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Searching approximate polynomial dependencies among the derivatives of the alternating zeta function

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Abstract. It is well-known that the Riemann zeta function does not satisfy any *exact* polynomial differential equation. Here we present numerical evidence for the existence of *approximate* algebraic dependencies between the values of the alternating zeta function and its initial derivatives calculated at a single point or at several points in general position.

A number of conjectures is stated.

40 tables, 1 figure.

Key words: Riemann's zeta function, alternating zeta function.

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1 Introduction

The *Riemann zeta function* can be defined by a *Dirichlet series*, namely,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.1}$$

The series converges for $\mathrm{Re}(s) > 1$ but the zeta function can be analytically extended to the whole complex plane with the exception of the simple pole at $s = 1$.

The Riemann zeta function was studed already by L. Euler (for real values of s only). Likewise, he considered the *alternating zeta function* (known also as *Dirichet eta function*)

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}. \tag{1.2}$$

The two functions, (1.1) and (1.2), are interconnected:

$$\eta(s) = (1 - 2 \times 2^{-s})\zeta(s). \tag{1.3}$$

The factor $1 - 2 \times 2^{-s}$ vanishes at $s = 1$, and thus $\eta(s)$ is an entire function. Thanks to this, often it is easier to work with $\eta(s)$. Moreover, the series (1.2) converges in a larger half-plane $\mathrm{Re}(s) > 0$.

The zeta function is one of the most mysterious mathematical objects. The celebrated *Riemann Hypothesis* remains unproved for more than a century and a half. But besides this long expected feature, the zeta function has many other remarkable properties. In this paper we consider (on numerical examples) an experimentally discovered class of relationships among the values of the (alternating) zeta function and its derivatives. These relationships seem to be new. Probably, they remained unnoticed because of their inexact, approximate character.

We also state a number of conjectures about exact equalities involving some limiting values.

One's dream could be to find a polynomial P with numerical coefficients such that for all a (not equal to 1)

$$P(a, \zeta(a), \zeta'(a), \ldots, \zeta^{\langle N \rangle}(a)) = 0. \tag{1.4}$$

However, this is impossible. D. Hilbert [3] asserted that the zeta function does not satisfy any algebraic differential equation. A proof was given later by D. D. Mordukhai-Boltovskoi [8] (see also [6, 7]).

In [4] (see also [5, Algorithm A1 and (3.6)]) the author defined a sequence of linear polynomials with numerical coefficients $L_1(x_1),\ \ldots,\ L_N(x_1,\ldots,x_N),\ \ldots$ such that

$$\zeta(a) \approx L_N(\zeta'(a),\ldots,\zeta^{\langle N\rangle}(a)), \tag{1.5}$$
$$\eta(a) \approx L_N(\eta'(a),\ldots,\eta^{\langle N\rangle}(a)). \tag{1.6}$$

The high accuracy of these approximations has been demonstrated on a number of numerical examples. (Other similar polynomials were proposed in Section 7 of [4] and in Section 9 of [5].)

In this paper we deal mainly with the alternating zeta function; the case of the classical Riemann's zeta function is briefly considered in Subsection 4.8. We define several series of non-linear polynomials

$$M_1(x_0,x_1),\ \ldots,\ M_N(x_0,\ldots,x_N),\ \ldots\ . \tag{1.7}$$

These polynomials have no constant terms, all their coefficients are integers, and for almost all a

$$M_N(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a)) \approx 0. \tag{1.8}$$

An approximate equality of the form (1.8) by itself is not very informative: the value of the polynomial might be small (in absolute value) just because the values of all its arguments are small. Actually, usually $M_N(x_0,\ldots,x_N)$ is (implicitly) defined via three polynomials, $M_N^{+}(x_0,\ldots,x_N)$, $M_N^{-}(x_0,\ldots,x_N)$, and $M_N^{*}(x_0,\ldots,x_N)$ such that

$$M_N(x_0,\ldots,x_N) = M_N^{+}(x_0,\ldots,x_N) - M_N^{-}(x_0,\ldots,x_N)M_N^{*}(x_0,\ldots,x_N). \tag{1.9}$$

Using this notation, we can rewrite (1.8) as

$$\frac{M_N^{+}(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a))}{M_N^{-}(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a))} \approx M_N^{*}(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a)). \tag{1.10}$$

The quality of approximations (1.8) and (1.10) will be characterized by the (small) value of

$$\left|\frac{M_N^{+}(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a))}{M_N^{-}(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a))M_N^{*}(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a))} - 1\right|. \tag{1.11}$$

In many cases

$$M_N^{*}(x_0,\ldots,x_N) = M^{**}(x_0,\ldots,x_{N^*}) \tag{1.12}$$

for certain fixed N^* and polynomial M^{**}; in such cases we usually state the conjecture that

$$\frac{M_N^+(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a))}{M_N^-(\eta(a),\eta'(a),\ldots,\eta^{\langle N\rangle}(a))} \underset{N\to\infty}{\longrightarrow} M^{**}(\eta(a),\eta'(a),\ldots,\eta^{\langle N^*\rangle}(a)). \tag{1.13}$$

Another way to show nontrivially of (1.8) is to consider polynomial equation

$$M_N(\eta(a),\eta'(a),\ldots,\eta^{\langle k-1\rangle}(a),x,\eta^{\langle k+1\rangle}(a),\ldots,\eta^{\langle N\rangle}(a))=0 \tag{1.14}$$

and demonstrate that the kth derivative of $\eta(s)$ at $s=a$ can be calculated with high accuracy as a root of equation (1.14).

Different series of polynomials with such properties are defined below by (3.1), (3.8), (3.18), (3.24), (3.28) for small k, (3.30) for k slightly less than N, (4.2), (4.4), (4.11), (4.12), (4.25), (4.29), (4.32), (4.40) for small k, and (4.41) for k slightly less than N.

Some of the relations (1.8) are the limiting cases of more general relations of the form

$$\begin{multline}M_N(\eta(a_0),\eta'(a_0),\ldots,\eta^{\langle N\rangle}(a_0),\ \ldots,\\ \eta(a_{N-1}),\eta'(a_{N-1}),\ldots,\eta^{\langle N\rangle}(a_{N-1}))\approx 0 \quad (1.15)\end{multline}$$

where $a_0,\ldots,a_{N-1}$ are points in general position. For details see Subsection 4.7.

Section 3 presents our main numerical discoveries. In Section 4 we outline possible variations. In particular, we demonstrate that the zeta function approximately satisfies the same equalities as the alternating zeta function but only when a is sufficiently far from the pole $s=1$.

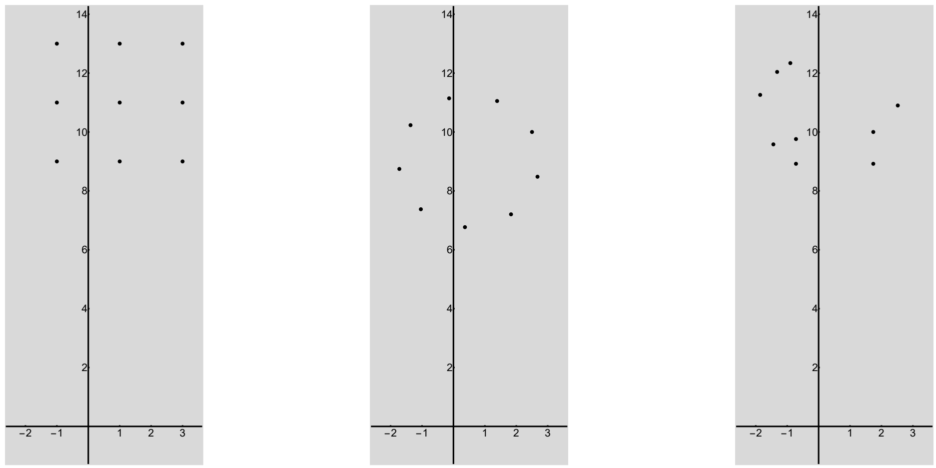
2 Notation

Let us fix a complex number a. We assume that a is generic, in particular, the denominators in all our formulas do not vanish. Consider Hankel matrices

$$H_{l,N}=\Big[\eta_{l+i+j}\Big]\Big|_{i=0}^{N-1}\Big|_{j=0}^{N-1}, \quad l=0,1,\ldots, \quad N=1,2,\ldots\ , \tag{2.1}$$

where

$$\eta_n=\frac{\mathrm{d}^n}{\mathrm{d}s^n}\eta(s)\Big|_{s=a}, \quad n=0,1,\ldots\ . \tag{2.2}$$



$\mathfrak{A}_{\mathrm{G}}(-1+9\mathrm{i},1,\mathrm{i},2,2)$ $\mathfrak{A}_{\mathrm{C}}(0.5+9\mathrm{i},2+\mathrm{i},9)$ $\mathfrak{A}_{\pi}(-3+7\mathrm{i},6,6\mathrm{i},100,3,9)$

Figure 1: Examples of sets of the three types defined by (2.4), (2.5), and (2.7)

For an arbitrary matrix M the *slightly perforated matrix*[1] $o_{i,j}(M)$ is defined as the matrix obtained from matrix M by replacing its entry on the crossing of ith row and jth column by zero (in spite of (2.1), we number rows and columns of matrices starting from 1). In the *highly perforated matrix*[2] $O_{i,j}(M)$ this entry remains unchanged but all other entries in the ith row and in the jth column are replaced by zero. Clearly, for a square matrix M we have decomposition

$$\det(M)=\det(o_{i,j}(M))+\det(O_{i,j}(M)). \tag{2.3}$$

For an arbitrary matrix M the *column-unified matrix*[3] $\mathbb{1}_{,j}(M)$ is defined as the matrix obtained from M by replacing all entries in the jth column by 1.

For an arbitrary square matrix M the *kth antidiagonal polynomial* $\mathrm{ADP}_k(M,x)$ is defined as follows: for every i and j such that $i+j=k+1$ the entry on the crossing of ith row and j column of matrix M is replaced by x and $\mathrm{ADP}_k(M,x)$ is equal to the determinant of the modified matrix.

In order to demonstrate numerical examples, we use the following notation for sets of particular forms.

For complex a, δ_1, δ_2 and non-negative integers N_1, N_2 the *grids* are defined as

$$\mathfrak{A}_{\mathrm{G}}(a,\delta_1,\delta_2,N_1,N_2)=\{a+k\delta_1+l\delta_2\,|\,k=0,\dots,N_1,l=0,\dots,N_2\}. \tag{2.4}$$

[1] In Russian: *слабо перфорированная матрица.*
[2] In Russian: *сильно перфорированная матрица.*
[3] In Russian: *столбцово унифицированная матрица.*

Fig. 1 (left) depicts an instance of a grid.

For complex c, r and positive integer N the *discrete circles* are defined as

$$\mathfrak{A}_{\mathrm{C}}(c, r, N) = \{c + \mathrm{e}^{2\pi \mathrm{i} k/N} r \mid k = 0, \ldots, N-1\}. \tag{2.5}$$

Fig. 1 (center) depicts an instance of a discrete circle.

To define a reproducible "random" set, we use sequences of pseudorandom integers. Let

$$\pi = 3 + \sum_{k=1}^{\infty} \pi_{m,k} m^{-k} \tag{2.6}$$

be base-m positional notation for the number π. For complex a, δ_1, δ_2 and positive integers m, n, N, we define (of cource, provided that $m \geq 2$)

$$\mathfrak{A}_{\pi}(a, \delta_1, \delta_2, m, n, N) = \\ \{a + (\pi_{m,n+2k}\delta_1 + \pi_{m,n+2k+1}\delta_2)/m \mid k = 0, \ldots, N-1\}. \tag{2.7}$$

Fig. 1 (right) depicts an instance of such a "random" set.

3 Main numerical discoveries

In this section we present some results of calculations, and on their basis made a number of conjectures.

3.1 First discovery: pricipal minor

In Table 1 the determinant of matrix $H_{0,N}$ is close to one of its principal minors, namely,

$$\frac{\det(H_{0,N})}{\det(H_{2,N-1})} \approx 1. \tag{3.1}$$

Table 1 and other calculations suggest the following.

Conjecture 1. *Except for countably many values of a,*

$$\frac{\det(H_{0,N})}{\det(H_{2,N-1})} \underset{N\to\infty}{\longrightarrow} 1. \tag{3.2}$$

The numerator in (3.1) is linear in η_0, and the denominator does not contain η_0. This allows us to calculate (an approximation to) the values of $\eta(a) = \eta_0$

via the values of higher derivatives of this function. To this end we solve the linear equation

$$\mathrm{ADP}_1(H_{0,N}, x) = \det(H_{2,N-1}) \tag{3.3}$$

and get

$$\eta_0 \approx 1 - \frac{\det(o_{1,1}(H_{0,N}))}{\det(H_{2,N-1})}. \tag{3.4}$$

Table 2 demonstrates the accuracy of such approximations. This table and other calculation suggest the following.

Conjecture 2. *Except for countably many values of a,*

$$1 - \frac{\det(o_{1,1}(H_{0,N}))}{\det(H_{2,N-1})} \underset{N\to\infty}{\longrightarrow} \eta_0. \tag{3.5}$$

The numerator and the denominator in (3.1) are both linear in η_{2N-2} (standing in the right bottom corners of matrices $H_{0,N}$ and $H_{2,N-1}$). This allows us to calculate (an approximation to) the value of the $(2N-2)$th derivative of $\eta(s)$ at $s = a$ via the values of the lower derivatives of this function. To this end we solve the linear equation

$$\mathrm{ADP}_{2N-1}(H_{0,N}, x) = \mathrm{ADP}_{2N-3}(H_{2,N-1}) \tag{3.6}$$

and get

$$\eta_{2N-2} \approx -\frac{\det(o_{N,N}(H_{0,N})) - \det(o_{N-1,N-1}(H_{2,N-1}))}{\det(H_{0,N-1}) - \det(H_{2,N-2})}. \tag{3.7}$$

Table (3) demonstrates the accuracy of such approximations.

3.2 Second discovery: perforated matrices

In Table 4 the determinants of strongly perforated matrix $O_{1,1}(H_{l,N})$ and sightly perforated matrix $o_{1,1}(H_{l,N})$ for several positive l are approximately equal up to the "sign":

$$\frac{\det(O_{1,1}(H_{l,N}))}{\det(o_{1,1}(H_{l,N}))} \approx -1. \tag{3.8}$$

Table 4 and other calculations suggest the following.

Conjecture 3. *Except for countably many values of a, for every positive integer l*

$$\frac{\det(O_{1,1}(H_{l,N}))}{\det(o_{1,1}(H_{l,N}))} \underset{N\to\infty}{\longrightarrow} -1. \tag{3.9}$$

Clearly, every zero of the lth derivative of the eta function is an exceptional value of a because in this case the numerator in (3.9) is identically equal to zero.

Let $l > 0$. The numerator in (3.8) is linear in η_l, and the denominator does not contain η_l. This allows us to calculate (an approximation to) the value of lth derivative of $\eta(s)$ at $s = a$ via the values of higher derivatives of this function. To this end we solve, according to (2.3), the linear equation

$$\mathrm{ADP}_1(H_{l,N}, x) = 0 \tag{3.10}$$

and get

$$\eta_l \approx -\frac{\det(o_{1,1}(H_{l,N}))}{\det(H_{l+2,N-1})}. \tag{3.11}$$

Table 5 demonstrates the accuracy of such approximations. This table and other calculation suggest the following.

Conjecture 4. *Except for countably many values of a, for every positive integer l*

$$-\frac{\det(o_{1,1}(H_{l,N}))}{\det(H_{l+2,N-1})} \underset{N\to\infty}{\longrightarrow} \eta_l. \tag{3.12}$$

The numerator and the denominator in (3.8) are both linear in η_{2N+l-2} (standing in the right bottom corners of matrices $H_{l,N}$ and $H_{l+2,N-1}$). This allows us to calculate (an approximation to) the value of the $(2N+l-2)$th derivative of $\eta(s)$ at $s = a$ via the values of some of the lower derivatives of this function. To this end we solve, according to (2.3), the linear equation

$$\mathrm{ADP}_{2N-1}(H_{l,N}, x) = 0 \tag{3.13}$$

and get

$$\eta_{2N+l-2} \approx -\frac{\det(o_{N,N}(H_{l,N}))}{\det(H_{l,N-1})}. \tag{3.14}$$

Table 6 demonstrates the accuracy of such approximations.

For an even l, the $(2N+l-2)$th derivative can be also calculated by (3.7) if one replaces N by $N + l/2$. However, unlike (3.14), this would require the knowledge of $\eta_0, \dots, \eta_{l-1}$.

3.3 Third discovery: equations of higher degrees. I

If $0 < k < 2N-2$, then the numerator in (3.1) is not linear in η_k. Nevertheless we can try to calculate this derivative via the values of other derivatives (both lower and higher than η_k) by solving polynomial equation

$$P(x) = 0 \tag{3.15}$$

where

$$P(x) = \mathrm{ADP}_{k+1}(H_{0,N}, x) - \mathrm{ADP}_{k-1}(H_{2,N-1}) = \sum_{m=0}^{d} p_m x^m. \tag{3.16}$$

Polynomial $P(x)$ has the degree $d = \min(k+1, 2N-k-1)$, and *a priori* there is a question: which of the d solutions of equation (3.15) is close to η_k? *A posteriori* we find that when d is small relative to N, then *all* roots of $P(x)$ are close to η_k. Table 7 demonstrates this phenomenon for small k and Table 8 does the same for k near $2N-2$.

Table 7 and other calculations suggest the following.

Conjecture 5. *Except for countably many values of a, for every non-negative integer k the maximal distance between the roots of polynomial $P(x)$ (defined by (3.16)) and η_k tends to zero as $N \longrightarrow \infty$.*

Conjecture 5 implies that if d is small (relative to N) then polynomial (3.16) has a special structure, namely,

$$P(x) \approx A(x - \eta_k)^d \tag{3.17}$$

for some A. This gives other ways to calculate (approximation to) η_k: for $m = 1, \ldots, d$

$$\eta_k \approx -\frac{d-m+1}{m} \cdot \frac{p_{m-1}}{p_m}. \tag{3.18}$$

Table 9 demonstrates this phenomenon for small k and Table 10 does the same for k near $2N-2$.

Table 9 and other calculations suggest the following.

Conjecture 6. *Except for countably many values of a, for every non-negative integer k and every positive integer m such that $m \le k+1$*

$$-\frac{k-m+2}{m} \cdot \frac{p_{m-1}}{p_m} \underset{N\to\infty}{\longrightarrow} \eta_k \tag{3.19}$$

where p_{m-1} and p_m are defined by (3.16).

Note that according to (3.16), p_{m-1} and p_m can be expressed as certain polynomials in $\eta_0, \ldots, \eta_{k-1}, \eta_{k+1}, \ldots, \eta_{2N-2}$.

3.4 Fourth discovery: equations of higher degrees. II.

If $l < k < 2N+l-2$, then the numerator in (3.8) is not linear in η_k. Nevertheless we can try to calculate this derivative via the values of other derivatives (both lower and higher than η_k) by solving corresponding equation. Namely, according to (2.3), relation (3.8) can be rewritten as

$$\det(H_{l,N}) \approx 0. \tag{3.20}$$

Respectively, in order to find η_k we solve equation

$$Q(x) = 0 \tag{3.21}$$

where

$$Q(x) = \mathrm{ADP}_{k-l+1}(H_{l,N}) = \sum_{m=0}^{d} q_m x^m. \tag{3.22}$$

Polynomial $Q(x)$ has the degree $d = \min(k-l+1, 2N-k+l-1)$, and *a priori* there is a question: which of the d solutions of equation (3.21) is close to η_k? *A posteriori* we find that when d is small relative to N, then all roots of $Q(x)$ are close to η_k. Tables 11–13 demonstrate this phenomenon for small k and Tables 14–16 do the same for k near $2N+l-2$.

Tables 11–13 and other calculations suggest the following.

Conjecture 7. *Except for countably many values of a, for every positive integers k and l such that $l \leq k$, the maximal distance between the roots of polynomial $Q(x)$ (defined by (3.22)) and η_k tends to zero as $N \longrightarrow \infty$.*

Conjecture 7 implies that if d is small (relative to N) then polynomial (3.22) has a special structure, namely,

$$Q(x) \approx A(x-\eta_k)^d \tag{3.23}$$

for some A. This gives other ways to calculate (approximation to) η_k: for $m = 1, \ldots, d$

$$\eta_k \approx -\frac{d-m+1}{m} \cdot \frac{q_{m-1}}{q_m}. \tag{3.24}$$

Tables 17–19 demonstrate this phenomenon for small k and Tables 20–22 do the same for k near $2N + l - 2$.

Tables 17–19 and other calculations suggest the following.

Conjecture 8. *Except for countably many values of a, for every positive integer k, l and m such that $l \leq k$ and $m \leq k+1$*

$$-\frac{k-m+2}{m} \cdot \frac{q_{m-1}}{q_m} \underset{N\to\infty}{\longrightarrow} \eta_k \tag{3.25}$$

where numbers q_{m-1} and q_m are defined by (3.22).

Note that according to (3.22), q_{m-1} and q_m can be expressed as certain polynomials in $\eta_l, \ldots, \eta_{k-1}, \eta_{k+1}, \ldots, \eta_{2N+l-2}$.

3.5 Fifth discovery: characteristic polynomials. I

Let $\mathrm{I_N}$ be the identity matrix of size $N \times N$. For a positive l consider the characteristic polynomials of perforated matrices $O_{1,1}(H_{l,N})$ and $o_{1,1}(H_{l,N})$:

$$R_{l,N} = \det(x\mathrm{I_N} - O_{1,1}(H_{l,N})) = r_{N,N}x^N + r_{N,N-1}x^{N-1} + \cdots + r_{N,0}, \tag{3.26}$$

$$S_{l,N} = \det(x\mathrm{I_N} - o_{1,1}(H_{l,N})) = s_{N,N}x^N + s_{N,N-1}x^{N-1} + \cdots + s_{N,0}. \tag{3.27}$$

With this notation Second discovery (3.8) can be stated as follows: if $k = 0$ then

$$\frac{r_{N,k}}{s_{N,k}} \approx -1. \tag{3.28}$$

In fact, the same holds for other k which are sufficiently small with respect to N. Tables 23–26 demonstrate these phenomenon. These tables and other calculations suggest the following generalization of Conjecture 2.

Conjecture 2G. *Except for countably many values of a, for every k and positive l*

$$\frac{r_{N,k}}{s_{N,k}} \underset{N\to\infty}{\longrightarrow} -1 \tag{3.29}$$

where $r_{N,k}$ and $s_{N,k}$ are defined by (3.26) and (3.27).

Note that according to (3.26) and (3.27), $r_{N,k}$ and $s_{N,k}$ can be expressed as certain polynomials in $\eta_l, \ldots, \eta_{2N+l-2}$.

3.6 Sixth discovery: characteristic polynomials. II

Clearly, relation (3.28) cannot hold for large k. For example, if $k = N$ then $r_{N,k} = 1 = s_{N,k}$. In Tables 23–26 we observe that if k is close to N, then

$$\frac{r_{N,k}}{s_{N,k}} \approx 1. \tag{3.30}$$

These tables and other calculations suggest the following.

Conjecture 9. *Except for countably many values of a, for every m and positive l*

$$\frac{r_{N,N-m}}{s_{N,N-m}} \underset{N\to\infty}{\longrightarrow} 1 \tag{3.31}$$

where $r_{N,N-m}$ and $s_{N,N-m}$ are defined by (3.26) *and* (3.27)*.*

Note that according to (3.26) and (3.27), $r_{N,N-m}$ and $s_{N,N-m}$ can be expressed as certain polynomials in $\eta_l, \dots, \eta_{2N+l-2}$.

4 Variations

In Section 3 our numerical discoveries were demonstrated for matrices $H_{l,N}$ (defined by (2.1)). These matrices have very regular structure of Hankel matrices, and their entries are the derivatives of the alternating zeta function. In fact, many other matrices have similar properties. In this section we outline some directions for possible generalizations and extensions. In many cases several such modifications can be applied simultaneously.

4.1 First variation: Taylor coefficients

In the definition (2.1) of matrices $H_{l,N}$, we can replace the derivatives (2.2) by their fractions, namely, by the Taylor coefficients of the eta function. Let

$$T_{l,N} = \left[\frac{\eta_{l+i+j}}{(l+i+j)!}\right]\Bigg|_{i=0}^{N-1}\Bigg|_{j=0}^{N-1}, \quad l = 0, 1, \dots, \quad N = 1, 2, \dots\ . \tag{4.1}$$

In general, matrices $T_{l,N}$ demonstrate properties similar to those of matrices $H_{l,N}$, but with lower accuracy.

In particular, the following counterpart of (3.1) holds:

$$\frac{\det(T_{0,N})}{\det(T_{2,N-1})} \approx 1. \tag{4.2}$$

Its accuracy is demonstrated by Table 27 which is a counterpart of Table 1. In spite of the lower accuracy, the calculations still allow one to state the following companion to Conjecture 1.

Conjecture 1T. *Except for countably many values of a,*

$$\frac{\det(T_{0,N})}{\det(T_{2,N-1})} \underset{N\to\infty}{\longrightarrow} 1. \tag{4.3}$$

Similar, the following counterpart of (3.8) holds: for $l > 0$

$$\frac{\det(O_{1,1}(T_{l,N}))}{\det(o_{1,1}(T_{l,N}))} \approx -1. \tag{4.4}$$

Its accuracy is demonstrated by Table 28 which is a counterpart of Table 4. Again, in spite of the lower accuracy, the calculations still allow one to state the following companion to Conjecture 2.

Conjecture 2T. *Except for countably many values of a, for every positive integer l*

$$\frac{\det(O_{1,1}(T_{l,N}))}{\det(o_{1,1}(T_{l,N}))} \underset{N\to\infty}{\longrightarrow} -1. \tag{4.5}$$

Conjectures 1T and 2T open new ways for calculation one of the derivatives of the eta function via other derivatives. This could be done either directly (by counterparts of (3.4), (3.7), (3.11), and (3.14)) or by solving the following counterparts of equations (3.15) and (3.21) with $P(x)$ and $Q(x)$ now defined (by analogy with (3.16) and (3.22)) as

$$P(x) = \mathrm{ADP}_{k+1}(T_{0,N}, x) - \mathrm{ADP}_{k-1}(T_{2,N-1}) = \sum_{m=0}^{d} p_m x^m. \tag{4.6}$$

and

$$Q(x) = \mathrm{ADP}_{k-l+1}(T_{l,N}) = \sum_{m=0}^{d} q_m x^m. \tag{4.7}$$

For these equations we can state the following conjectures which are companions to Conjectures 5–8:

Conjecture 5T. *Except for countably many values of a, for every non-negative integer k the maximal distance between the roots of polynomial $P(x)$ (defined by (4.6)) and η_k tends to zero as $N \longrightarrow \infty$.*

Conjecture 6T. *Except for countably many values of a, for every non-negative integer k and every positive integer m such that $m \le k+1$*

$$-\frac{k-m+2}{m} \cdot \frac{p_{m-1}}{p_m} \underset{N\to\infty}{\longrightarrow} \eta_k \tag{4.8}$$

where p_{m-1} and p_m are defined by (4.6).

Conjecture 7T. *Except for countably many values of a, for every positive integers k and l such that $l \le k$, the maximal distance between the roots of polynomial $Q(x)$ (defined by (4.7)) and η_k tends to zero as $N \longrightarrow \infty$.*

Conjecture 8T. *Except for countably many values of a, for every positive integers k, l, and m such that $l \le k$ and $m \le k+1$*

$$-\frac{k-m+2}{m} \cdot \frac{q_{m-1}}{q_m} \underset{N\to\infty}{\longrightarrow} \eta_k \tag{4.9}$$

where q_{m-1} and q_m are defined by (4.7).

4.2 Second variation: 'anti'-Taylor coefficients

Passing from (2.1) to (4.1), we divided the entries to matrix $H_{l,N}$ by corresponding factorials; now we multiply by these factorials. Let

$$A_{l,N} = \left[(l+i+j)!\eta_{l+i+j} \right] \Big|_{i=0}^{N-1} \Big|_{j=0}^{N-1}, \quad l = 0, 1, \dots, \quad N = 1, 2, \dots . \tag{4.10}$$

The entries to matrices $T_{l,N}$ have clear interpretation as the Taylor coefficients, but for the entries to matrices $A_{l,N}$ we cannot give similar interpretation. Our only justification for considering matrices $A_{l,N}$ is as follows: these matrices have some properties similar to those of matrices $H_{l,N}$ and $T_{l,N}$.

In particular, we have the following counterpart of (3.1) and (4.2),

$$\frac{\det(A_{0,N})}{\det(A_{2,N-1})} \approx 1, \tag{4.11}$$

and the following counterpart of (3.8) and (4.4),

$$\frac{\det(O_{1,1}(A_{l,N}))}{\det(o_{1,1}(A_{l,N}))} \approx -1. \tag{4.12}$$

However, the accuracy of approximate equalities (4.11) and (4.12) with matrices $A_{l,N}$ is a lot lower than the accuracy of similar approximate equalities for matrices $H_{l,N}$ and $T_{l,N}$ for the same l and N. In order to demonstrate a modest accuracy of (4.11) and (4.12) we use in Tables 29–30 matrices of rather large size.

4.3 Third variation: logarithmic derivatives

Suppose that instead of derivatives (2.2) we have at our disposal only their multiples,

$$\tilde{\eta}_n = A\eta_n, \qquad n = 0, 1, \dots \tag{4.13}$$

where A is a non-zero number unknown to us. We can determine its value according to the First discovery (3.1). Namely, if we define

$$\tilde{H}_{l,N} = \left[\tilde{\eta}_{l+i+j}\right]\Big|_{i=0}^{N-1}\Big|_{j=0}^{N-1}, \tag{4.14}$$

then by (3.1)

$$\frac{\det(\tilde{H}_{0,N})}{\det(\tilde{H}_{2,N-1})} = A\frac{\det(H_{0,N})}{\det(H_{2,N-1})} \approx A, \tag{4.15}$$

and by Conjecture 1

$$\frac{\det(\tilde{H}_{0,N})}{\det(\tilde{H}_{2,N-1})} \underset{N\to\infty}{\longrightarrow} A. \tag{4.16}$$

In general, the logarithmic derivative of a functions determines it up to a multiplicative constant only. However, in our case the situation is different. Namely, the above considerations show that a non-zero value of $\eta(a)$ can be found from the values of the Taylor expansion of the logarithmic derivative of $\eta(s)$ at $s = a$. Namely, if

$$\frac{\eta'(s)}{\eta(s)} = \sum_{n=0}^{\infty} \frac{\phi_n}{n!}(s-a)^n, \tag{4.17}$$

then

$$\eta_n = \eta(a)Z_n(\phi_0, \dots, \phi_{n-1}), \quad n = 1, 2, \dots \tag{4.18}$$

for some polynomials $Z_1(x_0),\ \dots,\ Z_n(x_0,\dots,x_{n-1}),\dots$ with integer coefficients. Setting $A = 1/\eta(a)$ and

$$\tilde{\eta}_n = \begin{cases} 1, & \text{if } n = 0, \\ Z_n(x_0,\dots,x_{n-1}), & \text{otherwise} \end{cases} \tag{4.19}$$

in (4.14), we get from (4.16) that

$$\frac{\det(\tilde{H}_{2,N-1})}{\det(\tilde{H}_{0,N})} \underset{N\to\infty}{\longrightarrow} \eta(a). \tag{4.20}$$

4.4 Fouth variation: submatrices

It was a natural choice to consider such "regular" matrices as Hankel matrices (2.1). However, this is not the only possibility. In fact, the square submatrices of matrices $H_{l,N}$ demonstrate similar properties, but with a lower accuracy.

Let

$$\mathfrak{M}_{\mathrm{R}} = \{i_1,\dots,i_m\} \tag{4.21}$$

and

$$\mathfrak{M}_{\mathrm{C}} = \{j_1,\dots,j_m\} \tag{4.22}$$

be two sets of integers such that

$$1 \le i_1 < \dots < i_k < i_{k+1} < \dots i_m \le N, \tag{4.23}$$

$$1 \le j_1 < \dots < j_k < j_{k+1} < \dots j_m \le N. \tag{4.24}$$

Let us delete rows with numbers $i_1,\dots,i_m$ and columns with numbers $j_1,\dots,j_m$ from matrix $H_{l,N}$; we denote the resulting matrix as $H_{l,N,\mathfrak{M}_{\mathrm{R}},\mathfrak{M}_{\mathrm{C}}}$.

With this notation we have the following generalization of the Second discovery (3.8):

$$\frac{\det(O_{1,1}(H_{l,N,\mathfrak{M}_{\mathrm{R}},\mathfrak{M}_{\mathrm{C}}}))}{\det(o_{1,1}(H_{l,N,\mathfrak{M}_{\mathrm{R}},\mathfrak{M}_{\mathrm{C}}}))} \approx -1. \tag{4.25}$$

Table 31 shows how the accuracy of (4.25) decreases when more and more rows and columns are deleted. For a fixed set of deleted rows and columns we can state the following.

Conjecture 2G. *Except for countably many values of a, for every positive integer l and every sets $\mathfrak{M}_{\mathrm{R}}$ and $\mathfrak{M}_{\mathrm{C}}$ of integers satisfying* (4.23) *and* (4.24)

$$\frac{\det(O_{1,1}(H_{l,N,\mathfrak{M}_{\mathrm{R}},\mathfrak{M}_{\mathrm{C}}}))}{\det(o_{1,1}(H_{l,N,\mathfrak{M}_{\mathrm{R}},\mathfrak{M}_{\mathrm{C}}}))} \underset{N\to\infty}{\longrightarrow} -1. \tag{4.26}$$

As for generalizing the First discovery, the matrices in the numerators and denominators in (3.1) and (3.2) are of different sizes, and this should be taken into account. For sets (4.21) and (4.22) let

$$\mathfrak{M}_{\mathrm{R}} - 1 = \{i_1 - 1, \dots, i_m - 1\} \tag{4.27}$$

and

$$\mathfrak{M}_{\mathrm{C}} - 1 = \{j_1 - 1, \dots, j_m - 1\}. \tag{4.28}$$

With this notation we have the following generalization of the First discovery (3.1): if $i_1 > 1$ and $j_1 > 1$ in (4.21) and (4.22), then

$$\frac{\det(H_{0,N,\mathfrak{M}_{\mathrm{R}},\mathfrak{M}_{\mathrm{C}}})}{\det(H_{2,N-1,\mathfrak{M}_{\mathrm{R}}-1,\mathfrak{M}_{\mathrm{C}}-1})} \approx 1. \tag{4.29}$$

Table 32 shows how the accuracy of (4.29) decreases when more and more rows and columns are deleted. For a fixed set of deleted rows and columns we can state the following.

Conjecture 1G. *Except for countably many values of a, for and every sets $\mathfrak{M}_{\mathrm{R}}$ and $\mathfrak{M}_{\mathrm{C}}$ of integers satisfying, besides (4.23) and (4.24), additional conditions $i_1 > 1$ and $j_1 > 1$,*

$$\frac{\det(H_{0,N,\mathfrak{M}_{\mathrm{R}},\mathfrak{M}_{\mathrm{C}}})}{\det(H_{2,N,\mathfrak{M}_{\mathrm{R}}-1,\mathfrak{M}_{\mathrm{C}}-1})} \underset{N\to\infty}{\longrightarrow} 1. \tag{4.30}$$

Very likely, Conjectures 1G and 2G admit further generalization. Namely, we need not fix $\mathfrak{M}_{\mathrm{R}}$ and $\mathfrak{M}_{\mathrm{C}}$ but can allow these sets to enlarge with N. However, at present it is not clear how we should bound the admissible size and/or the structure of $\mathfrak{M}_{\mathrm{R}}$ and $\mathfrak{M}_{\mathrm{C}}$ depending on N.

As an example, let $H^{=}_{l,N}$ be the result of deleting all even rows and columns (i.e., the 2nd, the 4th, and so on) from matrix $H_{l,2N}$. Matrix $H^{=}_{l,N}$ contains either only odd or only even derivatives of $\eta(s)$ depending on the parity of l. Tables 33 and 34 show that still

$$\frac{\det(H^{=}_{0,N})}{\det(H^{=}_{2,N-1})} \approx 1 \tag{4.31}$$

and

$$\frac{\det(O_{1,1}(H^{=}_{l,N}))}{\det(o_{1,1}(H^{=}_{l,N}))} \approx -1 \tag{4.32}$$

similar to the First discovery (3.1) and the Second discovery (3.8).

4.5 Fifth variation: other ways of perforating

The value $\eta_0 = \eta(a)$ standing in the topmost leftmost corner of matrix $H_{0,N}$ plays a special role for the First discovery (3.1). However, for the Second discovery (3.8) our choice of this corner for perforating is not critical. We can select any other entry to matrix $H_{l,N}$ and have the following extension of (3.8): if $1 \leq i \leq N$ and $1 \leq j \leq N$ then

$$\frac{\det(O_{i,j}(H_{l,N}))}{\det(o_{i,j}(H_{l,N}))} \approx -1. \tag{4.33}$$

Table 35 demonstrates this phenomenon. Respectively, Conjecture 2 can be extended.

Conjecture 2E. Except for countably many values of a, for every positive integer l, i, and j

$$\frac{\det(O_{i,j}(H_{l,N}))}{\det(o_{i,j}(H_{l,N}))} \underset{N\to\infty}{\longrightarrow} -1. \tag{4.34}$$

Relation (4.33) does not bring new relations of the form (1.15) because according to (2.3) for all i and j

$$\det(O_{i,j}(H_{l,N})) + \det(o_{i,j}(H_{l,N})) = \det(H_{l,N}). \tag{4.35}$$

4.6 Sixth variation: rearrangements

If we rearrange rows and columns of a square matrix, its determinant either remains the same or changes the "sign". Thus properties of the eta derivatives considered in Subsections 3.1–3.4 could be equivalently restated via rearranged matrices.

In contrast to the determinants, the characteristic polynomials can drastically change after a rearrangement of rows and columns of a matrix. Nevertheless, properties (3.28) and (3.30) "survive" under rearrangements of matrices $H_{l,N}$.

We formalize this discovery in a bit different way. Let J_N be a permutation matrix of size $N \times N$. By analogy with (3.26) and (3.27) we define

$$\begin{aligned} R_{l,N,J_N} &= \det(x\mathrm{J_N} - O_{1,1}(H_{l,N})) && (4.36)\\ &= r_{N,N,J_N}x^N + r_{N,N-1,J_N}x^{N-1} + \cdots + r_{N,0,J_N}, && (4.37)\end{aligned}$$

$$\begin{aligned} S_{l,N,J_N} &= \det(x\mathrm{J_N} - o_{1,1}(H_{l,N})) && (4.38)\\ &= s_{N,N,J_N}x^N + s_{N,N-1,J_N}x^{N-1} + \cdots + s_{N,0,J_N}. && (4.39)\end{aligned}$$

We have the following generalizations of (3.28) and (3.30): for k relatively small with respect to N

$$\frac{r_{N,k,J_N}}{s_{N,k,J_N}} \approx -1, \tag{4.40}$$

and for k close to N

$$\frac{r_{N,k,J_N}}{s_{N,k,J_N}} \approx 1. \tag{4.41}$$

Approximate equalities (4.40) and (4.41) are new properties of the derivatives of the zeta function (unless J_N is the identity matrix).

4.7 Seventh variation: derivatives at several points

Matrices $H_{l,N}$ (defined by (2.1)) are constructed from the derivatives of the eta function calculated at a *single* point. In fact, we can find polynomial relationship between the derivatives of the eta function calculated at *several different* points in general position.

Let $\mathfrak{A} = \{a_0, \dots, a_{N-1}\}$ be a generic set of N complex numbers. Consider matrices

$$H_{l,N,\mathfrak{A}} = \left[\eta_{i,l+j}\right]\Big|_{i=0}^{N-1}\Big|_{j=0}^{N-1}, \quad l = 0, 1, \dots \tag{4.42}$$

where

$$\eta_{i,n} = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\eta(s)\Big|_{s=a_i}. \tag{4.43}$$

With this notation we have the following generalization of the First discovery (3.1):

$$\frac{\det(H_{0,N,\mathfrak{A}})}{\det(\mathbb{1}_{,1}(H_{0,N,\mathfrak{A}}))} \approx 1 \tag{4.44}$$

where operator $\mathbb{1}_{,1}$ is defined in Section 2. This is indeed a generalization, namely, approximate equality (3.1) is the limiting case of (4.44) when all points a_k tend to a. In order to see this, let $a_k = a + k\varepsilon$, $k = 0, \dots, N-1$. In this case

$$\det(H_{0,N,\mathfrak{A}}) = \det(H_{0,N})\varepsilon^{\frac{N(N-1)}{2}} + O(\varepsilon^{\frac{N(N-1)}{2}+1}) \tag{4.45}$$

and

$$\det(\mathbb{1}_{,1}(H_{0,N,\mathfrak{A}})) = \det(H_{2,N-1})\varepsilon^{\frac{N(N-1)}{2}} + O(\varepsilon^{\frac{N(N-1)}{2}+1}), \tag{4.46}$$

hence

$$\frac{\det(H_{0,N})}{\det((H_{2,N-1}))} = \frac{\det(H_{0,N,\mathfrak{A}})}{\det(\mathbb{1}_{,1}(H_{0,N,\mathfrak{A}}))} + O(\varepsilon). \tag{4.47}$$

Tables 36–38 present the accuracy of (4.44) for several sets of the three types defined in Section 2.

Approximate equalities (3.1) and (4.44) are just two extreme cases. In general case we can take a set $\mathfrak{A}^* = \{b_1, \dots, b_{N^*}\}$ where $N^* < N$ and define $\mathfrak{A}$ as

$$\mathfrak{A} = \{b_i + j\varepsilon | i = 1, \dots, N^*, j = 0, \dots, N_j - 1^*\} \tag{4.48}$$

with $N_1^* + \cdots + N_{N^*}^* = N$. Letting $\varepsilon \to 0$ we get from (4.44) a new approximate equality of the form

$$\begin{multline} M_N(\eta(b_1), \eta'(b_1), \dots, \eta^{\langle N_1 - 1^* \rangle}(b_1),\ \dots, \\ \eta(b_{N^*}), \eta'(b_{N^*}), \dots, \eta^{\langle N_{N^*} - 1 \rangle}(b_{N^*})) \approx 0. \end{multline} \tag{4.49}$$

4.8 Eighth variation: Riemann's zeta function

By analogy with (2.1) and (2.2) we define matrices

$$G_{l,N} = \left[\zeta_{l+i+j} \right] \Big|_{i=0}^{N-1} \Big|_{j=0}^{N-1}, \quad l = 0, 1, \dots, \quad N = 1, 2, \dots, \tag{4.50}$$

where

$$\zeta_n = \frac{\mathrm{d}^n}{\mathrm{d}s^n} \zeta(s) \Big|_{s=a}, \quad n = 0, 1, \dots . \tag{4.51}$$

Tables 39–40 demonstrate that when a is sufficiently far from the pole of the zeta function, we have counterparts of the First discovery (3.1) and the Second discovery (3.8), that is,

$$\frac{\det(G_{0,N})}{\det(G_{2,N-1})} \approx 1, \tag{4.52}$$

$$\frac{\det(O_{1,1}(G_{l,N}))}{\det(o_{1,1}(G_{l,N}))} \approx -1. \tag{4.53}$$

However, when a is relatively close to the pole and the sizes of matrices are not extremely large, the left hand sides in (4.52) and (4.53) are sufficiently different from 1 and -1. Computations performed so far do not allow us to state counterparts to Conjectures 1 and 2 for the case of matrices $G_{l,N}$.

Acknowledgments

Numerical data for Tables 1–40 were computed by programs written in JULIA programming language [1] with package NEMO [2].

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a	$\lvert \det(H_{0,N})/\det(H_{2,N-1}) - 1\rvert$		
	$N=10$	$N=50$	$N=300$
-6	$2.3075\ldots\cdot 10^{-4}$	$1.4277\ldots\cdot 10^{-68}$	$1.4799\ldots\cdot 10^{-582}$
-5	$1.8825\ldots\cdot 10^{-5}$	$3.0648\ldots\cdot 10^{-70}$	$5.6382\ldots\cdot 10^{-585}$
-4	$1.5891\ldots\cdot 10^{-6}$	$6.5972\ldots\cdot 10^{-72}$	$2.1485\ldots\cdot 10^{-587}$
-3	$1.3826\ldots\cdot 10^{-7}$	$1.4237\ldots\cdot 10^{-73}$	$8.1896\ldots\cdot 10^{-590}$
-2	$1.2352\ldots\cdot 10^{-8}$	$3.0802\ldots\cdot 10^{-75}$	$3.1223\ldots\cdot 10^{-592}$
-1	$1.1290\ldots\cdot 10^{-9}$	$6.6797\ldots\cdot 10^{-77}$	$1.1907\ldots\cdot 10^{-594}$
$-1+10\mathrm{i}$	$5.0456\ldots\cdot 10^{-10}$	$5.9856\ldots\cdot 10^{-77}$	$1.1763\ldots\cdot 10^{-594}$
$-1+50\mathrm{i}$	$6.0891\ldots\cdot 10^{-8}$	$1.1224\ldots\cdot 10^{-77}$	$8.8233\ldots\cdot 10^{-595}$
$-1+300\mathrm{i}$	$5.1521\ldots\cdot 10^{-5}$	$6.9048\ldots\cdot 10^{-62}$	$7.9710\ldots\cdot 10^{-597}$
$-1+1500\mathrm{i}$	$1.0721\ldots\cdot 10^{-3}$	$4.8864\ldots\cdot 10^{-49}$	$5.1731\ldots\cdot 10^{-509}$
-0.8	$7.0142\ldots\cdot 10^{-10}$	$3.1053\ldots\cdot 10^{-77}$	$3.9092\ldots\cdot 10^{-595}$
$-0.8+10\mathrm{i}$	$3.2396\ldots\cdot 10^{-10}$	$2.7859\ldots\cdot 10^{-77}$	$3.8621\ldots\cdot 10^{-595}$
$-0.8+50\mathrm{i}$	$4.0710\ldots\cdot 10^{-8}$	$5.3554\ldots\cdot 10^{-78}$	$2.8989\ldots\cdot 10^{-595}$
$-0.8+100\mathrm{i}$	$5.4254\ldots\cdot 10^{-7}$	$3.5450\ldots\cdot 10^{-76}$	$1.2429\ldots\cdot 10^{-595}$
$-0.8+300\mathrm{i}$	$3.6530\ldots\cdot 10^{-5}$	$2.9730\ldots\cdot 10^{-62}$	$2.6751\ldots\cdot 10^{-597}$
$-0.8+1500\mathrm{i}$	$7.4566\ldots\cdot 10^{-4}$	$2.1951\ldots\cdot 10^{-49}$	$7.0923\ldots\cdot 10^{-510}$
0	$1.0522\ldots\cdot 10^{-10}$	$1.4517\ldots\cdot 10^{-78}$	$4.5420\ldots\cdot 10^{-597}$
$10\mathrm{i}$	$5.4915\ldots\cdot 10^{-11}$	$1.3085\ldots\cdot 10^{-78}$	$4.4878\ldots\cdot 10^{-597}$
$50\mathrm{i}$	$7.4580\ldots\cdot 10^{-9}$	$2.7720\ldots\cdot 10^{-79}$	$3.3781\ldots\cdot 10^{-597}$
$300\mathrm{i}$	$7.6348\ldots\cdot 10^{-6}$	$9.7983\ldots\cdot 10^{-64}$	$3.3932\ldots\cdot 10^{-599}$
0.2	$6.5598\ldots\cdot 10^{-11}$	$6.7519\ldots\cdot 10^{-79}$	$1.4912\ldots\cdot 10^{-597}$
$0.2+10\mathrm{i}$	$3.5213\ldots\cdot 10^{-11}$	$6.0928\ldots\cdot 10^{-79}$	$1.4735\ldots\cdot 10^{-597}$
$0.2+50\mathrm{i}$	$4.7540\ldots\cdot 10^{-9}$	$1.3218\ldots\cdot 10^{-79}$	$1.1099\ldots\cdot 10^{-597}$
$0.2+100\mathrm{i}$	$6.0238\ldots\cdot 10^{-8}$	$9.0140\ldots\cdot 10^{-78}$	$4.8119\ldots\cdot 10^{-598}$
$0.2+300\mathrm{i}$	$4.7803\ldots\cdot 10^{-6}$	$4.1428\ldots\cdot 10^{-64}$	$1.1386\ldots\cdot 10^{-599}$
$0.2+1500\mathrm{i}$	$6.5380\ldots\cdot 10^{-5}$	$4.5486\ldots\cdot 10^{-51}$	$2.2342\ldots\cdot 10^{-511}$
0.5	$3.2326\ldots\cdot 10^{-11}$	$2.1419\ldots\cdot 10^{-79}$	$2.8054\ldots\cdot 10^{-598}$
$0.5+10\mathrm{i}$	$1.8072\ldots\cdot 10^{-11}$	$1.9361\ldots\cdot 10^{-79}$	$2.7722\ldots\cdot 10^{-598}$
$0.5+50\mathrm{i}$	$2.3738\ldots\cdot 10^{-9}$	$4.3515\ldots\cdot 10^{-80}$	$2.0904\ldots\cdot 10^{-598}$
$0.5+300\mathrm{i}$	$2.1451\ldots\cdot 10^{-6}$	$1.1355\ldots\cdot 10^{-64}$	$2.2134\ldots\cdot 10^{-600}$
$0.5+1500\mathrm{i}$	$3.1818\ldots\cdot 10^{-5}$	$1.4557\ldots\cdot 10^{-51}$	$5.6042\ldots\cdot 10^{-512}$
0.8	$1.5951\ldots\cdot 10^{-11}$	$6.7960\ldots\cdot 10^{-80}$	$5.2780\ldots\cdot 10^{-599}$
$0.8+10\mathrm{i}$	$9.2696\ldots\cdot 10^{-12}$	$6.1536\ldots\cdot 10^{-80}$	$5.2156\ldots\cdot 10^{-599}$
$0.8+50\mathrm{i}$	$1.1618\ldots\cdot 10^{-9}$	$1.4322\ldots\cdot 10^{-80}$	$3.9371\ldots\cdot 10^{-599}$
$0.8+100\mathrm{i}$	$1.7304\ldots\cdot 10^{-8}$	$9.9368\ldots\cdot 10^{-79}$	$1.7181\ldots\cdot 10^{-599}$
$0.8+300\mathrm{i}$	$8.2211\ldots\cdot 10^{-7}$	$3.1098\ldots\cdot 10^{-65}$	$4.3024\ldots\cdot 10^{-601}$
$0.8+1500\mathrm{i}$	$1.6062\ldots\cdot 10^{-5}$	$4.5508\ldots\cdot 10^{-52}$	$1.3385\ldots\cdot 10^{-512}$
$1+10\mathrm{i}$	$5.9377\ldots\cdot 10^{-12}$	$2.8661\ldots\cdot 10^{-80}$	$1.7125\ldots\cdot 10^{-599}$
$1+50\mathrm{i}$	$7.1500\ldots\cdot 10^{-10}$	$6.8263\ldots\cdot 10^{-81}$	$1.2936\ldots\cdot 10^{-599}$
$1+300\mathrm{i}$	$4.0568\ldots\cdot 10^{-7}$	$1.3132\ldots\cdot 10^{-65}$	$1.4436\ldots\cdot 10^{-601}$
$1+1500\mathrm{i}$	$1.0261\ldots\cdot 10^{-5}$	$2.0145\ldots\cdot 10^{-52}$	$5.0636\ldots\cdot 10^{-513}$

Table 1: Accuracy of (3.1).

a	$\left\vert \left(1-\frac{\det(o_{1,1}(H_{0,N}))}{\det(H_{2,N-1})}\right) \Big/ \eta(a) - 1 \right\vert$ $N=10$	$N=50$	$N=250$
-5	$7.5301\ldots\cdot 10^{-5}$	$1.2259\ldots\cdot 10^{-69}$	$3.1608\ldots\cdot 10^{-476}$
-3	$1.1061\ldots\cdot 10^{-6}$	$1.1390\ldots\cdot 10^{-72}$	$1.3133\ldots\cdot 10^{-480}$
-1	$4.5163\ldots\cdot 10^{-9}$	$2.6718\ldots\cdot 10^{-76}$	$1.3660\ldots\cdot 10^{-485}$
$-1+10\mathrm{i}$	$6.3900\ldots\cdot 10^{-11}$	$7.5804\ldots\cdot 10^{-78}$	$4.2601\ldots\cdot 10^{-487}$
$-1+50\mathrm{i}$	$6.9226\ldots\cdot 10^{-10}$	$1.2761\ldots\cdot 10^{-79}$	$2.6800\ldots\cdot 10^{-488}$
$-1+300\mathrm{i}$	$4.2298\ldots\cdot 10^{-8}$	$5.6688\ldots\cdot 10^{-65}$	$1.0444\ldots\cdot 10^{-490}$
$-1+1500\mathrm{i}$	$7.6186\ldots\cdot 10^{-8}$	$3.4721\ldots\cdot 10^{-53}$	$1.7345\ldots\cdot 10^{-408}$
-0.8	$2.3164\ldots\cdot 10^{-9}$	$1.0255\ldots\cdot 10^{-76}$	$3.8377\ldots\cdot 10^{-486}$
$-0.8+10\mathrm{i}$	$5.2145\ldots\cdot 10^{-11}$	$4.4842\ldots\cdot 10^{-78}$	$1.8425\ldots\cdot 10^{-487}$
$-0.8+50\mathrm{i}$	$8.1505\ldots\cdot 10^{-10}$	$1.0722\ldots\cdot 10^{-79}$	$1.6077\ldots\cdot 10^{-488}$
$-0.8+100\mathrm{i}$	$4.8194\ldots\cdot 10^{-9}$	$3.1491\ldots\cdot 10^{-78}$	$2.5694\ldots\cdot 10^{-489}$
$-0.8+300\mathrm{i}$	$7.6112\ldots\cdot 10^{-8}$	$6.1944\ldots\cdot 10^{-65}$	$9.2674\ldots\cdot 10^{-491}$
$-0.8+1500\mathrm{i}$	$1.8583\ldots\cdot 10^{-7}$	$5.4708\ldots\cdot 10^{-53}$	$2.1761\ldots\cdot 10^{-408}$
0	$2.1045\ldots\cdot 10^{-10}$	$2.9034\ldots\cdot 10^{-78}$	$3.1164\ldots\cdot 10^{-488}$
$10\mathrm{i}$	$2.3190\ldots\cdot 10^{-11}$	$5.5258\ldots\cdot 10^{-79}$	$6.4831\ldots\cdot 10^{-489}$
$50\mathrm{i}$	$1.6860\ldots\cdot 10^{-9}$	$6.2667\ldots\cdot 10^{-80}$	$2.4448\ldots\cdot 10^{-489}$
$300\mathrm{i}$	$7.6673\ldots\cdot 10^{-7}$	$9.8400\ldots\cdot 10^{-65}$	$6.6736\ldots\cdot 10^{-491}$
0.2	$1.2060\ldots\cdot 10^{-10}$	$1.2413\ldots\cdot 10^{-78}$	$9.7493\ldots\cdot 10^{-489}$
$0.2+10\mathrm{i}$	$1.8843\ldots\cdot 10^{-11}$	$3.2604\ldots\cdot 10^{-79}$	$2.7958\ldots\cdot 10^{-489}$
$0.2+50\mathrm{i}$	$2.1979\ldots\cdot 10^{-9}$	$6.1112\ldots\cdot 10^{-80}$	$1.7033\ldots\cdot 10^{-489}$
$0.2+100\mathrm{i}$	$1.7477\ldots\cdot 10^{-8}$	$2.6153\ldots\cdot 10^{-78}$	$3.9120\ldots\cdot 10^{-490}$
$0.2+300\mathrm{i}$	$1.4406\ldots\cdot 10^{-6}$	$1.2485\ldots\cdot 10^{-64}$	$7.0012\ldots\cdot 10^{-491}$
$0.2+1500\mathrm{i}$	$1.0880\ldots\cdot 10^{-5}$	$7.5695\ldots\cdot 10^{-52}$	$6.1065\ldots\cdot 10^{-408}$
0.5	$5.3440\ldots\cdot 10^{-11}$	$3.5409\ldots\cdot 10^{-79}$	$1.7403\ldots\cdot 10^{-489}$
$0.5+10\mathrm{i}$	$1.3511\ldots\cdot 10^{-11}$	$1.4475\ldots\cdot 10^{-79}$	$7.7551\ldots\cdot 10^{-490}$
$0.5+50\mathrm{i}$	$2.8893\ldots\cdot 10^{-9}$	$5.2963\ldots\cdot 10^{-80}$	$8.9160\ldots\cdot 10^{-490}$
$0.5+300\mathrm{i}$	$3.4054\ldots\cdot 10^{-6}$	$1.8026\ldots\cdot 10^{-64}$	$7.6224\ldots\cdot 10^{-491}$
$0.5+1500\mathrm{i}$	$2.5364\ldots\cdot 10^{-5}$	$1.1604\ldots\cdot 10^{-51}$	$5.1066\ldots\cdot 10^{-408}$
0.8	$2.4173\ldots\cdot 10^{-11}$	$1.0299\ldots\cdot 10^{-79}$	$3.1672\ldots\cdot 10^{-490}$
$0.8+10\mathrm{i}$	$9.1448\ldots\cdot 10^{-12}$	$6.0708\ldots\cdot 10^{-80}$	$2.0316\ldots\cdot 10^{-490}$
$0.8+50\mathrm{i}$	$1.2768\ldots\cdot 10^{-9}$	$1.5739\ldots\cdot 10^{-80}$	$1.6008\ldots\cdot 10^{-490}$
$0.8+100\mathrm{i}$	$3.4791\ldots\cdot 10^{-8}$	$1.9978\ldots\cdot 10^{-78}$	$1.0820\ldots\cdot 10^{-490}$
$0.8+300\mathrm{i}$	$1.3073\ldots\cdot 10^{-6}$	$4.9453\ldots\cdot 10^{-65}$	$1.5779\ldots\cdot 10^{-491}$
$0.8+1500\mathrm{i}$	$1.7628\ldots\cdot 10^{-5}$	$4.9944\ldots\cdot 10^{-52}$	$1.1985\ldots\cdot 10^{-408}$
$1+10\mathrm{i}$	$6.6839\ldots\cdot 10^{-12}$	$3.2263\ldots\cdot 10^{-80}$	$7.8895\ldots\cdot 10^{-491}$
$1+50\mathrm{i}$	$6.8406\ldots\cdot 10^{-10}$	$6.5310\ldots\cdot 10^{-81}$	$4.7478\ldots\cdot 10^{-491}$
$1+300\mathrm{i}$	$6.2134\ldots\cdot 10^{-7}$	$2.0113\ldots\cdot 10^{-65}$	$5.3122\ldots\cdot 10^{-492}$
$1+1500\mathrm{i}$	$1.0491\ldots\cdot 10^{-5}$	$2.0596\ldots\cdot 10^{-52}$	$3.4171\ldots\cdot 10^{-409}$

Table 2: Accuracy of calculation of $\eta(a)=\eta_0$ via higher derivatives by (3.4).

a	$\left\| -\frac{\det(o_{N,N}(H_{0,N}))-\det(o_{N-1,N-1}(H_{2,N-1}))}{\det(H_{0,N-1})-\det(H_{2,N-2})} \Big/ \eta_{2N-2} - 1 \right\|$		
	$N=10$	$N=50$	$N=250$
-5	$1.3436\ldots\cdot 10^{-6}$	$6.0581\ldots\cdot 10^{-62}$	$1.4627\ldots\cdot 10^{-415}$
-3	$3.4638\ldots\cdot 10^{-7}$	$3.7248\ldots\cdot 10^{-63}$	$5.5681\ldots\cdot 10^{-417}$
-1	$6.8628\ldots\cdot 10^{-6}$	$4.3915\ldots\cdot 10^{-64}$	$3.2403\ldots\cdot 10^{-418}$
$-1+10\mathrm{i}$	$3.4745\ldots\cdot 10^{-11}$	$1.4430\ldots\cdot 10^{-68}$	$2.2426\ldots\cdot 10^{-423}$
$-1+50\mathrm{i}$	$4.7607\ldots\cdot 10^{-13}$	$4.1178\ldots\cdot 10^{-81}$	$2.8844\ldots\cdot 10^{-442}$
$-1+300\mathrm{i}$	$5.3962\ldots\cdot 10^{-13}$	$1.3409\ldots\cdot 10^{-80}$	$2.3580\ldots\cdot 10^{-502}$
$-1+1500\mathrm{i}$	$1.6149\ldots\cdot 10^{-13}$	$2.5937\ldots\cdot 10^{-76}$	$1.4030\ldots\cdot 10^{-466}$
-0.8	$2.8153\ldots\cdot 10^{-7}$	$2.8840\ldots\cdot 10^{-64}$	$1.9215\ldots\cdot 10^{-418}$
$-0.8+10\mathrm{i}$	$3.2097\ldots\cdot 10^{-11}$	$1.1497\ldots\cdot 10^{-68}$	$1.6356\ldots\cdot 10^{-423}$
$-0.8+50\mathrm{i}$	$5.2253\ldots\cdot 10^{-13}$	$3.6350\ldots\cdot 10^{-81}$	$2.1527\ldots\cdot 10^{-442}$
$-0.8+100\mathrm{i}$	$3.5280\ldots\cdot 10^{-13}$	$9.2338\ldots\cdot 10^{-86}$	$2.0288\ldots\cdot 10^{-461}$
$-0.8+300\mathrm{i}$	$8.4538\ldots\cdot 10^{-13}$	$1.4181\ldots\cdot 10^{-80}$	$2.0953\ldots\cdot 10^{-502}$
$-0.8+1500\mathrm{i}$	$2.6271\ldots\cdot 10^{-13}$	$3.8861\ldots\cdot 10^{-76}$	$1.6594\ldots\cdot 10^{-466}$
0	$3.2940\ldots\cdot 10^{-8}$	$9.9375\ldots\cdot 10^{-65}$	$5.0285\ldots\cdot 10^{-419}$
$10\mathrm{i}$	$2.2954\ldots\cdot 10^{-11}$	$4.6062\ldots\cdot 10^{-69}$	$4.6179\ldots\cdot 10^{-424}$
$50\mathrm{i}$	$7.0287\ldots\cdot 10^{-13}$	$2.1968\ldots\cdot 10^{-81}$	$6.6666\ldots\cdot 10^{-443}$
$300\mathrm{i}$	$3.4591\ldots\cdot 10^{-12}$	$1.7054\ldots\cdot 10^{-80}$	$1.3051\ldots\cdot 10^{-502}$
0.2	$2.4966\ldots\cdot 10^{-8}$	$8.7749\ldots\cdot 10^{-65}$	$4.2864\ldots\cdot 10^{-419}$
$0.2+10\mathrm{i}$	$2.1014\ldots\cdot 10^{-11}$	$3.6590\ldots\cdot 10^{-69}$	$3.3643\ldots\cdot 10^{-424}$
$0.2+50\mathrm{i}$	$7.4415\ldots\cdot 10^{-13}$	$1.9347\ldots\cdot 10^{-81}$	$4.9708\ldots\cdot 10^{-443}$
$0.2+100\mathrm{i}$	$9.2070\ldots\cdot 10^{-13}$	$7.6778\ldots\cdot 10^{-86}$	$5.5639\ldots\cdot 10^{-462}$
$0.2+300\mathrm{i}$	$4.1178\ldots\cdot 10^{-12}$	$1.7743\ldots\cdot 10^{-80}$	$1.1592\ldots\cdot 10^{-502}$
$0.2+1500\mathrm{i}$	$5.5561\ldots\cdot 10^{-12}$	$3.3953\ldots\cdot 10^{-75}$	$2.6687\ldots\cdot 10^{-466}$
0.5	$1.7814\ldots\cdot 10^{-8}$	$8.5506\ldots\cdot 10^{-65}$	$4.2323\ldots\cdot 10^{-419}$
$0.5+10\mathrm{i}$	$1.8347\ldots\cdot 10^{-11}$	$2.5877\ldots\cdot 10^{-69}$	$2.0911\ldots\cdot 10^{-424}$
$0.5+50\mathrm{i}$	$8.0236\ldots\cdot 10^{-13}$	$1.5975\ldots\cdot 10^{-81}$	$3.1993\ldots\cdot 10^{-443}$
$0.5+300\mathrm{i}$	$4.7272\ldots\cdot 10^{-12}$	$1.8811\ldots\cdot 10^{-80}$	$9.7024\ldots\cdot 10^{-503}$
$0.5+1500\mathrm{i}$	$1.6269\ldots\cdot 10^{-11}$	$6.1649\ldots\cdot 10^{-75}$	$2.8429\ldots\cdot 10^{-466}$
0.8	$1.3762\ldots\cdot 10^{-8}$	$1.2648\ldots\cdot 10^{-64}$	$9.9698\ldots\cdot 10^{-419}$
$0.8+10\mathrm{i}$	$1.5956\ldots\cdot 10^{-11}$	$1.8275\ldots\cdot 10^{-69}$	$1.2991\ldots\cdot 10^{-424}$
$0.8+50\mathrm{i}$	$8.5634\ldots\cdot 10^{-13}$	$1.3177\ldots\cdot 10^{-81}$	$2.0582\ldots\cdot 10^{-443}$
$0.8+100\mathrm{i}$	$1.6841\ldots\cdot 10^{-12}$	$6.8529\ldots\cdot 10^{-86}$	$2.5554\ldots\cdot 10^{-462}$
$0.8+300\mathrm{i}$	$5.2215\ldots\cdot 10^{-12}$	$1.9982\ldots\cdot 10^{-80}$	$8.1191\ldots\cdot 10^{-503}$
$0.8+1500\mathrm{i}$	$4.4151\ldots\cdot 10^{-11}$	$1.0055\ldots\cdot 10^{-74}$	$3.0005\ldots\cdot 10^{-466}$
$1+10\mathrm{i}$	$1.4507\ldots\cdot 10^{-11}$	$1.4482\ldots\cdot 10^{-69}$	$9.4560\ldots\cdot 10^{-425}$
$1+50\mathrm{i}$	$8.9045\ldots\cdot 10^{-13}$	$1.1583\ldots\cdot 10^{-81}$	$1.5335\ldots\cdot 10^{-443}$
$1+300\mathrm{i}$	$5.7557\ldots\cdot 10^{-12}$	$2.0861\ldots\cdot 10^{-80}$	$7.2091\ldots\cdot 10^{-503}$
$1+1500\mathrm{i}$	$7.2337\ldots\cdot 10^{-11}$	$1.3177\ldots\cdot 10^{-74}$	$3.1141\ldots\cdot 10^{-466}$

Table 3: Accuracy of calculation of η_{2N-2} via lower derivatives by (3.7).

a	$\left\lvert \frac{\det(O_{1,1}(H_{l,N}))}{\det(o_{1,1}(H_{l,N}))} + 1 \right\rvert$ $l = 1$	$l = 2$	$l = 3$
-6	$1.2570\ldots \cdot 10^{-473}$	$4.6911\ldots \cdot 10^{-473}$	$8.8868\ldots \cdot 10^{-472}$
-5	$3.0660\ldots \cdot 10^{-475}$	$4.7281\ldots \cdot 10^{-475}$	$1.3692\ldots \cdot 10^{-474}$
-4	$7.8997\ldots \cdot 10^{-478}$	$6.0206\ldots \cdot 10^{-477}$	$1.6580\ldots \cdot 10^{-476}$
-3	$7.5908\ldots \cdot 10^{-479}$	$1.7979\ldots \cdot 10^{-479}$	$2.9281\ldots \cdot 10^{-478}$
-2	$1.9066\ldots \cdot 10^{-482}$	$1.7984\ldots \cdot 10^{-481}$	$7.4646\ldots \cdot 10^{-481}$
-1	$6.9890\ldots \cdot 10^{-485}$	$1.5418\ldots \cdot 10^{-482}$	$6.0125\ldots \cdot 10^{-483}$
$-1+10$i	$1.8786\ldots \cdot 10^{-486}$	$8.5257\ldots \cdot 10^{-486}$	$3.6760\ldots \cdot 10^{-485}$
$-1+50$i	$5.1478\ldots \cdot 10^{-488}$	$1.0065\ldots \cdot 10^{-487}$	$1.9705\ldots \cdot 10^{-487}$
$-1+300$i	$1.1839\ldots \cdot 10^{-490}$	$1.3490\ldots \cdot 10^{-490}$	$1.5417\ldots \cdot 10^{-490}$
$-1+1500$i	$1.3957\ldots \cdot 10^{-408}$	$1.1161\ldots \cdot 10^{-408}$	$8.8674\ldots \cdot 10^{-409}$
-0.8	$2.4055\ldots \cdot 10^{-485}$	$1.4821\ldots \cdot 10^{-483}$	$2.4768\ldots \cdot 10^{-483}$
$-0.8+10$i	$8.0737\ldots \cdot 10^{-487}$	$3.7016\ldots \cdot 10^{-486}$	$1.6108\ldots \cdot 10^{-485}$
$-0.8+50$i	$3.0514\ldots \cdot 10^{-488}$	$5.9634\ldots \cdot 10^{-488}$	$1.1675\ldots \cdot 10^{-487}$
$-0.8+100$i	$3.9064\ldots \cdot 10^{-489}$	$5.9417\ldots \cdot 10^{-489}$	$9.0828\ldots \cdot 10^{-489}$
$-0.8+300$i	$1.0442\ldots \cdot 10^{-490}$	$1.1863\ldots \cdot 10^{-490}$	$1.3528\ldots \cdot 10^{-490}$
$-0.8+1500$i	$1.7643\ldots \cdot 10^{-408}$	$1.4218\ldots \cdot 10^{-408}$	$1.1386\ldots \cdot 10^{-408}$
0	$3.7455\ldots \cdot 10^{-487}$	$7.5211\ldots \cdot 10^{-486}$	$1.0612\ldots \cdot 10^{-484}$
10i	$2.7027\ldots \cdot 10^{-488}$	$1.2935\ldots \cdot 10^{-487}$	$5.8482\ldots \cdot 10^{-487}$
50i	$3.7612\ldots \cdot 10^{-489}$	$7.3299\ldots \cdot 10^{-489}$	$1.4365\ldots \cdot 10^{-488}$
300i	$6.6469\ldots \cdot 10^{-491}$	$7.2979\ldots \cdot 10^{-491}$	$8.1827\ldots \cdot 10^{-491}$
0.2	$1.3500\ldots \cdot 10^{-487}$	$2.4114\ldots \cdot 10^{-486}$	$5.9658\ldots \cdot 10^{-485}$
$0.2+10$i	$1.1502\ldots \cdot 10^{-488}$	$5.5684\ldots \cdot 10^{-488}$	$2.5426\ldots \cdot 10^{-487}$
$0.2+50$i	$2.2223\ldots \cdot 10^{-489}$	$4.3326\ldots \cdot 10^{-489}$	$8.5006\ldots \cdot 10^{-489}$
$0.2+100$i	$6.2742\ldots \cdot 10^{-490}$	$9.9145\ldots \cdot 10^{-490}$	$1.4813\ldots \cdot 10^{-489}$
$0.2+300$i	$6.1020\ldots \cdot 10^{-491}$	$6.5501\ldots \cdot 10^{-491}$	$7.2715\ldots \cdot 10^{-491}$
$0.2+1500$i	$5.3054\ldots \cdot 10^{-408}$	$4.2124\ldots \cdot 10^{-408}$	$3.4382\ldots \cdot 10^{-408}$
0.5	$2.9557\ldots \cdot 10^{-488}$	$4.6088\ldots \cdot 10^{-487}$	$5.6212\ldots \cdot 10^{-485}$
$0.5+10$i	$3.1801\ldots \cdot 10^{-489}$	$1.5672\ldots \cdot 10^{-488}$	$7.2669\ldots \cdot 10^{-488}$
$0.5+50$i	$1.0002\ldots \cdot 10^{-489}$	$1.9614\ldots \cdot 10^{-489}$	$3.8629\ldots \cdot 10^{-489}$
$0.5+300$i	$5.4870\ldots \cdot 10^{-491}$	$5.6936\ldots \cdot 10^{-491}$	$6.1546\ldots \cdot 10^{-491}$
$0.5+1500$i	$9.1374\ldots \cdot 10^{-408}$	$6.1492\ldots \cdot 10^{-408}$	$4.7074\ldots \cdot 10^{-408}$
0.8	$6.5519\ldots \cdot 10^{-489}$	$9.2061\ldots \cdot 10^{-488}$	$6.3007\ldots \cdot 10^{-486}$
$0.8+10$i	$8.7460\ldots \cdot 10^{-490}$	$4.3916\ldots \cdot 10^{-489}$	$2.0689\ldots \cdot 10^{-488}$
$0.8+50$i	$4.3820\ldots \cdot 10^{-490}$	$8.7933\ldots \cdot 10^{-490}$	$1.7487\ldots \cdot 10^{-489}$
$0.8+100$i	$1.5554\ldots \cdot 10^{-490}$	$3.3560\ldots \cdot 10^{-490}$	$5.2884\ldots \cdot 10^{-490}$
$0.8+300$i	$3.8570\ldots \cdot 10^{-491}$	$5.1347\ldots \cdot 10^{-491}$	$5.3172\ldots \cdot 10^{-491}$
$0.8+1500$i	$1.0273\ldots \cdot 10^{-407}$	$1.3983\ldots \cdot 10^{-407}$	$7.0962\ldots \cdot 10^{-408}$
$1+10$i	$3.6872\ldots \cdot 10^{-490}$	$1.8757\ldots \cdot 10^{-489}$	$8.9345\ldots \cdot 10^{-489}$
$1+50$i	$2.4484\ldots \cdot 10^{-490}$	$5.0953\ldots \cdot 10^{-490}$	$1.0267\ldots \cdot 10^{-489}$
$1+300$i	$2.0018\ldots \cdot 10^{-491}$	$4.6927\ldots \cdot 10^{-491}$	$4.9105\ldots \cdot 10^{-491}$
$1+1500$i	$2.9114\ldots \cdot 10^{-408}$	$3.0689\ldots \cdot 10^{-407}$	$1.1453\ldots \cdot 10^{-407}$

Table 4: Accuracy of (3.8) for $N = 250$.

a	$\left\| \frac{\det(o_{1,1}(H_{l,N}))}{\det(H_{l+2,N-1})} \Big/ \eta_l + 1 \right\|$ $l=1$	$l=2$	$l=3$
-5	$3.0660\ldots \cdot 10^{-475}$	$4.7281\ldots \cdot 10^{-475}$	$1.3692\ldots \cdot 10^{-474}$
-3	$7.5908\ldots \cdot 10^{-479}$	$1.7979\ldots \cdot 10^{-479}$	$2.9281\ldots \cdot 10^{-478}$
-1	$6.9890\ldots \cdot 10^{-485}$	$1.5418\ldots \cdot 10^{-482}$	$6.0125\ldots \cdot 10^{-483}$
$-1+10i$	$1.8786\ldots \cdot 10^{-486}$	$8.5257\ldots \cdot 10^{-486}$	$3.6760\ldots \cdot 10^{-485}$
$-1+50i$	$5.1478\ldots \cdot 10^{-488}$	$1.0065\ldots \cdot 10^{-487}$	$1.9705\ldots \cdot 10^{-487}$
$-1+300i$	$1.1839\ldots \cdot 10^{-490}$	$1.3490\ldots \cdot 10^{-490}$	$1.5417\ldots \cdot 10^{-490}$
$-1+1500i$	$1.3957\ldots \cdot 10^{-408}$	$1.1161\ldots \cdot 10^{-408}$	$8.8674\ldots \cdot 10^{-409}$
-0.8	$2.4055\ldots \cdot 10^{-485}$	$1.4821\ldots \cdot 10^{-483}$	$2.4768\ldots \cdot 10^{-483}$
$-0.8+10i$	$8.0737\ldots \cdot 10^{-487}$	$3.7016\ldots \cdot 10^{-486}$	$1.6108\ldots \cdot 10^{-485}$
$-0.8+50i$	$3.0514\ldots \cdot 10^{-488}$	$5.9634\ldots \cdot 10^{-488}$	$1.1675\ldots \cdot 10^{-487}$
$-0.8+100i$	$3.9064\ldots \cdot 10^{-489}$	$5.9417\ldots \cdot 10^{-489}$	$9.0828\ldots \cdot 10^{-489}$
$-0.8+300i$	$1.0442\ldots \cdot 10^{-490}$	$1.1863\ldots \cdot 10^{-490}$	$1.3528\ldots \cdot 10^{-490}$
$-0.8+1500i$	$1.7643\ldots \cdot 10^{-408}$	$1.4218\ldots \cdot 10^{-408}$	$1.1386\ldots \cdot 10^{-408}$
0	$3.7455\ldots \cdot 10^{-487}$	$7.5211\ldots \cdot 10^{-486}$	$1.0612\ldots \cdot 10^{-484}$
$10i$	$2.7027\ldots \cdot 10^{-488}$	$1.2935\ldots \cdot 10^{-487}$	$5.8482\ldots \cdot 10^{-487}$
$50i$	$3.7612\ldots \cdot 10^{-489}$	$7.3299\ldots \cdot 10^{-489}$	$1.4365\ldots \cdot 10^{-488}$
$300i$	$6.6469\ldots \cdot 10^{-491}$	$7.2979\ldots \cdot 10^{-491}$	$8.1827\ldots \cdot 10^{-491}$
0.2	$1.3500\ldots \cdot 10^{-487}$	$2.4114\ldots \cdot 10^{-486}$	$5.9658\ldots \cdot 10^{-485}$
$0.2+10i$	$1.1502\ldots \cdot 10^{-488}$	$5.5684\ldots \cdot 10^{-488}$	$2.5426\ldots \cdot 10^{-487}$
$0.2+50i$	$2.2223\ldots \cdot 10^{-489}$	$4.3326\ldots \cdot 10^{-489}$	$8.5006\ldots \cdot 10^{-489}$
$0.2+100i$	$6.2742\ldots \cdot 10^{-490}$	$9.9145\ldots \cdot 10^{-490}$	$1.4813\ldots \cdot 10^{-489}$
$0.2+300i$	$6.1020\ldots \cdot 10^{-491}$	$6.5501\ldots \cdot 10^{-491}$	$7.2715\ldots \cdot 10^{-491}$
$0.2+1500i$	$5.3054\ldots \cdot 10^{-408}$	$4.2124\ldots \cdot 10^{-408}$	$3.4382\ldots \cdot 10^{-408}$
0.5	$2.9557\ldots \cdot 10^{-488}$	$4.6088\ldots \cdot 10^{-487}$	$5.6212\ldots \cdot 10^{-485}$
$0.5+10i$	$3.1801\ldots \cdot 10^{-489}$	$1.5672\ldots \cdot 10^{-488}$	$7.2669\ldots \cdot 10^{-488}$
$0.5+50i$	$1.0002\ldots \cdot 10^{-489}$	$1.9614\ldots \cdot 10^{-489}$	$3.8629\ldots \cdot 10^{-489}$
$0.5+300i$	$5.4870\ldots \cdot 10^{-491}$	$5.6936\ldots \cdot 10^{-491}$	$6.1546\ldots \cdot 10^{-491}$
$0.5+1500i$	$9.1374\ldots \cdot 10^{-408}$	$6.1492\ldots \cdot 10^{-408}$	$4.7074\ldots \cdot 10^{-408}$
0.8	$6.5519\ldots \cdot 10^{-489}$	$9.2061\ldots \cdot 10^{-488}$	$6.3007\ldots \cdot 10^{-486}$
$0.8+10i$	$8.7460\ldots \cdot 10^{-490}$	$4.3916\ldots \cdot 10^{-489}$	$2.0689\ldots \cdot 10^{-488}$
$0.8+50i$	$4.3820\ldots \cdot 10^{-490}$	$8.7933\ldots \cdot 10^{-490}$	$1.7487\ldots \cdot 10^{-489}$
$0.8+100i$	$1.5554\ldots \cdot 10^{-490}$	$3.3560\ldots \cdot 10^{-490}$	$5.2884\ldots \cdot 10^{-490}$
$0.8+300i$	$3.8570\ldots \cdot 10^{-491}$	$5.1347\ldots \cdot 10^{-491}$	$5.3172\ldots \cdot 10^{-491}$
$0.8+1500i$	$1.0273\ldots \cdot 10^{-407}$	$1.3983\ldots \cdot 10^{-407}$	$7.0962\ldots \cdot 10^{-408}$
$1+10i$	$3.6872\ldots \cdot 10^{-490}$	$1.8757\ldots \cdot 10^{-489}$	$8.9345\ldots \cdot 10^{-489}$
$1+50i$	$2.4484\ldots \cdot 10^{-490}$	$5.0953\ldots \cdot 10^{-490}$	$1.0267\ldots \cdot 10^{-489}$
$1+300i$	$2.0018\ldots \cdot 10^{-491}$	$4.6927\ldots \cdot 10^{-491}$	$4.9105\ldots \cdot 10^{-491}$
$1+1500i$	$2.9114\ldots \cdot 10^{-408}$	$3.0689\ldots \cdot 10^{-407}$	$1.1453\ldots \cdot 10^{-407}$

Table 5: Accuracy of calculation of η_l via higher derivatives by (3.11) for $N = 250$.

a	$\left\lvert \frac{\det(o_{N,N}(H_{l,N}))}{\det(H_{l,N-1})} \Big/ \eta_{2N+l-2} + 1 \right\rvert$		
	$l=1$	$l=2$	$l=3$
-5	$1.7929\ldots \cdot 10^{-415}$	$2.4518\ldots \cdot 10^{-415}$	$3.7466\ldots \cdot 10^{-415}$
-3	$8.4727\ldots \cdot 10^{-417}$	$1.4822\ldots \cdot 10^{-416}$	$3.3034\ldots \cdot 10^{-416}$
-1	$7.1541\ldots \cdot 10^{-418}$	$3.4916\ldots \cdot 10^{-417}$	$2.8423\ldots \cdot 10^{-417}$
$-1+10\mathrm{i}$	$3.0597\ldots \cdot 10^{-423}$	$4.1729\ldots \cdot 10^{-423}$	$5.6889\ldots \cdot 10^{-423}$
$-1+50\mathrm{i}$	$3.8823\ldots \cdot 10^{-442}$	$5.2237\ldots \cdot 10^{-442}$	$7.0261\ldots \cdot 10^{-442}$
$-1+300\mathrm{i}$	$2.6564\ldots \cdot 10^{-502}$	$2.9923\ldots \cdot 10^{-502}$	$3.3704\ldots \cdot 10^{-502}$
$-1+1500\mathrm{i}$	$1.1777\ldots \cdot 10^{-466}$	$9.7267\ldots \cdot 10^{-467}$	$7.9210\ldots \cdot 10^{-467}$
-0.8	$3.4929\ldots \cdot 10^{-418}$	$8.4371\ldots \cdot 10^{-418}$	$9.0130\ldots \cdot 10^{-417}$
$-0.8+10\mathrm{i}$	$2.2322\ldots \cdot 10^{-423}$	$3.0453\ldots \cdot 10^{-423}$	$4.1529\ldots \cdot 10^{-423}$
$-0.8+50\mathrm{i}$	$2.8983\ldots \cdot 10^{-442}$	$3.9007\ldots \cdot 10^{-442}$	$5.2479\ldots \cdot 10^{-442}$
$-0.8+100\mathrm{i}$	$2.6510\ldots \cdot 10^{-461}$	$3.4631\ldots \cdot 10^{-461}$	$4.5226\ldots \cdot 10^{-461}$
$-0.8+300\mathrm{i}$	$2.3607\ldots \cdot 10^{-502}$	$2.6594\ldots \cdot 10^{-502}$	$2.9956\ldots \cdot 10^{-502}$
$-0.8+1500\mathrm{i}$	$1.4223\ldots \cdot 10^{-466}$	$1.1964\ldots \cdot 10^{-466}$	$9.9004\ldots \cdot 10^{-467}$
0	$6.2498\ldots \cdot 10^{-419}$	$8.6588\ldots \cdot 10^{-419}$	$1.3410\ldots \cdot 10^{-418}$
$10\mathrm{i}$	$6.3100\ldots \cdot 10^{-424}$	$8.6187\ldots \cdot 10^{-424}$	$1.1767\ldots \cdot 10^{-423}$
$50\mathrm{i}$	$8.9849\ldots \cdot 10^{-443}$	$1.2105\ldots \cdot 10^{-442}$	$1.6303\ldots \cdot 10^{-442}$
$300\mathrm{i}$	$1.4710\ldots \cdot 10^{-502}$	$1.6577\ldots \cdot 10^{-502}$	$1.8680\ldots \cdot 10^{-502}$
0.2	$4.8293\ldots \cdot 10^{-419}$	$6.1614\ldots \cdot 10^{-419}$	$8.7491\ldots \cdot 10^{-419}$
$0.2+10\mathrm{i}$	$4.5984\ldots \cdot 10^{-424}$	$6.2828\ldots \cdot 10^{-424}$	$8.5810\ldots \cdot 10^{-424}$
$0.2+50\mathrm{i}$	$6.7011\ldots \cdot 10^{-443}$	$9.0306\ldots \cdot 10^{-443}$	$1.2165\ldots \cdot 10^{-442}$
$0.2+100\mathrm{i}$	$7.2778\ldots \cdot 10^{-462}$	$9.5169\ldots \cdot 10^{-462}$	$1.2441\ldots \cdot 10^{-461}$
$0.2+300\mathrm{i}$	$1.3066\ldots \cdot 10^{-502}$	$1.4727\ldots \cdot 10^{-502}$	$1.6596\ldots \cdot 10^{-502}$
$0.2+1500\mathrm{i}$	$2.5326\ldots \cdot 10^{-466}$	$2.3725\ldots \cdot 10^{-466}$	$2.1861\ldots \cdot 10^{-466}$
0.5	$3.7828\ldots \cdot 10^{-419}$	$4.1666\ldots \cdot 10^{-419}$	$5.2258\ldots \cdot 10^{-419}$
$0.5+10\mathrm{i}$	$2.8595\ldots \cdot 10^{-424}$	$3.9087\ldots \cdot 10^{-424}$	$5.3409\ldots \cdot 10^{-424}$
$0.5+50\mathrm{i}$	$4.3146\ldots \cdot 10^{-443}$	$5.8168\ldots \cdot 10^{-443}$	$7.8393\ldots \cdot 10^{-443}$
$0.5+300\mathrm{i}$	$1.0938\ldots \cdot 10^{-502}$	$1.2329\ldots \cdot 10^{-502}$	$1.3896\ldots \cdot 10^{-502}$
$0.5+1500\mathrm{i}$	$2.7338\ldots \cdot 10^{-466}$	$2.6105\ldots \cdot 10^{-466}$	$2.4667\ldots \cdot 10^{-466}$
0.8	$3.9356\ldots \cdot 10^{-419}$	$3.3536\ldots \cdot 10^{-419}$	$3.6054\ldots \cdot 10^{-419}$
$0.8+10\mathrm{i}$	$1.7773\ldots \cdot 10^{-424}$	$2.4305\ldots \cdot 10^{-424}$	$3.3226\ldots \cdot 10^{-424}$
$0.8+50\mathrm{i}$	$2.7769\ldots \cdot 10^{-443}$	$3.7452\ldots \cdot 10^{-443}$	$5.0494\ldots \cdot 10^{-443}$
$0.8+100\mathrm{i}$	$3.3446\ldots \cdot 10^{-462}$	$4.3764\ldots \cdot 10^{-462}$	$5.7249\ldots \cdot 10^{-462}$
$0.8+300\mathrm{i}$	$9.1542\ldots \cdot 10^{-503}$	$1.0320\ldots \cdot 10^{-502}$	$1.1633\ldots \cdot 10^{-502}$
$0.8+1500\mathrm{i}$	$2.8976\ldots \cdot 10^{-466}$	$2.7933\ldots \cdot 10^{-466}$	$2.6798\ldots \cdot 10^{-466}$
$1+10\mathrm{i}$	$1.2940\ldots \cdot 10^{-424}$	$1.7702\ldots \cdot 10^{-424}$	$2.4207\ldots \cdot 10^{-424}$
$1+50\mathrm{i}$	$2.0695\ldots \cdot 10^{-443}$	$2.7918\ldots \cdot 10^{-443}$	$3.7650\ldots \cdot 10^{-443}$
$1+300\mathrm{i}$	$8.1289\ldots \cdot 10^{-503}$	$9.1652\ldots \cdot 10^{-503}$	$1.0332\ldots \cdot 10^{-502}$
$1+1500\mathrm{i}$	$3.0032\ldots \cdot 10^{-466}$	$2.8999\ldots \cdot 10^{-466}$	$2.7958\ldots \cdot 10^{-466}$

Table 6: Accuracy of calculation of η_{2N+l-2} via some of the lower derivatives by (3.14) for $N = 250$.

N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	1	$1.1377\ldots \cdot 10^{-8}$	$1.3768\ldots \cdot 10^{-8}$	$1.2225\ldots \cdot 10^{-7}$
	2	$2.7517\ldots \cdot 10^{-6}$	$3.3683\ldots \cdot 10^{-6}$	$5.8080\ldots \cdot 10^{-5}$
	3	$2.1547\ldots \cdot 10^{-4}$	$3.7669\ldots \cdot 10^{-4}$	$2.3603\ldots \cdot 10^{-4}$
	4	$3.8367\ldots \cdot 10^{-3}$	$4.3449\ldots \cdot 10^{-3}$	$3.3790\ldots \cdot 10^{-3}$
	5	$1.5643\ldots \cdot 10^{-2}$	$9.0714\ldots \cdot 10^{-3}$	$9.5718\ldots \cdot 10^{-3}$
20	1	$4.4077\ldots \cdot 10^{-23}$	$1.8088\ldots \cdot 10^{-23}$	$1.0648\ldots \cdot 10^{-23}$
	2	$8.2766\ldots \cdot 10^{-20}$	$2.3273\ldots \cdot 10^{-20}$	$1.2923\ldots \cdot 10^{-20}$
	3	$9.0780\ldots \cdot 10^{-17}$	$1.8254\ldots \cdot 10^{-17}$	$9.2353\ldots \cdot 10^{-18}$
	4	$6.2715\ldots \cdot 10^{-14}$	$8.9987\ldots \cdot 10^{-15}$	$4.1990\ldots \cdot 10^{-15}$
	5	$2.8211\ldots \cdot 10^{-11}$	$2.8009\ldots \cdot 10^{-12}$	$1.2775\ldots \cdot 10^{-12}$
30	1	$2.0015\ldots \cdot 10^{-39}$	$7.8669\ldots \cdot 10^{-40}$	$2.7528\ldots \cdot 10^{-40}$
	2	$9.1204\ldots \cdot 10^{-36}$	$2.5441\ldots \cdot 10^{-36}$	$7.7022\ldots \cdot 10^{-37}$
	3	$2.6140\ldots \cdot 10^{-32}$	$5.4848\ldots \cdot 10^{-33}$	$1.3437\ldots \cdot 10^{-33}$
	4	$5.1463\ldots \cdot 10^{-29}$	$8.3011\ldots \cdot 10^{-30}$	$1.5845\ldots \cdot 10^{-30}$
	5	$7.3092\ldots \cdot 10^{-26}$	$9.1458\ldots \cdot 10^{-27}$	$1.3276\ldots \cdot 10^{-27}$
40	1	$8.3351\ldots \cdot 10^{-57}$	$3.2716\ldots \cdot 10^{-57}$	$1.1409\ldots \cdot 10^{-57}$
	2	$6.6383\ldots \cdot 10^{-53}$	$1.8815\ldots \cdot 10^{-53}$	$5.7668\ldots \cdot 10^{-54}$
	3	$3.4166\ldots \cdot 10^{-49}$	$7.4426\ldots \cdot 10^{-50}$	$1.8822\ldots \cdot 10^{-50}$
	4	$1.2436\ldots \cdot 10^{-45}$	$2.1396\ldots \cdot 10^{-46}$	$4.3185\ldots \cdot 10^{-47}$
	5	$3.3707\ldots \cdot 10^{-42}$	$4.6547\ldots \cdot 10^{-43}$	$7.3611\ldots \cdot 10^{-44}$
50	1	$7.2502\ldots \cdot 10^{-75}$	$2.8187\ldots \cdot 10^{-75}$	$1.0400\ldots \cdot 10^{-75}$
	2	$8.6785\ldots \cdot 10^{-71}$	$2.4551\ldots \cdot 10^{-71}$	$8.0795\ldots \cdot 10^{-72}$
	3	$6.8087\ldots \cdot 10^{-67}$	$1.4937\ldots \cdot 10^{-67}$	$4.1265\ldots \cdot 10^{-68}$
	4	$3.8344\ldots \cdot 10^{-63}$	$6.7152\ldots \cdot 10^{-64}$	$1.5105\ldots \cdot 10^{-64}$
	5	$1.6333\ldots \cdot 10^{-59}$	$2.3248\ldots \cdot 10^{-60}$	$4.1963\ldots \cdot 10^{-61}$

Table 7: The maximal distance between a root of equation (3.15) and η_k.

N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	12	$1.8326\ldots \cdot 10^{-1}$	$1.6970\ldots \cdot 10^{-1}$	$1.4110\ldots \cdot 10^{-1}$
	13	$1.1540\ldots \cdot 10^{-2}$	$1.4505\ldots \cdot 10^{-2}$	$6.8455\ldots \cdot 10^{-3}$
	14	$4.3628\ldots \cdot 10^{-4}$	$3.5381\ldots \cdot 10^{-4}$	$2.1989\ldots \cdot 10^{-4}$
	15	$1.0039\ldots \cdot 10^{-5}$	$5.5894\ldots \cdot 10^{-6}$	$9.3600\ldots \cdot 10^{-6}$
	16	$1.3698\ldots \cdot 10^{-7}$	$5.2669\ldots \cdot 10^{-8}$	$9.3835\ldots \cdot 10^{-8}$
	17	$1.0103\ldots \cdot 10^{-9}$	$2.7187\ldots \cdot 10^{-10}$	$3.6740\ldots \cdot 10^{-10}$
20	32	$8.6967\ldots \cdot 10^{-12}$	$5.0177\ldots \cdot 10^{-13}$	$5.8393\ldots \cdot 10^{-14}$
	33	$1.1573\ldots \cdot 10^{-13}$	$5.3793\ldots \cdot 10^{-15}$	$4.3771\ldots \cdot 10^{-16}$
	34	$1.1060\ldots \cdot 10^{-15}$	$4.2498\ldots \cdot 10^{-17}$	$2.4661\ldots \cdot 10^{-18}$
	35	$7.3442\ldots \cdot 10^{-18}$	$2.3872\ldots \cdot 10^{-19}$	$1.0093\ldots \cdot 10^{-20}$
	36	$3.2016\ldots \cdot 10^{-20}$	$8.9807\ldots \cdot 10^{-22}$	$2.8290\ldots \cdot 10^{-23}$
	37	$8.2051\ldots \cdot 10^{-23}$	$2.0203\ldots \cdot 10^{-24}$	$4.8482\ldots \cdot 10^{-26}$
30	52	$3.4903\ldots \cdot 10^{-24}$	$4.6565\ldots \cdot 10^{-26}$	$2.8602\ldots \cdot 10^{-28}$
	53	$1.9161\ldots \cdot 10^{-26}$	$2.3639\ldots \cdot 10^{-28}$	$1.2219\ldots \cdot 10^{-30}$
	54	$7.9706\ldots \cdot 10^{-29}$	$9.1562\ldots \cdot 10^{-31}$	$4.0406\ldots \cdot 10^{-33}$
	55	$2.4224\ldots \cdot 10^{-31}$	$2.6064\ldots \cdot 10^{-33}$	$9.9484\ldots \cdot 10^{-36}$
	56	$5.0636\ldots \cdot 10^{-34}$	$5.1289\ldots \cdot 10^{-36}$	$1.7130\ldots \cdot 10^{-38}$
	57	$6.4959\ldots \cdot 10^{-37}$	$6.2212\ldots \cdot 10^{-39}$	$1.8370\ldots \cdot 10^{-41}$
40	72	$6.6609\ldots \cdot 10^{-38}$	$4.5103\ldots \cdot 10^{-40}$	$6.3070\ldots \cdot 10^{-43}$
	73	$2.0396\ldots \cdot 10^{-40}$	$1.3257\ldots \cdot 10^{-42}$	$1.7032\ldots \cdot 10^{-45}$
	74	$4.8672\ldots \cdot 10^{-43}$	$3.0438\ldots \cdot 10^{-45}$	$3.6120\ldots \cdot 10^{-48}$
	75	$8.7104\ldots \cdot 10^{-46}$	$5.2512\ldots \cdot 10^{-48}$	$5.7834\ldots \cdot 10^{-51}$
	76	$1.0984\ldots \cdot 10^{-48}$	$6.3952\ldots \cdot 10^{-51}$	$6.5648\ldots \cdot 10^{-54}$
	77	$8.6944\ldots \cdot 10^{-52}$	$4.8965\ldots \cdot 10^{-54}$	$4.7030\ldots \cdot 10^{-57}$
50	92	$1.7317\ldots \cdot 10^{-52}$	$7.6758\ldots \cdot 10^{-55}$	$4.4867\ldots \cdot 10^{-58}$
	93	$3.4695\ldots \cdot 10^{-55}$	$1.4974\ldots \cdot 10^{-57}$	$8.3177\ldots \cdot 10^{-61}$
	94	$5.5036\ldots \cdot 10^{-58}$	$2.3153\ldots \cdot 10^{-60}$	$1.2248\ldots \cdot 10^{-63}$
	95	$6.6455\ldots \cdot 10^{-61}$	$2.7275\ldots \cdot 10^{-63}$	$1.3770\ldots \cdot 10^{-66}$
	96	$5.7342\ldots \cdot 10^{-64}$	$2.2980\ldots \cdot 10^{-66}$	$1.1093\ldots \cdot 10^{-69}$
	97	$3.1470\ldots \cdot 10^{-67}$	$1.2324\ldots \cdot 10^{-69}$	$5.6981\ldots \cdot 10^{-73}$

Table 8: The maximal distance between a root of equation (3.15) and η_k.

N	k	$\max_{1\le m\le d}\left\lvert -\frac{d-m+1}{m}\cdot\frac{p_{m-1}}{p_m}\Big/\eta_k - 1\right\rvert$		
		$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	1	$5.6888\ldots\cdot 10^{-9}$	$6.8841\ldots\cdot 10^{-9}$	$6.1124\ldots\cdot 10^{-8}$
	2	$9.1729\ldots\cdot 10^{-7}$	$1.1226\ldots\cdot 10^{-6}$	$1.9360\ldots\cdot 10^{-5}$
	3	$5.3864\ldots\cdot 10^{-5}$	$9.4138\ldots\cdot 10^{-5}$	$5.8369\ldots\cdot 10^{-5}$
	4	$7.6509\ldots\cdot 10^{-4}$	$8.6715\ldots\cdot 10^{-4}$	$6.7403\ldots\cdot 10^{-4}$
	5	$2.5413\ldots\cdot 10^{-3}$	$1.4327\ldots\cdot 10^{-3}$	$1.5154\ldots\cdot 10^{-3}$
20	1	$2.2038\ldots\cdot 10^{-23}$	$9.0445\ldots\cdot 10^{-24}$	$5.3242\ldots\cdot 10^{-24}$
	2	$2.7589\ldots\cdot 10^{-20}$	$7.7580\ldots\cdot 10^{-21}$	$4.3077\ldots\cdot 10^{-21}$
	3	$2.2695\ldots\cdot 10^{-17}$	$4.5637\ldots\cdot 10^{-18}$	$2.3088\ldots\cdot 10^{-18}$
	4	$1.2543\ldots\cdot 10^{-14}$	$1.7998\ldots\cdot 10^{-15}$	$8.3978\ldots\cdot 10^{-16}$
	5	$4.7022\ldots\cdot 10^{-12}$	$4.6684\ldots\cdot 10^{-13}$	$2.1292\ldots\cdot 10^{-13}$
30	1	$1.0007\ldots\cdot 10^{-39}$	$3.9334\ldots\cdot 10^{-40}$	$1.3764\ldots\cdot 10^{-40}$
	2	$3.0401\ldots\cdot 10^{-36}$	$8.4805\ldots\cdot 10^{-37}$	$2.5674\ldots\cdot 10^{-37}$
	3	$6.5351\ldots\cdot 10^{-33}$	$1.3712\ldots\cdot 10^{-33}$	$3.3594\ldots\cdot 10^{-34}$
	4	$1.0292\ldots\cdot 10^{-29}$	$1.6602\ldots\cdot 10^{-30}$	$3.1690\ldots\cdot 10^{-31}$
	5	$1.2182\ldots\cdot 10^{-26}$	$1.5243\ldots\cdot 10^{-27}$	$2.2128\ldots\cdot 10^{-28}$
40	1	$4.1675\ldots\cdot 10^{-57}$	$1.6358\ldots\cdot 10^{-57}$	$5.7047\ldots\cdot 10^{-58}$
	2	$2.2127\ldots\cdot 10^{-53}$	$6.2717\ldots\cdot 10^{-54}$	$1.9222\ldots\cdot 10^{-54}$
	3	$8.5416\ldots\cdot 10^{-50}$	$1.8606\ldots\cdot 10^{-50}$	$4.7057\ldots\cdot 10^{-51}$
	4	$2.4873\ldots\cdot 10^{-46}$	$4.2793\ldots\cdot 10^{-47}$	$8.6371\ldots\cdot 10^{-48}$
	5	$5.6179\ldots\cdot 10^{-43}$	$7.7579\ldots\cdot 10^{-44}$	$1.2268\ldots\cdot 10^{-44}$
50	1	$3.6251\ldots\cdot 10^{-75}$	$1.4093\ldots\cdot 10^{-75}$	$5.2002\ldots\cdot 10^{-76}$
	2	$2.8928\ldots\cdot 10^{-71}$	$8.1838\ldots\cdot 10^{-72}$	$2.6931\ldots\cdot 10^{-72}$
	3	$1.7021\ldots\cdot 10^{-67}$	$3.7344\ldots\cdot 10^{-68}$	$1.0316\ldots\cdot 10^{-68}$
	4	$7.6689\ldots\cdot 10^{-64}$	$1.3430\ldots\cdot 10^{-64}$	$3.0211\ldots\cdot 10^{-65}$
	5	$2.7221\ldots\cdot 10^{-60}$	$3.8747\ldots\cdot 10^{-61}$	$6.9940\ldots\cdot 10^{-62}$

Table 9: The accuracy of (3.18) for small k; $d = k + 1$

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N	k	$\max_{1\le m\le d}\left\|-\frac{d-m+1}{m}\cdot\frac{p_{m-1}}{p_m}\Big/\eta_k-1\right\|$		
		$a=0.2+14\mathrm{i}$	$a=0.5+20\mathrm{i}$	$a=0.6+30\mathrm{i}$
10	12	$2.6280\ldots\cdot10^{-2}$	$2.3850\ldots\cdot10^{-2}$	$2.0564\ldots\cdot10^{-2}$
	13	$1.9165\ldots\cdot10^{-3}$	$2.4103\ldots\cdot10^{-3}$	$1.1364\ldots\cdot10^{-3}$
	14	$8.7135\ldots\cdot10^{-5}$	$7.0664\ldots\cdot10^{-5}$	$4.3811\ldots\cdot10^{-5}$
	15	$2.5086\ldots\cdot10^{-6}$	$1.3965\ldots\cdot10^{-6}$	$2.3390\ldots\cdot10^{-6}$
	17	$5.0518\ldots\cdot10^{-10}$	$1.3592\ldots\cdot10^{-10}$	$1.8368\ldots\cdot10^{-10}$
20	32	$1.2426\ldots\cdot10^{-12}$	$7.1680\ldots\cdot10^{-14}$	$8.3406\ldots\cdot10^{-15}$
	33	$1.9292\ldots\cdot10^{-14}$	$8.9657\ldots\cdot10^{-16}$	$7.2945\ldots\cdot10^{-17}$
	34	$2.2123\ldots\cdot10^{-16}$	$8.4999\ldots\cdot10^{-18}$	$4.9319\ldots\cdot10^{-19}$
	35	$1.8362\ldots\cdot10^{-18}$	$5.9682\ldots\cdot10^{-20}$	$2.5233\ldots\cdot10^{-21}$
	37	$4.1026\ldots\cdot10^{-23}$	$1.0101\ldots\cdot10^{-24}$	$2.4240\ldots\cdot10^{-26}$
30	52	$4.9866\ldots\cdot10^{-25}$	$6.6526\ldots\cdot10^{-27}$	$4.0860\ldots\cdot10^{-29}$
	53	$3.1938\ldots\cdot10^{-27}$	$3.9401\ldots\cdot10^{-29}$	$2.0365\ldots\cdot10^{-31}$
	54	$1.5942\ldots\cdot10^{-29}$	$1.8313\ldots\cdot10^{-31}$	$8.0813\ldots\cdot10^{-34}$
	55	$6.0562\ldots\cdot10^{-32}$	$6.5162\ldots\cdot10^{-34}$	$2.4871\ldots\cdot10^{-36}$
	57	$3.2479\ldots\cdot10^{-37}$	$3.1106\ldots\cdot10^{-39}$	$9.1854\ldots\cdot10^{-42}$
40	72	$9.5160\ldots\cdot10^{-39}$	$6.4436\ldots\cdot10^{-41}$	$9.0103\ldots\cdot10^{-44}$
	73	$3.3995\ldots\cdot10^{-41}$	$2.2096\ldots\cdot10^{-43}$	$2.8388\ldots\cdot10^{-46}$
	74	$9.7347\ldots\cdot10^{-44}$	$6.0877\ldots\cdot10^{-46}$	$7.2241\ldots\cdot10^{-49}$
	75	$2.1776\ldots\cdot10^{-46}$	$1.3128\ldots\cdot10^{-48}$	$1.4458\ldots\cdot10^{-51}$
	77	$4.3472\ldots\cdot10^{-52}$	$2.4482\ldots\cdot10^{-54}$	$2.3515\ldots\cdot10^{-57}$
50	92	$2.4740\ldots\cdot10^{-53}$	$1.0965\ldots\cdot10^{-55}$	$6.4097\ldots\cdot10^{-59}$
	93	$5.7826\ldots\cdot10^{-56}$	$2.4958\ldots\cdot10^{-58}$	$1.3863\ldots\cdot10^{-61}$
	94	$1.1007\ldots\cdot10^{-58}$	$4.6307\ldots\cdot10^{-61}$	$2.4497\ldots\cdot10^{-64}$
	95	$1.6613\ldots\cdot10^{-61}$	$6.8188\ldots\cdot10^{-64}$	$3.4426\ldots\cdot10^{-67}$
	97	$1.5735\ldots\cdot10^{-67}$	$6.1622\ldots\cdot10^{-70}$	$2.8490\ldots\cdot10^{-73}$

Table 10: The accuracy of (3.18) for k near $2N-2$; $d=2N-k-1$.

N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	2	$1.4304\ldots \cdot 10^{-8}$	$1.2462\ldots \cdot 10^{-8}$	$1.1310\ldots \cdot 10^{-7}$
	3	$3.1380\ldots \cdot 10^{-6}$	$3.1013\ldots \cdot 10^{-6}$	$2.6709\ldots \cdot 10^{-5}$
	4	$1.9164\ldots \cdot 10^{-4}$	$3.0488\ldots \cdot 10^{-4}$	$2.0045\ldots \cdot 10^{-4}$
	5	$2.3677\ldots \cdot 10^{-3}$	$1.8184\ldots \cdot 10^{-3}$	$1.1053\ldots \cdot 10^{-3}$
	6	$9.8669\ldots \cdot 10^{-3}$	$4.8354\ldots \cdot 10^{-3}$	$5.5543\ldots \cdot 10^{-3}$
20	2	$8.1042\ldots \cdot 10^{-23}$	$2.3604\ldots \cdot 10^{-23}$	$1.2220\ldots \cdot 10^{-23}$
	3	$1.5395\ldots \cdot 10^{-19}$	$3.2541\ldots \cdot 10^{-20}$	$1.4942\ldots \cdot 10^{-20}$
	4	$1.6629\ldots \cdot 10^{-16}$	$2.5637\ldots \cdot 10^{-17}$	$1.0407\ldots \cdot 10^{-17}$
	5	$1.1174\ldots \cdot 10^{-13}$	$1.2346\ldots \cdot 10^{-14}$	$4.5701\ldots \cdot 10^{-15}$
	6	$4.8623\ldots \cdot 10^{-11}$	$3.6859\ldots \cdot 10^{-12}$	$1.3382\ldots \cdot 10^{-12}$
30	2	$4.2246\ldots \cdot 10^{-39}$	$1.2027\ldots \cdot 10^{-39}$	$3.7544\ldots \cdot 10^{-40}$
	3	$1.9637\ldots \cdot 10^{-35}$	$4.2271\ldots \cdot 10^{-36}$	$1.0734\ldots \cdot 10^{-36}$
	4	$5.5939\ldots \cdot 10^{-32}$	$9.3167\ldots \cdot 10^{-33}$	$1.8550\ldots \cdot 10^{-33}$
	5	$1.0820\ldots \cdot 10^{-28}$	$1.4094\ldots \cdot 10^{-29}$	$2.1505\ldots \cdot 10^{-30}$
	6	$1.5021\ldots \cdot 10^{-25}$	$1.5371\ldots \cdot 10^{-26}$	$1.7669\ldots \cdot 10^{-27}$
40	2	$1.9111\ldots \cdot 10^{-56}$	$5.4693\ldots \cdot 10^{-57}$	$1.7236\ldots \cdot 10^{-57}$
	3	$1.5570\ldots \cdot 10^{-52}$	$3.4314\ldots \cdot 10^{-53}$	$8.9571\ldots \cdot 10^{-54}$
	4	$7.9879\ldots \cdot 10^{-49}$	$1.3935\ldots \cdot 10^{-49}$	$2.9157\ldots \cdot 10^{-50}$
	5	$2.8654\ldots \cdot 10^{-45}$	$4.0228\ldots \cdot 10^{-46}$	$6.6295\ldots \cdot 10^{-47}$
	6	$7.6147\ldots \cdot 10^{-42}$	$8.7072\ldots \cdot 10^{-43}$	$1.1185\ldots \cdot 10^{-43}$
50	2	$1.7629\ldots \cdot 10^{-74}$	$5.0109\ldots \cdot 10^{-75}$	$1.6811\ldots \cdot 10^{-75}$
	3	$2.1619\ldots \cdot 10^{-70}$	$4.7696\ldots \cdot 10^{-71}$	$1.3460\ldots \cdot 10^{-71}$
	4	$1.6931\ldots \cdot 10^{-66}$	$2.9847\ldots \cdot 10^{-67}$	$6.8749\ldots \cdot 10^{-68}$
	5	$9.4109\ldots \cdot 10^{-63}$	$1.3499\ldots \cdot 10^{-63}$	$2.5015\ldots \cdot 10^{-64}$
	6	$3.9364\ldots \cdot 10^{-59}$	$4.6592\ldots \cdot 10^{-60}$	$6.9005\ldots \cdot 10^{-61}$

Table 11: The maximal distance between a root of equation (3.21) and η_k for $l = 1$.

N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	3	$1.8309\ldots \cdot 10^{-8}$	$1.2251\ldots \cdot 10^{-8}$	$1.1251\ldots \cdot 10^{-7}$
	4	$3.4684\ldots \cdot 10^{-6}$	$2.9066\ldots \cdot 10^{-6}$	$1.1855\ldots \cdot 10^{-5}$
	5	$1.5447\ldots \cdot 10^{-4}$	$2.3092\ldots \cdot 10^{-4}$	$1.6673\ldots \cdot 10^{-4}$
	6	$1.4008\ldots \cdot 10^{-3}$	$7.5213\ldots \cdot 10^{-4}$	$3.8741\ldots \cdot 10^{-4}$
	7	$6.6896\ldots \cdot 10^{-3}$	$2.9325\ldots \cdot 10^{-3}$	$2.9280\ldots \cdot 10^{-3}$
20	3	$1.5349\ldots \cdot 10^{-22}$	$3.3810\ldots \cdot 10^{-23}$	$1.4495\ldots \cdot 10^{-23}$
	4	$2.8756\ldots \cdot 10^{-19}$	$4.6998\ldots \cdot 10^{-20}$	$1.7317\ldots \cdot 10^{-20}$
	5	$3.0257\ldots \cdot 10^{-16}$	$3.6371\ldots \cdot 10^{-17}$	$1.1678\ldots \cdot 10^{-17}$
	6	$1.9692\ldots \cdot 10^{-13}$	$1.6945\ldots \cdot 10^{-14}$	$4.9447\ldots \cdot 10^{-15}$
	7	$8.2807\ldots \cdot 10^{-11}$	$4.8247\ldots \cdot 10^{-12}$	$1.3924\ldots \cdot 10^{-12}$
30	3	$9.1808\ldots \cdot 10^{-39}$	$2.0198\ldots \cdot 10^{-39}$	$5.3039\ldots \cdot 10^{-40}$
	4	$4.2428\ldots \cdot 10^{-35}$	$7.2615\ldots \cdot 10^{-36}$	$1.5037\ldots \cdot 10^{-36}$
	5	$1.1878\ldots \cdot 10^{-31}$	$1.6007\ldots \cdot 10^{-32}$	$2.5583\ldots \cdot 10^{-33}$
	6	$2.2465\ldots \cdot 10^{-28}$	$2.3989\ldots \cdot 10^{-29}$	$2.9131\ldots \cdot 10^{-30}$
	7	$3.0424\ldots \cdot 10^{-25}$	$2.5795\ldots \cdot 10^{-26}$	$2.3465\ldots \cdot 10^{-27}$
40	3	$4.5106\ldots \cdot 10^{-56}$	$1.0041\ldots \cdot 10^{-56}$	$2.6984\ldots \cdot 10^{-57}$
	4	$3.6636\ldots \cdot 10^{-52}$	$6.4694\ldots \cdot 10^{-53}$	$1.3989\ldots \cdot 10^{-53}$
	5	$1.8524\ldots \cdot 10^{-48}$	$2.6387\ldots \cdot 10^{-49}$	$4.5150\ldots \cdot 10^{-50}$
	6	$6.5164\ldots \cdot 10^{-45}$	$7.5808\ldots \cdot 10^{-46}$	$1.0165\ldots \cdot 10^{-46}$
	7	$1.6944\ldots \cdot 10^{-41}$	$1.6258\ldots \cdot 10^{-42}$	$1.6976\ldots \cdot 10^{-43}$
50	3	$4.4122\ldots \cdot 10^{-74}$	$9.7826\ldots \cdot 10^{-75}$	$2.8161\ldots \cdot 10^{-75}$
	4	$5.4016\ldots \cdot 10^{-70}$	$9.5781\ldots \cdot 10^{-71}$	$2.2551\ldots \cdot 10^{-71}$
	5	$4.1756\ldots \cdot 10^{-66}$	$6.0309\ldots \cdot 10^{-67}$	$1.1450\ldots \cdot 10^{-67}$
	6	$2.2793\ldots \cdot 10^{-62}$	$2.7196\ldots \cdot 10^{-63}$	$4.1380\ldots \cdot 10^{-64}$
	7	$9.3433\ldots \cdot 10^{-59}$	$9.3193\ldots \cdot 10^{-60}$	$1.1335\ldots \cdot 10^{-60}$

Table 12: The maximal distance between a root of equation (3.21) and η_k for $l = 2$.

N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	4	$2.3206\ldots \cdot 10^{-8}$	$1.2301\ldots \cdot 10^{-8}$	$1.2115\ldots \cdot 10^{-7}$
	5	$3.6195\ldots \cdot 10^{-6}$	$2.7103\ldots \cdot 10^{-6}$	$5.3247\ldots \cdot 10^{-6}$
	6	$1.1139\ldots \cdot 10^{-4}$	$1.4326\ldots \cdot 10^{-4}$	$8.8647\ldots \cdot 10^{-5}$
	7	$8.2799\ldots \cdot 10^{-4}$	$3.2325\ldots \cdot 10^{-4}$	$1.7032\ldots \cdot 10^{-4}$
	8	$4.8330\ldots \cdot 10^{-3}$	$1.9618\ldots \cdot 10^{-3}$	$1.4856\ldots \cdot 10^{-3}$
20	4	$2.9188\ldots \cdot 10^{-22}$	$5.0037\ldots \cdot 10^{-23}$	$1.7244\ldots \cdot 10^{-23}$
	5	$5.3336\ldots \cdot 10^{-19}$	$6.8595\ldots \cdot 10^{-20}$	$1.9998\ldots \cdot 10^{-20}$
	6	$5.4421\ldots \cdot 10^{-16}$	$5.1655\ldots \cdot 10^{-17}$	$1.3035\ldots \cdot 10^{-17}$
	7	$3.4257\ldots \cdot 10^{-13}$	$2.3170\ldots \cdot 10^{-14}$	$5.3173\ldots \cdot 10^{-15}$
	8	$1.3930\ldots \cdot 10^{-10}$	$6.2688\ldots \cdot 10^{-12}$	$1.4389\ldots \cdot 10^{-12}$
30	4	$2.0018\ldots \cdot 10^{-38}$	$3.5068\ldots \cdot 10^{-39}$	$7.5320\ldots \cdot 10^{-40}$
	5	$9.0952\ldots \cdot 10^{-35}$	$1.2617\ldots \cdot 10^{-35}$	$2.1049\ldots \cdot 10^{-36}$
	6	$2.4904\ldots \cdot 10^{-31}$	$2.7571\ldots \cdot 10^{-32}$	$3.5223\ldots \cdot 10^{-33}$
	7	$4.5961\ldots \cdot 10^{-28}$	$4.0766\ldots \cdot 10^{-29}$	$3.9389\ldots \cdot 10^{-30}$
	8	$6.0691\ldots \cdot 10^{-25}$	$4.3132\ldots \cdot 10^{-26}$	$3.1097\ldots \cdot 10^{-27}$
40	4	$1.0679\ldots \cdot 10^{-55}$	$1.9059\ldots \cdot 10^{-56}$	$4.2480\ldots \cdot 10^{-57}$
	5	$8.5503\ldots \cdot 10^{-52}$	$1.2335\ldots \cdot 10^{-52}$	$2.1843\ldots \cdot 10^{-53}$
	6	$4.2402\ldots \cdot 10^{-48}$	$5.0080\ldots \cdot 10^{-49}$	$6.9837\ldots \cdot 10^{-50}$
	7	$1.4596\ldots \cdot 10^{-44}$	$1.4259\ldots \cdot 10^{-45}$	$1.5571\ldots \cdot 10^{-46}$
	8	$3.7115\ldots \cdot 10^{-41}$	$3.0240\ldots \cdot 10^{-42}$	$2.5739\ldots \cdot 10^{-43}$
50	4	$1.1075\ldots \cdot 10^{-73}$	$1.9740\ldots \cdot 10^{-74}$	$4.7437\ldots \cdot 10^{-75}$
	5	$1.3384\ldots \cdot 10^{-69}$	$1.9449\ldots \cdot 10^{-70}$	$3.7771\ldots \cdot 10^{-71}$
	6	$1.0162\ldots \cdot 10^{-65}$	$1.2211\ldots \cdot 10^{-66}$	$1.9051\ldots \cdot 10^{-67}$
	7	$5.4367\ldots \cdot 10^{-62}$	$5.4681\ldots \cdot 10^{-63}$	$6.8379\ldots \cdot 10^{-64}$
	8	$2.1825\ldots \cdot 10^{-58}$	$1.8564\ldots \cdot 10^{-59}$	$1.8601\ldots \cdot 10^{-60}$

Table 13: The maximal distance between a root of equation (3.21) and η_k for $l = 3$.

N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	13	$1.7136\ldots \cdot 10^{-1}$	$1.1112\ldots \cdot 10^{-1}$	$1.2182\ldots \cdot 10^{-1}$
	14	$1.0896\ldots \cdot 10^{-2}$	$1.3409\ldots \cdot 10^{-2}$	$5.9966\ldots \cdot 10^{-3}$
	15	$4.2441\ldots \cdot 10^{-4}$	$3.3975\ldots \cdot 10^{-4}$	$1.6190\ldots \cdot 10^{-4}$
	16	$1.0123\ldots \cdot 10^{-5}$	$5.4575\ldots \cdot 10^{-6}$	$7.9494\ldots \cdot 10^{-6}$
	17	$1.4350\ldots \cdot 10^{-7}$	$5.2669\ldots \cdot 10^{-8}$	$8.7727\ldots \cdot 10^{-8}$
	18	$1.0995\ldots \cdot 10^{-9}$	$2.7892\ldots \cdot 10^{-10}$	$3.5193\ldots \cdot 10^{-10}$
20	33	$9.9885\ldots \cdot 10^{-12}$	$5.4749\ldots \cdot 10^{-13}$	$5.9951\ldots \cdot 10^{-14}$
	34	$1.3608\ldots \cdot 10^{-13}$	$5.9940\ldots \cdot 10^{-15}$	$4.5586\ldots \cdot 10^{-16}$
	35	$1.3284\ldots \cdot 10^{-15}$	$4.8316\ldots \cdot 10^{-17}$	$2.6056\ldots \cdot 10^{-18}$
	36	$8.9913\ldots \cdot 10^{-18}$	$2.7661\ldots \cdot 10^{-19}$	$1.0820\ldots \cdot 10^{-20}$
	37	$3.9877\ldots \cdot 10^{-20}$	$1.0593\ldots \cdot 10^{-21}$	$3.0766\ldots \cdot 10^{-23}$
	38	$1.0379\ldots \cdot 10^{-22}$	$2.4231\ldots \cdot 10^{-24}$	$5.3478\ldots \cdot 10^{-26}$
30	53	$4.4100\ldots \cdot 10^{-24}$	$5.6774\ldots \cdot 10^{-26}$	$3.2620\ldots \cdot 10^{-28}$
	54	$2.4464\ldots \cdot 10^{-26}$	$2.9152\ldots \cdot 10^{-28}$	$1.4097\ldots \cdot 10^{-30}$
	55	$1.0275\ldots \cdot 10^{-28}$	$1.1411\ldots \cdot 10^{-30}$	$4.7137\ldots \cdot 10^{-33}$
	56	$3.1509\ldots \cdot 10^{-31}$	$3.2806\ldots \cdot 10^{-33}$	$1.1729\ldots \cdot 10^{-35}$
	57	$6.6414\ldots \cdot 10^{-34}$	$6.5156\ldots \cdot 10^{-36}$	$2.0404\ldots \cdot 10^{-38}$
	58	$8.5860\ldots \cdot 10^{-37}$	$7.9718\ldots \cdot 10^{-39}$	$2.2095\ldots \cdot 10^{-41}$
40	73	$8.7201\ldots \cdot 10^{-38}$	$5.7723\ldots \cdot 10^{-40}$	$7.6880\ldots \cdot 10^{-43}$
	74	$2.6863\ldots \cdot 10^{-40}$	$1.7079\ldots \cdot 10^{-42}$	$2.0917\ldots \cdot 10^{-45}$
	75	$6.4473\ldots \cdot 10^{-43}$	$3.9456\ldots \cdot 10^{-45}$	$4.4676\ldots \cdot 10^{-48}$
	76	$1.1600\ldots \cdot 10^{-45}$	$6.8473\ldots \cdot 10^{-48}$	$7.2020\ldots \cdot 10^{-51}$
	77	$1.4704\ldots \cdot 10^{-48}$	$8.3859\ldots \cdot 10^{-51}$	$8.2284\ldots \cdot 10^{-54}$
	78	$1.1695\ldots \cdot 10^{-51}$	$6.4549\ldots \cdot 10^{-54}$	$5.9313\ldots \cdot 10^{-57}$
50	93	$2.3059\ldots \cdot 10^{-52}$	$1.0060\ldots \cdot 10^{-54}$	$5.6762\ldots \cdot 10^{-58}$
	94	$4.6389\ldots \cdot 10^{-55}$	$1.9713\ldots \cdot 10^{-57}$	$1.0575\ldots \cdot 10^{-60}$
	95	$7.3875\ldots \cdot 10^{-58}$	$3.0609\ldots \cdot 10^{-60}$	$1.5648\ldots \cdot 10^{-63}$
	96	$8.9537\ldots \cdot 10^{-61}$	$3.6204\ldots \cdot 10^{-63}$	$1.7673\ldots \cdot 10^{-66}$
	97	$7.7536\ldots \cdot 10^{-64}$	$3.0621\ldots \cdot 10^{-66}$	$1.4300\ldots \cdot 10^{-69}$
	98	$4.2700\ldots \cdot 10^{-67}$	$1.6483\ldots \cdot 10^{-69}$	$7.3771\ldots \cdot 10^{-73}$

Table 14: The maximal distance between a root of equation (3.21) and η_k for $l = 1$.

N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	14	$1.5794\ldots \cdot 10^{-1}$	$8.5851\ldots \cdot 10^{-2}$	$1.0889\ldots \cdot 10^{-1}$
	15	$1.0092\ldots \cdot 10^{-2}$	$1.1137\ldots \cdot 10^{-2}$	$5.3100\ldots \cdot 10^{-3}$
	16	$4.0399\ldots \cdot 10^{-4}$	$3.2249\ldots \cdot 10^{-4}$	$1.3139\ldots \cdot 10^{-4}$
	17	$9.9930\ldots \cdot 10^{-6}$	$5.2716\ldots \cdot 10^{-6}$	$5.8205\ldots \cdot 10^{-6}$
	18	$1.4740\ldots \cdot 10^{-7}$	$5.2048\ldots \cdot 10^{-8}$	$8.4073\ldots \cdot 10^{-8}$
	19	$1.1755\ldots \cdot 10^{-9}$	$2.8269\ldots \cdot 10^{-10}$	$3.3575\ldots \cdot 10^{-10}$
20	34	$1.1379\ldots \cdot 10^{-11}$	$5.9254\ldots \cdot 10^{-13}$	$6.1230\ldots \cdot 10^{-14}$
	35	$1.5882\ldots \cdot 10^{-13}$	$6.6283\ldots \cdot 10^{-15}$	$4.7225\ldots \cdot 10^{-16}$
	36	$1.5843\ldots \cdot 10^{-15}$	$5.4541\ldots \cdot 10^{-17}$	$2.7387\ldots \cdot 10^{-18}$
	37	$1.0935\ldots \cdot 10^{-17}$	$3.1839\ldots \cdot 10^{-19}$	$1.1539\ldots \cdot 10^{-20}$
	38	$4.9358\ldots \cdot 10^{-20}$	$1.2418\ldots \cdot 10^{-21}$	$3.3289\ldots \cdot 10^{-23}$
	39	$1.3052\ldots \cdot 10^{-22}$	$2.8892\ldots \cdot 10^{-24}$	$5.8699\ldots \cdot 10^{-26}$
30	54	$5.5482\ldots \cdot 10^{-24}$	$6.8938\ldots \cdot 10^{-26}$	$3.7062\ldots \cdot 10^{-28}$
	55	$3.1105\ldots \cdot 10^{-26}$	$3.5807\ldots \cdot 10^{-28}$	$1.6205\ldots \cdot 10^{-30}$
	56	$1.3192\ldots \cdot 10^{-28}$	$1.4167\ldots \cdot 10^{-30}$	$5.4797\ldots \cdot 10^{-33}$
	57	$4.0821\ldots \cdot 10^{-31}$	$4.1138\ldots \cdot 10^{-33}$	$1.3783\ldots \cdot 10^{-35}$
	58	$8.6767\ldots \cdot 10^{-34}$	$8.2470\ldots \cdot 10^{-36}$	$2.4224\ldots \cdot 10^{-38}$
	59	$1.1305\ldots \cdot 10^{-36}$	$1.0178\ldots \cdot 10^{-38}$	$2.6490\ldots \cdot 10^{-41}$
40	74	$1.1381\ldots \cdot 10^{-37}$	$7.3662\ldots \cdot 10^{-40}$	$9.3475\ldots \cdot 10^{-43}$
	75	$3.5275\ldots \cdot 10^{-40}$	$2.1940\ldots \cdot 10^{-42}$	$2.5624\ldots \cdot 10^{-45}$
	76	$8.5151\ldots \cdot 10^{-43}$	$5.1005\ldots \cdot 10^{-45}$	$5.5123\ldots \cdot 10^{-48}$
	77	$1.5405\ldots \cdot 10^{-45}$	$8.9044\ldots \cdot 10^{-48}$	$8.9472\ldots \cdot 10^{-51}$
	78	$1.9627\ldots \cdot 10^{-48}$	$1.0966\ldots \cdot 10^{-50}$	$1.0289\ldots \cdot 10^{-53}$
	79	$1.5687\ldots \cdot 10^{-51}$	$8.4869\ldots \cdot 10^{-54}$	$7.4632\ldots \cdot 10^{-57}$
50	94	$3.0632\ldots \cdot 10^{-52}$	$1.3157\ldots \cdot 10^{-54}$	$7.1669\ldots \cdot 10^{-58}$
	95	$6.1878\ldots \cdot 10^{-55}$	$2.5895\ldots \cdot 10^{-57}$	$1.3420\ldots \cdot 10^{-60}$
	96	$9.8931\ldots \cdot 10^{-58}$	$4.0378\ldots \cdot 10^{-60}$	$1.9953\ldots \cdot 10^{-63}$
	97	$1.2036\ldots \cdot 10^{-60}$	$4.7953\ldots \cdot 10^{-63}$	$2.2640\ldots \cdot 10^{-66}$
	98	$1.0460\ldots \cdot 10^{-63}$	$4.0716\ldots \cdot 10^{-66}$	$1.8401\ldots \cdot 10^{-69}$
	99	$5.7808\ldots \cdot 10^{-67}$	$2.1999\ldots \cdot 10^{-69}$	$9.5331\ldots \cdot 10^{-73}$

Table 15: The maximal distance between a root of equation (3.21) and η_k for $l = 2$.

N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	15	$1.4061\ldots \cdot 10^{-1}$	$7.1436\ldots \cdot 10^{-2}$	$8.1392\ldots \cdot 10^{-2}$
	16	$9.1613\ldots \cdot 10^{-3}$	$8.2545\ldots \cdot 10^{-3}$	$4.6299\ldots \cdot 10^{-3}$
	17	$3.7594\ldots \cdot 10^{-4}$	$3.0005\ldots \cdot 10^{-4}$	$1.1613\ldots \cdot 10^{-4}$
	18	$9.6450\ldots \cdot 10^{-6}$	$5.0353\ldots \cdot 10^{-6}$	$3.4977\ldots \cdot 10^{-6}$
	19	$1.4832\ldots \cdot 10^{-7}$	$5.0820\ldots \cdot 10^{-8}$	$8.3962\ldots \cdot 10^{-8}$
	20	$1.2343\ldots \cdot 10^{-9}$	$2.8298\ldots \cdot 10^{-10}$	$3.2015\ldots \cdot 10^{-10}$
20	35	$1.2862\ldots \cdot 10^{-11}$	$6.3607\ldots \cdot 10^{-13}$	$6.2206\ldots \cdot 10^{-14}$
	36	$1.8400\ldots \cdot 10^{-13}$	$7.2740\ldots \cdot 10^{-15}$	$4.8663\ldots \cdot 10^{-16}$
	37	$1.8768\ldots \cdot 10^{-15}$	$6.1133\ldots \cdot 10^{-17}$	$2.8632\ldots \cdot 10^{-18}$
	38	$1.3214\ldots \cdot 10^{-17}$	$3.6408\ldots \cdot 10^{-19}$	$1.2241\ldots \cdot 10^{-20}$
	39	$6.0720\ldots \cdot 10^{-20}$	$1.4468\ldots \cdot 10^{-21}$	$3.5835\ldots \cdot 10^{-23}$
	40	$1.6316\ldots \cdot 10^{-22}$	$3.4251\ldots \cdot 10^{-24}$	$6.4112\ldots \cdot 10^{-26}$
30	55	$6.9509\ldots \cdot 10^{-24}$	$8.3370\ldots \cdot 10^{-26}$	$4.1948\ldots \cdot 10^{-28}$
	56	$3.9387\ldots \cdot 10^{-26}$	$4.3809\ldots \cdot 10^{-28}$	$1.8560\ldots \cdot 10^{-30}$
	57	$1.6870\ldots \cdot 10^{-28}$	$1.7522\ldots \cdot 10^{-30}$	$6.3478\ldots \cdot 10^{-33}$
	58	$5.2680\ldots \cdot 10^{-31}$	$5.1397\ldots \cdot 10^{-33}$	$1.6141\ldots \cdot 10^{-35}$
	59	$1.1292\ldots \cdot 10^{-33}$	$1.0401\ldots \cdot 10^{-35}$	$2.8666\ldots \cdot 10^{-38}$
	60	$1.4829\ldots \cdot 10^{-36}$	$1.2950\ldots \cdot 10^{-38}$	$3.1661\ldots \cdot 10^{-41}$
40	75	$1.4809\ldots \cdot 10^{-37}$	$9.3738\ldots \cdot 10^{-40}$	$1.1336\ldots \cdot 10^{-42}$
	76	$4.6185\ldots \cdot 10^{-40}$	$2.8106\ldots \cdot 10^{-42}$	$3.1313\ldots \cdot 10^{-45}$
	77	$1.1213\ldots \cdot 10^{-42}$	$6.5755\ldots \cdot 10^{-45}$	$6.7849\ldots \cdot 10^{-48}$
	78	$2.0398\ldots \cdot 10^{-45}$	$1.1548\ldots \cdot 10^{-47}$	$1.1089\ldots \cdot 10^{-50}$
	79	$2.6124\ldots \cdot 10^{-48}$	$1.4304\ldots \cdot 10^{-50}$	$1.2836\ldots \cdot 10^{-53}$
	80	$2.0984\ldots \cdot 10^{-51}$	$1.1129\ldots \cdot 10^{-53}$	$9.3694\ldots \cdot 10^{-57}$
50	95	$4.0597\ldots \cdot 10^{-52}$	$1.7169\ldots \cdot 10^{-54}$	$9.0313\ldots \cdot 10^{-58}$
	96	$8.2348\ldots \cdot 10^{-55}$	$3.3942\ldots \cdot 10^{-57}$	$1.6996\ldots \cdot 10^{-60}$
	97	$1.3218\ldots \cdot 10^{-57}$	$5.3151\ldots \cdot 10^{-60}$	$2.5393\ldots \cdot 10^{-63}$
	98	$1.6142\ldots \cdot 10^{-60}$	$6.3379\ldots \cdot 10^{-63}$	$2.8948\ldots \cdot 10^{-66}$
	99	$1.4081\ldots \cdot 10^{-63}$	$5.4026\ldots \cdot 10^{-66}$	$2.3633\ldots \cdot 10^{-69}$
	100	$7.8087\ldots \cdot 10^{-67}$	$2.9300\ldots \cdot 10^{-69}$	$1.2296\ldots \cdot 10^{-72}$

Table 16: The maximal distance between a root of equation (3.21) and η_k for $l = 3$.

		$\max_{1\le m\le d}\left\lvert -\frac{d-m+1}{m}\cdot\frac{q_{m-1}}{q_m}\Big/\eta_k - 1\right\rvert$		
N	k	$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	2	$7.1526\ldots\cdot 10^{-9}$	$6.2309\ldots\cdot 10^{-9}$	$5.6549\ldots\cdot 10^{-8}$
	3	$1.0460\ldots\cdot 10^{-6}$	$1.0336\ldots\cdot 10^{-6}$	$8.9044\ldots\cdot 10^{-6}$
	4	$4.7906\ldots\cdot 10^{-5}$	$7.6188\ldots\cdot 10^{-5}$	$5.0019\ldots\cdot 10^{-5}$
	5	$4.7126\ldots\cdot 10^{-4}$	$3.6278\ldots\cdot 10^{-4}$	$2.1939\ldots\cdot 10^{-4}$
	6	$1.6019\ldots\cdot 10^{-3}$	$7.6128\ldots\cdot 10^{-4}$	$9.1339\ldots\cdot 10^{-4}$
20	2	$4.0521\ldots\cdot 10^{-23}$	$1.1802\ldots\cdot 10^{-23}$	$6.1100\ldots\cdot 10^{-24}$
	3	$5.1320\ldots\cdot 10^{-20}$	$1.0847\ldots\cdot 10^{-20}$	$4.9807\ldots\cdot 10^{-21}$
	4	$4.1574\ldots\cdot 10^{-17}$	$6.4094\ldots\cdot 10^{-18}$	$2.6018\ldots\cdot 10^{-18}$
	5	$2.2350\ldots\cdot 10^{-14}$	$2.4694\ldots\cdot 10^{-15}$	$9.1401\ldots\cdot 10^{-16}$
	6	$8.1044\ldots\cdot 10^{-12}$	$6.1435\ldots\cdot 10^{-13}$	$2.2303\ldots\cdot 10^{-13}$
30	2	$2.1123\ldots\cdot 10^{-39}$	$6.0137\ldots\cdot 10^{-40}$	$1.8772\ldots\cdot 10^{-40}$
	3	$6.5459\ldots\cdot 10^{-36}$	$1.4090\ldots\cdot 10^{-36}$	$3.5782\ldots\cdot 10^{-37}$
	4	$1.3985\ldots\cdot 10^{-32}$	$2.3291\ldots\cdot 10^{-33}$	$4.6375\ldots\cdot 10^{-34}$
	5	$2.1641\ldots\cdot 10^{-29}$	$2.8188\ldots\cdot 10^{-30}$	$4.3010\ldots\cdot 10^{-31}$
	6	$2.5036\ldots\cdot 10^{-26}$	$2.5619\ldots\cdot 10^{-27}$	$2.9449\ldots\cdot 10^{-28}$
40	2	$9.5556\ldots\cdot 10^{-57}$	$2.7346\ldots\cdot 10^{-57}$	$8.6183\ldots\cdot 10^{-58}$
	3	$5.1902\ldots\cdot 10^{-53}$	$1.1438\ldots\cdot 10^{-53}$	$2.9857\ldots\cdot 10^{-54}$
	4	$1.9970\ldots\cdot 10^{-49}$	$3.4837\ldots\cdot 10^{-50}$	$7.2893\ldots\cdot 10^{-51}$
	5	$5.7309\ldots\cdot 10^{-46}$	$8.0458\ldots\cdot 10^{-47}$	$1.3259\ldots\cdot 10^{-47}$
	6	$1.2691\ldots\cdot 10^{-42}$	$1.4512\ldots\cdot 10^{-43}$	$1.8641\ldots\cdot 10^{-44}$
50	2	$8.8149\ldots\cdot 10^{-75}$	$2.5054\ldots\cdot 10^{-75}$	$8.4059\ldots\cdot 10^{-76}$
	3	$7.2063\ldots\cdot 10^{-71}$	$1.5898\ldots\cdot 10^{-71}$	$4.4868\ldots\cdot 10^{-72}$
	4	$4.2328\ldots\cdot 10^{-67}$	$7.4619\ldots\cdot 10^{-68}$	$1.7187\ldots\cdot 10^{-68}$
	5	$1.8822\ldots\cdot 10^{-63}$	$2.6999\ldots\cdot 10^{-64}$	$5.0030\ldots\cdot 10^{-65}$
	6	$6.5608\ldots\cdot 10^{-60}$	$7.7654\ldots\cdot 10^{-61}$	$1.1500\ldots\cdot 10^{-61}$

Table 17: The accuracy of (3.24) for $l = 1$ and small k; $d = k - l + 1$.

N	k	$\max_{1\le m\le d}\left\lvert -\frac{d-m+1}{m}\cdot\frac{q_{m-1}}{q_m}\Big/\eta_k - 1\right\rvert$		
		$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	3	$9.1552\ldots\cdot 10^{-9}$	$6.1257\ldots\cdot 10^{-9}$	$5.6256\ldots\cdot 10^{-8}$
	4	$1.1562\ldots\cdot 10^{-6}$	$9.6874\ldots\cdot 10^{-7}$	$3.9532\ldots\cdot 10^{-6}$
	5	$3.8606\ldots\cdot 10^{-5}$	$5.7703\ldots\cdot 10^{-5}$	$4.1711\ldots\cdot 10^{-5}$
	6	$2.7810\ldots\cdot 10^{-4}$	$1.4963\ldots\cdot 10^{-4}$	$7.5972\ldots\cdot 10^{-5}$
	7	$1.0909\ldots\cdot 10^{-3}$	$4.7103\ldots\cdot 10^{-4}$	$4.8935\ldots\cdot 10^{-4}$
20	3	$7.6750\ldots\cdot 10^{-23}$	$1.6905\ldots\cdot 10^{-23}$	$7.2477\ldots\cdot 10^{-24}$
	4	$9.5857\ldots\cdot 10^{-20}$	$1.5666\ldots\cdot 10^{-20}$	$5.7726\ldots\cdot 10^{-21}$
	5	$7.5646\ldots\cdot 10^{-17}$	$9.0929\ldots\cdot 10^{-18}$	$2.9195\ldots\cdot 10^{-18}$
	6	$3.9386\ldots\cdot 10^{-14}$	$3.3892\ldots\cdot 10^{-15}$	$9.8893\ldots\cdot 10^{-16}$
	7	$1.3802\ldots\cdot 10^{-11}$	$8.0416\ldots\cdot 10^{-13}$	$2.3207\ldots\cdot 10^{-13}$
30	3	$4.5904\ldots\cdot 10^{-39}$	$1.0099\ldots\cdot 10^{-39}$	$2.6520\ldots\cdot 10^{-40}$
	4	$1.4142\ldots\cdot 10^{-35}$	$2.4205\ldots\cdot 10^{-36}$	$5.0126\ldots\cdot 10^{-37}$
	5	$2.9696\ldots\cdot 10^{-32}$	$4.0020\ldots\cdot 10^{-33}$	$6.3960\ldots\cdot 10^{-34}$
	6	$4.4930\ldots\cdot 10^{-29}$	$4.7979\ldots\cdot 10^{-30}$	$5.8264\ldots\cdot 10^{-31}$
	7	$5.0708\ldots\cdot 10^{-26}$	$4.2993\ldots\cdot 10^{-27}$	$3.9109\ldots\cdot 10^{-28}$
40	3	$2.2553\ldots\cdot 10^{-56}$	$5.0209\ldots\cdot 10^{-57}$	$1.3492\ldots\cdot 10^{-57}$
	4	$1.2212\ldots\cdot 10^{-52}$	$2.1564\ldots\cdot 10^{-53}$	$4.6633\ldots\cdot 10^{-54}$
	5	$4.6312\ldots\cdot 10^{-49}$	$6.5969\ldots\cdot 10^{-50}$	$1.1287\ldots\cdot 10^{-50}$
	6	$1.3033\ldots\cdot 10^{-45}$	$1.5161\ldots\cdot 10^{-46}$	$2.0331\ldots\cdot 10^{-47}$
	7	$2.8241\ldots\cdot 10^{-42}$	$2.7097\ldots\cdot 10^{-43}$	$2.8294\ldots\cdot 10^{-44}$
50	3	$2.2061\ldots\cdot 10^{-74}$	$4.8913\ldots\cdot 10^{-75}$	$1.4080\ldots\cdot 10^{-75}$
	4	$1.8005\ldots\cdot 10^{-70}$	$3.1927\ldots\cdot 10^{-71}$	$7.5172\ldots\cdot 10^{-72}$
	5	$1.0439\ldots\cdot 10^{-66}$	$1.5077\ldots\cdot 10^{-67}$	$2.8626\ldots\cdot 10^{-68}$
	6	$4.5587\ldots\cdot 10^{-63}$	$5.4392\ldots\cdot 10^{-64}$	$8.2760\ldots\cdot 10^{-65}$
	7	$1.5572\ldots\cdot 10^{-59}$	$1.5532\ldots\cdot 10^{-60}$	$1.8892\ldots\cdot 10^{-61}$

Table 18: The accuracy of (3.24) for $l = 2$ and small k; $d = k - l + 1$.

N	k	$\max_{1\le m\le d}\left\lvert -\frac{d-m+1}{m}\cdot\frac{q_{m-1}}{q_m}\Big/\eta_k-1\right\rvert$ $a=0.2+14\mathrm{i}$	$a=0.5+20\mathrm{i}$	$a=0.6+30\mathrm{i}$
10	4	$1.1603\ldots\cdot 10^{-8}$	$6.1506\ldots\cdot 10^{-9}$	$6.0577\ldots\cdot 10^{-8}$
	5	$1.2066\ldots\cdot 10^{-6}$	$9.0330\ldots\cdot 10^{-7}$	$1.7764\ldots\cdot 10^{-6}$
	6	$2.7831\ldots\cdot 10^{-5}$	$3.5800\ldots\cdot 10^{-5}$	$2.2170\ldots\cdot 10^{-5}$
	7	$1.6396\ldots\cdot 10^{-4}$	$6.3418\ldots\cdot 10^{-5}$	$3.2803\ldots\cdot 10^{-5}$
	8	$7.9282\ldots\cdot 10^{-4}$	$3.2011\ldots\cdot 10^{-4}$	$2.4917\ldots\cdot 10^{-4}$
20	4	$1.4594\ldots\cdot 10^{-22}$	$2.5019\ldots\cdot 10^{-23}$	$8.6223\ldots\cdot 10^{-24}$
	5	$1.7779\ldots\cdot 10^{-19}$	$2.2865\ldots\cdot 10^{-20}$	$6.6660\ldots\cdot 10^{-21}$
	6	$1.3605\ldots\cdot 10^{-16}$	$1.2914\ldots\cdot 10^{-17}$	$3.2588\ldots\cdot 10^{-18}$
	7	$6.8518\ldots\cdot 10^{-14}$	$4.6342\ldots\cdot 10^{-15}$	$1.0634\ldots\cdot 10^{-15}$
	8	$2.3219\ldots\cdot 10^{-11}$	$1.0448\ldots\cdot 10^{-12}$	$2.3981\ldots\cdot 10^{-13}$
30	4	$1.0009\ldots\cdot 10^{-38}$	$1.7534\ldots\cdot 10^{-39}$	$3.7660\ldots\cdot 10^{-40}$
	5	$3.0317\ldots\cdot 10^{-35}$	$4.2058\ldots\cdot 10^{-36}$	$7.0164\ldots\cdot 10^{-37}$
	6	$6.2261\ldots\cdot 10^{-32}$	$6.8929\ldots\cdot 10^{-33}$	$8.8060\ldots\cdot 10^{-34}$
	7	$9.1923\ldots\cdot 10^{-29}$	$8.1533\ldots\cdot 10^{-30}$	$7.8780\ldots\cdot 10^{-31}$
	8	$1.0115\ldots\cdot 10^{-25}$	$7.1888\ldots\cdot 10^{-27}$	$5.1829\ldots\cdot 10^{-28}$
40	4	$5.3395\ldots\cdot 10^{-56}$	$9.5298\ldots\cdot 10^{-57}$	$2.1240\ldots\cdot 10^{-57}$
	5	$2.8501\ldots\cdot 10^{-52}$	$4.1117\ldots\cdot 10^{-53}$	$7.2811\ldots\cdot 10^{-54}$
	6	$1.0600\ldots\cdot 10^{-48}$	$1.2520\ldots\cdot 10^{-49}$	$1.7459\ldots\cdot 10^{-50}$
	7	$2.9193\ldots\cdot 10^{-45}$	$2.8519\ldots\cdot 10^{-46}$	$3.1143\ldots\cdot 10^{-47}$
	8	$6.1860\ldots\cdot 10^{-42}$	$5.0400\ldots\cdot 10^{-43}$	$4.2899\ldots\cdot 10^{-44}$
50	4	$5.5378\ldots\cdot 10^{-74}$	$9.8703\ldots\cdot 10^{-75}$	$2.3718\ldots\cdot 10^{-75}$
	5	$4.4616\ldots\cdot 10^{-70}$	$6.4833\ldots\cdot 10^{-71}$	$1.2590\ldots\cdot 10^{-71}$
	6	$2.5405\ldots\cdot 10^{-66}$	$3.0530\ldots\cdot 10^{-67}$	$4.7627\ldots\cdot 10^{-68}$
	7	$1.0873\ldots\cdot 10^{-62}$	$1.0936\ldots\cdot 10^{-63}$	$1.3675\ldots\cdot 10^{-64}$
	8	$3.6375\ldots\cdot 10^{-59}$	$3.0941\ldots\cdot 10^{-60}$	$3.1002\ldots\cdot 10^{-61}$

Table 19: The accuracy of (3.24) for $l=3$ and small k; $d=k-l+1$.

N	k	$\max_{1\le m\le d}\left\lvert -\frac{d-m+1}{m}\cdot\frac{q_{m-1}}{q_m}\Big/\eta_k - 1\right\rvert$		
		$a = 0.2 + 14\mathrm{i}$	$a = 0.5 + 20\mathrm{i}$	$a = 0.6 + 30\mathrm{i}$
10	13	$2.4551\ldots\cdot 10^{-2}$	$1.5593\ldots\cdot 10^{-2}$	$1.7444\ldots\cdot 10^{-2}$
	14	$1.8089\ldots\cdot 10^{-3}$	$2.2274\ldots\cdot 10^{-3}$	$9.9628\ldots\cdot 10^{-4}$
	15	$8.4750\ldots\cdot 10^{-5}$	$6.7853\ldots\cdot 10^{-5}$	$3.2211\ldots\cdot 10^{-5}$
	16	$2.5295\ldots\cdot 10^{-6}$	$1.3635\ldots\cdot 10^{-6}$	$1.9870\ldots\cdot 10^{-6}$
	17	$4.7829\ldots\cdot 10^{-8}$	$1.7551\ldots\cdot 10^{-8}$	$2.9234\ldots\cdot 10^{-8}$
	18	$5.4975\ldots\cdot 10^{-10}$	$1.3945\ldots\cdot 10^{-10}$	$1.7595\ldots\cdot 10^{-10}$
20	33	$1.4272\ldots\cdot 10^{-12}$	$7.8211\ldots\cdot 10^{-14}$	$8.5630\ldots\cdot 10^{-15}$
	34	$2.2685\ldots\cdot 10^{-14}$	$9.9902\ldots\cdot 10^{-16}$	$7.5969\ldots\cdot 10^{-17}$
	35	$2.6571\ldots\cdot 10^{-16}$	$9.6634\ldots\cdot 10^{-18}$	$5.2110\ldots\cdot 10^{-19}$
	36	$2.2480\ldots\cdot 10^{-18}$	$6.9155\ldots\cdot 10^{-20}$	$2.7049\ldots\cdot 10^{-21}$
	37	$1.3293\ldots\cdot 10^{-20}$	$3.5312\ldots\cdot 10^{-22}$	$1.0255\ldots\cdot 10^{-23}$
	38	$5.1900\ldots\cdot 10^{-23}$	$1.2116\ldots\cdot 10^{-24}$	$2.6739\ldots\cdot 10^{-26}$
30	53	$6.3007\ldots\cdot 10^{-25}$	$8.1112\ldots\cdot 10^{-27}$	$4.6601\ldots\cdot 10^{-29}$
	54	$4.0777\ldots\cdot 10^{-27}$	$4.8589\ldots\cdot 10^{-29}$	$2.3496\ldots\cdot 10^{-31}$
	55	$2.0551\ldots\cdot 10^{-29}$	$2.2823\ldots\cdot 10^{-31}$	$9.4276\ldots\cdot 10^{-34}$
	56	$7.8776\ldots\cdot 10^{-32}$	$8.2018\ldots\cdot 10^{-34}$	$2.9325\ldots\cdot 10^{-36}$
	57	$2.2138\ldots\cdot 10^{-34}$	$2.1719\ldots\cdot 10^{-36}$	$6.8014\ldots\cdot 10^{-39}$
	58	$4.2930\ldots\cdot 10^{-37}$	$3.9859\ldots\cdot 10^{-39}$	$1.1047\ldots\cdot 10^{-41}$
40	73	$1.2457\ldots\cdot 10^{-38}$	$8.2465\ldots\cdot 10^{-41}$	$1.0983\ldots\cdot 10^{-43}$
	74	$4.4774\ldots\cdot 10^{-41}$	$2.8466\ldots\cdot 10^{-43}$	$3.4863\ldots\cdot 10^{-46}$
	75	$1.2895\ldots\cdot 10^{-43}$	$7.8914\ldots\cdot 10^{-46}$	$8.9353\ldots\cdot 10^{-49}$
	76	$2.9002\ldots\cdot 10^{-46}$	$1.7118\ldots\cdot 10^{-48}$	$1.8005\ldots\cdot 10^{-51}$
	77	$4.9014\ldots\cdot 10^{-49}$	$2.7953\ldots\cdot 10^{-51}$	$2.7428\ldots\cdot 10^{-54}$
	78	$5.8477\ldots\cdot 10^{-52}$	$3.2274\ldots\cdot 10^{-54}$	$2.9656\ldots\cdot 10^{-57}$
50	93	$3.2943\ldots\cdot 10^{-53}$	$1.4372\ldots\cdot 10^{-55}$	$8.1091\ldots\cdot 10^{-59}$
	94	$7.7316\ldots\cdot 10^{-56}$	$3.2856\ldots\cdot 10^{-58}$	$1.7626\ldots\cdot 10^{-61}$
	95	$1.4775\ldots\cdot 10^{-58}$	$6.1220\ldots\cdot 10^{-61}$	$3.1296\ldots\cdot 10^{-64}$
	96	$2.2384\ldots\cdot 10^{-61}$	$9.0512\ldots\cdot 10^{-64}$	$4.4184\ldots\cdot 10^{-67}$
	97	$2.5845\ldots\cdot 10^{-64}$	$1.0207\ldots\cdot 10^{-66}$	$4.7669\ldots\cdot 10^{-70}$
	98	$2.1350\ldots\cdot 10^{-67}$	$8.2417\ldots\cdot 10^{-70}$	$3.6885\ldots\cdot 10^{-73}$

Table 20: The accuracy of (3.24) for $l = 1$ and k near $2N+l-2$; $d = 2N-k+l-1$.

N	k	$\max_{1\le m\le d}\left\lvert -\frac{d-m+1}{m}\cdot\frac{q_{m-1}}{q_m}\Big/\eta_k - 1\right\rvert$ $a = 0.2+14\mathrm{i}$	$a = 0.5+20\mathrm{i}$	$a = 0.6+30\mathrm{i}$
10	14	$2.2523\ldots\cdot 10^{-2}$	$1.2109\ldots\cdot 10^{-2}$	$1.5436\ldots\cdot 10^{-2}$
	15	$1.6750\ldots\cdot 10^{-3}$	$1.8491\ldots\cdot 10^{-3}$	$8.8277\ldots\cdot 10^{-4}$
	16	$8.0659\ldots\cdot 10^{-5}$	$6.4402\ldots\cdot 10^{-5}$	$2.6173\ldots\cdot 10^{-5}$
	17	$2.4966\ldots\cdot 10^{-6}$	$1.3170\ldots\cdot 10^{-6}$	$1.4553\ldots\cdot 10^{-6}$
	18	$4.9125\ldots\cdot 10^{-8}$	$1.7344\ldots\cdot 10^{-8}$	$2.8017\ldots\cdot 10^{-8}$
	19	$5.8776\ldots\cdot 10^{-10}$	$1.4133\ldots\cdot 10^{-10}$	$1.6786\ldots\cdot 10^{-10}$
20	34	$1.6259\ldots\cdot 10^{-12}$	$8.4646\ldots\cdot 10^{-14}$	$8.7456\ldots\cdot 10^{-15}$
	35	$2.6475\ldots\cdot 10^{-14}$	$1.1047\ldots\cdot 10^{-15}$	$7.8701\ldots\cdot 10^{-17}$
	36	$3.1691\ldots\cdot 10^{-16}$	$1.0908\ldots\cdot 10^{-17}$	$5.4771\ldots\cdot 10^{-19}$
	37	$2.7340\ldots\cdot 10^{-18}$	$7.9602\ldots\cdot 10^{-20}$	$2.8847\ldots\cdot 10^{-21}$
	38	$1.6453\ldots\cdot 10^{-20}$	$4.1396\ldots\cdot 10^{-22}$	$1.1096\ldots\cdot 10^{-23}$
	39	$6.5262\ldots\cdot 10^{-23}$	$1.4446\ldots\cdot 10^{-24}$	$2.9349\ldots\cdot 10^{-26}$
30	54	$7.9268\ldots\cdot 10^{-25}$	$9.8490\ldots\cdot 10^{-27}$	$5.2946\ldots\cdot 10^{-29}$
	55	$5.1846\ldots\cdot 10^{-27}$	$5.9681\ldots\cdot 10^{-29}$	$2.7009\ldots\cdot 10^{-31}$
	56	$2.6386\ldots\cdot 10^{-29}$	$2.8336\ldots\cdot 10^{-31}$	$1.0959\ldots\cdot 10^{-33}$
	57	$1.0205\ldots\cdot 10^{-31}$	$1.0285\ldots\cdot 10^{-33}$	$3.4459\ldots\cdot 10^{-36}$
	58	$2.8923\ldots\cdot 10^{-34}$	$2.7490\ldots\cdot 10^{-36}$	$8.0748\ldots\cdot 10^{-39}$
	59	$5.6526\ldots\cdot 10^{-37}$	$5.0893\ldots\cdot 10^{-39}$	$1.3245\ldots\cdot 10^{-41}$
40	74	$1.6259\ldots\cdot 10^{-38}$	$1.0523\ldots\cdot 10^{-40}$	$1.3353\ldots\cdot 10^{-43}$
	75	$5.8794\ldots\cdot 10^{-41}$	$3.6568\ldots\cdot 10^{-43}$	$4.2709\ldots\cdot 10^{-46}$
	76	$1.7030\ldots\cdot 10^{-43}$	$1.0201\ldots\cdot 10^{-45}$	$1.1024\ldots\cdot 10^{-48}$
	77	$3.8513\ldots\cdot 10^{-46}$	$2.2261\ldots\cdot 10^{-48}$	$2.2368\ldots\cdot 10^{-51}$
	78	$6.5425\ldots\cdot 10^{-49}$	$3.6556\ldots\cdot 10^{-51}$	$3.4297\ldots\cdot 10^{-54}$
	79	$7.8439\ldots\cdot 10^{-52}$	$4.2434\ldots\cdot 10^{-54}$	$3.7316\ldots\cdot 10^{-57}$
50	94	$4.3762\ldots\cdot 10^{-53}$	$1.8796\ldots\cdot 10^{-55}$	$1.0238\ldots\cdot 10^{-58}$
	95	$1.0313\ldots\cdot 10^{-55}$	$4.3160\ldots\cdot 10^{-58}$	$2.2367\ldots\cdot 10^{-61}$
	96	$1.9786\ldots\cdot 10^{-58}$	$8.0758\ldots\cdot 10^{-61}$	$3.9906\ldots\cdot 10^{-64}$
	97	$3.0090\ldots\cdot 10^{-61}$	$1.1988\ldots\cdot 10^{-63}$	$5.6601\ldots\cdot 10^{-67}$
	98	$3.4869\ldots\cdot 10^{-64}$	$1.3572\ldots\cdot 10^{-66}$	$6.1338\ldots\cdot 10^{-70}$
	99	$2.8904\ldots\cdot 10^{-67}$	$1.0999\ldots\cdot 10^{-69}$	$4.7665\ldots\cdot 10^{-73}$

Table 21: The accuracy of (3.24) for $l = 2$ and k near $2N+l-2$; $d = 2N-k+l-1$.

N	k	$\max_{1\le m\le d}\left\lvert -\frac{d-m+1}{m}\cdot\frac{q_{m-1}}{q_m}\Big/\eta_k - 1\right\rvert$ $a=0.2+14\mathrm{i}$	$a=0.5+20\mathrm{i}$	$a=0.6+30\mathrm{i}$
10	15	$1.9911\ldots\cdot10^{-2}$	$1.0115\ldots\cdot10^{-2}$	$1.1515\ldots\cdot10^{-2}$
	16	$1.5200\ldots\cdot10^{-3}$	$1.3691\ldots\cdot10^{-3}$	$7.6999\ldots\cdot10^{-4}$
	17	$7.5045\ldots\cdot10^{-5}$	$5.9915\ldots\cdot10^{-5}$	$2.3177\ldots\cdot10^{-5}$
	18	$2.4095\ldots\cdot10^{-6}$	$1.2579\ldots\cdot10^{-6}$	$8.7479\ldots\cdot10^{-7}$
	19	$4.9430\ldots\cdot10^{-8}$	$1.6935\ldots\cdot10^{-8}$	$2.7981\ldots\cdot10^{-8}$
	20	$6.1716\ldots\cdot10^{-10}$	$1.4147\ldots\cdot10^{-10}$	$1.6006\ldots\cdot10^{-10}$
20	35	$1.8378\ldots\cdot10^{-12}$	$9.0863\ldots\cdot10^{-14}$	$8.8849\ldots\cdot10^{-15}$
	36	$3.0673\ldots\cdot10^{-14}$	$1.2123\ldots\cdot10^{-15}$	$8.1097\ldots\cdot10^{-17}$
	37	$3.7541\ldots\cdot10^{-16}$	$1.2227\ldots\cdot10^{-17}$	$5.7261\ldots\cdot10^{-19}$
	38	$3.3038\ldots\cdot10^{-18}$	$9.1024\ldots\cdot10^{-20}$	$3.0602\ldots\cdot10^{-21}$
	39	$2.0241\ldots\cdot10^{-20}$	$4.8229\ldots\cdot10^{-22}$	$1.1945\ldots\cdot10^{-23}$
	40	$8.1586\ldots\cdot10^{-23}$	$1.7126\ldots\cdot10^{-24}$	$3.2055\ldots\cdot10^{-26}$
30	55	$9.9309\ldots\cdot10^{-25}$	$1.1910\ldots\cdot10^{-26}$	$5.9927\ldots\cdot10^{-29}$
	56	$6.5650\ldots\cdot10^{-27}$	$7.3020\ldots\cdot10^{-29}$	$3.0933\ldots\cdot10^{-31}$
	57	$3.3742\ldots\cdot10^{-29}$	$3.5046\ldots\cdot10^{-31}$	$1.2695\ldots\cdot10^{-33}$
	58	$1.3170\ldots\cdot10^{-31}$	$1.2849\ldots\cdot10^{-33}$	$4.0355\ldots\cdot10^{-36}$
	59	$3.7642\ldots\cdot10^{-34}$	$3.4671\ldots\cdot10^{-36}$	$9.5555\ldots\cdot10^{-39}$
	60	$7.4148\ldots\cdot10^{-37}$	$6.4754\ldots\cdot10^{-39}$	$1.5830\ldots\cdot10^{-41}$
40	75	$2.1157\ldots\cdot10^{-38}$	$1.3391\ldots\cdot10^{-40}$	$1.6195\ldots\cdot10^{-43}$
	76	$7.6977\ldots\cdot10^{-41}$	$4.6846\ldots\cdot10^{-43}$	$5.2190\ldots\cdot10^{-46}$
	77	$2.2427\ldots\cdot10^{-43}$	$1.3151\ldots\cdot10^{-45}$	$1.3570\ldots\cdot10^{-48}$
	78	$5.0997\ldots\cdot10^{-46}$	$2.8871\ldots\cdot10^{-48}$	$2.7722\ldots\cdot10^{-51}$
	79	$8.7083\ldots\cdot10^{-49}$	$4.7681\ldots\cdot10^{-51}$	$4.2788\ldots\cdot10^{-54}$
	80	$1.0492\ldots\cdot10^{-51}$	$5.5647\ldots\cdot10^{-54}$	$4.6847\ldots\cdot10^{-57}$
50	95	$5.7997\ldots\cdot10^{-53}$	$2.4527\ldots\cdot10^{-55}$	$1.2902\ldots\cdot10^{-58}$
	96	$1.3725\ldots\cdot10^{-55}$	$5.6571\ldots\cdot10^{-58}$	$2.8328\ldots\cdot10^{-61}$
	97	$2.6437\ldots\cdot10^{-58}$	$1.0630\ldots\cdot10^{-60}$	$5.0788\ldots\cdot10^{-64}$
	98	$4.0357\ldots\cdot10^{-61}$	$1.5845\ldots\cdot10^{-63}$	$7.2371\ldots\cdot10^{-67}$
	99	$4.6937\ldots\cdot10^{-64}$	$1.8008\ldots\cdot10^{-66}$	$7.8779\ldots\cdot10^{-70}$
	100	$3.9043\ldots\cdot10^{-67}$	$1.4650\ldots\cdot10^{-69}$	$6.1482\ldots\cdot10^{-73}$

Table 22: The accuracy of (3.24) for $l=3$ and k near $2N+l-2$; $d=2N-k+l-1$.

k	r_k	r_k/s_k
0	$-8.9397\ldots\cdot 10^{-118} +2.9041\ldots\cdot 10^{-118}\mathrm{i}$	$-1-1.2923\ldots\cdot 10^{-26} +9.2369\ldots\cdot 10^{-27}\mathrm{i}$
1	$-2.9674\ldots\cdot 10^{-89} +4.1582\ldots\cdot 10^{-89}\mathrm{i}$	$-1+9.2530\ldots\cdot 10^{-24} +5.2705\ldots\cdot 10^{-26}\mathrm{i}$
2	$-2.2639\ldots\cdot 10^{-64} -2.8626\ldots\cdot 10^{-64}\mathrm{i}$	$-1-3.6958\ldots\cdot 10^{-21} -2.8439\ldots\cdot 10^{-21}\mathrm{i}$
3	$-3.9155\ldots\cdot 10^{-43} +7.0563\ldots\cdot 10^{-43}\mathrm{i}$	$-1+4.7714\ldots\cdot 10^{-19} +1.9455\ldots\cdot 10^{-18}\mathrm{i}$
4	$-8.2461\ldots\cdot 10^{-25} +5.5196\ldots\cdot 10^{-25}\mathrm{i}$	$-1+3.1823\ldots\cdot 10^{-16} -6.4602\ldots\cdot 10^{-16}\mathrm{i}$
5	$7.4035\ldots\cdot 10^{-10} -8.2499\ldots\cdot 10^{-10}\mathrm{i}$	$-1-1.9555\ldots\cdot 10^{-13} +8.1133\ldots\cdot 10^{-14}\mathrm{i}$
6	$-4.5466\ldots\cdot 10^{2} -1.7165\ldots\cdot 10^{3}\mathrm{i}$	$-1+4.6519\ldots\cdot 10^{-11} +1.6579\ldots\cdot 10^{-11}\mathrm{i}$
7	$5.8908\ldots\cdot 10^{12} -2.6426\ldots\cdot 10^{12}\mathrm{i}$	$-1-4.1342\ldots\cdot 10^{-9} -7.7187\ldots\cdot 10^{-9}\mathrm{i}$
8	$-8.4380\ldots\cdot 10^{19} -2.4850\ldots\cdot 10^{19}\mathrm{i}$	$-1-2.1770\ldots\cdot 10^{-7} +1.0801\ldots\cdot 10^{-6}\mathrm{i}$
9	$-8.5003\ldots\cdot 10^{24} -5.0703\ldots\cdot 10^{24}\mathrm{i}$	$-1+4.3258\ldots\cdot 10^{-5} -5.8534\ldots\cdot 10^{-5}\mathrm{i}$
10	$4.2788\ldots\cdot 10^{28} -2.0061\ldots\cdot 10^{28}\mathrm{i}$	$-1-1.0404\ldots\cdot 10^{-3} -1.7398\ldots\cdot 10^{-4}\mathrm{i}$
11	$-6.8205\ldots\cdot 10^{30} +5.8104\ldots\cdot 10^{30}\mathrm{i}$	$-1-1.1056\ldots\cdot 10^{-2} +8.0431\ldots\cdot 10^{-3}\mathrm{i}$
12	$8.1297\ldots\cdot 10^{31} -3.9259\ldots\cdot 10^{31}\mathrm{i}$	$-1-6.6808\ldots\cdot 10^{-2} +1.3896\ldots\cdot 10^{-1}\mathrm{i}$
13	$-7.2775\ldots\cdot 10^{31} +3.0580\ldots\cdot 10^{30}\mathrm{i}$	$-3.2165\ldots\cdot 10^{-1} +1.0116\ldots\cdot 10^{0}\mathrm{i}$
14	$1.2953\ldots\cdot 10^{31} +5.8025\ldots\cdot 10^{30}\mathrm{i}$	$+4.9858\ldots\cdot 10^{-1} +3.6297\ldots\cdot 10^{-1}\mathrm{i}$
15	$2.2202\ldots\cdot 10^{30} -1.1503\ldots\cdot 10^{30}\mathrm{i}$	$+1+4.8168\ldots\cdot 10^{-2} -1.6900\ldots\cdot 10^{-2}\mathrm{i}$
16	$-2.8220\ldots\cdot 10^{28} +1.2915\ldots\cdot 10^{28}\mathrm{i}$	$+1+1.9011\ldots\cdot 10^{-4} -4.7152\ldots\cdot 10^{-4}\mathrm{i}$
17	$1.0370\ldots\cdot 10^{24} -5.6431\ldots\cdot 10^{24}\mathrm{i}$	$+1-1.0007\ldots\cdot 10^{-6} -3.8151\ldots\cdot 10^{-6}\mathrm{i}$
18	$9.4085\ldots\cdot 10^{18} -9.1736\ldots\cdot 10^{17}\mathrm{i}$	$+1-1.5173\ldots\cdot 10^{-8} -1.0305\ldots\cdot 10^{-8}\mathrm{i}$
19	$6.7082\ldots\cdot 10^{10} +2.5062\ldots\cdot 10^{10}\mathrm{i}$	$+1-3.3558\ldots\cdot 10^{-11} +6.6263\ldots\cdot 10^{-12}\mathrm{i}$

Table 23: Accuracy of (3.28) and (3.30) for $a = 1/3 + 14\mathrm{i}$, $N = 20$, and $l = 1$.

k	r_k	r_k/s_k
0	$-4.7569\ldots\cdot 10^{-112} -3.6658\ldots\cdot 10^{-112}\mathrm{i}$	$-1-2.2818\ldots\cdot 10^{-26} +1.8679\ldots\cdot 10^{-26}\mathrm{i}$
1	$1.1308\ldots\cdot 10^{-83} +6.5179\ldots\cdot 10^{-85}\mathrm{i}$	$-1+1.6849\ldots\cdot 10^{-23} -1.1015\ldots\cdot 10^{-24}\mathrm{i}$
2	$1.0711\ldots\cdot 10^{-59} -2.6538\ldots\cdot 10^{-59}\mathrm{i}$	$-1-6.9960\ldots\cdot 10^{-21} -4.5633\ldots\cdot 10^{-21}\mathrm{i}$
3	$2.2822\ldots\cdot 10^{-38} +9.3372\ldots\cdot 10^{-40}\mathrm{i}$	$-1+1.1299\ldots\cdot 10^{-18} +3.3293\ldots\cdot 10^{-18}\mathrm{i}$
4	$-8.6842\ldots\cdot 10^{-21} -5.6686\ldots\cdot 10^{-21}\mathrm{i}$	$-1+4.3471\ldots\cdot 10^{-16} -1.1569\ldots\cdot 10^{-15}\mathrm{i}$
5	$-4.0297\ldots\cdot 10^{-6} -1.7352\ldots\cdot 10^{-6}\mathrm{i}$	$-1-3.0974\ldots\cdot 10^{-13} +1.7217\ldots\cdot 10^{-13}\mathrm{i}$
6	$2.4241\ldots\cdot 10^{6} -1.2757\ldots\cdot 10^{6}\mathrm{i}$	$-1+7.8678\ldots\cdot 10^{-11} +1.6408\ldots\cdot 10^{-11}\mathrm{i}$
7	$-1.9716\ldots\cdot 10^{15} -3.4906\ldots\cdot 10^{15}\mathrm{i}$	$-1-8.5112\ldots\cdot 10^{-9} -1.0879\ldots\cdot 10^{-8}\mathrm{i}$
8	$8.6811\ldots\cdot 10^{21} -2.0768\ldots\cdot 10^{22}\mathrm{i}$	$-1+7.4979\ldots\cdot 10^{-8} +1.6999\ldots\cdot 10^{-6}\mathrm{i}$
9	$-1.2154\ldots\cdot 10^{27} +4.6440\ldots\cdot 10^{26}\mathrm{i}$	$-1+3.3973\ldots\cdot 10^{-6} -9.1336\ldots\cdot 10^{-5}\mathrm{i}$
10	$-1.6266\ldots\cdot 10^{30} +5.2495\ldots\cdot 10^{30}\mathrm{i}$	$-1-7.6524\ldots\cdot 10^{-4} +5.1846\ldots\cdot 10^{-5}\mathrm{i}$
11	$2.9074\ldots\cdot 10^{31} +8.5761\ldots\cdot 10^{32}\mathrm{i}$	$-1-7.5872\ldots\cdot 10^{-3} +5.6661\ldots\cdot 10^{-3}\mathrm{i}$
12	$-6.8889\ldots\cdot 10^{32} +7.1047\ldots\cdot 10^{33}\mathrm{i}$	$-1-5.8373\ldots\cdot 10^{-2} +8.6271\ldots\cdot 10^{-2}\mathrm{i}$
13	$-1.7147\ldots\cdot 10^{33} +3.5042\ldots\cdot 10^{33}\mathrm{i}$	$-1+4.9975\ldots\cdot 10^{-1} +1.0845\ldots\cdot 10^{0}\mathrm{i}$
14	$-4.5204\ldots\cdot 10^{32} +2.1418\ldots\cdot 10^{32}\mathrm{i}$	$+3.1882\ldots\cdot 10^{-1} +5.4686\ldots\cdot 10^{-1}\mathrm{i}$
15	$-3.2995\ldots\cdot 10^{31} -7.6923\ldots\cdot 10^{31}\mathrm{i}$	$+1+3.7369\ldots\cdot 10^{-2} -2.3298\ldots\cdot 10^{-2}\mathrm{i}$
16	$-3.6520\ldots\cdot 10^{29} -4.3255\ldots\cdot 10^{29}\mathrm{i}$	$+1+6.1180\ldots\cdot 10^{-5} -4.4287\ldots\cdot 10^{-4}\mathrm{i}$
17	$-5.0260\ldots\cdot 10^{25} +2.2114\ldots\cdot 10^{25}\mathrm{i}$	$+1-1.6807\ldots\cdot 10^{-6} -2.7284\ldots\cdot 10^{-6}\mathrm{i}$
18	$3.8003\ldots\cdot 10^{19} +2.3561\ldots\cdot 10^{19}\mathrm{i}$	$+1-1.3221\ldots\cdot 10^{-8} -4.0128\ldots\cdot 10^{-9}\mathrm{i}$
19	$-1.2322\ldots\cdot 10^{11} -1.0075\ldots\cdot 10^{11}\mathrm{i}$	$+1-2.1575\ldots\cdot 10^{-11} +1.1718\ldots\cdot 10^{-11}\mathrm{i}$

Table 24: Accuracy of (3.28) and (3.30) for $a = 1/3 + 14\mathrm{i}$, $N = 20$, and $l = 2$.

k	r_k	r_k/s_k
0	$1.2739\ldots\cdot10^{-281}-1.8005\ldots\cdot10^{-281}\mathrm{i}$	$-1-3.3046\ldots\cdot10^{-43}-8.7177\ldots\cdot10^{-44}\mathrm{i}$
1	$-4.4174\ldots\cdot10^{-234}-6.3356\ldots\cdot10^{-234}\mathrm{i}$	$-1+2.5555\ldots\cdot10^{-40}+2.0450\ldots\cdot10^{-40}\mathrm{i}$
2	$1.1617\ldots\cdot10^{-190}-9.5490\ldots\cdot10^{-191}\mathrm{i}$	$-1-1.3037\ldots\cdot10^{-37}-2.5735\ldots\cdot10^{-37}\mathrm{i}$
3	$-3.3242\ldots\cdot10^{-151}-1.2833\ldots\cdot10^{-151}\mathrm{i}$	$-1+7.4185\ldots\cdot10^{-36}+2.3283\ldots\cdot10^{-34}\mathrm{i}$
4	$-9.4420\ldots\cdot10^{-116}-1.4071\ldots\cdot10^{-115}\mathrm{i}$	$-1+6.9158\ldots\cdot10^{-32}-1.5691\ldots\cdot10^{-31}\mathrm{i}$
5	$1.0022\ldots\cdot10^{-83}+2.1617\ldots\cdot10^{-83}\mathrm{i}$	$-1-8.7586\ldots\cdot10^{-29}+7.3636\ldots\cdot10^{-29}\mathrm{i}$
6	$9.2840\ldots\cdot10^{-55}+1.0155\ldots\cdot10^{-54}\mathrm{i}$	$-1+6.6936\ldots\cdot10^{-26}-1.5927\ldots\cdot10^{-26}\mathrm{i}$
7	$-4.3724\ldots\cdot10^{-29}-2.3160\ldots\cdot10^{-30}\mathrm{i}$	$-1-3.5972\ldots\cdot10^{-23}-8.7905\ldots\cdot10^{-24}\mathrm{i}$
8	$-3.6452\ldots\cdot10^{-7}+9.4035\ldots\cdot10^{-7}\mathrm{i}$	$-1+1.3324\ldots\cdot10^{-20}+1.1662\ldots\cdot10^{-20}\mathrm{i}$
9	$-2.1337\ldots\cdot10^{13}-4.4880\ldots\cdot10^{12}\mathrm{i}$	$-1-2.6942\ldots\cdot10^{-18}-6.9601\ldots\cdot10^{-18}\mathrm{i}$
10	$2.3898\ldots\cdot10^{29}+5.1719\ldots\cdot10^{29}\mathrm{i}$	$-1-3.0761\ldots\cdot10^{-16}+2.7315\ldots\cdot10^{-15}\mathrm{i}$
11	$-6.6625\ldots\cdot10^{41}+2.3084\ldots\cdot10^{43}\mathrm{i}$	$-1+4.8477\ldots\cdot10^{-13}-7.3041\ldots\cdot10^{-13}\mathrm{i}$
12	$7.6432\ldots\cdot10^{52}-1.8616\ldots\cdot10^{54}\mathrm{i}$	$-1-2.0707\ldots\cdot10^{-10}+1.2104\ldots\cdot10^{-10}\mathrm{i}$
13	$-1.3552\ldots\cdot10^{62}-3.5924\ldots\cdot10^{62}\mathrm{i}$	$-1+5.5519\ldots\cdot10^{-8}-5.9522\ldots\cdot10^{-9}\mathrm{i}$
14	$2.3741\ldots\cdot10^{68}+1.0841\ldots\cdot10^{68}\mathrm{i}$	$-1-1.0477\ldots\cdot10^{-5}-3.0526\ldots\cdot10^{-6}\mathrm{i}$
15	$-1.0896\ldots\cdot10^{72}+1.0418\ldots\cdot10^{70}\mathrm{i}$	$-1-1.2157\ldots\cdot10^{-3}+5.1981\ldots\cdot10^{-5}\mathrm{i}$
16	$-1.5657\ldots\cdot10^{75}-1.0118\ldots\cdot10^{74}\mathrm{i}$	$-1+3.2841\ldots\cdot10^{-4}-1.3436\ldots\cdot10^{-3}\mathrm{i}$
17	$1.2937\ldots\cdot10^{77}-4.9423\ldots\cdot10^{76}\mathrm{i}$	$-1-2.4058\ldots\cdot10^{-2}+1.0630\ldots\cdot10^{-2}\mathrm{i}$
18	$-5.6156\ldots\cdot10^{77}+4.4065\ldots\cdot10^{77}\mathrm{i}$	$-1-2.4918\ldots\cdot10^{-1}+1.7097\ldots\cdot10^{-1}\mathrm{i}$
19	$2.4875\ldots\cdot10^{77}-3.0182\ldots\cdot10^{77}\mathrm{i}$	$+1+1.2703\ldots\cdot10^{-1}+1.3860\ldots\cdot10^{0}\mathrm{i}$
20	$-6.6992\ldots\cdot10^{76}+1.5375\ldots\cdot10^{76}\mathrm{i}$	$-1.9990\ldots\cdot10^{-1}+6.3279\ldots\cdot10^{-1}\mathrm{i}$
21	$1.4169\ldots\cdot10^{76}+1.1826\ldots\cdot10^{76}\mathrm{i}$	$+1-4.0056\ldots\cdot10^{-1}+5.1770\ldots\cdot10^{-1}\mathrm{i}$
22	$2.3074\ldots\cdot10^{75}-1.4521\ldots\cdot10^{75}\mathrm{i}$	$+1-2.3574\ldots\cdot10^{-2}+2.3880\ldots\cdot10^{-2}\mathrm{i}$
23	$1.4781\ldots\cdot10^{73}-1.8278\ldots\cdot10^{73}\mathrm{i}$	$+1+1.4895\ldots\cdot10^{-4}+3.9724\ldots\cdot10^{-4}\mathrm{i}$
24	$-2.6316\ldots\cdot10^{69}-2.2490\ldots\cdot10^{69}\mathrm{i}$	$+1+3.9057\ldots\cdot10^{-6}+1.7653\ldots\cdot10^{-6}\mathrm{i}$
25	$1.7445\ldots\cdot10^{63}+6.0084\ldots\cdot10^{63}\mathrm{i}$	$+1+3.3332\ldots\cdot10^{-8}-1.1325\ldots\cdot10^{-8}\mathrm{i}$
26	$-5.7673\ldots\cdot10^{55}-7.5906\ldots\cdot10^{55}\mathrm{i}$	$+1+8.4745\ldots\cdot10^{-11}-1.8716\ldots\cdot10^{-10}\mathrm{i}$
27	$7.7141\ldots\cdot10^{45}-2.8544\ldots\cdot10^{45}\mathrm{i}$	$+1-3.1741\ldots\cdot10^{-13}-6.9038\ldots\cdot10^{-13}\mathrm{i}$
28	$2.0795\ldots\cdot10^{33}+1.5114\ldots\cdot10^{33}\mathrm{i}$	$+1-1.5361\ldots\cdot10^{-15}-4.3714\ldots\cdot10^{-16}\mathrm{i}$
29	$1.2348\ldots\cdot10^{18}+1.1357\ldots\cdot10^{18}\mathrm{i}$	$+1-1.2204\ldots\cdot10^{-18}+8.0136\ldots\cdot10^{-19}\mathrm{i}$

Table 25: Accuracy of (3.28) and (3.30) for $a = 1/3 + 14\mathrm{i}$, $N = 30$, and $l = 1$.

k	r_k	r_k/s_k
0	$9.6587\ldots\cdot10^{-271} -7.4222\ldots\cdot10^{-272}\mathrm{i}$	$-1-7.0683\ldots\cdot10^{-43} -1.5032\ldots\cdot10^{-43}\mathrm{i}$
1	$-2.9344\ldots\cdot10^{-224} +9.7993\ldots\cdot10^{-224}\mathrm{i}$	$-1+5.5638\ldots\cdot10^{-40} +4.0098\ldots\cdot10^{-40}\mathrm{i}$
2	$5.8956\ldots\cdot10^{-181} +1.4146\ldots\cdot10^{-181}\mathrm{i}$	$-1-2.9839\ldots\cdot10^{-37} -5.1917\ldots\cdot10^{-37}\mathrm{i}$
3	$1.1039\ldots\cdot10^{-142} +4.2762\ldots\cdot10^{-142}\mathrm{i}$	$-1+4.1745\ldots\cdot10^{-35} +4.7701\ldots\cdot10^{-34}\mathrm{i}$
4	$2.4674\ldots\cdot10^{-107} -6.0378\ldots\cdot10^{-107}\mathrm{i}$	$-1+1.2178\ldots\cdot10^{-31} -3.2696\ldots\cdot10^{-31}\mathrm{i}$
5	$1.5716\ldots\cdot10^{-75} -2.4116\ldots\cdot10^{-75}\mathrm{i}$	$-1-1.6678\ldots\cdot10^{-28} +1.5895\ldots\cdot10^{-28}\mathrm{i}$
6	$-1.5997\ldots\cdot10^{-47} +5.0262\ldots\cdot10^{-47}\mathrm{i}$	$-1+1.3090\ldots\cdot10^{-25} -4.0529\ldots\cdot10^{-26}\mathrm{i}$
7	$2.2314\ldots\cdot10^{-22} +4.9028\ldots\cdot10^{-22}\mathrm{i}$	$-1-7.1899\ldots\cdot10^{-23} -1.2178\ldots\cdot10^{-23}\mathrm{i}$
8	$-4.0209\ldots\cdot10^{0} +2.4978\ldots\cdot10^{-1}\mathrm{i}$	$-1+2.7631\ldots\cdot10^{-20} +2.0605\ldots\cdot10^{-20}\mathrm{i}$
9	$4.8092\ldots\cdot10^{18} +2.8213\ldots\cdot10^{19}\mathrm{i}$	$-1-6.3121\ldots\cdot10^{-18} -1.2895\ldots\cdot10^{-17}\mathrm{i}$
10	$-1.8297\ldots\cdot10^{35} +1.6841\ldots\cdot10^{35}\mathrm{i}$	$-1-8.7634\ldots\cdot10^{-17} +5.2247\ldots\cdot10^{-15}\mathrm{i}$
11	$3.3199\ldots\cdot10^{48} -6.9226\ldots\cdot10^{47}\mathrm{i}$	$-1+7.5896\ldots\cdot10^{-13} -1.4621\ldots\cdot10^{-12}\mathrm{i}$
12	$9.2542\ldots\cdot10^{58} -9.4811\ldots\cdot10^{57}\mathrm{i}$	$-1-3.5488\ldots\cdot10^{-10} +2.7025\ldots\cdot10^{-10}\mathrm{i}$
13	$-6.0688\ldots\cdot10^{66} +2.5229\ldots\cdot10^{66}\mathrm{i}$	$-1+9.9871\ldots\cdot10^{-8} -2.5202\ldots\cdot10^{-8}\mathrm{i}$
14	$-7.8749\ldots\cdot10^{71} +1.4303\ldots\cdot10^{72}\mathrm{i}$	$-1-1.8692\ldots\cdot10^{-5} -2.7240\ldots\cdot10^{-6}\mathrm{i}$
15	$-5.1103\ldots\cdot10^{75} +8.9364\ldots\cdot10^{75}\mathrm{i}$	$-1-4.2223\ldots\cdot10^{-4} +2.5424\ldots\cdot10^{-4}\mathrm{i}$
16	$2.9095\ldots\cdot10^{78} -4.6154\ldots\cdot10^{77}\mathrm{i}$	$-1-1.3649\ldots\cdot10^{-3} -5.1157\ldots\cdot10^{-3}\mathrm{i}$
17	$2.1415\ldots\cdot10^{80} -2.6666\ldots\cdot10^{80}\mathrm{i}$	$-1-1.1805\ldots\cdot10^{-2} +4.6212\ldots\cdot10^{-3}\mathrm{i}$
18	$5.1081\ldots\cdot10^{80} -1.5654\ldots\cdot10^{81}\mathrm{i}$	$-1-1.5345\ldots\cdot10^{-1} +7.8730\ldots\cdot10^{-2}\mathrm{i}$
19	$-1.8402\ldots\cdot10^{80} -5.9001\ldots\cdot10^{80}\mathrm{i}$	$+3+1.4969\ldots\cdot10^{-1} -1.0759\ldots\cdot10^{0}\mathrm{i}$
20	$4.5486\ldots\cdot10^{79} -2.4154\ldots\cdot10^{79}\mathrm{i}$	$-2.3427\ldots\cdot10^{-1} +1.2115\ldots\cdot10^{-1}\mathrm{i}$
21	$2.5114\ldots\cdot10^{79} +1.0510\ldots\cdot10^{78}\mathrm{i}$	$+1-2.3325\ldots\cdot10^{-1} +2.5461\ldots\cdot10^{-1}\mathrm{i}$
22	$-8.2778\ldots\cdot10^{77} +9.3212\ldots\cdot10^{77}\mathrm{i}$	$+1-1.3618\ldots\cdot10^{-2} +2.8424\ldots\cdot10^{-2}\mathrm{i}$
23	$-5.9468\ldots\cdot10^{74} -6.1404\ldots\cdot10^{75}\mathrm{i}$	$+1+2.0727\ldots\cdot10^{-4} +3.0616\ldots\cdot10^{-4}\mathrm{i}$
24	$4.2392\ldots\cdot10^{71} -1.8748\ldots\cdot10^{71}\mathrm{i}$	$+1+3.5199\ldots\cdot10^{-6} +4.5743\ldots\cdot10^{-7}\mathrm{i}$
25	$3.9980\ldots\cdot10^{65} -7.0375\ldots\cdot10^{64}\mathrm{i}$	$+1+2.1473\ldots\cdot10^{-8} -1.6271\ldots\cdot10^{-8}\mathrm{i}$
26	$1.8379\ldots\cdot10^{57} -2.2101\ldots\cdot10^{57}\mathrm{i}$	$+1+1.5914\ldots\cdot10^{-11} -1.4708\ldots\cdot10^{-10}\mathrm{i}$
27	$-8.8086\ldots\cdot10^{46} -6.7567\ldots\cdot10^{46}\mathrm{i}$	$+1-3.5539\ldots\cdot10^{-13} -3.8674\ldots\cdot10^{-13}\mathrm{i}$
28	$3.9854\ldots\cdot10^{33} +1.4476\ldots\cdot10^{34}\mathrm{i}$	$+1-1.0687\ldots\cdot10^{-15} +3.6303\ldots\cdot10^{-17}\mathrm{i}$
29	$-1.9341\ldots\cdot10^{18} -3.6291\ldots\cdot10^{18}\mathrm{i}$	$+1-5.9570\ldots\cdot10^{-19} +7.4042\ldots\cdot10^{-19}\mathrm{i}$

Table 26: Accuracy of (3.28) and (3.30) for $a = 1/3 + 14\mathrm{i}$, $N = 30$, and $l = 2$.

a	$\lvert \det(T_{0,N})/\det(T_{2,N-1}) - 1\rvert$		
	$N = 10$	$N = 50$	$N = 300$
-6	$4.2612\ldots\cdot 10^{1}$	$1.1799\ldots\cdot 10^{2}$	$6.6484\ldots\cdot 10^{-4}$
-5	$6.3520\ldots\cdot 10^{1}$	$2.4046\ldots\cdot 10^{2}$	$1.2441\ldots\cdot 10^{-4}$
-4	$9.6199\ldots\cdot 10^{0}$	$3.0066\ldots\cdot 10^{0}$	$1.1859\ldots\cdot 10^{-5}$
-3	$2.2126\ldots\cdot 10^{0}$	$2.9567\ldots\cdot 10^{-1}$	$1.0821\ldots\cdot 10^{-5}$
-2	$3.0811\ldots\cdot 10^{0}$	$1.4383\ldots\cdot 10^{-1}$	$5.7055\ldots\cdot 10^{-7}$
-1	$5.1095\ldots\cdot 10^{-1}$	$7.0189\ldots\cdot 10^{-3}$	$9.1201\ldots\cdot 10^{-8}$
$-1+10\mathrm{i}$	$1.1824\ldots\cdot 10^{0}$	$7.6855\ldots\cdot 10^{-2}$	$5.2291\ldots\cdot 10^{-8}$
$-1+50\mathrm{i}$	$8.8365\ldots\cdot 10^{1}$	$1.2891\ldots\cdot 10^{1}$	$1.0838\ldots\cdot 10^{-6}$
$-1+300\mathrm{i}$	$1.3717\ldots\cdot 10^{3}$	$1.3207\ldots\cdot 10^{3}$	$1.1627\ldots\cdot 10^{3}$
-0.8	$3.6093\ldots\cdot 10^{-1}$	$5.6840\ldots\cdot 10^{-3}$	$6.4104\ldots\cdot 10^{-8}$
$-0.8+10\mathrm{i}$	$8.9539\ldots\cdot 10^{-1}$	$6.7158\ldots\cdot 10^{-2}$	$3.0255\ldots\cdot 10^{-8}$
$-0.8+50\mathrm{i}$	$5.0872\ldots\cdot 10^{1}$	$8.3247\ldots\cdot 10^{0}$	$6.8354\ldots\cdot 10^{-7}$
$-0.8+100\mathrm{i}$	$1.2444\ldots\cdot 10^{2}$	$1.2193\ldots\cdot 10^{2}$	$4.3438\ldots\cdot 10^{-4}$
$-0.8+300\mathrm{i}$	$5.5682\ldots\cdot 10^{2}$	$5.3128\ldots\cdot 10^{2}$	$4.7434\ldots\cdot 10^{2}$
0	$1.1165\ldots\cdot 10^{-1}$	$1.1862\ldots\cdot 10^{-2}$	$1.0474\ldots\cdot 10^{-7}$
$10\mathrm{i}$	$2.4161\ldots\cdot 10^{-1}$	$5.9512\ldots\cdot 10^{-3}$	$3.6726\ldots\cdot 10^{-8}$
$50\mathrm{i}$	$5.6514\ldots\cdot 10^{0}$	$1.6116\ldots\cdot 10^{0}$	$7.0751\ldots\cdot 10^{-8}$
$300\mathrm{i}$	$1.5537\ldots\cdot 10^{1}$	$1.3861\ldots\cdot 10^{1}$	$1.3111\ldots\cdot 10^{1}$
0.2	$8.7210\ldots\cdot 10^{-2}$	$1.2535\ldots\cdot 10^{-2}$	$4.1954\ldots\cdot 10^{-8}$
$0.2+10\mathrm{i}$	$1.6186\ldots\cdot 10^{-1}$	$4.1468\ldots\cdot 10^{-3}$	$2.6930\ldots\cdot 10^{-8}$
$0.2+50\mathrm{i}$	$3.2900\ldots\cdot 10^{0}$	$1.0757\ldots\cdot 10^{0}$	$4.3350\ldots\cdot 10^{-8}$
$0.2+100\mathrm{i}$	$3.8034\ldots\cdot 10^{0}$	$4.0733\ldots\cdot 10^{0}$	$3.2153\ldots\cdot 10^{-5}$
$0.2+300\mathrm{i}$	$6.3822\ldots\cdot 10^{0}$	$5.5388\ldots\cdot 10^{0}$	$5.3432\ldots\cdot 10^{0}$
0.5	$6.4161\ldots\cdot 10^{-2}$	$8.4996\ldots\cdot 10^{-3}$	$1.0585\ldots\cdot 10^{-8}$
$0.5+10\mathrm{i}$	$8.9909\ldots\cdot 10^{-2}$	$3.1341\ldots\cdot 10^{-3}$	$1.6436\ldots\cdot 10^{-8}$
$0.5+50\mathrm{i}$	$1.4861\ldots\cdot 10^{0}$	$5.8511\ldots\cdot 10^{-1}$	$2.6381\ldots\cdot 10^{-8}$
$0.5+300\mathrm{i}$	$1.6680\ldots\cdot 10^{0}$	$1.3713\ldots\cdot 10^{0}$	$1.3888\ldots\cdot 10^{0}$
0.8	$5.3468\ldots\cdot 10^{-2}$	$6.1407\ldots\cdot 10^{-3}$	$4.1760\ldots\cdot 10^{-9}$
$0.8+10\mathrm{i}$	$7.0071\ldots\cdot 10^{-2}$	$2.8319\ldots\cdot 10^{-3}$	$8.1363\ldots\cdot 10^{-9}$
$0.8+50\mathrm{i}$	$6.9648\ldots\cdot 10^{-1}$	$3.1958\ldots\cdot 10^{-1}$	$1.8955\ldots\cdot 10^{-8}$
$0.8+100\mathrm{i}$	$4.6403\ldots\cdot 10^{-1}$	$5.7041\ldots\cdot 10^{-1}$	$4.5812\ldots\cdot 10^{-6}$
$0.8+300\mathrm{i}$	$4.2792\ldots\cdot 10^{-1}$	$3.1600\ldots\cdot 10^{-1}$	$3.5902\ldots\cdot 10^{-1}$
$1+10\mathrm{i}$	$7.3817\ldots\cdot 10^{-2}$	$2.6329\ldots\cdot 10^{-3}$	$4.4885\ldots\cdot 10^{-9}$
$1+50\mathrm{i}$	$4.3467\ldots\cdot 10^{-1}$	$2.1544\ldots\cdot 10^{-1}$	$1.2980\ldots\cdot 10^{-8}$
$1+300\mathrm{i}$	$1.8468\ldots\cdot 10^{-1}$	$1.0633\ldots\cdot 10^{-1}$	$1.4385\ldots\cdot 10^{-1}$

Table 27: Accuracy of (4.2).

a	$\left\lvert \frac{\det(O_{1,1}(T_{l,N}))}{\det(o_{1,1}(T_{l,N}))} + 1 \right\rvert$ $l = 1$	$l = 2$	$l = 3$
-6	$4.2767\ldots \cdot 10^{-6}$	$1.6987\ldots \cdot 10^{-7}$	$2.1697\ldots \cdot 10^{-8}$
-5	$5.5836\ldots \cdot 10^{-5}$	$2.8834\ldots \cdot 10^{-7}$	$3.3280\ldots \cdot 10^{-9}$
-4	$2.0460\ldots \cdot 10^{-7}$	$7.6939\ldots \cdot 10^{-8}$	$1.0495\ldots \cdot 10^{-8}$
-3	$3.4034\ldots \cdot 10^{-4}$	$1.9598\ldots \cdot 10^{-8}$	$4.7623\ldots \cdot 10^{-10}$
-2	$1.5106\ldots \cdot 10^{-7}$	$8.1489\ldots \cdot 10^{-9}$	$2.3314\ldots \cdot 10^{-10}$
-1	$1.1236\ldots \cdot 10^{-8}$	$5.0549\ldots \cdot 10^{-9}$	$1.1921\ldots \cdot 10^{-11}$
$-1 + 10\mathrm{i}$	$6.9024\ldots \cdot 10^{-10}$	$4.5556\ldots \cdot 10^{-12}$	$6.8856\ldots \cdot 10^{-14}$
$-1 + 50\mathrm{i}$	$2.3203\ldots \cdot 10^{-10}$	$1.1582\ldots \cdot 10^{-12}$	$2.4854\ldots \cdot 10^{-14}$
$-1 + 300\mathrm{i}$	$3.8909\ldots \cdot 10^{-3}$	$3.0171\ldots \cdot 10^{-5}$	$3.5050\ldots \cdot 10^{-7}$
$-1 + 1500\mathrm{i}$	$4.1398\ldots \cdot 10^{-3}$	$3.2639\ldots \cdot 10^{-5}$	$3.8570\ldots \cdot 10^{-7}$
-0.8	$1.0968\ldots \cdot 10^{-8}$	$9.4580\ldots \cdot 10^{-10}$	$7.6924\ldots \cdot 10^{-12}$
$-0.8 + 10\mathrm{i}$	$5.7120\ldots \cdot 10^{-10}$	$1.0315\ldots \cdot 10^{-11}$	$6.6540\ldots \cdot 10^{-14}$
$-0.8 + 50\mathrm{i}$	$2.8513\ldots \cdot 10^{-10}$	$9.1923\ldots \cdot 10^{-13}$	$1.2972\ldots \cdot 10^{-14}$
$-0.8 + 100\mathrm{i}$	$7.6449\ldots \cdot 10^{-8}$	$2.3616\ldots \cdot 10^{-10}$	$1.0847\ldots \cdot 10^{-12}$
$-0.8 + 300\mathrm{i}$	$3.9888\ldots \cdot 10^{-3}$	$3.0835\ldots \cdot 10^{-5}$	$3.5743\ldots \cdot 10^{-7}$
$-0.8 + 1500\mathrm{i}$	$4.2040\ldots \cdot 10^{-3}$	$3.2984\ldots \cdot 10^{-5}$	$3.8851\ldots \cdot 10^{-7}$
0	$5.5094\ldots \cdot 10^{-9}$	$1.4024\ldots \cdot 10^{-10}$	$2.5979\ldots \cdot 10^{-13}$
$10\mathrm{i}$	$5.5542\ldots \cdot 10^{-10}$	$3.7006\ldots \cdot 10^{-12}$	$7.9216\ldots \cdot 10^{-14}$
$50\mathrm{i}$	$9.3711\ldots \cdot 10^{-10}$	$7.4365\ldots \cdot 10^{-12}$	$1.4800\ldots \cdot 10^{-14}$
$300\mathrm{i}$	$4.6384\ldots \cdot 10^{-3}$	$3.4680\ldots \cdot 10^{-5}$	$3.9509\ldots \cdot 10^{-7}$
0.2	$8.2207\ldots \cdot 10^{-9}$	$1.3653\ldots \cdot 10^{-10}$	$1.1811\ldots \cdot 10^{-12}$
$0.2 + 10\mathrm{i}$	$4.6928\ldots \cdot 10^{-10}$	$4.6885\ldots \cdot 10^{-12}$	$4.0950\ldots \cdot 10^{-14}$
$0.2 + 50\mathrm{i}$	$1.2297\ldots \cdot 10^{-9}$	$1.4242\ldots \cdot 10^{-11}$	$5.3311\ldots \cdot 10^{-14}$
$0.2 + 100\mathrm{i}$	$2.7451\ldots \cdot 10^{-7}$	$1.0945\ldots \cdot 10^{-9}$	$1.0539\ldots \cdot 10^{-11}$
$0.2 + 300\mathrm{i}$	$4.9506\ldots \cdot 10^{-3}$	$3.6217\ldots \cdot 10^{-5}$	$4.0849\ldots \cdot 10^{-7}$
$0.2 + 1500\mathrm{i}$	$5.7523\ldots \cdot 10^{-3}$	$4.0261\ldots \cdot 10^{-5}$	$4.4197\ldots \cdot 10^{-7}$
0.5	$2.2978\ldots \cdot 10^{-8}$	$1.6268\ldots \cdot 10^{-10}$	$3.7873\ldots \cdot 10^{-12}$
$0.5 + 10\mathrm{i}$	$2.8433\ldots \cdot 10^{-10}$	$4.8533\ldots \cdot 10^{-12}$	$5.7369\ldots \cdot 10^{-14}$
$0.5 + 50\mathrm{i}$	$3.2338\ldots \cdot 10^{-9}$	$1.6214\ldots \cdot 10^{-11}$	$1.3539\ldots \cdot 10^{-13}$
$0.5 + 300\mathrm{i}$	$5.5794\ldots \cdot 10^{-3}$	$3.9551\ldots \cdot 10^{-5}$	$4.3429\ldots \cdot 10^{-7}$
$0.5 + 1500\mathrm{i}$	$8.7189\ldots \cdot 10^{-3}$	$5.0293\ldots \cdot 10^{-5}$	$5.0223\ldots \cdot 10^{-7}$
0.8	$1.4756\ldots \cdot 10^{-7}$	$1.4369\ldots \cdot 10^{-10}$	$6.8037\ldots \cdot 10^{-12}$
$0.8 + 10\mathrm{i}$	$1.4442\ldots \cdot 10^{-10}$	$3.6589\ldots \cdot 10^{-12}$	$7.6624\ldots \cdot 10^{-14}$
$0.8 + 50\mathrm{i}$	$5.6967\ldots \cdot 10^{-9}$	$2.4632\ldots \cdot 10^{-11}$	$3.6208\ldots \cdot 10^{-13}$
$0.8 + 100\mathrm{i}$	$7.0873\ldots \cdot 10^{-7}$	$2.7082\ldots \cdot 10^{-9}$	$9.9229\ldots \cdot 10^{-12}$
$0.8 + 300\mathrm{i}$	$4.9165\ldots \cdot 10^{-3}$	$4.4927\ldots \cdot 10^{-5}$	$4.7204\ldots \cdot 10^{-7}$
$0.8 + 1500\mathrm{i}$	$8.7404\ldots \cdot 10^{-3}$	$1.0111\ldots \cdot 10^{-4}$	$6.5932\ldots \cdot 10^{-7}$
$1 + 10\mathrm{i}$	$8.1492\ldots \cdot 10^{-11}$	$2.7526\ldots \cdot 10^{-12}$	$7.6733\ldots \cdot 10^{-14}$
$1 + 50\mathrm{i}$	$3.2344\ldots \cdot 10^{-9}$	$3.5468\ldots \cdot 10^{-11}$	$3.2191\ldots \cdot 10^{-13}$
$1 + 300\mathrm{i}$	$2.9709\ldots \cdot 10^{-3}$	$4.8061\ldots \cdot 10^{-5}$	$5.0883\ldots \cdot 10^{-7}$
$1 + 1500\mathrm{i}$	$2.2121\ldots \cdot 10^{-3}$	$2.0298\ldots \cdot 10^{-4}$	$9.8642\ldots \cdot 10^{-7}$

Table 28: Accuracy of (4.4) for $N = 250$.

a	$\frac{\det(A_{0,N})}{\det(A_{2,N-1})}$
10i	$0.98807\ldots + 0.00994\ldots\mathrm{i}$
50i	$0.99103\ldots + 0.00538\ldots\mathrm{i}$
0.2	$0.99748\ldots$
$0.2 + 10\mathrm{i}$	$0.99023\ldots + 0.00823\ldots\mathrm{i}$
$0.2 + 50\mathrm{i}$	$0.99612\ldots + 0.00539\ldots\mathrm{i}$
0.5	$0.99657\ldots$
$0.5 + 10\mathrm{i}$	$0.99275\ldots + 0.00618\ldots\mathrm{i}$

Table 29: Accuracy of (4.11) for $N = 600$.

a	$\left\lvert \frac{\det(O_{1,1}(A_{l,N}))}{\det(o_{1,1}(A_{l,N}))} + 1 \right\rvert$		
	$l = 1$	$l = 2$	$l = 3$
10i	$5.411\ldots \cdot 10^{-5}$	$7.605\ldots \cdot 10^{-7}$	$1.333\ldots \cdot 10^{-8}$
50i	$8.546\ldots \cdot 10^{-6}$	$5.780\ldots \cdot 10^{-8}$	$5.239\ldots \cdot 10^{-10}$
0.2	$1.891\ldots \cdot 10^{-4}$	$1.878\ldots \cdot 10^{-5}$	$8.522\ldots \cdot 10^{-7}$
$0.2 + 10\mathrm{i}$	$5.542\ldots \cdot 10^{-5}$	$7.850\ldots \cdot 10^{-7}$	$1.394\ldots \cdot 10^{-8}$
$0.2 + 50\mathrm{i}$	$9.330\ldots \cdot 10^{-6}$	$6.250\ldots \cdot 10^{-8}$	$5.626\ldots \cdot 10^{-10}$
0.5	$2.195\ldots \cdot 10^{-4}$	$7.464\ldots \cdot 10^{-6}$	$6.527\ldots \cdot 10^{-6}$
$0.5 + 10\mathrm{i}$	$5.761\ldots \cdot 10^{-5}$	$8.245\ldots \cdot 10^{-7}$	$1.483\ldots \cdot 10^{-8}$

Table 30: Accuracy of (4.12) for $N = 590$.

$\mathfrak{A}_{\mathrm{R}}$	$\mathfrak{A}_{\mathrm{C}}$	$N = 50$	$N = 200$
$\{\}$	$\{\}$	$9.5786\ldots \cdot 10^{-79}$	$9.2324\ldots \cdot 10^{-382}$
$\{2\}$	$\{3\}$	$1.7947\ldots \cdot 10^{-73}$	$4.3678\ldots \cdot 10^{-375}$
$\{2,3\}$	$\{3,5\}$	$8.0235\ldots \cdot 10^{-69}$	$5.1109\ldots \cdot 10^{-369}$
$\{2,3,5\}$	$\{3,5,6\}$	$1.3660\ldots \cdot 10^{-64}$	$2.3663\ldots \cdot 10^{-363}$
$\{2,3,5,6\}$	$\{3,5,6,7\}$	$1.4933\ldots \cdot 10^{-60}$	$7.2424\ldots \cdot 10^{-358}$
$\{2,3,5,6,8\}$	$\{3,5,6,7,9\}$	$8.1963\ldots \cdot 10^{-57}$	$1.1716\ldots \cdot 10^{-352}$
$\{2,3,5,6,8,9\}$	$\{3,5,6,7,9,10\}$	$3.3574\ldots \cdot 10^{-53}$	$1.4616\ldots \cdot 10^{-347}$
$\{2,3,5,6,8,9,11\}$	$\{3,5,6,7,9,10,11\}$	$9.3569\ldots \cdot 10^{-50}$	$1.2983\ldots \cdot 10^{-342}$

Table 31: Accuracy of (4.25) for $a = 0.2 + 10\mathrm{i}$ and $l = 1$.

$\mathfrak{A}_{\mathrm{R}}$	$\mathfrak{A}_{\mathrm{C}}$	$N = 50$	$N = 200$
$\{\}$	$\{\}$	$6.0928\ldots \cdot 10^{-79}$	$4.3685\ldots \cdot 10^{-382}$
$\{2\}$	$\{3\}$	$1.1462\ldots \cdot 10^{-73}$	$2.0684\ldots \cdot 10^{-375}$
$\{2,3\}$	$\{3,5\}$	$5.1452\ldots \cdot 10^{-69}$	$2.4223\ldots \cdot 10^{-369}$
$\{2,3,5\}$	$\{3,5,6\}$	$8.7963\ldots \cdot 10^{-65}$	$1.1224\ldots \cdot 10^{-363}$
$\{2,3,5,6\}$	$\{3,5,6,7\}$	$9.6566\ldots \cdot 10^{-61}$	$3.4382\ldots \cdot 10^{-358}$
$\{2,3,5,6,8\}$	$\{3,5,6,7,9\}$	$5.3225\ldots \cdot 10^{-57}$	$5.5671\ldots \cdot 10^{-353}$
$\{2,3,5,6,8,9\}$	$\{3,5,6,7,9,10\}$	$2.1895\ldots \cdot 10^{-53}$	$6.9509\ldots \cdot 10^{-348}$
$\{2,3,5,6,8,9,11\}$	$\{3,5,6,7,9,10,11\}$	$6.1286\ldots \cdot 10^{-50}$	$6.1796\ldots \cdot 10^{-343}$

Table 32: Accuracy of (4.29) for $a = 0.2 + 10\mathrm{i}$.

a	$\left\lvert \frac{\det(H^{=}_{0,N})}{\det(H^{=}_{4,N-1})} - 1 \right\rvert$		
	$N = 20$	$N = 60$	$N = 200$
0.2	$4.1225\ldots \cdot 10^{-10}$	$1.0087\ldots \cdot 10^{-51}$	$1.0971\ldots \cdot 10^{-236}$
$0.2 + 1500\mathrm{i}$	$1.3078\ldots \cdot 10^{-3}$	$1.6565\ldots \cdot 10^{-27}$	$5.6012\ldots \cdot 10^{-176}$
0.5	$1.5933\ldots \cdot 10^{-10}$	$2.9298\ldots \cdot 10^{-52}$	$2.2720\ldots \cdot 10^{-237}$
$0.5 + 10\mathrm{i}$	$6.4582\ldots \cdot 10^{-11}$	$2.3485\ldots \cdot 10^{-52}$	$2.1627\ldots \cdot 10^{-237}$
$0.5 + 50\mathrm{i}$	$7.3449\ldots \cdot 10^{-12}$	$1.7220\ldots \cdot 10^{-54}$	$6.6994\ldots \cdot 10^{-238}$
$0.5 + 300\mathrm{i}$	$2.1187\ldots \cdot 10^{-4}$	$4.4155\ldots \cdot 10^{-42}$	$1.4075\ldots \cdot 10^{-244}$
$0.5 + 1500\mathrm{i}$	$6.2954\ldots \cdot 10^{-4}$	$8.3262\ldots \cdot 10^{-28}$	$1.9769\ldots \cdot 10^{-176}$
$0.8 + 10\mathrm{i}$	$2.5421\ldots \cdot 10^{-11}$	$6.8358\ldots \cdot 10^{-53}$	$4.4799\ldots \cdot 10^{-238}$
$0.8 + 50\mathrm{i}$	$3.2435\ldots \cdot 10^{-12}$	$5.2695\ldots \cdot 10^{-55}$	$1.3927\ldots \cdot 10^{-238}$
$0.8 + 300\mathrm{i}$	$8.7181\ldots \cdot 10^{-6}$	$1.7994\ldots \cdot 10^{-42}$	$3.1837\ldots \cdot 10^{-245}$
$0.8 + 1500\mathrm{i}$	$3.0144\ldots \cdot 10^{-4}$	$3.7334\ldots \cdot 10^{-28}$	$6.7410\ldots \cdot 10^{-177}$

Table 33: Accuracy of (4.31).

a	$\left\lvert \frac{\det(O_{1,1}(H^{=}_{l,N}))}{\det(o_{1,1}(H^{=}_{l,N}))} + 1 \right\rvert$ $l = 1$	$l = 2$	$l = 3$
0.2	$1.9747 \ldots \cdot 10^{-99}$	$3.0245 \ldots \cdot 10^{-98}$	$6.4168 \ldots \cdot 10^{-97}$
0.2 + 10i	$1.5186 \ldots \cdot 10^{-100}$	$6.3012 \ldots \cdot 10^{-100}$	$2.4664 \ldots \cdot 10^{-99}$
0.2 + 50i	$2.6920 \ldots \cdot 10^{-102}$	$4.4481 \ldots \cdot 10^{-102}$	$7.3979 \ldots \cdot 10^{-102}$
0.2 + 300i	$2.1532 \ldots \cdot 10^{-91}$	$2.0001 \ldots \cdot 10^{-91}$	$1.9040 \ldots \cdot 10^{-91}$
0.2 + 1500i	$1.3102 \ldots \cdot 10^{-62}$	$6.9132 \ldots \cdot 10^{-63}$	$3.7874 \ldots \cdot 10^{-63}$
0.5	$5.4899 \ldots \cdot 10^{-100}$	$7.3389 \ldots \cdot 10^{-99}$	$7.6748 \ldots \cdot 10^{-97}$
0.5 + 10i	$5.3344 \ldots \cdot 10^{-101}$	$2.2528 \ldots \cdot 10^{-100}$	$8.9528 \ldots \cdot 10^{-100}$
0.5 + 50i	$1.5614 \ldots \cdot 10^{-102}$	$2.5947 \ldots \cdot 10^{-102}$	$4.3313 \ldots \cdot 10^{-102}$
0.5 + 300i	$2.3518 \ldots \cdot 10^{-91}$	$2.1387 \ldots \cdot 10^{-91}$	$2.0051 \ldots \cdot 10^{-91}$
0.5 + 1500i	$3.8917 \ldots \cdot 10^{-62}$	$1.8079 \ldots \cdot 10^{-62}$	$9.1664 \ldots \cdot 10^{-63}$
0.8	$1.5455 \ldots \cdot 10^{-100}$	$1.8614 \ldots \cdot 10^{-99}$	$1.0921 \ldots \cdot 10^{-97}$
0.8 + 10i	$1.8642 \ldots \cdot 10^{-101}$	$8.0205 \ldots \cdot 10^{-101}$	$3.2379 \ldots \cdot 10^{-100}$
0.8 + 50i	$8.8170 \ldots \cdot 10^{-103}$	$1.4991 \ldots \cdot 10^{-102}$	$2.5265 \ldots \cdot 10^{-102}$
0.8 + 300i	$1.9778 \ldots \cdot 10^{-91}$	$2.3360 \ldots \cdot 10^{-91}$	$2.1252 \ldots \cdot 10^{-91}$
0.8 + 1500i	$6.4364 \ldots \cdot 10^{-62}$	$6.9098 \ldots \cdot 10^{-62}$	$2.4823 \ldots \cdot 10^{-62}$

Table 34: Accuracy of (4.32) for $N = 100$.

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
$i = 1$	$4.42 \ldots \cdot 10^{-6}$	$6.93 \ldots \cdot 10^{-7}$	$2.82 \ldots \cdot 10^{-7}$	$2.36 \ldots \cdot 10^{-7}$	$4.05 \ldots \cdot 10^{-7}$	$1.76 \ldots \cdot 10^{-6}$
$i = 2$	$6.93 \ldots \cdot 10^{-7}$	$1.07 \ldots \cdot 10^{-7}$	$4.33 \ldots \cdot 10^{-8}$	$3.59 \ldots \cdot 10^{-8}$	$6.07 \ldots \cdot 10^{-8}$	$2.61 \ldots \cdot 10^{-7}$
$i = 3$	$2.82 \ldots \cdot 10^{-7}$	$4.33 \ldots \cdot 10^{-8}$	$1.72 \ldots \cdot 10^{-8}$	$1.41 \ldots \cdot 10^{-8}$	$2.35 \ldots \cdot 10^{-8}$	$1.00 \ldots \cdot 10^{-7}$
$i = 4$	$2.36 \ldots \cdot 10^{-7}$	$3.59 \ldots \cdot 10^{-8}$	$1.41 \ldots \cdot 10^{-8}$	$1.13 \ldots \cdot 10^{-8}$	$1.87 \ldots \cdot 10^{-8}$	$7.86 \ldots \cdot 10^{-8}$
$i = 5$	$4.05 \ldots \cdot 10^{-7}$	$6.07 \ldots \cdot 10^{-8}$	$2.35 \ldots \cdot 10^{-8}$	$1.87 \ldots \cdot 10^{-8}$	$3.05 \ldots \cdot 10^{-8}$	$1.26 \ldots \cdot 10^{-7}$
$i = 6$	$1.76 \ldots \cdot 10^{-6}$	$2.61 \ldots \cdot 10^{-7}$	$1.00 \ldots \cdot 10^{-7}$	$7.86 \ldots \cdot 10^{-8}$	$1.26 \ldots \cdot 10^{-7}$	$5.13 \ldots \cdot 10^{-7}$

Table 35: Accuracy of (4.33) for $a = 1/3 + 16\mathrm{i}$, $N = 6$, and $l = 1$.

a	δ_1	δ_2	N_1	N_2	N	$\left\lvert \frac{\det(H_{0,N,\mathfrak{A}_{\mathrm{G}}(a,\delta_1,\delta_2,N_1,N_2)})}{\det(\mathbb{1}_1(H_{0,N,\mathfrak{A}_{\mathrm{G}}(a,\delta_1,\delta_2,N_1,N_2)}))} - 1 \right\rvert$
10i	0.1	0	11	0	12	$2.1806\ldots \cdot 10^{-14}$
2	1	0	11	0	12	$3.9020\ldots \cdot 10^{-22}$
2	2	0	11	0	12	$2.3319\ldots \cdot 10^{-29}$
3	2	0	11	0	12	$1.9178\ldots \cdot 10^{-30}$
$-1+10000$i	2	2i	6	6	49	$4.0222\ldots \cdot 10^{-50}$
10i	0.01	0.01i	6	6	49	$8.1705\ldots \cdot 10^{-77}$
$5+10$i	3	3i	6	6	49	$9.7505\ldots \cdot 10^{-100}$
$-50+10$i	3	3i	6	6	49	$5.8706\ldots \cdot 10^{-8}$
$-20+20$i	1	0	40	0	41	$9.7954\ldots \cdot 10^{-67}$
$0.5+10$i	i	0	40	0	41	$4.9512\ldots \cdot 10^{-58}$
$-20+10$i	5	5i	5	5	36	$2.0188\ldots \cdot 10^{-41}$
10i	0.1i	0	11	0	12	$8.0268\ldots \cdot 10^{-14}$
10i	10	10i	6	6	49	$1.5113\ldots \cdot 10^{-128}$
10i	1	10i	1	18	38	$2.2140\ldots \cdot 10^{-27}$
1000i	0.1i	0	11	0	12	$2.2052\ldots \cdot 10^{-6}$

Table 36: Accuracy of (4.44) for grids.

c	r	N	$\left\lvert \frac{\det(H_{0,N,\mathfrak{A}_{\mathrm{C}}(c,r,N)})}{\det(\mathbb{1}_{,1}(H_{0,N,\mathfrak{A}_{\mathrm{C}}(c,r,N)}))} - 1 \right\rvert$
10i	1	12	$8.3348\ldots \cdot 10^{-14}$
1 + 10i	1	12	$7.4285\ldots \cdot 10^{-15}$
10i	1	12	$8.3348\ldots \cdot 10^{-14}$
10i	1	12	$8.3348\ldots \cdot 10^{-14}$
10i	10 + 5i	12	$1.2529\ldots \cdot 10^{-13}$
10i	50	12	$1.4654\ldots \cdot 10^{-36}$
10i	120	12	$4.0214\ldots \cdot 10^{-83}$
0.5 + 13i	10 + 5i	12	$3.2934\ldots \cdot 10^{-14}$
2 + 15i	50	12	$3.6442\ldots \cdot 10^{-38}$
−1 + 20i	120	12	$4.5719\ldots \cdot 10^{-82}$
10i	10	31	$1.6742\ldots \cdot 10^{-44}$
10i	50	31	$7.9971\ldots \cdot 10^{-70}$
0.8 + 10i	120	31	$1.3800\ldots \cdot 10^{-137}$
10i	120	31	$1.2585\ldots \cdot 10^{-136}$
0.2 + 40i	50	31	$3.5771\ldots \cdot 10^{-69}$
100i	10	31	$2.1256\ldots \cdot 10^{-39}$
1	50	61	$2.4090\ldots \cdot 10^{-113}$
0	120	61	$4.2161\ldots \cdot 10^{-188}$
1 + i	10	61	$6.7542\ldots \cdot 10^{-101}$
10i	10	61	$3.4604\ldots \cdot 10^{-99}$
10i	50	61	$1.5260\ldots \cdot 10^{-111}$
10i	120	61	$1.3813\ldots \cdot 10^{-188}$

Table 37: Accuracy of (4.44) for discrete circles.

a	δ_1	δ_2	m	n	N	$\frac{\det(H_{0,N,\mathfrak{A}_\pi(a,\delta_1,\delta_2,m,n,N)})}{\det(\mathbb{1}_{,1}(H_{0,N,\mathfrak{A}_\pi(a,\delta_1,\delta_2,m,n,N)}))} - 1$
$-3+7\mathrm{i}$	6	$6\mathrm{i}$	100	3	9	$1.2998\ldots \cdot 10^{-9}$
$-1+10000\mathrm{i}$	6	$6\mathrm{i}$	100	3	9	$8.4518\ldots \cdot 10^{-6}$
$-10+10\mathrm{i}$	1	i	1000	3	9	$4.4429\ldots \cdot 10^{-2}$
$0.5+10\mathrm{i}$	0.1	$0.1\mathrm{i}$	1000	3	9	$3.9104\ldots \cdot 10^{-10}$
$-10+10\mathrm{i}$	1	i	1000	3	19	$2.1200\ldots \cdot 10^{-12}$
$0.5+10\mathrm{i}$	0.1	$0.1\mathrm{i}$	1000	3	19	$3.6990\ldots \cdot 10^{-25}$
$-10+10\mathrm{i}$	1	i	1000	3	29	$2.6063\ldots \cdot 10^{-27}$
$0.5+10\mathrm{i}$	0.1	$0.1\mathrm{i}$	1000	3	29	$7.7092\ldots \cdot 10^{-42}$
$-10+10\mathrm{i}$	1	i	1000	3	49	$4.7281\ldots \cdot 10^{-61}$
$0.2+100\mathrm{i}$	1	i	100	3	49	$1.6165\ldots \cdot 10^{-76}$

Table 38: Accuracy of (4.44) for pseudo-random sets.

a	$\lvert \det(G_{0,N})/\det(G_{2,N-1}) - 1\rvert$ $N=10$	$N=50$	$N=300$
-1	$9.7978\ldots \cdot 10^{-1}$	$9.0366\ldots \cdot 10^{-1}$	$2.6337\ldots \cdot 10^{-1}$
$-1+10\mathrm{i}$	$1.8225\ldots \cdot 10^{-6}$	$1.4541\ldots \cdot 10^{-22}$	$1.3356\ldots \cdot 10^{-75}$
$-1+50\mathrm{i}$	$7.3987\ldots \cdot 10^{-11}$	$3.5816\ldots \cdot 10^{-64}$	$5.6367\ldots \cdot 10^{-223}$
$-1+300\mathrm{i}$	$1.8624\ldots \cdot 10^{-5}$	$3.0383\ldots \cdot 10^{-74}$	$2.0475\ldots \cdot 10^{-526}$
$-1+1500\mathrm{i}$	$2.6831\ldots \cdot 10^{-4}$	$1.1149\ldots \cdot 10^{-53}$	$6.6917\ldots \cdot 10^{-594}$
0.2	$7.8320\ldots \cdot 10^{-1}$	$9.6794\ldots \cdot 10^{-1}$	$5.2395\ldots \cdot 10^{-1}$
$0.2+10\mathrm{i}$	$1.3078\ldots \cdot 10^{-7}$	$1.6826\ldots \cdot 10^{-25}$	$1.4341\ldots \cdot 10^{-83}$
$0.2+50\mathrm{i}$	$5.8038\ldots \cdot 10^{-12}$	$5.8227\ldots \cdot 10^{-66}$	$3.5624\ldots \cdot 10^{-227}$
$0.2+300\mathrm{i}$	$9.4264\ldots \cdot 10^{-7}$	$3.9372\ldots \cdot 10^{-76}$	$4.8954\ldots \cdot 10^{-529}$
$0.2+1500\mathrm{i}$	$2.4046\ldots \cdot 10^{-5}$	$8.2049\ldots \cdot 10^{-56}$	$8.9624\ldots \cdot 10^{-597}$
0.5	$9.6480\ldots \cdot 10^{-1}$	$1.0421\ldots \cdot 10^{0}$	$6.7185\ldots \cdot 10^{1}$
$0.5+10\mathrm{i}$	$6.6169\ldots \cdot 10^{-8}$	$2.9504\ldots \cdot 10^{-26}$	$1.2635\ldots \cdot 10^{-85}$
$0.5+50\mathrm{i}$	$3.0645\ldots \cdot 10^{-12}$	$2.0645\ldots \cdot 10^{-66}$	$3.1292\ldots \cdot 10^{-228}$
$0.5+300\mathrm{i}$	$3.5617\ldots \cdot 10^{-7}$	$1.3292\ldots \cdot 10^{-76}$	$1.0810\ldots \cdot 10^{-529}$
$0.5+1500\mathrm{i}$	$1.1243\ldots \cdot 10^{-5}$	$2.4251\ldots \cdot 10^{-56}$	$1.7144\ldots \cdot 10^{-597}$
0.8	$1.0776\ldots \cdot 10^{0}$	$1.0324\ldots \cdot 10^{0}$	$9.5644\ldots \cdot 10^{-1}$
$0.8+10\mathrm{i}$	$3.3197\ldots \cdot 10^{-8}$	$5.0780\ldots \cdot 10^{-27}$	$1.0554\ldots \cdot 10^{-87}$
$0.8+50\mathrm{i}$	$1.6167\ldots \cdot 10^{-12}$	$7.2997\ldots \cdot 10^{-67}$	$2.7325\ldots \cdot 10^{-229}$
$0.8+300\mathrm{i}$	$1.3776\ldots \cdot 10^{-7}$	$4.4967\ldots \cdot 10^{-77}$	$2.3858\ldots \cdot 10^{-530}$
$0.8+1500\mathrm{i}$	$5.1472\ldots \cdot 10^{-6}$	$6.8743\ldots \cdot 10^{-57}$	$3.2796\ldots \cdot 10^{-598}$

Table 39: Accuracy of (4.52).

a	$\left\lvert \frac{\det(O_{1,1}(G_{l,N}))}{\det(o_{1,1}(G_{l,N}))} + 1 \right\rvert$ $l=1$	$l=2$	$l=3$
-2	$4.7286\ldots \cdot 10^{-1}$	$5.5491\ldots \cdot 10^{-2}$	$3.0985\ldots \cdot 10^{-2}$
-1	$1.8611\ldots \cdot 10^{-1}$	$3.9061\ldots \cdot 10^{-2}$	$1.5019\ldots \cdot 10^{-2}$
$-1+10i$	$3.7249\ldots \cdot 10^{-66}$	$6.0181\ldots \cdot 10^{-65}$	$8.8328\ldots \cdot 10^{-64}$
$-1+50i$	$2.4353\ldots \cdot 10^{-199}$	$1.0203\ldots \cdot 10^{-198}$	$4.3210\ldots \cdot 10^{-198}$
$-1+300i$	$1.8857\ldots \cdot 10^{-456}$	$2.5196\ldots \cdot 10^{-456}$	$3.3575\ldots \cdot 10^{-456}$
$-1+1500i$	$6.2882\ldots \cdot 10^{-469}$	$7.0867\ldots \cdot 10^{-469}$	$8.2329\ldots \cdot 10^{-469}$
-0.8	$1.5804\ldots \cdot 10^{-1}$	$6.2433\ldots \cdot 10^{-2}$	$1.0394\ldots \cdot 10^{-2}$
$-0.8+10i$	$2.8163\ldots \cdot 10^{-67}$	$4.5708\ldots \cdot 10^{-66}$	$6.7616\ldots \cdot 10^{-65}$
$-0.8+50i$	$8.7133\ldots \cdot 10^{-200}$	$3.6344\ldots \cdot 10^{-199}$	$1.5312\ldots \cdot 10^{-198}$
$-0.8+100i$	$1.5111\ldots \cdot 10^{-291}$	$4.0787\ldots \cdot 10^{-291}$	$1.0954\ldots \cdot 10^{-290}$
$-0.8+300i$	$1.5352\ldots \cdot 10^{-456}$	$2.0558\ldots \cdot 10^{-456}$	$2.7455\ldots \cdot 10^{-456}$
$-0.8+1500i$	$5.7471\ldots \cdot 10^{-469}$	$6.2496\ldots \cdot 10^{-469}$	$7.0351\ldots \cdot 10^{-469}$
0	$2.4178\ldots \cdot 10^{-1}$	$6.9068\ldots \cdot 10^{-3}$	$3.3397\ldots \cdot 10^{-4}$
$10i$	$7.2065\ldots \cdot 10^{-72}$	$1.1841\ldots \cdot 10^{-70}$	$1.8039\ldots \cdot 10^{-69}$
$50i$	$1.3939\ldots \cdot 10^{-201}$	$5.8289\ldots \cdot 10^{-201}$	$2.3977\ldots \cdot 10^{-200}$
$300i$	$6.6504\ldots \cdot 10^{-457}$	$8.9264\ldots \cdot 10^{-457}$	$1.2051\ldots \cdot 10^{-456}$
0.2	$7.6790\ldots \cdot 10^{-2}$	$1.5904\ldots \cdot 10^{-2}$	$3.2672\ldots \cdot 10^{-4}$
$0.2+10i$	$4.8349\ldots \cdot 10^{-73}$	$7.9574\ldots \cdot 10^{-72}$	$1.2203\ldots \cdot 10^{-70}$
$0.2+50i$	$4.8788\ldots \cdot 10^{-202}$	$2.0721\ldots \cdot 10^{-201}$	$8.4779\ldots \cdot 10^{-201}$
$0.2+100i$	$6.6424\ldots \cdot 10^{-293}$	$1.8416\ldots \cdot 10^{-292}$	$5.0086\ldots \cdot 10^{-292}$
$0.2+300i$	$5.4084\ldots \cdot 10^{-457}$	$7.2051\ldots \cdot 10^{-457}$	$9.7520\ldots \cdot 10^{-457}$
$0.2+1500i$	$6.4198\ldots \cdot 10^{-469}$	$5.6106\ldots \cdot 10^{-469}$	$5.2791\ldots \cdot 10^{-469}$
0.5	$3.8918\ldots \cdot 10^{-2}$	$5.3373\ldots \cdot 10^{-3}$	$8.2529\ldots \cdot 10^{-5}$
$0.5+10i$	$8.0624\ldots \cdot 10^{-75}$	$1.3287\ldots \cdot 10^{-73}$	$2.0566\ldots \cdot 10^{-72}$
$0.5+50i$	$9.8015\ldots \cdot 10^{-203}$	$4.3790\ldots \cdot 10^{-202}$	$1.7837\ldots \cdot 10^{-201}$
$0.5+300i$	$4.0484\ldots \cdot 10^{-457}$	$5.2113\ldots \cdot 10^{-457}$	$7.0590\ldots \cdot 10^{-457}$
$0.5+1500i$	$1.0500\ldots \cdot 10^{-468}$	$6.8359\ldots \cdot 10^{-469}$	$5.7445\ldots \cdot 10^{-469}$
0.8	$1.4142\ldots \cdot 10^{-3}$	$9.7770\ldots \cdot 10^{-5}$	$3.4807\ldots \cdot 10^{-6}$
$0.8+10i$	$1.2821\ldots \cdot 10^{-76}$	$2.1135\ldots \cdot 10^{-75}$	$3.2983\ldots \cdot 10^{-74}$
$0.8+50i$	$1.8564\ldots \cdot 10^{-203}$	$9.1651\ldots \cdot 10^{-203}$	$3.7576\ldots \cdot 10^{-202}$
$0.8+100i$	$9.4564\ldots \cdot 10^{-294}$	$2.7776\ldots \cdot 10^{-293}$	$7.6595\ldots \cdot 10^{-293}$
$0.8+300i$	$3.2680\ldots \cdot 10^{-457}$	$3.7816\ldots \cdot 10^{-457}$	$5.0750\ldots \cdot 10^{-457}$
$0.8+1500i$	$7.6360\ldots \cdot 10^{-469}$	$1.2462\ldots \cdot 10^{-468}$	$7.1589\ldots \cdot 10^{-469}$
$1+10i$	$7.9061\ldots \cdot 10^{-78}$	$1.3026\ldots \cdot 10^{-76}$	$2.0428\ldots \cdot 10^{-75}$
$1+50i$	$5.8564\ldots \cdot 10^{-204}$	$3.1922\ldots \cdot 10^{-203}$	$1.3309\ldots \cdot 10^{-202}$
$1+300i$	$3.1999\ldots \cdot 10^{-457}$	$3.0863\ldots \cdot 10^{-457}$	$4.0606\ldots \cdot 10^{-457}$
$1+1500i$	$2.0002\ldots \cdot 10^{-469}$	$2.6164\ldots \cdot 10^{-468}$	$9.2831\ldots \cdot 10^{-469}$

Table 40: Accuracy of (4.53) for $N = 250$.