function (a Rayleigh wave with an "escaping" cut-off function of width $O(1)$) yields a Rayleigh quotient asymptotics $c_R^2 k^2 + O(1)$, insufficient to guarantee a discrete eigenvalue below $c_R^2 k^2$. However, this gap was recently closed by A.A. Matskovskiy and G.L. Zavorokhin [12], who proved the existence of a localized waveguide mode for the right-angle case using a modified variational construction.

Building upon the method developed in [11] for an infinite elastic wedge, we propose a more sophisticated ansatz for the test functions, in contrast to that of [1]. As a result, we prove the existence of localized wedge waves within a certain range of angles greater than $\pi/2$. Furthermore, our ansatz enables the distinction between the existence of symmetric and antisymmetric modes. For some (negative) values of Poisson's ratio $\sigma = \frac{\lambda}{2(\lambda+\mu)}$, the existence of both types of waves is established.

2 Problem Statement. A result due to V.M. Babich

We assume that Ξ is a domain in $\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_i \in \mathbb{R},\ i = 1, 2, 3\}$, described by the inequality $x_2 < f(x_1)$, where $f(x_1)$ is a nonnegative continuous function defined for $x_1 \in \mathbb{R}$ and vanishing outside a finite interval $[a, b]$. Moreover, the graph of $f(x_1)$ for $a \le x_1 \le b$ is assumed to consist of two rectilinear segments $[aM]$ and $[Mb]$ ($f(a) = f(b) = 0$, $M = (x_0, f(x_0))$, $x_0 \in (a, b)$, $f(x_0) > 0$) forming an angle φ at M. The domain Ξ is a topological product: $\Xi = \Omega \times \mathbb{R}$, where Ω is the domain $x_2 < f(x_1)$ on the (x_1, x_2)-plane.

Let $\mathbf{U}(t, x_1, x_2, x_3) = (U_1, U_2, U_3)$ be a solution of the elastodynamic equations in Ξ,

$$\rho \mathbf{U}_{tt} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{U}) + \mu \Delta \mathbf{U} = 0,$$

$$\rho, \lambda, \mu = \text{const}, \quad \rho > 0,\ \mu > 0,\ \lambda + \frac{2}{3}\mu > 0,$$

that has the form

$$\mathbf{U} = e^{-i\omega t + ikx_3}\mathbf{u}(x_1, x_2), \qquad \omega, k = \text{const},\ \omega > 0,\ k > 0,$$

and satisfies the traction-free boundary condition on $\partial\Xi$. We get the following equation for $\mathbf{u}$ in Ω:

$$\rho\omega^2 \mathbf{u} = \mathcal{L}(ik)\mathbf{u}, \tag{1}$$

where

$$\mathcal{L}(ik)\mathbf{u} = -(\lambda + \mu)\nabla(\nabla \cdot \mathbf{U}) - \mu\Delta\mathbf{U}.$$

On the right-hand side of the last identity, the differentiation $\frac{\partial}{\partial x_3}$ is assumed to be replaced by multiplication by ik.

We subject the vector (2.2) to the traction-free boundary condition on $\partial\Omega$ (i. e., for $x_2 = f(x_1)$). We arrive at the identity

$$\mathcal{N}(ik)\mathbf{u} = 0, \tag{2}$$

where $\mathcal{N}(ik)$ is a first order differential operator depending on k.

Later, we shall describe the meaning of the equation and boundary conditions more precisely. We use the notation employed in [1]. Without loss of generality, we put the density $\rho = 1$.

If a function $\mathbf{u} \not\equiv 0$ satisfies equation (2.3) and the boundary conditions (2.4), and belongs to $(L^2(\Omega))^3$ (i. e., $\mathbf{u}$ is a vector-valued eigenfunction of the operator $\mathcal{L}(ik)$ that corresponds to the boundary conditions (2.4)), then the vector-valued function (2.2) is called a *localized waveguide mode*. This paper provides a rigorous proof of the existence of localized waveguide modes.

The variational (weak) formulation of the problem (1)–(2) refers to the integral identity

$$a_\Omega(ik; u, v) := \int_\Omega \sigma_{nm}[u]\overline{\partial_m v_n}\, dx_1 dx_2 = \omega^2 (u, v)_\Omega, \quad v \in \left(H^1(\Omega)\right)^3, \tag{3}$$

where bar stands for complex conjugation and $(\cdot,\cdot)_\Omega$ for the natural inner product in $(L^2(\Omega))^3$.The quadratic form $a_\Omega(ik; u, u)$ in $(L^2(\Omega))^3$ with domain $\left(H^1(\Omega)\right)^3$ is symmetric, positive and closed. Therefore, the boundary value problem (1)–(2) is associated with an unbounded positive definite self-adjoint operator $\mathcal{A}(ik)$ in the space $\left(L^2(\Omega)\right)^3$ (see, e.g., [13, Ch. 10]).

It is shown in [9] that the essential spectrum of this operator coincides with the ray $[\mathbf{c}_R^2 k^2, +\infty)$. Consequently, only discrete spectrum may occur below the cutoff point $\omega_R^2 = \mathbf{c}_R^2 k^2$. If this spectrum is non-empty, the corresponding eigenvector generates a localized wedge mode propagating along the edge with a velocity less than $\mathbf{c}_R$. Since the problem possesses a plane of symmetry, the operator $\mathcal{A}(ik)$ can be decomposed by splitting the space $(H^1(\Omega))^3$ into subspaces of antisymmetric ($\mathcal{H}a$) and symmetric ($\mathcal{H}s$) displacements. This decomposition creates the fundamental possibility for the existence of both antisymmetric (flexural) and symmetric modes.

According to the variational principle, the lower bound of the spectrum of the operator $\mathcal{A}(ik)$ is equal to $\inf\limits_{(H^1(\Omega))^3} \Phi(u)$, where

$$\Phi(u) := \frac{a_\Omega(ik; u, u)}{(u, u)_\Omega} \tag{4}$$

is the Rayleigh quotient.

To prove the existence of the discrete spectrum, it suffices to present a function $u^{test} \in (H^1(\Omega))^3$ such that

$$\Phi(u^{test}) < c_R^2 k^2. \tag{5}$$

In [1], to construct this test function, the coordinate system was placed as shown in Fig. 3. On the interval $[0, d]$, the part of the boundary $\partial\Omega$ not lying on the x_1-axis is the rectilinear segment described by the equation $x_2 = x_1 \mathrm{tg}(\varphi), 0 < \varphi < \frac{\pi}{2}$.

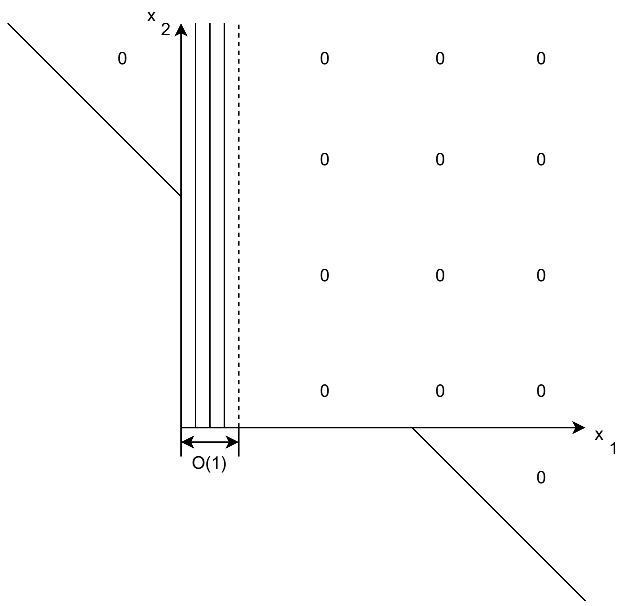

Fig. 2. Waveguide cross-section.

Рис. 2: Waveguide cross-section.

As a test function it was taken

$$u_k^{test}(x_1, x_2) = \begin{cases} \eta(x_1) u^R(x_2), & \text{in} \quad \Omega_1, \\ 0, & \text{in} \quad \Omega \setminus \Omega_1, \end{cases} \tag{6}$$

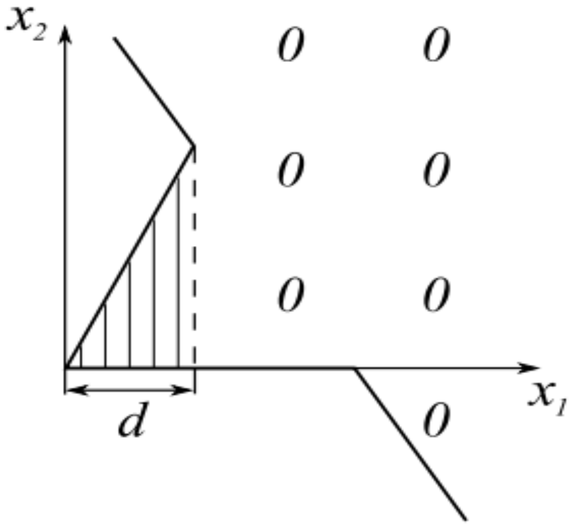

Рис. 3: Waveguide cross-section. Domain Ω_1.

where Ω_1 is the triangle

$$0 \le x_1 \le d, \quad 0 \le x_2 \le x_1 \mathrm{tg}\varphi$$

(see Fig. 3),

$$\eta(x_1) = \begin{cases} 1, & \text{if} \quad 0 \le x_1 < d', \\ 0, & \text{if} \quad x_1 > d, \end{cases} \tag{7}$$

is the cut-off function defined for $x_1 \ge 0$,

$$0 < d' < d, \quad d, d' = \text{const}, \quad \eta \ge 0, \quad \frac{d\eta}{dx} \le 0, \quad \eta \in C^\infty(\mathbb{R}_+),$$

and

$$U^R(x_1, x_2, x_3, t) = e^{-i\omega t + ikx_3} u^R(x_2), \tag{8}$$

is the Rayleigh wave [2] traveling in the direction (0,0,1) and leaving the plane $x_2 = 0$ free of stresses.

It turns out that

$$\Phi(u_k^{test}) = c_R^2 k^2 - ck + O(1), \quad k \to +\infty, \tag{9}$$

where $c > 0$ for $\varphi < \frac{\pi}{2}$. Then (9) implies (5) for k sufficiently large. This proves the existence of a guided mode for $\varphi < 0$ and for all admissible values of σ.

3 A generalization of V.M. Babich's result

Together with the Rayleigh wave (8) we consider another Rayleigh wave

$$V^R(x_1, x_2, x_3, t) = e^{-i\omega t + ikx_3} u^R\big(\sin\varphi\, x_1 - \cos\varphi\, x_2\big), \tag{10}$$

which propagates in the direction $(0, 0, 1)$ and leaves the plane $\sin\varphi\, x_1 - \cos\varphi\, x_2 = 0$ free of traction.

As a test function we take a linear combination of the "cut-off" Rayleigh-wave profiles (8) and (10):

$$v_n^{\text{test}}(x_1,x_2) = \alpha\,\eta(x_1)\,u^R(x_2) + \beta\,\eta\big(\cos\varphi\,x_1 + \sin\varphi\,x_2\big)\,u^R\big(\sin\varphi\,x_1 - \cos\varphi\,x_2\big), \qquad (11)$$

where for $\varphi < \pi/2$ each term vanishes outside the corresponding right triangle adjacent to the apex (see Fig. 3), while for $\varphi \geq \pi/2$ it vanishes outside the corresponding semi-infinite strip bounded by the wedge face (see Fig. 2).

It is clear that

$$\Phi\big(v_n^{\text{test}}\big) = \frac{\mathfrak{A}_n(\alpha,\beta)}{\mathfrak{B}_n(\alpha,\beta)},$$

where $\mathfrak{A}_n$ and $\mathfrak{B}_n$ are quadratic forms on $\mathbb{R}^2$. Hence, the inequality $\Phi(v_n^{\text{test}}) < c_R^2k^2$ for some α,β is equivalent to the presence of a negative eigenvalue of the form $\mathfrak{A}_n - c_R^2k^2\mathfrak{B}_n$.

A direct calculation gives

$$\mathfrak{A}_n(\alpha,\beta) - c_R^2k^2\mathfrak{B}_n(\alpha,\beta) = \mathfrak{M}(\alpha,\beta) + O\left(\frac{1}{k}\right), \qquad k\to\infty,$$

where $\mathfrak{M}$ is a quadratic form whose matrix M:

$$\begin{bmatrix} \mathfrak{m}_1 & \mathfrak{m}_2 \\ \mathfrak{m}_2 & \mathfrak{m}_1 \end{bmatrix}$$

does not depend on k.

Obviously the form $\mathfrak{M}$ has eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$. It is also easy to see that if the negative eigenvalue corresponds to the eigenvector $\begin{bmatrix}1\\-1\end{bmatrix}$, then for sufficiently large k the inequality $\Phi(v_n^{\text{test}}) < c_R^2k^2$ is attained on an antisymmetric function, which yields $\inf_{\mathcal{H}_a}\Phi(u) < c_R^2k^2$ and thus proves the existence of an antisymmetric mode. Similarly, the presence of a negative eigenvalue with eigenvector $\begin{bmatrix}1\\1\end{bmatrix}$ proves the existence of a symmetric mode.

Using elementary facts from linear algebra, we obtain the following criteria for the existence of guided modes[1]:

I	$\lvert\mathfrak{m}_1\rvert < \mathfrak{m}_2$	antisymmetric mode
II	$\lvert\mathfrak{m}_1\rvert < -\mathfrak{m}_2$	symmetric mode
III	$\lvert\mathfrak{m}_2\rvert < -\mathfrak{m}_1$	both antisymmetric and symmetric modes

The results of the computations are shown in Fig. 4. From Fig. 4 one can see that the fundamental antisymmetric mode (I) can propagate as a localized wave in a wedge with angle φ up to approximately 101°. For the symmetric mode (II) we were able to prove existence only for small positive values of σ, besides the unrealistic (negative) values of σ. The intersection of the regions (III) corresponds to the case where both modes exist, because for this range of angles $\varphi < \pi/2$ the determinant $\det M > 0$ and the trace

[1]The boundary between zones I and III (respectively, II and III) belongs to zone I (respectively, II).

$\operatorname{tr} M < 0$. It should be noted that the determinant of the matrix M vanishes for $\lambda = 0$ and $\varphi = \pi/2$ (see [10]); that is, in the right-angle case with $\sigma = 0$, we could not prove the existence of any mode.

Further use of the ansatz

$$\underline{\sigma} = \inf \Phi\Big(\sum_{m=1}^{l} \alpha_m \eta_m u_m^R\Big), \qquad u_m \in (H^1(\Omega))^3,\ u_m \neq 0,$$

$$\Phi\Big(\sum_m \alpha_m \eta_m u_m^R\Big) := \frac{a_\Omega\big(ik; \sum_{m=1}^{l} \alpha_m \eta_m u_m^R, \sum_{m=1}^{l} \alpha_m \eta_m u_m^R\big)}{\big\|\sum_{m=1}^{l} \alpha_m \eta_m u_m^R; (L_2(\Omega))^3\big\|^2},$$

with several “cut-off"Rayleigh-waves profiles (α_m being their amplitudes)

$$U_m^R(x_1, x_2, x_3, t) = e^{-i\omega t + ikx_3} u_m^R\big(\sin\beta_m x_1 + \cos\beta_m x_2\big), \quad m = 1, 2, \ldots, l,$$

which travel in the direction $(0, 0, 1)$ and leave the planes $\sin\beta_m x_1 + \cos\beta_m x_2 = 0$ traction-free (here β_m are angles measured clockwise from the positive x_1-axis), did not enlarge the existence regions of the modes. The only slight improvement was a modest extension of the symmetric-mode region (II) when three Rayleigh waves were taken: two waves travelling along the two wedge faces and one wave “propagating” along the plane $\sin\big((\pi - \varphi)/2\big)\, x_1 + \cos\big((\pi - \varphi)/2\big)\, x_2 = 0$.

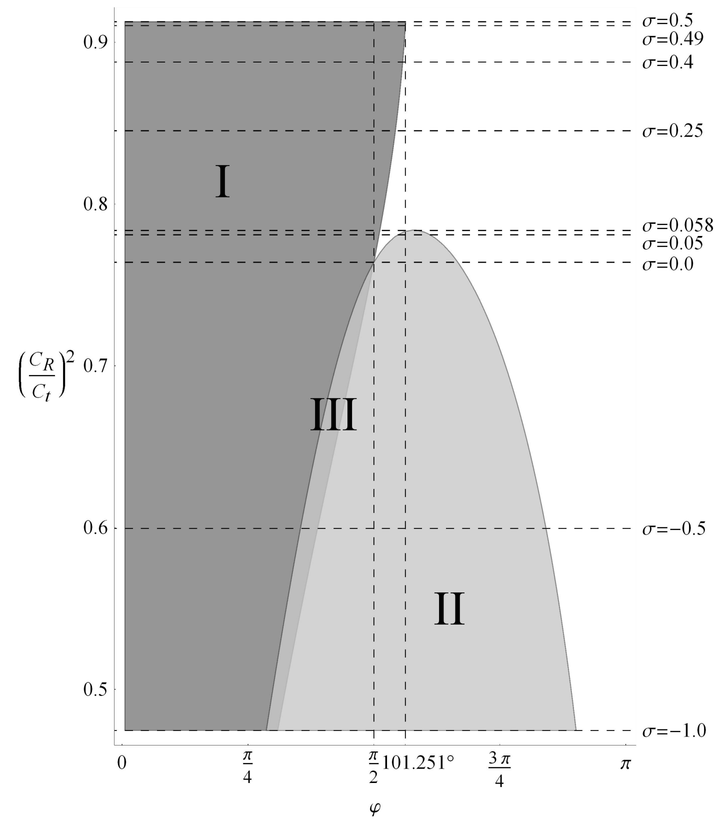

Рис. 4: Existence regions of antisymmetric (I) and symmetric (II) localized guided modes.

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