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Central Measures of Continuous Graded Graphs: the Case of Unequal Frequencies

A. M. Vershik[1], F. V. Petrov[2]

[1]St Petersburg Department of V. A. Steklov Mathematical Institute of Russian Academy of Sciences, St Petersburg State University, Institute for Information Transmission Problems of Russian Academy of Sciences (Kharkevich Institute);

e-mail: vershik@pdmi.ras.ru

[2]St Petersburg Department of V. A. Steklov Mathematical Institute of Russian Academy of Sciences, St Petersburg State University

e-mail: fedyapetrov@pdmi.ras.ru

Abstract. A class of continuous graded graphs of Gelfand–Tsetlin schemes type is defined, and the set of all ergodic central measures of discrete type on the path spaces of such graphs is described. The main observation is that the central ergodic measure on a subgraph of "Bernoulli" graph is often obtained as the restriction of the standard Bernoulli measure to the path space of the subgraph. It dramatically changes the approach to finding central measures also on discrete graphs, like the famous Young graph.

The simplest example of this type is given by the theorem on the weak limit of normalized Lebesgue measures on simplices — the so called Cesaro measures, which are concentrated on the sequences with prescribed Cesaro limits (this limit parametrizes the measures).

More complicated examples are the graphs of continuous Young tableaux with fixed number of rows, and the graphs of spectra of infinite Hermitian matrices of finite rank. We prove existence and uniqueness theorems for ergodic central measures and describe their structure. In particular these results:

1) give a new spectral description of the so called infinite-dimensional Wishart measures [9] — ergodic unitary invariant measures of discrete type on the set of infinite Hermitian matrices;

2) describe the structure of continuous analogues of measures on discrete graded graphs.

New problems and connections which appear are to be considered in new publications.

Key words and phrases: continuous graded graphs, central measures, Bernoulli measures, cocicles.

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ПРЕПРИНТЫ
Санкт-Петербургского отделения
Математического института им. В. А. Стеклова
Российской академии наук

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1. Introduction

Locally finite $\mathbb{N}$-graded graphs, often called Bratteli diagrams, proved to be a useful instrument both for algebraic and stochastic problems. It is important to stress that the geometry of such graphs is related to the theory of discrete topological Markov chains, although the usual questions about Markov chains differ from the Bratteli diagrams theory questions. Uniting these two theories may yield interesting results. In Markov chains theory it is a well known and fruitful to consider not finitely or countably many states, but continuously many states. This is well developed in probability theory, although some generalizations of notions and results are non-trivial. But in the theory of graded graphs such a change is not developed yet. In this work we want to start the first examples of results concerning the so called continuous graded graphs. We recall that the theory of continuous graphs is a new and active branch [12], but the topic of graded graphs has its own problems which are somehow different from the usual graph theory problems.

The basic notion for a graph is the path space, possibly structured. For a graded graph, we consider the "graded" paths started in the first level and going down. In this paper the levels correspond to convex cones in Euclidean spaces, and so are the edge sets between consecutive levels. Such graphs are called to be of *Gelfand – Tsetlin type*.

The easiest such graph has a semiline $\mathbb{R}_+$ as a vertex set at each level, and the edges from x to y go iff $x \leqslant y$. This makes possible to identify the path in such graph and a series with non-negative terms. Call it *Cesáro* graph. We are especially interested in the following two generalizations of Cesáro graph. At first, this is the continuous Gelfand–Tsetlin graph: at level n, the vertices are increasing sequences $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ of real numbers, and the edges between two sequences $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ and $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_{n+1}$ at level $n+1$ correspond to interlacing

$$y_1 \leqslant x_1 \leqslant y_2 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant y_{n+1}$$

of these sequences. It has a subgraph of non-negative sequences ("positive Gelfand–Tsetlin graph"). An important intimately related graph is the so called *rank d Gelfand–Tsetlin graph*, in which the vertices at each level n are the increasing sequences $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_d$ of d non-negative numbers, and the edges correspond to interlacing

$$x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant \ldots \leqslant x_d \leqslant y_d.$$

This may be considered as a subgraph of the positive Gelfand–Tsetlin graph, if you consider the levels starting from the d-th and sequences of form $(0, 0, \ldots, 0, x_1, \ldots, x_d)$ (with $n-d$ initial zeros). These graphs correspond to the spectra of corners of Hermitian matrices (arbitrary, or non-negative definite, or non-negative definite of rank at most d.)

For an infinite non-negative definite Hermitian matrix A of rank at most d we may define A_n, $n \geqslant d$, as the principal $n \times n$ minor, and let $x_1 \leqslant \ldots \leqslant x_d$ be d largest eigenvalues of A_n (other eigenvalues are equal to 0). So we get a path in the Gelfand–Tsetlin graph of rank d. The pushforward of the measures on on non-negative definite infinite Hermitian matrices of rank d are, therefore, measures on the path space of the Gelfand–Tsetlin graph of rank d, and $U(\infty)$-invariant measure on matrices correspond to central measures on the path space. For matrices we get the so called *Wishart measures*. The Wishart measure of rank d with frequencies

$\lambda_1, \ldots, \lambda_d$ is defined as follows: take d i.i.d. standard complex Gaussian vectors $\xi_1, \xi_2, \ldots, \xi_d \in \mathbb{C}^n$ and consider the matrix $\sum \lambda_i \xi_i \cdot \xi_i^*$.

Possibly the first paper in which the GUE is considered in the framework of central measures on graded graphs is [5]. The whole list of invariant measures for Hermitian infinite matrices (and other such series) is known for a while and was obtained by different methods by Pickrel [13] and Olshansky and Vershik [4] (where ergodic method was also used but very differently from what we mean here).

This problem was considered in the old paper of the first author [2] about ergodic method, but Wishart list is missed there. Invariant measures on positive definite Hermitian matrices are easier to find, this was addressed in [8, 7].

The Gaussian measures GUE, GOE on matrices are especially popular. But we study these measures in terms of the graph of spectra of matrices rather than the graph of matrices themselves (for the GUE measures this is done in [5]). The spectral approach is important not only in itself, but also because it allows us to discover more subtle properties of invariant measures. Note, moreover, that this point of view is also possible for other fields (see, for example, [14]) — self-adjoint and quaternionic matrices. See more on the relation between graphs of matrices and spectra in [11].

Another important generalization is the graph of increasing sequences $x_1 \leqslant \ldots \leqslant x_d$ of non-negative real numbers with edges corresponding to the inequalities $x_i \leqslant y_i$ for all $i = 1, \ldots, d$. In other words, the vertices are continuous Young diagrams with d rows, and an edge corresponds to one diagram being contained in another.

A probabilistic Markov chain is determined by transition probabilities, but in the theory of Bratteli diagrams the basic notion if the *cotransition probability*, as in other problems in mathematics and physics [3]. The cotransition, or cocycle, is the main additional structure on the graded graph, and here we consider the cotransition probability measures as *normalized Lebesgue measures on the phase spaces of cotransitions*. These cotransitions arise in the theory of invariant measures on Hermitian matrices [5]. Lebesgue cocycle is a continuous analogue of the uniform cocycle for discrete graded graphs. A measure on the path space of a graded graph with Lebesgue cocycle is called *central measure*.

Next, a remarkable problem on describing all central measures for a given graph arises. We discuss it for the concrete cases. For many countable graphs (graphs of Pascal, Young, Schur, dynamical graphs on groups and random walks) this problem was considered in recent decades.

A natural method for solving this problem is the ergodic method [2, 3], see the details further. Applying it for finding the central measures of continuous graphs is especially spectacular because of geometric nature of the cocycle.

In many cases, as in our case, the list of ergodic central measures is conjecturally specified (or even already found by other methods). For example, in our case, each such measure is given by a set of finite or countable frequencies of natural objects. The problem is to explain why frequencies uniquely define an ergodic central measure and why they are defined by it, and, most importantly, to explain how the frequency description of the measure is related to the corresponding product measure (in particular Bernoulli's measure). Up to now there has been no such description, not only for continuous graphs, but even for the Young graph. This is what is done further for the chosen class of graded graphs and for some discrete graphs.

Here we outline the contents of the paper. We begin with the fundamental example of the continuous Cesáro graph. A cocycle here is a set of normalzied Lebesgue measures on simplexes $\{x_i \geqslant 0, \sum x_i = S\}$ of of growing dimension with a sum S that grows linearly with dimension. It is easy to understand that a central measure is a weak limit of Lebesgue measures on simplexes, but it is more difficult to prove that there exists unique central measure for each growth factor, i.e. frequency. The structure of the measure is clear — it is a measure on the Cesáro sequences, which explains the name. Cesáro's measures here play the same role as Bernoulli's measures for discrete graphs.

For the graphs more general than Cesáro, that is, finite rank Gelfand–Tsetlin graph and the continuous Young tableaux graph, *the central measures with unequal frequencies are the restrictions of the direct product of Cesáro measures to the corresponding set of paths.* In the case of equal frequencies some regularization of the weak limits is needed. But quite unexpectedly it turns out that exactly the same and even simpler answer is given to the problem of description of central measures of discrete type for many countable graphs, for example for the Young graph.

Note that in [1] all central measures for the Young graph were realized by means of the generalized RSK algorithm as homomorphic images of Bernoulli measures, and in the following works [10] it was proved that this homomorphism is isomorphism. However, this approach, while important, does not prove the completeness of the list of central measures. It turns out that the completeness of the list in this and other cases follows from an entirely different relation between Bernoulli measures and central measures: *the latter are not only factors of Bernoulli measures, but also restrictions on their proper subsets.* For some reason this fact has so far, to the authors' knowledge, been bypassed by mathematicians.

The class of continuous graphs under consideration is wide enough and it is convenient to describe infinite paths in these graphs by *vector positive series.* Dimension of the vector being equal to 1 is a case of Cesáro graph.

We emphasize that the main conclusion from our considerations is that the ergodic central measure for a rather wide class of continuous graphs are defined by their frequencies, i.e. only the growth rates for specific coordinates. More precisely, the central measure is a Markov measure on positive vector series and the behavior of these random series reflects the asymptotic behavior of spectra.

It is important to emphasize that for the Gaussian ensemble GUE (for the Plancherel measure in the Young graph case) the picture is quite different since all the limiting frequencies are zero and GUE (the Plancherel measure) owes its existence to a very non-trivial interaction between the positive and negative parts of the spectra, each of which has sublinear (rather than linear) asymptotics.

A remarkable parallel between these examples will be discussed in another paper. We are grateful to a referee who paid our attention to the interesting paper [15] where the connection between these two measures is established via the values of the moments. But we mean a different type of parallelism.

We emphasize that the calculations associated with the ergodic method are sometimes quite nontrivial. In this paper we note the very fact that the central measures are defined by frequencies, but explicit calculations of central measures and their definition of their properties can be found in few cases. Some of these calculations will be published in a forthcoming publication.

2. Setup

An $\mathbb{N}$-graded graph has a vertex set

$$\Gamma = \sqcup_{n=0}^{\infty} \Gamma_n$$

graduated by levels, every edge joins two vertices of consecutive levels. Here we assume that

1. Euclideness. All levels $\Gamma_n, n = 0, 1, 2 \dots$ are identified with closed finite dimensional convex polyhedral cones $\Gamma_n \subset \mathbb{R}^{d_n}$ with non-empty interior (but as vertices of the graph, the levels are disjoint, so, rigorously speaking, we had to write something like $\Gamma_n \subset \mathbb{R}^{d_n} \times \{n\}$). Dimensions d_n may depend on n or not.

2. Convex Incidence. Two vertices $x \in \Gamma_n, y \in \Gamma_{n+1}$ of consecutive levels are joined by an edge iff the pair x, y belongs to a certain closed convex polyhedral cone

$$D_n \subset \Gamma_n \times \Gamma_{n+1}, n = 0, 1, \dots.$$

We require that the natural projections $D_n \to \Gamma_n$ and $D_n \to \Gamma_{n+1}$ are surjective, in other words, every vertex in Γ_0 has a neighbor in Γ_1, and every vertex in $\Gamma_n, n > 0$ has neighbors in both levels Γ_{n-1} and Γ_{n+1}.

An infinite path is defined as the sequence of vertices $(v_0, v_1, \dots)$, $v_i \in V_i$, such that $(v_i, v_{i+1}) \in D_i$ for all $i = 0, 1, \dots$. The path space is the set of all infinite paths endowed with a natural (weak) topology.

3. Non-degeneracy. For all interior points $x \in \Gamma_n, z \in \Gamma_{n+1}$ the section $D_n^+(x) = \{y \in \Gamma_{n+1} : (x, y) \in D_n\}$ (the set of second endpoints of the edges from x to Γ_{n+1}) is a non-degenerated convex polyhedral set not containing lines; and the section $D_n^-(z) = \{y \in \Gamma_n : (y, z) \in D_n\}$ (the set of starting points of the edges from Γ_n to z) is a non-degenerated convex polytope. (Non-degeneracy means the maximal dimension). This property may be weakened, but it specifies an important special class of continuous graphs.

A certain class of continuous graphs is defined already. We proceed to specify the class of graphs under consideration by imposing new conditions on the graph structure.

4. Homogeneity. The following condition concerns the case when all levels are isomorphic. A graph is called *homogeneous*, if the incidence cone D_n is the same for all n.

The graphs satisfying aforementioned properties 1-3 are called *Gelfand–Tsetlin type continuous graphs*, and if all levels are the same and property 4) holds, we call it *homogeneous Gelfand–Tsetlin type continuous graphs*.

3. Cocyles

We proceed with defining the cotransition probabilities and cocycle for graded graphs. It is introduced in the same manner (even simpler) as for discrete graded graphs.

For every interior vertex $x \in \Gamma_n$, $n > 0$, the paths from Γ_0 to x form a compact set $\mathcal{P}(x)$ of dimension $d_0 + d_1 + \dots + d_{n-1}$, which is naturally stratified onto sets $\mathcal{P}(y)$ for $y \in D_n^-(x) \subset \Gamma_{n-1}$. Assume that there is already fixed the probability measure μ_y on each $\mathcal{P}(y)$, where $y \in D_n^-(x) \subset \Gamma_{n-1}$. Then, if we fix a probability measure on $D_n^-(x)$ (the so called cotransition probability measure), it defines the probability measure μ_x on $\mathcal{P}(x)$. So, by obvious induction the system of cotransition measures defines the measure on paths to x for every vertex x of Γ. More

generally, it defines the probability measure on paths from y to x for every two vertices x and y such that there exists a path from y to x. Hereafter we assume that the measures are absolutely continuous with respect to Lebesgue measures in corresponding Euclidean spaces. Then, for two tail-equivalent paths P_1, P_2 (that is, infinite paths which coincide from certain place) we may define the *cocycle* $c(P_1, P_2)$ as follows: take large number n such that P_1, P_2 coincide after n-th level, and cut the paths P_1, P_2 at this level. We get two paths $P_1(n)$, $P_2(n)$ belonging to the same set $\mathcal{P}(x)$ for certain $x \in \Gamma_n$. The ratio of densities of μ_x at $P_2(n)$ and $P_1(n)$ is denoted by $c(P_1, P_2)$. It is straightforward that this number does not depend on the choice of the level number n.

For the discrete case, it is natural to consider the uniform distributions on the sets $\mathcal{P}(x)$, which corresponds to the the *central cocycle* $c(P_1, P_2) \equiv 1$ (instead of densities, we may simply divide the probabilities). The Borel measures on infinite paths with such cocycle are called *central measures*. They are Markov measures (with time corresponding to the grading), as follows from the definition, but not viceversa. The set of ergodic (indecomposable) central measures is called the *absolute* of the graph. Finding the absolute is the most important problem of the theory of graded graphs.

For the introduced class of continuous Gelfand–Tsetlin type graphs we introduce the natural analogue of the central cocycle, which we call *Lebesgue cocycle*. It corresponds to the normalized Lebesgue measure on each $\mathcal{P}(x)$, in other words, again $c \equiv 1$. The measures with such cocycle are again called central measures, and the main problem is again to describe all ergodic central measures, i.e., *absolute* of Gelfand–Tsetlin type graphs.

More general cocycles are to be considered separately.

4. Ergodic method

To describe the central and invariant measures, the so called ergodic method was proposed in [2]. It essentially follows from individual ergodic theorems or the martingale convergence theorem. To formulate it, let us introduce the necessary notion of weak convergence of measures on the path space. A sequence of measures on a path space of a graph is weakly convergent if for any cylindrical set there is convergence of values of measures of this set. In our situation the ergodic method can be described as follows (cf. [2, 3]).

Theorem 1. *For every ergodic central measure μ on the path space of Gelfand–Tsetlin type graph Γ there exists a path, i.e. a sequence of vertices $\{x_n\}_n$, $\quad x_n \in D_n^-(x_{n+1}) n = 1, 2, \dots,$ for which the sequence of Lebesgue measures on the sets $\mathcal{P}(x_n)$ weakly converges to μ.*

Although formally the continuous setting is different from what was considered earlier, the usual proof holds verbatim.

This theorem reduces the problem of describing the absolute to concrete calculations, which may be laborious in practice. But on the other hand, the weak convergence conditions in concrete cases may be already restrictive enough and allow to describe the absolute explicitly.

The ergodic method was used in [1] for describing the characters of S_∞, in [4] for describing the invariant measures on infinite Hermitian matrices and in many other problems on invariant or central measures.

5. Frequencies

Here we reformulate the main problem of describing the absolute of Gelfand–Tsetlin type graph in concrete coordinate terms.

Let each level have dimension d, and each level be the cone

$$\mathcal{C} := \{(x_1, \dots, x_d) \in \mathbb{R}^d \colon 0 \leqslant x_1 \leqslant x_2 \leqslant \dots \leqslant x_d\}.$$

A convex cone D_n (it is also the same for all n) is defined by several linear inequalities enjoyed by two vertices $x = (x_1, \dots, x_n) \in \Gamma_n$ and $y = (y_1, \dots, y_n) \in \Gamma_{n+1}$.

Two main examples for us are the following:

(i) Gelfand–Tsetlin rank d graph is defined by the interlacing inequalities

$$D_n = \mathcal{D}_{GT} := \{x, y \colon 0 \leqslant x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant \dots \leqslant x_d \leqslant y_d\}.$$

This graph arises in the random matrix theory. Namely, consider the $n \times n$ Hermitian matrix with eigenvalues 0 (of multiplicity $n - d$), $y_1, \dots, y_d$, where $0 \leqslant y_1 \leqslant \dots \leqslant y_d$. Its principal minor of order $n - 1$ has eigenvalues 0 (of multiplicity $n - d - 1$) and $x_1, \dots, x_d$, where $0 \leqslant x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant \dots \leqslant x_d \leqslant y_d$. Moreover, the unitary invariant (in natural sense) measure on the orbit of matrices with fixed eigenvalues and fixed eigenvalues of a minor of order k, $d \leqslant k \leqslant n$, is Lebesgue measure on the corresponding polytope. This is proved by Baryshnikov [5] in greater generality.

So, along with the Gelfand–Tsetlin graph, we may consider a "covering" graph of matrices: on k-th level the vertices are the Hermitian matrices of rank at most d and size $k \times k$, and we join by an edge a $k \times k$ matrix and its principal $(k-1) \times (k-1)$ minor.

A usual Gelfand–Tsetlin graph is defined similarly, but without the rank restriction.

Note that the definition of our graphs is via inequalities between elements only, and thus it may be automatically considered over arbitrary linearly ordered set (for example, a segment, not a semiline) and even poset.

In the discrete case of the linearly ordered set $\mathbb{Z}$ it is related to the representation theory of unitary groups, see, for example, [6].

(ii) Young jumps graph. Here D_n is the following cone:

$$\mathcal{D}_Y := \{x, y \in \mathcal{C} \colon x_1 \leqslant y_1, x_2 \leqslant y_2, \dots, x_d \leqslant y_d\}.$$

If we identify a point $(x_1, \dots, x_d)$ and the Young diagrams with d rows of (not necessarily integer) lengths $x_1, \dots, x_d$, $(x, y) \in \mathcal{D}_Y$ means that one diagram is contained in another. In the usual discrete Young graph, an edge corresponds to a relation "one diagram is contained in another and differs from it by removing exactly one box". Here we may remove (or add, if we consider moving along a forward path) arbitrarily many "boxes".

Let $x(n) = (x_1(n), x_2(n), \dots, x_d(n)) \in \Gamma_n$, $n = 1, 2, \dots$ be a path defining a central probability measure μ on the homogeneous Gelfand–Tsetlin type graph Γ in view of Theorem 1. If $\lim x_i(n)/n = \lambda_i \in [0, \infty]$, we say that λ_i's are *frequencies* of the corresponding measure. We always may pass to a subsequence of the sequence $x(n)$ for which the frequencies exist. The importance of this notion is explained

by the fact that in many cases (and we conjecture that for homogeneous Gelfand–Tsetlin type graph always) the frequencies uniquely determine the ergodic central measure.

6. Restrictions and restorations

Let Γ be a graded graph with levels Γ_n, and $\tilde{D}_n \subset D_n$ be (measurable) subsets of edge sets. Denote by $\tilde{\Gamma}$ the subgraph induced on these subsets.

Let μ be a central probability measure on the path space of Γ. Assume that with positive μ-probability a random path is a path in $\tilde{\Gamma}$, that is, it goes only along the edges in $\sqcup \tilde{D}_n$.

Our crucial observation is the following almost obvious

Theorem 2. *The restriction of μ to the path space of $\tilde{\Gamma}$ is also a central measure.*

Proof. Fix a level n_0. As every central measure, μ enjoys the Markov property: that is, it is characterized by the distribution on the n_0-th level Γ_{n_0} and the conditional measures on the tails after this level. The initial segments before Γ_{n_0} are distributed Lebesgue-uniformly on the corresponding polytopes $\mathcal{P}(x)$, $x \in \Gamma_{n_0}$ and do not depend on the tails. This all is preserved after restriction, since the restriction of Lebesgue measure is again Lebesgue measure. □

Note that Theorem does not give a receipt how to get specific formulae for the restricted measure, and this may be quite non-trivial. For example, consider the usual (discrete) Young graph of Young diagrams with at most k rows. It may be viewed as a subgraph of decreasing integer sequences $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k \geqslant 0$ of the Pascal graph $\mathbb{Z}^k_{\geqslant 0}$. The Bernoulli measure with frequencies $p_1 > p_2 > \ldots > p_k$ on $\mathbb{Z}^k_{\geqslant 0}$ (that means that on each step we increase the i-th coordinate by 1 with probability p_i, and do not change other coordinates) restricted to Young graph gives a Thoma measure with parameters $p_1, \ldots, p_k$: the probability of the diagram with rows lengths $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k \geqslant 0$ is the value of the corresponding Schur function at the point $(p_1, \ldots, p_k)$.

A counterpart of Theorem 2 is that a finite measure ν on the path space of the subgraph $\tilde{\Gamma}$ may be extended to a measure μ on the path space of the whole graph Γ such that the restriction of μ on the path space of $\tilde{\Gamma}$ coincides with ν (but this μ is not always finite, this is a delicate question).

Indeed, the centrality property allows to define μ on the images of the path space of $\tilde{\Gamma}$ under transforms which preserve central measures. For example, we may consider the transforms which fix the tail after n_0-th level and change the initial segments accordingly (in the Lebesgue-preserving manner). This defines μ on the set of paths which are eventually in $\tilde{\Gamma}$. It is not hard to see that this is consistent, and we get a measure on the path space of Γ, supported on the paths which eventually are in $\tilde{\Gamma}$.

If the obtained measure is finite, then the initial measure on the path space of $\tilde{\Gamma}$ is obtained by restriction (and normalization). For homogeneous Gelfand–Tsetlin type graphs, aswell as for discrete Young graph, this corresponds to the case of *unequal frequencies*.

7. One-dimensional case: Cesáro measure

Here we deal with the $d = 1$ case of Gelfand–Tsetlin graph (Young jumps graph is the same as Gelfand–Tsetlin for $d = 1$) and compute the central measures of this graph.

The path in our graph is simply a sequence $0 \leqslant x_0 \leqslant x_1 \leqslant \ldots$ of non-negative numbers, and for the vertex $a \in \Gamma_n = [0, \infty)$ the path space $\mathcal{P}(a)$ is the n-dimensional simplex $\Delta_n(a) := \{0 \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = a\}$. We should study the distribution of finitely many coordinates $x_0, \ldots, x_{k-1}$ (k is fixed and n is large) with respect to the normalized Lebesgue measure μ_a on this simplex.

Consider an ergodic central measure on the 1-dimensional Gelfand–Tsetlin graph. By Theorem 1, it corresponds to a certain sequence of vertices (even to a path, but it is more convenient to consider a sequence: the difference is that in the sequence some levels may be skipped). Passing to a subsequence, we may suppose that the vertices $a_i \in \mathbb{R}_+$ at levels n_i satisfy $\lim a_i/n_i = \lambda$, where $\lambda \in [0, \infty]$.

Thus the following theorem yields that every ergodic central measure on the 1-dimensional Gelfand–Tsetlin graph is an exponential random walk by the levels of the graph, as is seen from the following theorem.

Theorem 3. *Assume that $\lambda > 0$ is constant and $\Gamma_n \ni a = \lambda n + o(n)$ is a vertex of n-th level of the 1-dimensional Gelfand–Tsetlin graph. Then for each fixed positive integer k the measure induced by μ_a on the sequences $x_0 < x_1 < \ldots < x_{k-1}$ converges to the exponential random walk with mean λ: $x_0, x_1 - x_0, \ldots, x_{k-1} - x_{k-2}$ are $Exp(\lambda)$ i.i.d.*

If $a = o(n)$, the corresponding distribution weakly converges to a stationary distribution at 0; if $a/n \to \infty$, the corresponding measures weakly converge to zero.

Because of importance of this Theorem and its relation to generalizations, we give three proofs. We concentrate on the case $0 < \lambda < \infty$, the proofs for $\lambda = 0$ and $\lambda = \infty$ are similar and only simpler.

1. Denote $y_0 = x_0$, $y_i = x_i - x_{i-1}$ for $i = 1, \ldots, n-1$. Then our simplex is defined as $\{y_i \geqslant 0, \sum y_i \leqslant a\}$. Its volume equals $a^n/n!$. If we restrict the distribution to the first k coordinates $(y_0, \ldots, y_{k-1})$, then the density at the above point would be proportional to the volume of the corresponding section $\{y_k + \ldots + y_{n-1} \leqslant a - (y_0 + \ldots + y_{k-1})\}$. Denoting $s = y_0 + \ldots + y_{k-1}$, we get that this volume equals $(a-s)^{n-k}/(n-k)!$. Thus the density equals

$$\frac{(a-s)^{n-k}}{x^{n-k} \cdot a^k} \cdot n!(n-k)! = (1 - s/a)^{n-k} \cdot \frac{n(n-1)\ldots(n-k+1)}{a^k}$$
$$\sim e^{-s/\lambda}\lambda^{-k} = \prod_{i=1}^{k} \lambda^{-1} e^{-y_i/\lambda}$$

that proves the claim. □

2. It is more convenient to study the distribution of $a - x_{n-1}$, $x_{n-1} - x_{n-2}$, …, $x_{n-k+1} - x_{n-k}$. Of course it has the same distribution as $x_0, x_1 - x_0, \ldots, x_{k-1} - x_{k-2}$, as the measure-preserving automorphism $(x_0, \ldots, x_{n-1}) \to (a - x_{n-1}, \ldots, a - x_0)$ of $\Delta_n(a)$ shows.

Consider the following map from $\Delta_n(a)$ to the unit cube $[0,1]^n$, which maps the normalized Lebesgue measure on the simplex to the Lebesgue measure on the cube

$$\Phi_n\colon (x_0, x_1, \ldots, x_{n-1}) \to \left(\frac{x_0}{x_1}, \frac{x_1^2}{x_2^2}, \ldots, \frac{x_{n-2}^{n-1}}{x_{n-1}^{n-1}}, \frac{x_{n-1}^n}{a^n}\right) =: (t_0, \ldots, t_{n-1}).$$

The Jacobi matrix of Φ_n is triangular, that immediately yields that Jacobian equals $n!$. Thus Φ indeed maps the uniform distribution on $\Delta_n(x)$ to the uniform distribution on $[0,1]^n$. Thus

$$\begin{aligned}
a - x_{n-1} &= a\left(1 - t_{n-1}^{1/n}\right) \sim -\frac{a}{n}\log t_{n-1} \\
x_{n-1} - x_{n-2} &= x_{n-1}\left(1 - t_{n-2}^{1/(n-1)}\right) = x \cdot t_{n-1}^{1/n} \cdot \left(1 - t_{n-2}^{1/(n-1)}\right) \sim -\frac{x}{n}\log t_{n-2} \\
x_{n-2} - x_{n-3} &= x_{n-2}\left(1 - t_{n-3}^{1/(n-3)}\right) \sim -\frac{a}{n}\log t_{n-3} \\
&\ldots\ldots\ldots
\end{aligned}$$

that yields the result, as a/n goes to λ and $-\lambda \log T \in Exp(\lambda)$ when $T \in Unif(0,1)$. □

3. This proof, based on Law of Large Numbers, is ideologically close to the proof of Maxwell–Poincaré lemma stating that the distribution of coordinates of a Lebesgue-random point on the n-dimensional sphere of radius $\sqrt{n}$ is asymptotically standard normal.

We start with the distribution η_n of vector $x \in (0,\infty)^n$ which is the product of $Exp(\lambda)$ distributions over all n coordinates, and prove that the distribution of the first k coordinates of this vector is close to that of a random point in our simplex $\Delta_n(a)$. That is, the density at a point $y = (y_1, \ldots, y_n)$ equals

$$p(y) = \prod_{i=1}^{d} \lambda^{-1} e^{-y_i/\lambda} = \lambda^{-n} e^{-(y_1+\ldots+y_n)/\lambda}.$$

Thus, we see that $p(y)$ depends only on the sum $y_1 + \ldots + y_n$ of coordinates. In other words, the random point y may be obtained by choosing at random (according to a certain distribution, which is a Gamma distribution but we do not use this) the value of $x = y_1 + \ldots + y_n$ and then choosing uniformly a random point in the simplex $S_x := \{y_i \geqslant 0, \sum y_i = x\}$. Now fix a set $\Omega \subset \mathbb{R}^k$ which is a product of intervals. The probability that $(y_1, \ldots, y_k) \in \Omega$ is equal to the expectation against x of the probability that $(y_1, \ldots, y_k) \in \Omega$ for a random $(y_1, \ldots, y_n) \in S_x$. Note that $(y_1, \ldots, y_k) \in \Omega$ if and only if $\frac{\lambda n}{x}(y_1, \ldots, y_k) \in \frac{\lambda n}{x}\Omega$. And the point $\frac{\lambda n}{x}(y_1, \ldots, y_n)$ is uniformly distributed in $S_{\lambda n}$. Now note that by law of Large numbers, for any specific $\varepsilon > 0$ the probability that $\frac{\lambda n}{x} \notin [1-\varepsilon, 1+\varepsilon]$ tends to 0. When $\tau \in [1-\varepsilon, 1+\varepsilon]$, the sets $\tau\Omega$ slightly vary. In particular, their intersection contains a set $\Omega_-(\varepsilon)$ and their union is contained in $\Omega_+(\varepsilon)$ such that the η_k-measures of both are close to each other. This yields the required weak convergence. □

Thus, any ergodic measure on one-dimensional Gelfand–Tsetlin graph is concentrated on the series $x_0 + (x_1 - x_0) + (x_2 - x_1) + \ldots$ which converge in Cesáro sense to a certain number $\lambda > 0$. That's why we suggest to call it *Cesáro measure*.

8. Multi-dimensional case

Here we explain how combining the one-dimensional Cesáro case and Theorem 2 to handle the case of finite rank Gelfand–Tsetlin graph and Young jumps graph.

Now we may note that the rank d Gelfand–Tsetlin graph is a subgraph of $\mathbf{Ces}^d$, where $\mathbf{Ces}$ is a Cesáro graph. On $\mathbf{Ces}^d$ there exists a Bernoulli measure which is a product of Cesáro measures with frequencies $p_1 < p_2 < \ldots < p_d$. With positive probability, any vertex $(x_1, \ldots, x_d)$ forms an increasing sequence, and any two consecutive vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ interlace (with probability 1 this happens eventually by Law of Large Numbers, it easily yields that with positive probability this "eventually" is from the very beginning.) Thus by our restoration procedure we get a unique ergodic central measure on the path space of Gelfand–Tsetlin graph with given distinct frequencies.

A measure with equal frequencies may be obtained as a weak limit of measures with distinct frequencies, but we skip the details here. All the same holds for Young jumps graph: the ergodic central measures are in one-to-one correspondence with arrays of frequencies, and for distinct frequencies the corresponding measures are obtained by restriction of the Cesáro–Bernoulli measures.

To summarize, we provided a new way to study the central measures, which is based not on computations and estimates but on intrinsic properties of these measures. Of course it may be applied to discrete graphs also, but quite surprisingly it was realized for the continuous graphs. Applications to discrete graphs are to be considered in a separate paper. The calculations of transition probabilities also get a new interpretation.

For a Gelfand–Tsetlin graph of growing rank (graph of interlacing sequences), we have countably many frequencies with finite sum of absolute values, and if they all are distinct, the same restriction argument works.

This method reduces explicit evaluation of the central measures with given frequencies to a probabilistic problem of finding the "Cesáro–Bernoulli" measures of corresponding cones (in order to compute its restriction). Alternative, more standard way is applying a continuous version of Lindstrom–Gessel–Viennot lemma.

Concerning the general Gelfand–Tsetlin type graphs, we also conjecture that the central measures are also uniquely determined by the limit frequencies.

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