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An approach to the Lindelöf hypothesis for Dirichlet L-functions

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Abstract. The suggested approach is based on a known representation of Dirichlet L-functions via the incomplete gamma functions. Some properties of the Taylor coefficients of the lower incomplete gamma function at infinity seem to be new. Specifically, these coefficients can be expressed in terms of Touchard polynomials. Furthermore, these same coefficients can be used to reformulate the functional equation for Dirichlet L-functions. This relationship "explains" why $|L_\chi(1/2+\mathrm{i}t)|$ should be small.

To present the new ideas in a nutshell, we start by giving (in Section 1) a "formula proof" of the Lindelöf hypothesis. This is not a genuine proof, as we are not concerned with the convergence of our series nor do we justify changing the order of summation.

In Section 2, we suggest some hypothetical ways of transforming the "proof" from Section 1 into a rigorous mathematical proof.

Sections 3–5 contain some technical detail and bibliographical references.

Key words: Lindelöf hypothesis, Dirichlet L-functions, functional equation, series acceleration, incomplete gamma function, Touchard polynomials

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1 "Formula proof"

We start by presenting what could be called a 'formula proof' of the Lindelöf hypothesis. In this section, we ignore the convergence of our series,nor do we justify changing the order of summation. In the next section, we discuss possible ways to transform this "proof" into a rigorous mathematical argument.

To simplify notation, this section deals with a specific function $L_3(s)$; a generalisation to an arbitrary Dirichlet L-function is considered in Section 5.

The function $L_3(s)$ is defined (for $\mathrm{Re}(s) > 0$) by the Dirichlet series.

$$L_3(s) = \sum_{n=1}^{\infty} \chi_3(n) n^{-s}, \tag{1}$$

where χ_3 is the only non-principal character modulo 3:

$$\chi_3(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3, \\ 1, & \text{if } n \equiv 1 \pmod 3, \\ -1, & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{2}$$

The Lindelöf hypothesis (for the case of $L_3(s)$). *For every positive ε*

$$L_3(\tfrac{1}{2} + \mathrm{i}t) = O_\varepsilon(t^\varepsilon) \tag{3}$$

as $t \to +\infty$.

The hypothesis can also be stated in terms of function.

$$\xi_3(s) = g_3(s) L_3(s), \tag{4}$$

where

$$g_3(s) = \left(\tfrac{\pi}{3}\right)^{-\frac{s+1}{2}} \Gamma\left(\tfrac{s+1}{2}\right). \tag{5}$$

Namely, for a positive a

$$|\Gamma(a + \mathrm{i}\,t)| = (1 + o_a(t))\sqrt{2\pi} t^{a-\frac{1}{2}} \mathrm{e}^{-\frac{\pi t}{2}} \quad \text{as } t \to +\infty. \tag{6}$$

Thus (3) is equivalent to

$$\xi_3(\tfrac{1}{2} + \mathrm{i}t) = O_\epsilon\Big(t^{\frac{1}{4}+\epsilon} \mathrm{e}^{-\frac{\pi t}{4}}\Big). \tag{7}$$

It is well known that the function $\xi_3(s)$ satisfies the *functional equation*

$$\xi_3(s) = \xi_3(1 - s). \tag{8}$$

This implies that

$$\xi_3(\tfrac{1}{2}+\mathrm{i}t) = \tfrac{1}{2}\xi_3(\tfrac{1}{2}+\mathrm{i}t) + \tfrac{1}{2}\xi_3(\tfrac{1}{2}-\mathrm{i}t). \tag{9}$$

Straightforward substitution of (1) and (5) into (9) gives the following:

$$\begin{aligned}\xi_3(\tfrac{1}{2}+\mathrm{i}t) = \frac{1}{2}\sum_{n=1}^{\infty}\chi_3(n)\left(\tfrac{\pi}{3}\right)^{-\frac{3}{4}-\frac{\mathrm{i}t}{2}}\Gamma\left(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}\right)n^{-\frac{1}{2}-\mathrm{i}t}+&\\ +\frac{1}{2}\sum_{n=1}^{\infty}\chi_3(n)\left(\tfrac{\pi}{3}\right)^{-\frac{3}{4}+\frac{\mathrm{i}t}{2}}\Gamma\left(\tfrac{3}{4}-\tfrac{\mathrm{i}t}{2}\right)n^{-\frac{1}{2}+\mathrm{i}t}.&\end{aligned} \tag{10}$$

It turns out (for details see Section 3) that we can drop the factors $\frac{1}{2}$ and replace the complete gamma function with the lower incomplete gamma function with a suitable second argument:

$$\begin{aligned}\xi_3(\tfrac{1}{2}+\mathrm{i}t) = \sum_{n=1}^{\infty}\chi_3(n)\left(\tfrac{\pi}{3}\right)^{-\frac{3}{4}-\frac{\mathrm{i}t}{2}}\gamma\left(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2},\tfrac{\pi n^2}{3}\right)n^{-\frac{1}{2}-\mathrm{i}t}+&\\ +\sum_{n=1}^{\infty}\chi_3(n)\left(\tfrac{\pi}{3}\right)^{-\frac{3}{4}+\frac{\mathrm{i}t}{2}}\gamma\left(\tfrac{3}{4}-\tfrac{\mathrm{i}t}{2},\tfrac{\pi n^2}{3}\right)n^{-\frac{1}{2}+\mathrm{i}t}.&\end{aligned} \tag{11}$$

The following expansion (due to E. E. Kummer) holds for the lower gamma function:

$$\gamma(w,m) = m^w\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(w+k)}m^k. \tag{12}$$

Assuming that

$$\frac{1}{w+k} = \sum_{l=1}^{\infty}(-1)^{l-1}\frac{k^{l-1}}{w^l}, \tag{13}$$

we have

$$\begin{aligned}\gamma(w,m) &= m^w\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+l-1}k^{l-1}}{k!w^l}m^k &(14)\\ &= m^w\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{w^l}\sum_{k=0}^{\infty}\frac{(-1)^k k^{l-1}}{k!}m^k &(15)\end{aligned}$$

(in (13)–(15) and in the sequel we assume that $0^0=1$). Further we have:

$$k^l = \sum_{j=0}^{l}\left\{\begin{matrix} l \\ j \end{matrix}\right\}k^{\underline{j}}, \tag{16}$$

where $\left\{ \begin{array}{c} l \\ j \end{array} \right\}$ is the *Stirling number of the second kind* and $k^{\underline{j}}$ is the *falling factorial power*,

$$k^{\underline{j}} = k(k-1)\dots(k-j+1) \tag{17}$$

(we use the notation from [2]).

Substituting (16) into (15), we get:

$$\begin{aligned}
\gamma(w,m) &= m^w \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{w^l} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} m^k \sum_{j=0}^{l-1} \left\{ \begin{array}{c} l-1 \\ j \end{array} \right\} k^{\underline{j}} && (18) \\
&= m^w \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{w^l} \sum_{j=0}^{l-1} \left\{ \begin{array}{c} l-1 \\ j \end{array} \right\} \sum_{k=0}^{\infty} (-1)^k \frac{k^{\underline{j}}}{k!} m^k && (19) \\
&= m^w \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{w^l} \sum_{j=0}^{l-1} (-1)^j \left\{ \begin{array}{c} l-1 \\ j \end{array} \right\} m^j \sum_{k=j}^{\infty} \frac{(-1)^{k-j}}{(k-j)!} m^{k-j} && (20) \\
&= m^w \mathrm{e}^{-m} \sum_{l=1}^{\infty} \frac{(-1)^{l-1} T_{l-1}(-m)}{w^l}, && (21)
\end{aligned}$$

where

$$T_n(x) = \sum_{j=0}^{n} \left\{ \begin{array}{c} n \\ j \end{array} \right\} x^j. \tag{22}$$

Polynomials (22) are known as *Touchard polynomials* (after [8]). They have been widely studied, but the author has not found the equality (21) in the literature. This equality can be viewed from different points:

- as the Taylor expansion of the lower gamma function at infinity;
- as a definition of Touchard polynomials with $m^{-w}\mathrm{e}^m\gamma(w,m)$ playing the role of a generating function.

Substituting $a + \mathrm{i}b$ for w in (21), we get that

$$\gamma(a+\mathrm{i}b, m) = m^{a+\mathrm{i}b} \mathrm{e}^{-m} \sum_{l=1}^{\infty} \frac{(-1)^{l-1} T_{l-1}(-m)}{(a+\mathrm{i}b)^l}. \tag{23}$$

Assuming that

$$\frac{1}{(a+\mathrm{i}b)^l} = (-\mathrm{i})^l \sum_{j=0}^{\infty} \binom{l+j-1}{j} \frac{(\mathrm{i}a)^j}{b^{l+j}}, \tag{24}$$

we get that

$$\gamma(a+\mathrm{i}b,m) = -m^{a+\mathrm{i}b}\mathrm{e}^{-m}\sum_{l=1}^{\infty}\mathrm{i}^l T_{l-1}(-m)\sum_{j=0}^{\infty}\binom{l+j-1}{j}\frac{(\mathrm{i}a)^j}{b^{l+j}} \tag{25}$$

$$= m^{a+\mathrm{i}b}\mathrm{e}^{-m}\sum_{k=1}^{\infty}\frac{U_k(a,m)}{b^k}, \tag{26}$$

where

$$U_k(a,m) = -\mathrm{i}^k\sum_{j=0}^{k-1}\binom{k-1}{j}a^j T_{k-j-1}(-m). \tag{27}$$

Let us set $a=\frac{3}{4}$ and $b=\pm\frac{t}{2}$ in (26). Substituting the resulting right-hand sides of (26) into (11), we get that

$$\xi_3(\tfrac{1}{2}+\mathrm{i}t) = \sum_{n=1}^{\infty}\chi_3(n)n\mathrm{e}^{-\frac{\pi n^2}{3}}\sum_{k=1}^{\infty}2^k U_k(\tfrac{3}{4},\tfrac{\pi n^2}{3})\left(\tfrac{1}{t^k}+\tfrac{1}{(-t)^k}\right) \tag{28}$$

$$= \sum_{n=1}^{\infty}\chi_3(n)n\mathrm{e}^{-\frac{\pi n^2}{3}}\sum_{k=1}^{\infty}U_{2k}(\tfrac{3}{4},\tfrac{\pi n^2}{3})\tfrac{2^{2k+1}}{t^{2k}} \tag{29}$$

$$= \sum_{k=1}^{\infty}\frac{2^{2k+1}}{t^{2k}}\sum_{n=1}^{\infty}\chi_3(n)nU_{2k}(\tfrac{3}{4},\tfrac{\pi n^2}{3})\mathrm{e}^{-\frac{\pi n^2}{3}}. \tag{30}$$

Our approach to the Lindelöf hypothesis is based on the following

Key discovery. *For every k*

$$\sum_{n=1}^{\infty}\chi_3(n)nU_{2k}(\tfrac{3}{4},\tfrac{\pi n^2}{3})\mathrm{e}^{-\frac{\pi n^2}{3}} = 0. \tag{31}$$

The equalities (31) are essentially equivalent to the functional equation (8) (see Section 4).

Corollary of (30) **and** (31)**.** *For every real t*

$$\xi_3(\tfrac{1}{2}+\mathrm{i}t) = 0. \tag{32}$$

The the desired equality (7) follows from the paradoxical identity (32). However, the latter is, of course, incorrect. Nevertheless, the above formal deduction of (32) suggests an approach to the Lindelöf hypothesis (see the next section).

2 Approaches to the Lindelöf hypothesis

Here we consider hypothetical ways in which considerations from the previous section could be transformed into rigorous estimates of $\left|\xi_3(\frac{1}{2}+\mathrm{i}t)\right|$.

Our first error was (13). The equality holds when $k < |w|$ only. Later, we did not justify the change of order of summation in (14)–(15), (18)–(19), (25)–(26), and (28)–(30). Nevertheless, (21) and (26) are asymptotic representations for the lower gamma function, that is,

$$\gamma(w,m) \quad = \quad m^w \mathrm{e}^{-m} \left(\sum_{l=1}^{L} \frac{(-1)^{l-1} T_{l-1}(-m)}{w^l} + O_{m,L}\left(\tfrac{1}{w^{L+1}}\right) \right) \tag{33}$$

and

$$\gamma(a+\mathrm{i}b,m) \quad = \quad m^{a+\mathrm{i}b} \mathrm{e}^{-m} \left(\sum_{k=1}^{K} \frac{U_k(a,m)}{b^k} + O_{a,m,K}\left(\tfrac{1}{b^{K+1}}\right) \right). \tag{34}$$

As a natural attempt to improve our "formula proof" we could introduce two bounds, T and K (depending on t, $N=N(t)$, $K=K(t)$), substitute them for the upper limits (equal to the infinity) in the summations in (11) and (28)–(30), and try to estimate the emerging errors.

Let (cf. (11))

$$\begin{aligned}\xi_{3,N}(\tfrac{1}{2}+\mathrm{i}t) = \sum_{n=1}^{N} \chi_3(n) \left(\tfrac{\pi}{3}\right)^{-\frac{3}{4}-\frac{\mathrm{i}t}{2}} \gamma\left(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}, \tfrac{\pi n^2}{3}\right) n^{-\frac{1}{2}-\mathrm{i}t} + \\ + \sum_{n=1}^{N} \chi_3(n) \left(\tfrac{\pi}{3}\right)^{-\frac{3}{4}+\frac{\mathrm{i}t}{2}} \gamma\left(\tfrac{3}{4}-\tfrac{\mathrm{i}t}{2}, \tfrac{\pi n^2}{3}\right) n^{-\frac{1}{2}+\mathrm{i}t}\end{aligned} \tag{35}$$

and (cf. (29)–(30))

$$\xi_{3,N,K}(\tfrac{1}{2}+\mathrm{i}t) \quad = \quad \sum_{n=1}^{N} \chi_3(n) n \mathrm{e}^{-\frac{\pi n^2}{3}} \sum_{k=1}^{K} U_{2k}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \tfrac{2^{2k+1}}{t^{2k}} \tag{36}$$

$$\quad = \quad \sum_{k=1}^{K} \tfrac{2^{2k+1}}{t^{2k}} \sum_{n=1}^{N} \chi_3(n) n U_{2k}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \mathrm{e}^{-\frac{\pi n^2}{3}}. \tag{37}$$

We have:

$$\left|\xi_3(\tfrac{1}{2}+\mathrm{i}t)\right| \leq$$

$$\left|\xi_3(\tfrac{1}{2}+\mathrm{i}t) - \xi_{3,N}(\tfrac{1}{2}+\mathrm{i}t)\right| + \tag{38}$$

$$+ \left|\xi_{3,N}(\tfrac{1}{2}+\mathrm{i}t) - \xi_{3,N,K}(\tfrac{1}{2}+\mathrm{i}t)\right| + \tag{39}$$

$$+ \left|\xi_{3,N,K}(\tfrac{1}{2}+\mathrm{i}t)\right|. \tag{40}$$

Thanks to the convergence of the series in (11), the value of (38) can be made arbitrarily small. However, this requires the selection of a sufficiently large N.

Thanks to (31), each inner sum in (37) tends to zero as $N \to \infty$. Hence for a fixed K the value of (40) can be made arbitrarily small, again at the cost of selecting sufficiently large N.

The choice of K is trickier. We have:

$$\xi_{3,N,K}(\tfrac{1}{2}+\mathrm{i}t) = \sum_{n=1}^{N} \chi_3(n) n \mathrm{e}^{-\frac{\pi n^2}{3}} \sum_{k=1}^{2K} U_k(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \tfrac{2^{2k}}{t^k} + \\ + \sum_{n=1}^{N} \chi_3(n) n \mathrm{e}^{-\frac{\pi n^2}{3}} \sum_{k=1}^{2K} U_k(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \tfrac{2^{2k}}{(-t)^k}, \quad (41)$$

thus

$$\xi_{3,N}\left(\tfrac{1}{2}+\mathrm{i}t\right) - \xi_{3,N,K}\left(\tfrac{1}{2}+\mathrm{i}t\right) = \sum_{n=1}^{N} \chi_3(n) \times \\ \times \Bigg(\pi^{-\left(\frac{3}{4}+\frac{\mathrm{i}t}{2}\right)} \gamma\left(\frac{3}{4}+\frac{\mathrm{i}t}{2}, \frac{\pi n^2}{3}\right) n^{-\left(\frac{1}{2}+\mathrm{i}t\right)} - \mathrm{e}^{-\frac{\pi n^2}{3}} \textstyle\sum_{k=1}^{2K} \frac{2^k n U_k\left(\frac{3}{4}, \frac{\pi n^2}{3}\right)}{t^k} + \\ + \pi^{-\left(\frac{3}{4}-\frac{\mathrm{i}t}{2}\right)} \gamma\left(\frac{3}{4}-\frac{\mathrm{i}t}{2}, \frac{\pi n^2}{3}\right) n^{-\left(\frac{1}{2}-\mathrm{i}t\right)} - \mathrm{e}^{-\frac{\pi n^2}{3}} \textstyle\sum_{k=1}^{2K} \frac{2^k n U_k\left(\frac{3}{4}, \frac{\pi n^2}{3}\right)}{(-t)^k}\Bigg). \quad (42)$$

According to (34) (with $a = \frac{3}{4}$, $b = \pm\frac{t}{2}$, and $m = \frac{\pi n^2}{3}$)

$$\pi^{-\left(\frac{3}{4}\pm\frac{\mathrm{i}t}{2}\right)} \gamma\left(\frac{3}{4} \pm \frac{\mathrm{i}t}{2}, \frac{\pi n^2}{3}\right) n^{-\left(\frac{1}{2}\pm\mathrm{i}t\right)} - \mathrm{e}^{-\frac{\pi n^2}{3}} \sum_{k=1}^{K} \frac{2^k n U_k\left(\frac{3}{4}, \frac{\pi n^2}{3}\right)}{(\pm t)^k} = O_{n,K}\left(\frac{1}{t^{K+1}}\right). \quad (43)$$

Using the explicit bounds (hidden in $O_{n,K}$ notation in (43)), we can estimate (39). However, it is unclear whether this (together with the estimations of (38) and (40)) would result in the desired estimate (7).

The following approach looks more promising.

There are numerous known methods for accelerating slowly convergent series or even obtaining 'correct' value of a divergent series. Many of these methods are linear: an infinite sum

$$\sum_{n=1}^{\infty} a_n \quad (44)$$

is very well approximated for relatively small N by a finite sum

$$\sum_{n=1}^{N} \nu_n(N) a_n \quad (45)$$

for suitable weights $\nu_n(N)$. We can select two sequences of weights, $\nu_n(N)$ and $\kappa_k(K)$, and redefine (35), (36), and (37) as weighted finite sums:

$$\xi_{3,N}(\tfrac{1}{2}+\mathrm{i}t) = \sum_{n=1}^{N} \nu_n(N)\chi_3(n)\left(\tfrac{\pi}{3}\right)^{-\frac{3}{4}-\frac{\mathrm{i}t}{2}}\gamma\left(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}, \tfrac{\pi n^2}{3}\right) n^{-\frac{1}{2}-\mathrm{i}t} + \\ + \sum_{n=1}^{N} \nu_n(N)\chi_3(n)\left(\tfrac{\pi}{3}\right)^{-\frac{3}{4}+\frac{\mathrm{i}t}{2}}\gamma\left(\tfrac{3}{4}-\tfrac{\mathrm{i}t}{2}, \tfrac{\pi n^2}{3}\right) n^{-\frac{1}{2}+\mathrm{i}t}, \quad (46)$$

$$\begin{aligned} \xi_{3,N,K}(\tfrac{1}{2}+\mathrm{i}t) &= \sum_{n=1}^{N} \nu_n(N)\chi_3(n) n \mathrm{e}^{-\frac{\pi n^2}{3}} \sum_{k=1}^{K} \kappa_k(K) U_{2k}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \tfrac{2^{2k+1}}{t^{2k}} &(47)\\ &= \sum_{k=1}^{K} \kappa_k(K) \tfrac{2^{2k+1}}{t^{2k}} \sum_{n=1}^{N} \nu_n(N)\chi_3(n) n U_{2k}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \mathrm{e}^{-\frac{\pi n^2}{3}}. &(48)\end{aligned}$$

The weights $\kappa_k(K)$ should provide a good approximation of the lower gamma function via a weighted finite sum (cf. (43)):

$$\pi^{-\left(\frac{3}{4}\pm\frac{\mathrm{i}t}{2}\right)}\gamma\left(\tfrac{3}{4}\pm\tfrac{\mathrm{i}t}{2}, \tfrac{\pi n^2}{3}\right) n^{-\left(\frac{1}{2}\pm\mathrm{i}t\right)} \approx \mathrm{e}^{-\frac{\pi n^2}{3}} \sum_{k=1}^{K} \kappa_k(K) \frac{2^k n U_k\left(\frac{3}{4}, \frac{\pi n^2}{3}\right)}{(\pm t)^k}. \quad (49)$$

The weights $\nu_n(N)$ should satisfy two conditions. Firstly, the finite sums from (46) should provide a good approximation to $\xi_3(\frac{1}{2}+\mathrm{i}t)$ (cf. (11)):

$$\xi_3(\tfrac{1}{2}+\mathrm{i}t) \approx \xi_{3,N}(\tfrac{1}{2}+\mathrm{i}t). \quad (50)$$

Secondly, the inner sums in (48) should be small (cf. (31)):

$$\sum_{n=1}^{N} \nu_n(N)\chi_3(n) n U_{2k}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \mathrm{e}^{-\frac{\pi n^2}{3}} \approx 0. \quad (51)$$

One can hope that for a suitable choice of the weights it will be possible to use values of N and K that are not too large, and deduce (7) from (38)–(40) (after the redefinitions (46)–(48)).

There are efficient non-linear methods for accelerating convergence. However, linearity is essential for us: we need both representations (47) and (48).

3 Representations of L-functions via incomplete gamma functions

A predecessor of (11) can already be found in the seminal paper of B. Riemann [6]. Naturally, it provides an expression for the zeta function. For Dirichlet L-functions, A. F. Lavrik obtained the following general representation.

Let $\chi(n)$ be a primitive character modulo some q, $q > 1$,

$$\xi_\chi(s) = g_\chi(s) L_\chi(s), \tag{52}$$

where

$$L_\chi(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \tag{53}$$

$$g_\chi(s) = \left(\frac{\pi}{q}\right)^{-\frac{s+\delta}{2}} \Gamma\left(\frac{s+\delta}{2}\right), \tag{54}$$

$$\delta = \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1. \end{cases} \tag{55}$$

The function $\xi_\chi(s)$ satisfies the functional equation

$$\xi_\chi(s) = \omega \xi_{\bar{\chi}}(1-s), \tag{56}$$

where

$$\bar{\chi}(n) = \overline{\chi(n)} \tag{57}$$

and

$$\omega = \frac{\sum_{k=1}^{q} \chi(k) \mathrm{e}^{\frac{2\pi \mathrm{i} k}{q}}}{\mathrm{i}^\delta \sqrt{q}}. \tag{58}$$

A. F. Lavrik proved in [9, Th. 1] that for all s

$$\begin{aligned} \xi_\chi(s) = \sum_{n=1}^{\infty} \chi(n) \left(\tfrac{\pi}{q}\right)^{-\frac{s+\delta}{2}} \Gamma\left(\tfrac{s+\delta}{2}, \tfrac{\pi n^2 \tau}{q}\right) n^{-s} + \\ + \omega \sum_{n=1}^{\infty} \bar{\chi}(n) \left(\tfrac{\pi}{q}\right)^{-\frac{1-s+\delta}{2}} \Gamma\left(\tfrac{1-s+\delta}{2}, \tfrac{\pi n^2}{q\tau}\right) n^{-(1-s)}, \end{aligned} \tag{59}$$

where τ is an arbitrary complex number with positive real part.

From (10) and the well-known identity

$$\Gamma(a) = \gamma(a, z) + \Gamma(a, z) \tag{60}$$

we obtain a dual representation in which the upper gamma function is replaced by the lower gamma function:

$$\xi_\chi(s) = \sum_{n=1}^{\infty} \chi(n) \left(\tfrac{\pi}{q}\right)^{-\frac{s+\delta}{2}} \gamma\left(\tfrac{s+\delta}{2}, \tfrac{\pi n^2 \tau}{q}\right) n^{-s} + \\ + \omega \sum_{n=1}^{\infty} \bar{\chi}(n) \left(\tfrac{\pi}{q}\right)^{-\frac{1-s+\delta}{2}} \gamma\left(\tfrac{1-s+\delta}{2}, \tfrac{\pi n^2}{q\tau}\right) n^{-(1-s)}. \quad (61)$$

Representation (11) is a special case of (61) with $q = 3$, $\chi = \chi_3$ (defined by (2)), $\delta = 1$, $\bar{\chi} = \chi_3$, $\omega = 1$ and $\tau = 1$.

From a computational point of view representation (59) is more efficient than (61). This is because the upper gamma decays quickly as the imaginary part of the second argument tends to infinity; in this case, the lower gamma function tends to a non-zero limit. However, for our purposes the case of the lower gamma function is more interesting because the polynomials $U_k(a, m)$ from representation (26) satisfy the identity (31).

The representation (59) was generalized in a number of ways. A. F. Lavrik [12] and M. Alzergani [1] gave (different) representations which involve a finite number of factors from the Euler product. For another generalisation see, for example, the survey by M. O. Rubinstein [7].

4 Corollaries of the functional equation for ξ_3

The functional equation (8) can be proved in many ways. A standard technique (see, for example, [10, Ch. 1, Sect. 4, Th. 1]) is based on the properties of Dirichlet θ-functions. Let us define $\theta_3(\tau)$ for $\mathrm{Re}(\tau) > 0$ as follows:

$$\theta_3(\tau) = \sum_{n=-\infty}^{\infty} \chi_3(n) n \mathrm{e}^{-\frac{\pi n^2}{3}\tau}. \quad (62)$$

This function satisfies ([10, Ch. 1, Sect. 4, Lemma 2]) the functional equation

$$\theta_3(\tau^{-1}) = \tau^{\frac{3}{2}} \theta_3(\tau). \quad (63)$$

The functional equation (8) can be deduced from (63). A. F. Lavrik shows in [11] that, *vice versa*, (63) can be deduced from (8).

We reformulate (63) in the following way. This *functional* equality (an identity in τ) implies *numerical* equalities

$$\frac{\mathrm{d}^m}{\mathrm{d}\tau^m} \left(\theta_3(\tau^{-1}) - \tau^{\frac{3}{2}} \theta_3(\tau)\right)\bigg|_{\tau=1} = 0, \qquad m = 1, 2, \dots . \quad (64)$$

In their turn, the numerical equalities (64), taken together, imply the functional equality (63).

The left-hand side in (64) can be written as

$$\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\left(\theta_3(\tau^{-1}) - \tau^{\frac{3}{2}}\theta_3(\tau)\right)\Big|_{\tau=1} = \sum_{n=1}^{\infty}\chi_3(n)G_m(n)\mathrm{e}^{-\frac{\pi n^2}{3}} \tag{65}$$

for certain polynomials $G_m(n)$. Some of these are presented in Table 2. Explicit representations for $G_m(n)$ are given by (81), (76) and (78).

In general, the polynomial $G_m(n)$ contains only odd powers of n. The degree of $G_m(n)$ is equal to $2m+1$ for an odd m and $2m-1$ for an even m.

With the new notation, the equalities (64) can be written as

$$\sum_{n=1}^{\infty}\chi_3(n)G_m(n)\mathrm{e}^{-\frac{\pi n^2}{3}} = 0. \tag{66}$$

These equalities are not independent. For example, $G_2(n) = 2G_1(n)$.

Let $\mathcal{G}_3(m)$ be the linear span of polynomials $G_1(n), \dots, G_m(n)$. For an even m the dimension of the space $\mathcal{G}_3(m)$ is equal to $m/2$ and

$$\mathcal{G}_3(m) = \mathcal{G}_3(m-1). \tag{67}$$

A basis of $\mathcal{G}_3(m)$ can be selected in many ways. For example, for an even m polynomials

$$G_1(n), G_3, \dots, G_{m-1}(n) \tag{68}$$

and polynomials

$$G_2(n), G_4, \dots, G_m(n) \tag{69}$$

span (67).

Clearly, if $P(n) \in \mathcal{G}_3(m)$, then

$$\sum_{n=1}^{\infty}\chi_3(n)P(n)\mathrm{e}^{-\frac{\pi n^2}{3}} = 0. \tag{70}$$

Identity (31) is a corollary of

Key discovery (stronger form). *For every k*

$$nU_{2k}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \in \mathcal{G}_3(2k). \tag{71}$$

Tables 3 and 4 present a few of the corresponding linear relations for bases (68) and (69) respectively.

The relations in Tables 3 and 4 can be inverted; Table 5 presents a few of the inverse relations.

In the general case, the polynomials

$$nU_2(\tfrac{3}{4},\tfrac{\pi n^2}{3}),\ nU_4(\tfrac{3}{4},\tfrac{\pi n^2}{3}),\ \ldots,\ nU_{2k}(\tfrac{3}{4},\tfrac{\pi n^2}{3}) \tag{72}$$

span $\mathcal{G}_3(2k)$. Thus the equalities (31) imply the equalities (66); the latter are identical to (64) and hence they imply the functional equations (63) and (8).

5 Corollaries of the functional equation in the general case

As was mentioned in the previous section, the functional identity (8) is equivalent to (66). This observation can be generalised to the case of an arbitrary Dirichlet primitive character χ modulo some q, where $q > 1$.

In Section 4 we deal with the character χ_3, which is defined by (2). It has two special features:

- χ_3 is a real character, that is, $\chi_3 = \bar{\chi}_3$;
- χ_3 is an odd character, that is, $\chi_3(-1) = -1$.

In the general case counterparts to (63)–(66) involve both χ and $\bar{\chi}$, as well as δ, the indicator of the parity of χ (defined by (55)).

More precisely, the functional equation takes the following form (cf. (63)):

$$\theta_\chi(\tau^{-1}) = \omega\tau^{\delta+\frac{1}{2}}\theta_{\bar{\chi}}(\tau), \tag{73}$$

where (cf.(62))

$$\theta_\chi(\tau) = \sum_{n=-\infty}^{\infty} \chi(n) n^\delta \mathrm{e}^{-\frac{\pi n^2}{q}\tau} \tag{74}$$

and ω is defined by (58) ([10, Ch. 1, Sect. 4, Lemma 2]).

Instead of a single series of polynomials, $G_m(n)$, we now introduce two series:

$$E_m(l) = \mathrm{e}^l \frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\mathrm{e}^{-\frac{l}{\tau}}\bigg|_{\tau=1} \tag{75}$$

$$= \sum_{k=0}^{m}(-1)^{m+k} m!k!\binom{m-1}{k-1} l^k, \tag{76}$$

$$F_{d,m}(l) = \mathrm{e}^{l}\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\tau^{d+\frac{1}{2}}\mathrm{e}^{-l\tau}\Big|_{\tau=1} \tag{77}$$

$$= \sum_{k=0}^{m}(-1)^k\binom{m}{k}\,\underline{(d+\tfrac{1}{2})^{m-k}}\,l^k, \tag{78}$$

where $d=0$ or $d=1$. The functional equation (73) is equivalent (cf. (64)) to the infinite set of numerical equalities

$$\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\left(\theta_\chi(\tau^{-1}) - \omega\tau^{\delta+\frac{1}{2}}\theta_{\bar\chi}(\tau)\right)\Big|_{\tau=1} = 0, \qquad m=0,1,\dots\ . \tag{79}$$

In the notation (75)–(77), the equalities (79) can be written as

$$\sum_{n=1}^{\infty}\left(\chi(n)n^\delta E_m\left(\tfrac{\pi n^2}{q}\right) - \omega\bar\chi(n)n^\delta F_{\delta,m}\left(\tfrac{\pi n^2}{q}\right)\right)\mathrm{e}^{-\frac{\pi n^2}{q}} = 0. \tag{80}$$

The equality (66) is a special case of (80): if $\chi=\chi_3$, then $q=3$, $\bar\chi=\chi$, $\delta=1$, $\omega=1$, and

$$G_m(n) = nE_m\left(\tfrac{\pi n^2}{3}\right) - nF_{1,m}\left(\tfrac{\pi n^2}{3}\right). \tag{81}$$

Let $\mathcal{F}_d(m)$ be the linear span of pairs of polynomials

$$\langle E_0(l),F_{d,0}(l)\rangle, \langle E_1(l),F_{d,1}(l)\rangle, \dots, \langle E_m(l),F_{d,m}(l)\rangle. \tag{82}$$

The dimension of $\mathcal{F}_d(m)$ is equal to $m+1$.
Clearly, if $\langle P(l),Q(l)\rangle\in\mathcal{F}_\delta(m)$ then (cf. (70))

$$\sum_{n=1}^{\infty}\left(\chi(n)n^\delta P\left(\tfrac{\pi n^2}{q}\right) - \omega\bar\chi(n)n^\delta Q\left(\tfrac{\pi n^2}{q}\right)\right)\mathrm{e}^{-\frac{\pi n^2}{q}} = 0. \tag{83}$$

Starting from the general representation (61), we can derive (by analogy with (30)) that

$$\xi_\chi(\tfrac{1}{2}+\mathrm{i}t) = \sum_{k=1}^{\infty}\frac{2^k}{t^k}\sum_{n=1}^{\infty}\Big(\chi(n)n^\delta U_k(\tfrac{\delta}{2}+\tfrac{1}{4},\tfrac{\pi n^2}{q}) + $$
$$+(-1)^k\omega\bar\chi(n)n^\delta U_k(\tfrac{\delta}{2}+\tfrac{1}{4},\tfrac{\pi n^2}{q})\Big)\,\mathrm{e}^{-\frac{\pi n^2}{q}}. \tag{84}$$

Key discovery (general case). *For $d=0$ and for $d=1$ for every k*

$$\langle U_k(\tfrac{d}{2}+\tfrac{1}{4},l),\ -(-1)^kU_k(\tfrac{d}{2}+\tfrac{1}{4},l)\rangle\in\mathcal{F}_d(k). \tag{85}$$

From (84) and (85) we obtain a generalisation of the (incorrect) equality (32): *for every primitive Dirichlet character χ modulo q ($q > 1$), for every real t*

$$\xi_\chi(\tfrac{1}{2} + \mathrm{i}t) = 0. \tag{86}$$

Thus, the speculations from Section 2 can extended to an arbitrary χ as well.

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$$
\begin{aligned}
U_2\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= -\tfrac{\pi}{3}n^2 + \tfrac{3}{4} \\
U_4\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= \tfrac{\pi^3}{27}n^6 - \tfrac{7\pi^2}{12}n^4 + \tfrac{79\pi}{48}n^2 - \tfrac{27}{64} \\
U_6\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= -\tfrac{\pi^5}{243}n^{10} + \tfrac{55\pi^4}{324}n^8 - \tfrac{425\pi^3}{216}n^6 + \tfrac{665\pi^2}{96}n^4 - \tfrac{4141\pi}{768}n^2 + \tfrac{243}{1024} \\
U_8\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= \tfrac{\pi^7}{2187}n^{14} - \tfrac{35\pi^6}{972}n^{12} + \tfrac{3689\pi^5}{3888}n^{10} - \tfrac{52745\pi^4}{5184}n^8 + \tfrac{299131\pi^3}{6912}n^6 - \\
&\quad -\tfrac{185857\pi^2}{3072}n^4 + \tfrac{205339\pi}{12288}n^2 - \tfrac{2187}{16384} \\
U_{10}\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= -\tfrac{\pi^9}{19683}n^{18} + \tfrac{19\pi^8}{2916}n^{16} - \tfrac{895\pi^7}{2916}n^{14} + \tfrac{2905\pi^6}{432}n^{12} - \tfrac{744317\pi^5}{10368}n^{10} + \\
&\quad +\tfrac{4961495\pi^4}{13824}n^8 - \tfrac{20501665\pi^3}{27648}n^6 + \tfrac{5930365\pi^2}{12288}n^4 - \tfrac{10083481\pi}{196608}n^2 + \tfrac{19683}{262144} \\
U_{12}\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= \tfrac{\pi^{11}}{177147}n^{22} - \tfrac{253\pi^{10}}{236196}n^{20} + \tfrac{8305\pi^9}{104976}n^{18} - \tfrac{136895\pi^8}{46656}n^{16} + \tfrac{5497327\pi^7}{93312}n^{14} - \\
&\quad -\tfrac{26765585\pi^6}{41472}n^{12} + \tfrac{1847382779\pi^5}{497664}n^{10} - \tfrac{6802994495\pi^4}{663552}n^8 + \\
&\quad +\tfrac{20311855861\pi^3}{1769472}n^6 - \tfrac{2930804107\pi^2}{786432}n^4 + \tfrac{494287399\pi}{3145728}n^2 - \tfrac{177147}{4194304} \\
U_{14}\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= -\tfrac{\pi^{13}}{1594323}n^{26} + \tfrac{13\pi^{12}}{78732}n^{24} - \tfrac{24947\pi^{11}}{1417176}n^{22} + \tfrac{1871441\pi^{10}}{1889568}n^{20} - \tfrac{54017249\pi^9}{1679616}n^{18} + \\
&\quad +\tfrac{463109933\pi^8}{746496}n^{16} - \tfrac{15833905945\pi^7}{2239488}n^{14} + \tfrac{45796856105\pi^6}{995328}n^{12} - \\
&\quad -\tfrac{2546294373623\pi^5}{15925248}n^{10} + \tfrac{5611157536985\pi^4}{21233664}n^8 - \tfrac{2399632288235\pi^3}{14155776}n^6 + \\
&\quad +\tfrac{178796540195\pi^2}{6291456}n^4 - \tfrac{24221854021\pi}{50331648}n^2 + \tfrac{1594323}{67108864} \\
U_{16}\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= \tfrac{\pi^{15}}{14348907}n^{30} - \tfrac{155\pi^{14}}{6377292}n^{28} + \tfrac{90125\pi^{13}}{25509168}n^{26} - \tfrac{117845\pi^{12}}{419904}n^{24} + \\
&\quad +\tfrac{609126973\pi^{11}}{45349632}n^{22} - \tfrac{24376821469\pi^{10}}{60466176}n^{20} + \tfrac{618247546345\pi^9}{80621568}n^{18} - \\
&\quad -\tfrac{364277794595\pi^8}{3981312}n^{16} + \tfrac{31762146891761\pi^7}{47775744}n^{14} - \tfrac{59284085936075\pi^6}{21233664}n^{12} + \\
&\quad +\tfrac{529152534925469\pi^5}{84934656}n^{10} - \tfrac{729958548564245\pi^4}{113246208}n^8 + \tfrac{1107751011830191\pi^3}{452984832}n^6 - \\
&\quad -\tfrac{43414094382757\pi^2}{201326592}n^4 + \tfrac{1186886790259\pi}{805306368}n^2 - \tfrac{14348907}{1073741824} \\
U_{18}\left(\tfrac{3}{4}, \tfrac{\pi n^2}{3}\right) &= -\tfrac{\pi^{17}}{129140163}n^{34} + \tfrac{595\pi^{16}}{172186884}n^{32} - \tfrac{18853\pi^{15}}{28697814}n^{30} + \tfrac{297755\pi^{14}}{4251528}n^{28} - \\
&\quad -\tfrac{157797689\pi^{13}}{34012224}n^{26} + \tfrac{1008323771\pi^{12}}{5038848}n^{24} - \tfrac{519715370239\pi^{11}}{90699264}n^{22} + \\
&\quad +\tfrac{13200567768817\pi^{10}}{120932352}n^{20} - \tfrac{293911332044473\pi^9}{214990848}n^{18} + \tfrac{1048386106345721\pi^8}{95551488}n^{16} - \\
&\quad -\tfrac{5171000831348105\pi^7}{95551488}n^{14} + \tfrac{6535078384876405\pi^6}{42467328}n^{12} - \\
&\quad -\tfrac{233211900264411091\pi^5}{1019215872}n^{10} + \tfrac{207084721195113025\pi^4}{1358954496}n^8 - \\
&\quad -\tfrac{315789329888804785\pi^3}{905969664}n^6 + \tfrac{657528343229765\pi^2}{402653184}n^4 - \tfrac{58157596211761\pi}{12884901888}n^2 + \\
&\quad +\tfrac{129140163}{17179869184}
\end{aligned}
$$

Table 1: Polynomials $U_k(a, m)$ defined by (27).

$$
\begin{aligned}
G_1(n) &= \tfrac{2\pi}{3}n^3 - \tfrac{3}{2}n \\
G_2(n) &= \tfrac{\pi}{3}n^3 - \tfrac{3}{4}n \\
G_3(n) &= \tfrac{2\pi^3}{27}n^7 - \tfrac{7\pi^2}{6}n^5 + \tfrac{11\pi}{4}n^3 + \tfrac{3}{8}n \\
G_4(n) &= -\tfrac{2\pi^3}{9}n^7 + \tfrac{7\pi^2}{2}n^5 - \tfrac{17\pi}{2}n^3 - \tfrac{9}{16}n \\
G_5(n) &= \tfrac{2\pi^5}{243}n^{11} - \tfrac{55\pi^4}{162}n^9 + \tfrac{85\pi^3}{18}n^7 - \tfrac{105\pi^2}{4}n^5 + \tfrac{655\pi}{16}n^3 + \tfrac{45}{32}n \\
G_6(n) &= -\tfrac{7\pi^5}{81}n^{11} + \tfrac{385\pi^4}{108}n^9 - \tfrac{805\pi^3}{18}n^7 + \tfrac{3185\pi^2}{16}n^5 - \tfrac{3885\pi}{16}n^3 - \tfrac{315}{64}n \\
G_7(n) &= \tfrac{2\pi^7}{2187}n^{15} - \tfrac{35\pi^6}{486}n^{13} + \tfrac{287\pi^5}{108}n^{11} - \tfrac{11165\pi^4}{216}n^9 + \tfrac{22435\pi^3}{48}n^7 - \tfrac{53655\pi^2}{32}n^5 + \\
&\quad + \tfrac{108255\pi}{64}n^3 + \tfrac{2835}{128}n \\
G_8(n) &= -\tfrac{44\pi^7}{2187}n^{15} + \tfrac{385\pi^6}{243}n^{13} - \tfrac{1309\pi^5}{27}n^{11} + \tfrac{156695\pi^4}{216}n^9 - \tfrac{62755\pi^3}{12}n^7 + \tfrac{250635\pi^2}{16}n^5 - \\
&\quad - \tfrac{215985\pi}{16}n^3 - \tfrac{31185}{256}n \\
G_9(n) &= \tfrac{2\pi^9}{19683}n^{19} - \tfrac{19\pi^8}{1458}n^{17} + \tfrac{227\pi^7}{243}n^{15} - \tfrac{6265\pi^6}{162}n^{13} + \tfrac{62741\pi^5}{72}n^{11} - \tfrac{501655\pi^4}{48}n^9 + \\
&\quad + \tfrac{1003765\pi^3}{16}n^7 - \tfrac{5158125\pi^2}{32}n^5 + \tfrac{31059315\pi}{256}n^3 + \tfrac{405405}{512}n \\
G_{10}(n) &= -\tfrac{25\pi^9}{6561}n^{19} + \tfrac{475\pi^8}{972}n^{17} - \tfrac{6725\pi^7}{243}n^{15} + \tfrac{188125\pi^6}{216}n^{13} - \tfrac{376355\pi^5}{24}n^{11} + \\
&\quad + \tfrac{15051575\pi^4}{96}n^9 - \tfrac{12903975\pi^3}{16}n^7 + \tfrac{464330475\pi^2}{256}n^5 - \tfrac{310333275\pi}{256}n^3 - \tfrac{6081075}{1024}n \\
G_{11}(n) &= \tfrac{2\pi^{11}}{177147}n^{23} - \tfrac{253\pi^{10}}{118098}n^{21} + \tfrac{6655\pi^9}{26244}n^{19} - \tfrac{105545\pi^8}{5832}n^{17} + \tfrac{164285\pi^7}{216}n^{15} - \\
&\quad - \tfrac{8278655\pi^6}{432}n^{13} + \tfrac{82793095\pi^5}{288}n^{11} - \tfrac{157690225\pi^4}{64}n^9 + \tfrac{2838718575\pi^3}{256}n^7 - \\
&\quad - \tfrac{11351634525\pi^2}{512}n^5 + \tfrac{13647231675\pi}{1024}n^3 + \tfrac{103378275}{2048}n \\
G_{12}(n) &= -\tfrac{38\pi^{11}}{59049}n^{23} + \tfrac{4807\pi^{10}}{39366}n^{21} - \tfrac{145255\pi^9}{13122}n^{19} + \tfrac{258115\pi^8}{432}n^{17} - \tfrac{2168375\pi^7}{108}n^{15} + \\
&\quad + \tfrac{182136185\pi^6}{432}n^{13} - \tfrac{86733955\pi^5}{16}n^{11} + \tfrac{10407745425\pi^4}{256}n^9 - \tfrac{20816697825\pi^3}{128}n^7 + \\
&\quad + \tfrac{149851981125\pi^2}{512}n^5 - \tfrac{81852984675\pi}{512}n^3 - \tfrac{1964187225}{4096}n \\
G_{13}(n) &= \tfrac{2\pi^{13}}{1594323}n^{27} - \tfrac{13\pi^{12}}{39366}n^{25} + \tfrac{767\pi^{11}}{13122}n^{23} - \tfrac{55913\pi^{10}}{8748}n^{21} + \tfrac{15101515\pi^9}{34992}n^{19} - \\
&\quad - \tfrac{48321845\pi^8}{2592}n^{17} + \tfrac{676520845\pi^7}{1296}n^{15} - \tfrac{902016115\pi^6}{96}n^{13} + \tfrac{27060798765\pi^5}{256}n^{11} - \\
&\quad - \tfrac{360804869425\pi^4}{512}n^9 + \tfrac{1298942619975\pi^3}{512}n^7 - \tfrac{4250531560275\pi^2}{1024}n^5 + \\
&\quad + \tfrac{8510470543575\pi}{4096}n^3 + \tfrac{41247931725}{8192}n
\end{aligned}
$$

Table 2: Polynomials $G_m(n)$ defined by (65).

$$nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = -\tfrac{1}{2}G_1(n)$$

$$nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = \tfrac{13}{32}G_1(n) + \tfrac{1}{2}G_3(n)$$

$$nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = \tfrac{359}{512}G_1(n) + \tfrac{85}{16}G_3(n) - \tfrac{1}{2}G_5(n)$$

$$nU_8(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = \tfrac{468133}{8192}G_1(n) + \tfrac{190351}{512}G_3(n) - \tfrac{1477}{32}G_5(n) + \tfrac{1}{2}G_7(n)$$

$$\begin{aligned} nU_{10}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = {} & \tfrac{1881623759}{131072}G_1(n) + \tfrac{193261745}{2048}G_3(n) - \tfrac{3156027}{256}G_5(n) + \\ & + \tfrac{1401}{8}G_7(n) - \tfrac{1}{2}G_9(n) \end{aligned}$$

$$\begin{aligned} nU_{12}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = {} & \tfrac{20907034805053}{2097152}G_1(n) + \tfrac{8596877726701}{131072}G_3(n) - \tfrac{35325623707}{4096}G_5(n) + \\ & + \tfrac{32737881}{256}G_7(n) - \tfrac{15015}{32}G_9(n) + \tfrac{1}{2}G_{11}(n) \end{aligned}$$

$$\begin{aligned} nU_{14}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = {} & \tfrac{519432419559176759}{33554432}G_1(n) + \tfrac{106803035312049895}{1048576}G_3(n) - \\ & - \tfrac{1756698214958687}{131072}G_5(n) + \tfrac{409249074377}{2048}G_7(n) - \tfrac{390116727}{512}G_9(n) + \\ & + \tfrac{16471}{16}G_{11}(n) - \tfrac{1}{2}G_{13}(n) \end{aligned}$$

$$\begin{aligned} nU_{16}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = {} & \tfrac{25147259164376391074773}{536870912}G_1(n) + \tfrac{1034138623884167524905 1}{33554432}G_3(n) - \\ & - \tfrac{850537881121760578 11}{2097152}G_5(n) + \tfrac{79307042206189083}{131072}G_7(n) - \\ & - \tfrac{18993638637545}{8192}G_9(n) + \tfrac{1662115273}{512}G_{11}(n) - \tfrac{63385}{32}G_{13}(n) + \tfrac{1}{2}G_{15}(n) \end{aligned}$$

$$\begin{aligned} nU_{18}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = {} & \tfrac{2153279642725416095838675359}{8589934592}G_1(n) + \tfrac{110687584346643190774486825}{67108864}G_3(n) - \\ & - \tfrac{1820736210067848438753775}{8388608}G_5(n) + \tfrac{848913796088552235639}{262144}G_7(n) - \\ & - \tfrac{813701365343805939}{65536}G_9(n) + \tfrac{17883551790767}{1024}G_{11}(n) - \\ & - \tfrac{1410882207}{128}G_{13}(n) + \tfrac{13889}{4}G_{15}(n) - \tfrac{1}{2}G_{17}(n) \end{aligned}$$

$$\begin{aligned} nU_{20}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) = {} & \tfrac{3034431637320576254277629526052 93}{137438953472}G_1(n) + \\ & + \tfrac{12478599807824804216713445146140 1}{8589934592}G_3(n) - \\ & - \tfrac{2565809007285260555089603449 67}{134217728}G_5(n) + \\ & + \tfrac{2392620854491399016865170 41}{8388608}G_7(n) - \tfrac{114675835780201454785845}{1048576}G_9(n) + \\ & + \tfrac{1008673281402153394 3}{65536}G_{11}(n) - \tfrac{199809728486035}{2048}G_{13}(n) + \\ & + \tfrac{4063696269}{128}G_{15}(n) - \tfrac{181659}{32}G_{17}(n) + \tfrac{1}{2}G_{19}(n) \end{aligned}$$

Table 3: Polynomials $nU_m\left(\frac{3}{4}, \frac{\pi n^2}{3}\right)$ via basis (68).

$$
\begin{aligned}
nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= -G_2(n) \\[6pt]
nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= \tfrac{11}{16}G_2(n) - \tfrac{1}{6}G_4(n) \\[6pt]
nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= -\tfrac{21}{256}G_2(n) - \tfrac{35}{48}G_4(n) + \tfrac{1}{21}G_6(n) \\[6pt]
nU_8(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= \tfrac{18271}{4096}G_2(n) - \tfrac{8337}{512}G_4(n) + \tfrac{85}{48}G_6(n) - \tfrac{1}{44}G_8(n) \\[6pt]
nU_{10}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= \tfrac{16180951}{65536}G_2(n) - \tfrac{6422435}{6144}G_4(n) + \tfrac{90943}{640}G_6(n) - \tfrac{543}{176}G_8(n) + \tfrac{1}{75}G_{10}(n) \\[6pt]
nU_{12}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= \tfrac{41997202611}{1048576}G_2(n) - \tfrac{68523014021}{393216}G_4(n) + \tfrac{1097088289}{43008}G_6(n) - \\
&\quad - \tfrac{342111}{512}G_8(n) + \tfrac{1133}{240}G_{10}(n) - \tfrac{1}{114}G_{12}(n) \\[6pt]
nU_{14}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= \tfrac{248669854673603}{16777216}G_2(n) - \tfrac{68052696874435}{1048576}G_4(n) + \tfrac{9532351392421}{983040}G_6(n) - \\
&\quad - \tfrac{1105778401}{4096}G_8(n) + \tfrac{43310267}{19200}G_{10}(n) - \tfrac{6097}{912}G_{12}(n) + \tfrac{1}{161}G_{14}(n) \\[6pt]
nU_{16}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= \tfrac{2909419346984333831}{268435456}G_2(n) - \tfrac{4783910393581167031}{100663296}G_4(n) + \\
&\quad + \tfrac{7483065768091849}{1048576}G_6(n) - \tfrac{52982924993193}{262144}G_8(n) + \tfrac{36494379493}{20480}G_{10}(n) - \\
&\quad - \tfrac{179981893}{29184}G_{12}(n) + \tfrac{3305}{368}G_{14}(n) - \tfrac{1}{216}G_{16}(n) \\[6pt]
nU_{18}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= \tfrac{6074473861721437644435}{4294967296}G_2(n) - \tfrac{1248901811595350624883 5}{201326592}G_4(n) + \\
&\quad + \tfrac{4107272868321959300807}{440401920}G_6(n) - \tfrac{15299189643156643 77}{5767168}G_8(n) + \\
&\quad + \tfrac{5835930414241499}{2457600}G_{10}(n) - \tfrac{503828721149}{58368}G_{12}(n) + \tfrac{21420901}{1472}G_{14}(n) - \\
&\quad - \tfrac{5015}{432}G_{16}(n) + \tfrac{1}{279}G_{18}(n) \\[6pt]
nU_{20}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) &= \tfrac{2099543857253492320406595931}{68719476736}G_2(n) - \tfrac{1151179945136215796266665347}{8589934592}G_4(n) + \\
&\quad + \tfrac{4057413215226496747266091}{201326592}G_6(n) - \tfrac{1059049402077805198050 61}{184549376}G_8(n) + \\
&\quad + \tfrac{40556259310967555519}{7864320}G_{10}(n) - \tfrac{3737235414418597}{196608}G_{12}(n) + \\
&\quad + \tfrac{790217331055}{23552}G_{14}(n) - \tfrac{709418143}{23040}G_{16}(n) + \tfrac{21679}{1488}G_{18}(n) - \tfrac{1}{350}G_{20}(n)
\end{aligned}
$$

Table 4: Polynomials $nU_m\left(\frac{3}{4}, \frac{\pi n^2}{3}\right)$ via basis (69).

$$G_1(n) = -2nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3})$$

$$G_2(n) = -nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3})$$

$$G_3(n) = -\tfrac{13}{8}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + 2nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3})$$

$$G_4(n) = -\tfrac{33}{8}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - 6nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3})$$

$$G_5(n) = -\tfrac{1851}{128}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \tfrac{85}{4}nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - 2nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3})$$

$$G_6(n) = -\tfrac{16611}{256}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{735}{8}nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + 21nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3})$$

$$G_7(n) = -\tfrac{727497}{2048}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \tfrac{60739}{128}nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{1477}{8}nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + 2nU_8(\tfrac{3}{4}, \tfrac{\pi n^2}{3})$$

$$\begin{aligned} G_8(n) = {} & -\tfrac{2351745}{1024}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{183029}{64}nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \tfrac{6545}{4}nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \\ & -44nU_8(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \end{aligned}$$

$$\begin{aligned} G_9(n) = {} & -\tfrac{561268215}{32768}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \tfrac{10094789}{512}nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{982527}{64}nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \\ & +\tfrac{1401}{2}nU_8(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - 2nU_{10}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \end{aligned}$$

$$\begin{aligned} G_{10}(n) = {} & -\tfrac{9488044485}{65536}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{156659325}{1024}nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \tfrac{19815705}{128}nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \\ & -\tfrac{40725}{4}nU_8(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + 75nU_{10}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \end{aligned}$$

$$\begin{aligned} G_{11}(n) = {} & -\tfrac{717105141765}{524288}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \tfrac{43195052901}{32768}nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{1724416375}{1024}nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \\ & +\tfrac{9334149}{64}nU_8(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{15015}{8}nU_{10}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + 2nU_{12}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \end{aligned}$$

$$\begin{aligned} G_{12}(n) = {} & -\tfrac{7488983814435}{524288}nU_2(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{408626277477}{32768}nU_4(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \\ & +\tfrac{20218928565}{1024}nU_6(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - \tfrac{136171233}{64}nU_8(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) + \\ & +\tfrac{322905}{8}nU_{10}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) - 114nU_{12}(\tfrac{3}{4}, \tfrac{\pi n^2}{3}) \end{aligned}$$

Table 5: Polynomials $G_m(n)$ via basis (72).