PDMI Preprint 01/2023

A possible method to calculate the value of the Riemann zeta function via its derivatives

Yu. V. Matiyasevich

St. Petersburg Department
of V. A. Steklov Institute of Mathematics
of Russian Academy of Sciences

yumat@pdmi.ras.ru

Abstract. We consider the following two problems. We are given the values of several initial derivatives of the Riemann zeta function calculated at some (unknown to us) point a.

- How could we calculate an approximate value of the function itself at the same point a without prior finding this number?
- How could we find an approximate value of a itself?

We suggest several algorithms for answering these questions and demonstrate their accuracy on a few numerical examples. The algorithms reveal some new properties of the zeta function.

Key words: zeta function, alternating zeta function

ПРЕПРИНТЫ ПОМИ РАН

Учредитель: Федеральное государственное бюджетное учреждение науки
Санкт-Петербургское отделение Математического института
им. В. А. Стеклова Российской академии наук

Свидетельство о регистрации средства массовой информации:
ЭЛ №ФС 77-33560 от 16 октября 2008 г.
Выдано Федеральной службой по надзору
в сфере связи и массовых коммуникаций

Контактные данные:

191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

телефоны: (812) 312-40-58; (812) 571-57-54

e-mail: `admin@pdmi.ras.ru`

`https://www.pdmi.ras.ru/preprint/`

Заведующая информационно-издательским сектором В. Н. Симонова

1 Our main problem

The *Riemann zeta function* can be defined for $\mathrm{Re}(s) > 1$ by Dirichlet series

$$\zeta(s) = 1^{-s} + 2^{-s} + \ldots + n^{-s} + \ldots \tag{1}$$

and analytically extended to the whole complex plane except for the point $s = 1$, which is a pole of the function.

Let us assume that numbers

$$d_1, d_2, \ldots, \tag{2}$$

are the values of the derivatives of the Riemann zeta function taken at some (unknown to us) point $s = a$,

$$d_k = \left.\frac{\mathrm{d}^k}{\mathrm{d}s^k}\zeta(s)\right|_{s=a}, \quad k = 1, 2, \ldots \tag{3}$$

For the given numbers (2) such an a is unique, hence the knowledge of the derivatives of the zeta function is sufficient for determining its value,

$$d_0 = \zeta(a). \tag{4}$$

Question 1. *How can we calculate $\zeta(a)$ from numbers (2)?*

2 First Method

In this section we demonstrate (by numerical examples only) a non-evident way to answer our Question 1 *without prior calculation of the number a itself.*

A finite calculation can involve only finitely many numbers from the infinite list (2), say $N - 1$ initial numbers. Let us consider a finite Dirichlet series with N summands

$$D_N(s) = \sum_{n=1}^{N} c_n n^{-s}. \tag{5}$$

We want $D_N(0)$ to be close to $\zeta(a)$, and to this end we demand that the first $N-1$ derivatives of $D_N(s)$ taken at point $s = 0$ should be equal to the corresponding derivatives of the zeta function taken at point $s = a$:

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}s^k}D_N(s)\right|_{s=0} = d_k, \quad k = 1, \ldots, N-1. \tag{6}$$

Clearly, this condition can be written as a system of linear equations,

$$\sum_{n=2}^{N} \bigl(-\ln(n)^k \bigr) c_n = d_k, \quad k = 1, \dots, N-1, \tag{7}$$

with unknowns

$$c_2, \dots, c_N. \tag{8}$$

The matrix of this system is essentially of Vandermond type, so the system has a solution.

Equations (7) do not involve c_1, and the crucial point is the selection of the value of this coefficient. Surprisingly, this choice can be done independently of the values of numbers (3). Namely, we put

$$c_1 = 1. \tag{9}$$

This choice justifies itself: we shall see on numerical examples how small (in absolute value) can the difference $D_N(0) - \zeta(a)$ be, provided that a is sufficiently far from the zeta function pole at $s = 1$.

Table 1 presents the values of the coefficients of $D_N(s)$ for $N = 20$ and $a = -1 + 17\mathrm{i}$. Futher calculations give:

$$\begin{aligned} D_N(0) &= 4.2514\,30819\,75980... + 3.9105\,73040\,34974...\mathrm{i}, && (10) \\ \zeta(a) &= 4.2514\,30819\,79612... + 3.9105\,73040\,36079...\mathrm{i}. && (11) \end{aligned}$$

We observe that the real and imaginary parts of $D_N(0)$ and $\zeta(a)$ have 11 common initial decimal digits.

Table 2 shows the relative error of the approximation of $\zeta(a)$ by $D_N(0)$ when a is "away" from the zeros of the zeta function and its pole. On the contrary, in Table 3 the zeta zeros were taken for the values of a and respectively $D_N(0)$ almost vanishes.

Remark. From Table 1 we can also observe that

$$\begin{aligned} c_2 &= 2^{-a} - 0.0000\,00068\,17796... - 0.00000\,00313\,13323...\mathrm{i}, && (12) \\ c_3 &= 3^{-a} + 0.0000\,12435\,91128... + 0.00000\,78012\,25876...\mathrm{i}, && (13) \\ c_4 &= 4^{-a} - 0.0006\,75348\,95498... - 0.00056\,08292\,92317...\mathrm{i}. && (14) \end{aligned}$$

Also by our choice (9)

$$c_1 = 1^{-a}. \tag{15}$$

In other words a few initial summands of the finite series (5) at $s = 0$ are approximately equal to the corresponding summands of the infinite (and divergent in our example) series (1) at $s = a$.

n	c_n	
2	1.417786386498559...	+1.410631589250944...i
3	2.955154840132295...	+0.516857267487458...i
4	0.019561513173469...	+3.999387979056508...i
5	-3.037128837211730...	-3.941624128897215...i
6	3.253894203754391...	+4.594841969201085...i
7	0.861951474783640...	-3.753251948643660...i
8	-12.006629512292581...	-16.341356533510580...i
9	20.926227144580919...	+106.181184994072795...i
10	15.170115497499071...	-348.870638305533251...i
11	-168.168120200587817...	+798.007204791309054...i
12	467.990056861655709...	-1365.913771566818394...i
13	-812.125851047954882...	+1792.560833222644864...i
14	995.566624737176037...	-1814.013879181722684...i
15	-891.778500043800322...	+1407.070268334236594...i
16	584.114689453315294...	-822.257728793268618...i
17	-273.514447113214965...	+350.547171775067432...i
18	86.942550916705601...	-102.943059194003426...i
19	-16.838044407943776...	+18.621875181574025...i
20	1.501538953490888...	-1.564374411153188...i

Table 1: Solution of system (7) for $N = 20$ when the right-hand sides are defined by (2) with $a = -1 + 17\mathrm{i}$.

3 Calculations in the neighborhood of a

Knowing numbers $d_0, \dots, d_N$ (defined by (4) and (2)), we can use the initial fragment of Taylor series,

$$T(s) = \sum_{k=0}^{N} \frac{d_k}{k!}(s-a)^k, \tag{16}$$

to calculate (approximate) value of $\zeta(s)$ for s in the vicinity of a,

$$\zeta(s) \approx T(s). \tag{17}$$

According to (6) the Taylor series for $D_N(s)$ at $s = 0$ has the same initial coefficients as (16) except for the first one (for $k = 0$). Hence if $D_N(0)$ is indeed close to $\zeta(a)$, then it is quite natural to expect that

$$\zeta(s) \approx D_N(s-a) \tag{18}$$

a	$\lvert D_N(0)/\zeta(a)-1\rvert$ $N=16$	$N=30$	$N=50$
$-2+14\mathrm{i}$	$2.0895\ldots\cdot 10^{-8}$	$2.3596\ldots\cdot 10^{-12}$	$1.3112\ldots\cdot 10^{-14}$
$14\mathrm{i}$	$4.9989\ldots\cdot 10^{-9}$	$2.8947\ldots\cdot 10^{-13}$	$1.0194\ldots\cdot 10^{-15}$
$0.5+14\mathrm{i}$	$8.2510\ldots\cdot 10^{-9}$	$3.8749\ldots\cdot 10^{-13}$	$1.1465\ldots\cdot 10^{-15}$
$1+14\mathrm{i}$	$8.8751\ldots\cdot 10^{-10}$	$3.3214\ldots\cdot 10^{-14}$	$8.0544\ldots\cdot 10^{-17}$
$2+14\mathrm{i}$	$4.6624\ldots\cdot 10^{-11}$	$1.0511\ldots\cdot 10^{-15}$	$1.5884\ldots\cdot 10^{-18}$
$-2+30\mathrm{i}$	$1.1240\ldots\cdot 10^{-11}$	$1.1109\ldots\cdot 10^{-20}$	$6.5682\ldots\cdot 10^{-30}$
$30\mathrm{i}$	$3.4877\ldots\cdot 10^{-12}$	$1.6334\ldots\cdot 10^{-21}$	$5.0058\ldots\cdot 10^{-31}$
$0.5+30\mathrm{i}$	$2.5063\ldots\cdot 10^{-12}$	$9.6335\ldots\cdot 10^{-22}$	$2.4698\ldots\cdot 10^{-31}$
$1+30\mathrm{i}$	$7.0833\ldots\cdot 10^{-13}$	$2.2247\ldots\cdot 10^{-22}$	$4.7448\ldots\cdot 10^{-32}$
$2+30\mathrm{i}$	$4.7578\ldots\cdot 10^{-14}$	$9.8496\ldots\cdot 10^{-24}$	$1.4292\ldots\cdot 10^{-33}$
$-2+100\mathrm{i}$	$2.7242\ldots\cdot 10^{-8}$	$3.5610\ldots\cdot 10^{-24}$	$6.3506\ldots\cdot 10^{-45}$
$100\mathrm{i}$	$1.3724\ldots\cdot 10^{-8}$	$9.6736\ldots\cdot 10^{-25}$	$9.6584\ldots\cdot 10^{-46}$
$0.5+100\mathrm{i}$	$7.7589\ldots\cdot 10^{-9}$	$4.6759\ldots\cdot 10^{-25}$	$4.0308\ldots\cdot 10^{-46}$
$1+100\mathrm{i}$	$2.9857\ldots\cdot 10^{-9}$	$1.5370\ldots\cdot 10^{-25}$	$1.1431\ldots\cdot 10^{-46}$
$2+100\mathrm{i}$	$2.2346\ldots\cdot 10^{-10}$	$8.3714\ldots\cdot 10^{-27}$	$4.6239\ldots\cdot 10^{-48}$
$-2+200\mathrm{i}$	$2.3721\ldots\cdot 10^{-3}$	$2.5974\ldots\cdot 10^{-16}$	$6.2881\ldots\cdot 10^{-41}$
$200\mathrm{i}$	$9.8287\ldots\cdot 10^{-4}$	$8.7790\ldots\cdot 10^{-17}$	$1.3128\ldots\cdot 10^{-41}$
$0.5+200\mathrm{i}$	$4.9123\ldots\cdot 10^{-4}$	$4.1704\ldots\cdot 10^{-17}$	$5.5235\ldots\cdot 10^{-42}$
$1+200\mathrm{i}$	$1.7331\ldots\cdot 10^{-4}$	$1.3986\ldots\cdot 10^{-17}$	$1.6399\ldots\cdot 10^{-42}$
$2+200\mathrm{i}$	$1.0311\ldots\cdot 10^{-5}$	$7.5195\ldots\cdot 10^{-19}$	$6.9029\ldots\cdot 10^{-44}$

Table 2: Series (5) well approximate $\zeta(a)$ when their coefficients are defined by (3), (7), and (9).

when s is close enough to a. The following observation is remarkable: approximation (18) can be much more accurate than (17). Here is an example: for $N=50$, $a=1+30\mathrm{i}$, and $s=4+40\mathrm{i}$

$$|T(s)/\zeta(s)-1| = 0.0112..., \tag{19}$$

$$|D_N(s)/\zeta(s)-1| = 5.6644...\times 10^{-27}. \tag{20}$$

Moreover, outside the circle of convergency of the Taylor series its finite fragment (16) can give absolutely unrealistic approximations to $\zeta(s)$ while approximations by $D_N(s)$ can be quite reasonable: for $N=50$, $a=1+30\mathrm{i}$, and $s=4+61\mathrm{i}$

$$|T(s)/\zeta(s)| = 1.3432...\times 10^{22}, \tag{21}$$

$$|D_N(s)/\zeta(s)-1| = 8.7339...\times 10^{-6}. \tag{22}$$

		$\|D_N(0)\|$		
m	$\mathrm{Im}(\rho_m)$	$N = 16$	$N = 30$	$N = 50$
1	$14.13472\ldots$	$7.7808\ldots \cdot 10^{-10}$	$3.1660\ldots \cdot 10^{-14}$	$7.6774\ldots \cdot 10^{-17}$
2	$21.02203\ldots$	$1.3440\ldots \cdot 10^{-11}$	$1.3062\ldots \cdot 10^{-18}$	$7.4006\ldots \cdot 10^{-25}$
3	$25.01085\ldots$	$3.7407\ldots \cdot 10^{-12}$	$2.4584\ldots \cdot 10^{-20}$	$3.3201\ldots \cdot 10^{-28}$
4	$30.42487\ldots$	$1.4197\ldots \cdot 10^{-12}$	$4.3805\ldots \cdot 10^{-22}$	$8.2480\ldots \cdot 10^{-32}$
5	$32.93506\ldots$	$1.1301\ldots \cdot 10^{-12}$	$1.0109\ldots \cdot 10^{-22}$	$3.3226\ldots \cdot 10^{-33}$
6	$37.58617\ldots$	$9.7731\ldots \cdot 10^{-13}$	$1.1201\ldots \cdot 10^{-23}$	$1.9702\ldots \cdot 10^{-35}$
7	$40.91871\ldots$	$1.0518\ldots \cdot 10^{-12}$	$3.2535\ldots \cdot 10^{-24}$	$8.6142\ldots \cdot 10^{-37}$
8	$43.32707\ldots$	$1.1952\ldots \cdot 10^{-12}$	$1.5431\ldots \cdot 10^{-24}$	$1.1376\ldots \cdot 10^{-37}$
9	$48.00515\ldots$	$1.7737\ldots \cdot 10^{-12}$	$4.8920\ldots \cdot 10^{-25}$	$3.6259\ldots \cdot 10^{-39}$
10	$49.77383\ldots$	$2.1488\ldots \cdot 10^{-12}$	$3.4683\ldots \cdot 10^{-25}$	$1.1418\ldots \cdot 10^{-39}$
11	$52.97032\ldots$	$3.1928\ldots \cdot 10^{-12}$	$2.0785\ldots \cdot 10^{-25}$	$1.6942\ldots \cdot 10^{-40}$
12	$56.44624\ldots$	$5.2226\ldots \cdot 10^{-12}$	$1.3743\ldots \cdot 10^{-25}$	$2.6964\ldots \cdot 10^{-41}$
13	$59.34704\ldots$	$8.2027\ldots \cdot 10^{-12}$	$1.0761\ldots \cdot 10^{-25}$	$6.8837\ldots \cdot 10^{-42}$
14	$60.83177\ldots$	$1.0462\ldots \cdot 10^{-11}$	$9.8054\ldots \cdot 10^{-26}$	$3.6127\ldots \cdot 10^{-42}$
15	$65.11254\ldots$	$2.1943\ldots \cdot 10^{-11}$	$8.3664\ldots \cdot 10^{-26}$	$6.7868\ldots \cdot 10^{-43}$
16	$67.07981\ldots$	$3.1350\ldots \cdot 10^{-11}$	$8.1754\ldots \cdot 10^{-26}$	$3.4290\ldots \cdot 10^{-43}$
17	$69.54640\ldots$	$4.9622\ldots \cdot 10^{-11}$	$8.2669\ldots \cdot 10^{-26}$	$1.5621\ldots \cdot 10^{-43}$
18	$72.06715\ldots$	$8.0253\ldots \cdot 10^{-11}$	$8.7256\ldots \cdot 10^{-26}$	$7.5390\ldots \cdot 10^{-44}$
19	$75.70469\ldots$	$1.6317\ldots \cdot 10^{-10}$	$1.0107\ldots \cdot 10^{-25}$	$2.9794\ldots \cdot 10^{-44}$
20	$77.14484\ldots$	$2.1698\ldots \cdot 10^{-10}$	$1.0937\ldots \cdot 10^{-25}$	$2.1417\ldots \cdot 10^{-44}$

Table 3: Series (5) almost vanish at $s = 0$ when their coefficients are defined by (3), (7), and (9) with $a = \rho_m$, the mth non-trivial zero of the zeta function.

4 Calculation of a

We saw in Section 2 that it is possible to approximately calculate $\zeta(a)$ from several initial numbers from (2) (defined by (3)). Now we use this knowledge to answer

Question 2. *How can we calculate a from numbers* (2)*?*

According to the Remark in Section 2,

$$c_2 \approx 2^{-a} \tag{23}$$

when c_2 is defined by system (7). Respectively,

$$\mathrm{Re}(a) \approx \sigma, \tag{24}$$

where

$$\sigma = -\mathrm{Re}(\log_2(c_2)). \tag{25}$$

As for the imaginary part of a, it is defined by (23) up to an integer multiple of $2\pi/\ln(2)$ only:

$$\mathrm{Im}(a) \approx -\mathrm{Im}(\log_2(c_2)) + \frac{2\pi\mathrm{i}}{\ln(2)}m \tag{26}$$

for some integer m. Thus we need to use a more involved technique for detering-ing $\mathrm{Im}(a)$.

We shall use a slightly modified method from [1]. It is based on the well-known *functional equation* satisfied by the zeta function. This identity can be written in many equivalent forms, we start from the following one:

$$\pi(-3+2\mathrm{i}t)\Gamma\big(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}\big)\zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big) = \pi^{\mathrm{i}t}(1-2\mathrm{i}t)\Gamma\big(\tfrac{7}{4}-\tfrac{\mathrm{i}t}{2}\big)\zeta\big(\tfrac{3}{2}-\mathrm{i}t\big). \tag{27}$$

We can eliminate the gamma function thanks to its functional equation; for our purpose this identity can be written as

$$(3+2t\mathrm{i})\Gamma\big(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}\big) = 4\Gamma\big(\tfrac{7}{4}+\tfrac{\mathrm{i}t}{2}\big). \tag{28}$$

Now, we replace t by $-t$ in (27) and in (28), multiply the resulting and the original identities side-by-side, and get the identities

$$\begin{multline}\pi^2(9+4t^2)\Gamma\big(\tfrac{3}{4}-\tfrac{\mathrm{i}t}{2}\big)\Gamma\big(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}\big)\zeta\big(-\tfrac{1}{2}-\mathrm{i}t\big)\zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big) = \\ (1+4t^2)\Gamma\big(\tfrac{7}{4}-\tfrac{\mathrm{i}t}{2}\big)\Gamma\big(\tfrac{7}{4}+\tfrac{\mathrm{i}t}{2}\big)\zeta\big(\tfrac{3}{2}-\mathrm{i}t\big)\zeta\big(\tfrac{3}{2}+\mathrm{i}t\big) \tag{29}\end{multline}$$

and

$$(9+4t^2)\Gamma\big(\tfrac{3}{4}-\tfrac{\mathrm{i}t}{2}\big)\Gamma\big(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}\big) = 16\Gamma\big(\tfrac{7}{4}-\tfrac{\mathrm{i}t}{2}\big)\Gamma\big(\tfrac{7}{4}+\tfrac{\mathrm{i}t}{2}\big). \tag{30}$$

Next, we divide (29) side-by-side by (30), and get the desired identity not containing the gamma function,

$$\pi^2\zeta\big(-\tfrac{1}{2}-\mathrm{i}t\big)\zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big) = (1+4t^2)\zeta\big(\tfrac{3}{2}-\mathrm{i}t\big)\zeta\big(\tfrac{3}{2}+\mathrm{i}t\big)/16. \tag{31}$$

Unless both sides in (31) vanish, this identity can be rewritten as

$$t^2 = T_2\left(\zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big), \zeta\big(\tfrac{3}{2}+\mathrm{i}t\big), \zeta\big(-\tfrac{1}{2}-\mathrm{i}t\big), \zeta\big(\tfrac{3}{2}-\mathrm{i}t\big)\right), \tag{32}$$

where

$$T_2(x_1, y_1, x_2, y_2) = \frac{4\pi^2 x_1 x_2}{y_1 y_2} - \frac{1}{4}. \tag{33}$$

Thus we can determine the value of t from the values of the zeta function calculated at four points, but up to a factor ± 1 only.

In order to resolve this ambiguity we can replace t by $t+u$ and get that

$$t^2+2tu+u^2= \\ T_2\left(\zeta\left(-\tfrac{1}{2}+\mathrm{i}u+\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}+\mathrm{i}u+\mathrm{i}t\right),\zeta\left(-\tfrac{1}{2}-\mathrm{i}u-\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}-\mathrm{i}u-\mathrm{i}t\right)\right). \quad (34)$$

Now from (32) and (34) we get that for a non-zero u

$$t=T_1\left(u,\zeta\left(-\tfrac{1}{2}+\mathrm{i}u+\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}+\mathrm{i}u+\mathrm{i}t\right),\zeta\left(-\tfrac{1}{2}+\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}+\mathrm{i}t\right),\right. \\ \left.\zeta\left(-\tfrac{1}{2}-\mathrm{i}u-\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}-\mathrm{i}u-\mathrm{i}t\right)\zeta\left(-\tfrac{1}{2}-\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}-\mathrm{i}t\right)\right), \quad (35)$$

where

$$T_1(u,v_1,w_1,x_1,y_1,v_2,w_2,x_2,y_2)= \\ \frac{T_2(v_1,w_1,v_2,w_2)-T_2(x_1,x_1,x_2,y_2)-u^2}{2u}. \quad (36)$$

Thus we can determine t from the values of the zeta function calculated at eight points. When t is real, the number of arguments of T_2 and hence of T_1 as well can be reduced:

$$t^2=T_2\left(\zeta\left(-\tfrac{1}{2}+\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}+\mathrm{i}t\right)\right), \quad (37)$$

where

$$T_2(x,y)=\frac{4\pi^2|x|^2}{|y|^2}-\frac{1}{4}, \quad (38)$$

and respectively

$$t=T_1\left(u,\zeta\left(-\tfrac{1}{2}+\mathrm{i}u+\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}+\mathrm{i}u+\mathrm{i}t\right),\zeta\left(-\tfrac{1}{2}+\mathrm{i}t\right),\zeta\left(\tfrac{3}{2}+\mathrm{i}t\right)\right), \quad (39)$$

where

$$T_1(u,v,w,x,y)=\frac{T_2(v,w)-T_2(x,y)-u^2}{2u}. \quad (40)$$

If a is not far from the critical line $\mathrm{Re}(s)=1/2$, then we can put $t=\mathrm{Im}(a)$ and approximate (according to (18)) the zeta function by the corresponding values of $D_N(s)$. In this way we get the following counterparts of (37) and (39):

$$\mathrm{Im}(a)^2\approx T_2\left(D_N\left(-\tfrac{1}{2}-\sigma\right),D_N\left(\tfrac{3}{2}-\sigma\right)\right), \quad (41)$$

$$\mathrm{Im}(a)\approx \\ T_1\left(u,D_N\left(-\tfrac{1}{2}+\mathrm{i}u-\sigma\right),D_N\left(\tfrac{3}{2}+\mathrm{i}u-\sigma\right),D_N\left(-\tfrac{1}{2}-\sigma\right),D_N\left(\tfrac{3}{2}-\sigma\right)\right), \quad (42)$$

where $T_1(u,v,w,x,y)$ and $T_2(x,y)$ are defined by (40) and (38), $D_N(s)$ is defined by (5), (9), (7), and (3), and σ is defined by (25).

Here is a numerical example. If $N=50$, $a=0.2+14\mathrm{i}$, and $u=1$, then we get from (24)–(25)and (42) the following values:

$$\begin{aligned} \mathrm{Re}(a) &\approx 0.2-3.237...\times 10^{-10}, &(43)\\ \mathrm{Im}(a) &\approx 14+3.872...\times 10^{-8}. &(44)\end{aligned}$$

5 Improving the accuracy

In fact, we can calculate the value of a with higher accuracy than it was done in the previous Sections *using the same initial data* (3).

At first, we improve the value of $\mathrm{Im}(a)$. Continuing our previous example for $N=50$ and $a=0.2+14\mathrm{i}$, we substitute the already found value (44) and the value

$$\mathrm{Im}(\log_2(c_2)) = -4.3294\ 40566\ 85430\ 74390\ 28357\ 72999... \tag{45}$$

into (26) and find that

$$m \approx 2.0000\ 00004\ 22171... \tag{46}$$

Since m should be an integer, we use the exact value $m=2$ in (26) and get an improved approximation

$$\mathrm{Im}(a) \approx 14+4.544...\times 10^{-10}. \tag{47}$$

In order to improve $\mathrm{Re}(a)$ we use the functional equation again. From two copies of (39), for $u=u_1$ and $u=u_2$, we get yet another identity satisfied by the zeta function:

$$\begin{aligned} &T_1\left(u_1, \zeta\big(-\tfrac{1}{2}+\mathrm{i}u_1+\mathrm{i}t\big), \zeta\big(\tfrac{3}{2}+\mathrm{i}u_1+\mathrm{i}t\big), \zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big), \zeta\big(\tfrac{3}{2}+\mathrm{i}t\big)\right) - \\ &\quad T_1\left(u_2, \zeta\big(-\tfrac{1}{2}+\mathrm{i}u_2+\mathrm{i}t\big), \zeta\big(\tfrac{3}{2}+\mathrm{i}u_2+\mathrm{i}t\big), \zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big), \zeta\big(\tfrac{3}{2}+\mathrm{i}t\big)\right) = 0. \end{aligned} \tag{48}$$

Note that in this identity, t occurs only in the arguments of the zeta function.

Respectively, according to (18) the difference

$$\begin{aligned} &T_1\left(u_1, D_N\big(-\tfrac{1}{2}+\mathrm{i}u_1-\sigma\big), D_N\big(\tfrac{3}{2}+\mathrm{i}u_1-\sigma\big), D_N\big(-\tfrac{1}{2}-\sigma\big), D_N\big(\tfrac{3}{2}-\sigma\big)\right) - \\ &T_1\left(u_2, D_N\big(-\tfrac{1}{2}+\mathrm{i}u_2-\sigma\big), D_N\big(\tfrac{3}{2}+\mathrm{i}u_2-\sigma\big), D_N\big(-\tfrac{1}{2}-\sigma\big), D_N\big(\tfrac{3}{2}-\sigma\big)\right) \end{aligned} \tag{49}$$

should be small. Indeed, for $N=50$ and $a=1+14\mathrm{i}$, $u_1=1$, and $u_2=-1$ the difference (49) is equal to $1.175...\times 10^{-7}$. Now we try to make it even smaller (in absolute value) via adjusting the value of σ by a tiny number ε.

Namely, we replace the numerical value (25) by series

$$\sigma = -\mathrm{Re}(\log_2(c_2)) + \varepsilon + O(\varepsilon^2). \tag{50}$$

Under this replacement the difference (49) is equal to

$$1.1752\ 10464\ 95536\ 62742... \times 10^{-7} - 363.03\ 04836\ 50806\ 01535...\varepsilon \tag{51}$$

up to terms of order $O(\varepsilon^2)$. The difference (51) vanishes when

$$\varepsilon = 3.2372\ 22541\ 58079\ 16360... \times 10^{-10}, \tag{52}$$

so, instead of (25), we redefine

$$\sigma = -\mathrm{Re}(\log_2(c_2)) + \varepsilon = 0.2 - 1.765... \times 10^{-17}. \tag{53}$$

Using this adjusted value of σ in (24) we get much better approximation to $\mathrm{Re}(a)$ than (43). Similar, now (42) with $u = 1$ gives

$$\mathrm{Im}(a) \approx 14 + 1.270... \times 10^{-16} \tag{54}$$

which is much better approximation to the imaginary part of a than (44) and (47).

6 The case of the alternating zeta function

All numerical examples presented above dealt with the Riemann zeta function. In fact, the same technique works for some other functions defined by Dirichlet series. In this section we consider the same questions but for the alternating zeta function

$$\eta(s) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s} = (1 - 2 \times 2^{-s})\zeta(s); \tag{55}$$

in this case an additional trick can by applied for increasing the accuracy of $\eta(a)$.

To save notation, in this section numbers (2) have (instead of (3)) the following values:

$$d_k = \left.\frac{\mathrm{d}^k}{\mathrm{d}s^k}\eta(s)\right|_{s=a}, \qquad k = 1, 2, \ldots \tag{56}$$

Respectively, the coefficients of the finite series (5) are defined as the solution of system (7) with the new meaning of the right-hand sides.

Previously, we choose the value of c_1 independently of our initial data (3). Now we use our knowledge of some zeros z_m of $\eta(s)$, namely, those that are due to the factor $(1 - 2 \times 2^{-s})$ in the definition (55): $\eta(s)$ vanishes when

$$s = \mathrm{z}_m = 1 + \frac{2\pi\mathrm{i}}{\ln(2)}m, \qquad m = \pm1, \pm2, \ldots \tag{57}$$

According to the definition (56),

$$\eta(s) = D_N(s-a) + \delta + O(s-a)^{N+1} \tag{58}$$

for a certain constant δ. To make this constant small (in absolute value), we choose such a c_1 that

$$D_N(\mathrm{z}_m - a) = 0 \tag{59}$$

holds for some m. Formally, we define

$$c_1 = -\sum_{n=2}^{N} c_n n^{-\mathrm{z}_m} \tag{60}$$

for an m such that z_m is close (ideally, the closest) to a among all numbers (57).

Thanks to m being an integer, for determining its value it is sufficient to have a rather rough approximation to a which we can find by the technique described in Section 4.

Here is a numerical example. Let $N = 25$ and $a = 0.8 + 10\mathrm{i}$. We temporally put $c_1 = 1$ and calculate that

$$\begin{aligned} D_N(0) = \quad & 0.20071\ 87631\ 27891\ 86587\ 07496\ 58506... + \\ & 0.99357\ 65348\ 46494\ 27220\ 22984\ 35417...\mathrm{i}, && (61) \\ \mathrm{Im}(a) = \quad & 10 - 1.862... \times 10^{-25} && (62) \end{aligned}$$

(the latter by (42) with $u = 1$). Respectively, we select $m = 1$ and calculate c_1 according to (60):

$$\begin{aligned} c_1 = 1 \quad & +3.2375\ 46899\ 02577\ 79956\ 81132... \times 10^{-21} \\ & -3.2788\ 82128\ 48951\ 81087\ 06971... \times 10^{-21}\mathrm{i}. \end{aligned} \tag{63}$$

Further, with this value of c_1 we calculate that

$$\begin{aligned} D_N(0) = \quad & 0.20071\ 87631\ 27891\ 86587\ 39872\ 05405\ 48743\ 51847... + \\ & 0.99357\ 65348\ 46494\ 27219\ 90195\ 53288\ 88745\ 51656...\mathrm{i}. \end{aligned} \tag{64}$$

Comparing with the true value

$$\begin{aligned} \eta(a) = \quad & 0.20071\ 87631\ 27891\ 86587\ 39872\ 05405\ 48743\ 52143... + \\ & 0.99357\ 65348\ 46494\ 27219\ 90195\ 53288\ 88745\ 50995...\mathrm{i}. \end{aligned} \tag{65}$$

we see that (64) has 36 correct decimal digits while (61) had only 20.

7 A generalization

There is no need to use *consecutive* initial derivatives of the zeta function in order to calculate its value and the value of the argument. Numbers (8) could be defined by the linear equations from (7) for any set of $N-1$ values of k; however, this would result in lower accuracy in the further calculations.

Here is a numerical example. Let $N = 20$ and $a = -1 + 17\mathrm{i}$ (as in the example in Section 2), and let numbers (8) be defined by the equations from (7) for k running now from 2 to N. In this case

$$D_N(0) = 4.2514\ 30818\ 87779... + 3.9105\ 73040\ 40740...\mathrm{i}. \tag{66}$$

Comparing it with the exact value (11) we see that the replacement of the 1st derivative by the 20th derivative resulted in lower accuracy of 9 correct decimal digits compared to 11 correct digits in (10).

The missing first derivative can be easily calculated as well:

$$\frac{\mathrm{d}}{\mathrm{d}s} D_N(s) \bigg|_{s=0} = -2.5966\ 08922\ 91986... - 4.0149\ 56947\ 25371...\mathrm{i} \tag{67}$$

while

$$\frac{\mathrm{d}}{\mathrm{d}s} \zeta(s) \bigg|_{s=a} = -2.5966\ 08923\ 00960... - 4.0149\ 56947\ 24784...\mathrm{i}. \tag{68}$$

References

[1] Yu. V. Matiyasevich. Hunting zeros of Dirichlet series by linear algebra. I. *POMI Preprints* 1, 2020, 18 pages; `http://www.pdmi.ras.ru/preprint/2020/20-01.html`, doi:10.13140/RG.2.2.29328.43528.