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Minimal ideal triangulations of hyperbolic 3-manifolds with geodesic boundary via $\mathbb{Z}_2$-homology

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ABSTRACT

The triangulation complexity $c_\Delta(M)$ of a compact 3-manifold M with non-empty boundary is the minimal number of tetrahedra in any ideal triangulation of M. We obtain a new lower bound on $c_\Delta(M)$ via $\mathbb{Z}_2$-homology, as well as a characterisation of minimal ideal triangulations of manifolds for which our complexity bound is achieved.

Keywords: 3-manifold, ideal triangulation, complexity

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1 Introduction

In this paper we will restrict our attention to connected compact 3-manifolds M with non-empty boundary. The *triangulation complexity* $c_\Delta(M)$ of M is the minimal number of tetrahedra in any ideal triangulation of M. There are remarkably few examples of exact computations of triangulation complexity of manifolds with non-empty boundary. The lower bound

$$c_\Delta(M) \geq \beta_0(\partial M; \mathbb{Z}_2) - \chi(M), \tag{1}$$

on complexity is known through work of Frigerio, Martelli and Petronio [1], where it is shown that the bound is attained by infinite families of manifolds $M_{g,k}$ with a totally geodesic boundary component of genus $g \geq 2$ and k cusps. In particular, each minimal ideal triangulation of $M_{g,0}$ [2] has only a single edge. An equivalent approach to complexity is via Matveev's theory of special spines. From this point of view, it is proved [3] that any ideal triangulation $\mathcal{T}$ with exactly two edges such that no 3-2 Pachner moves can be applied to $\mathcal{T}$ is minimal. Infinite families of such manifolds were described in [4, 5, 6]. Moreover, a census of connected compact 3-manifolds with non-empty boundary decomposed into at most 4 ideal tetrahedra was given in [7, 8].

In this paper we present a new lower bound for the triangulation complexity of connected compact 3-manifolds M with non-empty boundary via $\mathbb{Z}_2$-homology (Theorem 3.1). It is shown (Theorem 3.3) that this bound is stronger than that given by (1). We use $\mathbb{Z}_2$-homology to study the anatomy of minimal triangulations of M for which our lower bound is achieved (Theorem 4.1, Lemma 4.3). The class of such manifolds is denoted $\mathcal{M}_h$.

Our next task is to study manifolds in $\mathcal{M}_h$. We characterise edges and tetrahedra of minimal triangulations (Lemma 4.3), which yields a natural partition of the set of minimal triangulations into four classes. A natural question to ask does a manifold in $\mathcal{M}_h$ admit minimal triangulations of different classes. We prove (Theorem 5.1) that the answer is negative.

We are now left with two tasks. We must provide examples of manifolds in $\mathcal{M}_h$, and we can prove that each M in $\mathcal{M}_h$, with a few exceptions, is a hyperbolic manifold with totally geodesic boundary components and some cusps.

2 Duality between ideal triangulations and special spines

In this paper, we will translate freely back and forth between an ideal triangulation and its dual special spine. We begin by recalling some definitions.

2.1 Ideal triangulations

Let $\mathcal{T}$ be a cell complex made out of a pairwise disjoint collection of 3-simplices by gluing all of their 2-dimensional faces in pairs via simplicial maps, and $|\mathcal{T}|$ its underlying space. The simplices prior to identification, and their vertices, edges, and so on, are called *model cells*. We will denote the union of the vertices of $\mathcal{T}$ by $\mathcal{T}^{(0)}$.

Note that $|\mathcal{T}|$ may actually not be a 3-manifold, because the link of a vertex could be any surface, and the link of the midpoint of an edge could be a projective plane. If M is a compact 3-manifold with boundary, and if $|\mathcal{T}| \setminus |\mathcal{T}^{(0)}|$ is homeomorphic to the interior of M, then we say that $\mathcal{T}$ is an *ideal triangulation* of M. In this case the vertices of $\mathcal{T}$ are called *ideal vertices*.

2.2 Special spines

A *spine* of a compact 3-manifold M with boundary is a compact polyhedron $P \subset M$ such that $M \setminus P$ is homeomorphic to $\partial M \times [0, 1)$. A spine P carries much information about M. In particular, P is homotopy equivalent to M and hence determines the homotopy type of M.

We will restrict our class of spines to those called *special (or, standard) spines*. A compact two-dimensional polyhedron P is said to be *simple* if the link of every point x in P is homeomorphic either to a circle (such a point x is called *nonsingular*), to a graph consisting of two vertices and three edges joining them (such a point x is called a *triple point*), or to the complete graph K_4 with four vertices (such a point x is called a *true vertex*). Connected components of the set of all nonsingular points are called 2-*components* of P, while connected components of the set of all triple points are called *triple lines* of P. The set of singular points of P (that is, the union of all triple lines and all true vertices) is called a *singular graph* of P. A simple polyhedron is *special* if each of its triple line is an open 1-cell, and each of its 2-component is an open 2-cell. A singular graph of a special polyhedron has at least one true vertex and is a 4-regular graph. Therefore, it is natural to call the triple lines of a special polyhedron *edges*.

An ideal triangulation $\mathcal{T}$ of a compact 3-manifold M with boundary defines in a natural way a dual special polyhedron, which is in fact a spine of M. For each model tetrahedron Δ_i of $\mathcal{T}$, let R_i denote the union of the links of all four vertices of Δ_i in the first barycentric subdivision. Since the face-pairings are simplicial, gluing Δ_i's induces gluing the corresponding R_i's together. We get a special spine P of M. In fact, the assignment $\mathcal{T} \to P$ induces a bijection between ideal triangulations (considered up to equivalence) and special spines (considered up to homeomorphisms) [9].

For each 2-component ξ of a special polyhedron P, there is a characteristic map $f : D^2 \to P$, which carries the interior of the disc D^2 onto ξ homeomorphically and which restricts to a local embedding on $S^1 = \partial D^2$. We will call the curve $f|_{\partial D^2} : \partial D^2 \to P$ (and its image $f|_{\partial D^2}(\partial D^2)$) the *boundary curve* $\partial\xi$ of ξ.

3 Lower bounds for complexity of manifolds with boundary

3.1 Lower bounds via $\mathbb{Z}_2$-homology

Theorem 3.1. *Let M be a connected compact 3-manifold with boundary. Then*

$$c_\Delta(M) \geq \beta_1(M, \mathbb{Z}_2).$$

By $\mathtt{d}(P)$ and $\mathtt{v}(P)$ denote the number of 2-components and the number of true vertices of a special polyhedron P, respectively.

Lemma 3.2. *Let P be a special spine of a connected compact 3-manifold M with boundary. Then we have:*

1. $\mathtt{d}(P) - (\beta_2(M, \mathbb{Z}_2) + 1) = \mathtt{v}(P) - \beta_1(M, \mathbb{Z}_2)$;
2. $\mathtt{d}(P) \geq \beta_2(M, \mathbb{Z}_2) + 1$.

Proof. Since the singular graph of a special spine is 4-regular, $\chi(P) = \mathtt{d}(P) - \mathtt{v}(P)$. Since M is connected, we have $\beta_0(M, \mathbb{Z}_2) = 1$ and $\chi(M) = 1 - \beta_1(M, \mathbb{Z}_2) + \beta_2(M, \mathbb{Z}_2)$. The homotopy equivalence of P and M implies that $\chi(M) = \chi(P)$. Thus 1 holds.

In order to prove 2 we consider a part of the chain complex of P with $\mathbb{Z}_2$-coefficients:

$$C_2 \xrightarrow{\partial} C_1.$$

Recall that $B_1 = \partial C_2$ is the group of 1-dimensional boundaries. Notice, that

$$\partial(\alpha_0 + \ldots + \alpha_{\mathtt{d}(P)-1}) = \gamma_0 + \ldots + \gamma_{2\mathtt{v}(P)-1}, \tag{2}$$

where $\alpha_0, \ldots, \alpha_{\mathtt{d}(P)-1}$ are the 2-components and $\gamma_0, \ldots, \gamma_{2\mathtt{v}(P)-1}$ are the edges of P. Hence $\dim B_1 \geqslant 1$. Since P is 2-dimensional polyhedron, we obtain $\mathtt{d}(P) = \dim C_2$ and $\dim(\operatorname{Ker} \partial) = \beta_2(P, \mathbb{Z}_2)$. Again, from homotopy equivalence of P and M we have $\beta_2(P, \mathbb{Z}_2) = \beta_2(M, \mathbb{Z}_2)$. The following computation provides 2:

$$\mathtt{d}(P) = \dim C_2 = \dim(\operatorname{Ker} \partial) + \dim(\operatorname{Im} \partial) = \beta_2(P, \mathbb{Z}_2) + \dim B_1 \geq \beta_2(M, \mathbb{Z}_2) + 1. \tag{3}$$

□

Proof of Theorem 3.1. Let $\mathcal{T}$ be a minimal ideal triangulation of M. Consider the special polyhedron P that is dual to $\mathcal{T}$. It is clear that the number of tetrahedra in $\mathcal{T}$ is equal to $\mathtt{v}(P)$ due to the duality of P and $\mathcal{T}$. Hence $c_\Delta(M) = \mathtt{v}(P)$. Applying Lemma 3.2 to P we obtain $\mathtt{v}(P) \geqslant \beta_1(M, \mathbb{Z}_2)$. Hence $c_\Delta(M) \geqslant \beta_1(M, \mathbb{Z}_2)$. □

3.2 Comparing two lower bounds

Now we prove that the lower bound for $c_\Delta(M)$ in Theorem 3.1 is stronger than the Frigerio–Martelli–Petronio one (1).

Theorem 3.3. *Let M be a connected compact 3-manifold with boundary. Then*

$$\beta_1(M, \mathbb{Z}_2) \geqslant \beta_0(\partial M; \mathbb{Z}_2) - \chi(M).$$

The desired inequality is derived from the following lemma.

Lemma 3.4. *Let M be a connected compact 3-manifold with boundary. Then*

$$\beta_2(M; \mathbb{Z}_2) + 1 \geq \beta_0(\partial M; \mathbb{Z}_2).$$

Proof. Consider a part of the long exact sequence for the pair $(M, \partial M)$ with $\mathbb{Z}_2$-coefficients:

$$H_1(M, \partial M; \mathbb{Z}_2) \xrightarrow{\varphi} H_0(\partial M; \mathbb{Z}_2) \xrightarrow{\psi} H_0(M; \mathbb{Z}_2).$$

Since $\operatorname{Ker}\psi = \operatorname{Im}\varphi$ and M is connected, we have

$$\begin{aligned}\beta_0(\partial M; \mathbb{Z}_2) &= \dim(\operatorname{Im}\psi) + \dim(\operatorname{Im}\varphi) \leq \\ &\leq \dim(H_0(M; \mathbb{Z}_2)) + \dim(H_1(M, \partial M; \mathbb{Z}_2)) = \\ &= 1 + \beta_1(M, \partial M; \mathbb{Z}_2).\end{aligned}$$

Lefschetz duality gives a natural isomorphism $H_1(M, \partial M; \mathbb{Z}_2) \cong H^2(M; \mathbb{Z}_2)$. Since the homology group $H_2(M; \mathbb{Z}_2)$ is finitely generated, then the vector spaces $H_2(M; \mathbb{Z}_2)$ and $H^2(M; \mathbb{Z}_2)$ are finite-dimensional and mutually dual. In particular, they have the same dimension. Hence, $\beta_2(M; \mathbb{Z}_2) = \beta_1(M, \partial M; \mathbb{Z}_2)$. □

Proof of Theorem 3.3. Applying Lemma 3.4 we have:

$$\beta_1(M, \mathbb{Z}_2) = \beta_2(M; \mathbb{Z}_2) + 1 - \chi(M) \geqslant \beta_0(\partial M; \mathbb{Z}_2) - \chi(M)$$

□

4 Class $\mathcal{M}_h$ of 3-manifolds for which the lower bound in Theorem 3.1 is achieved: minimal triangulations

Let $\mathcal{M}_h$ denote the set of connected compact 3-manifolds M with boundary having an ideal triangulation $\mathcal{T}$ with $\beta_1(M, \mathbb{Z}_2)$ ideal tetrahedra. By theorem 3.1, $\mathcal{T}$ is a minimal ideal triangulation of M. In this section we study the anatomy of $\mathcal{T}$.

4.1 Minimal triangulations

We introduce two infinite sets $\mathscr{T}_o$ and $\mathscr{T}_e$ of ideal triangulations. Let $\mathcal{T}$ be an ideal triangulation and e be its edge. We say that e is *even* (resp., *odd*) if each model face contains even (resp., odd) number of pre-images of e. The set $\mathscr{T}_e$ consists of all the ideal triangulations with at least two edges, one of which is odd, and the others are even. The set $\mathscr{T}_o$ consists of all the ideal triangulations with odd edges only. By definition, $\mathscr{T}_o \cap \mathscr{T}_e = \emptyset$.

Theorem 4.1. *Let $\mathcal{T}$ be an ideal triangulation of a connected compact 3-manifold M with boundary. Then the following are equivalent:*

- *$\mathcal{T}$ belongs to the union $\mathscr{T}_o \cup \mathscr{T}_e$.*
- *M belongs to $\mathcal{M}_h$ and $\mathcal{T}$ is minimal.*

Let us study in more detail ideal triangulations that have only odd and even edges.

Lemma 4.2. *Let $\mathcal{T}$ be an ideal triangulation having only odd and even edges. Then $\mathcal{T}$ is connected and belongs to the union $\mathscr{T}_o \cup \mathscr{T}_e$. Moreover, if $\mathcal{T}$ belongs to $\mathscr{T}_o$ then it has either one or three edges.*

Proof. Since each model face contains exactly three model edges, $\mathcal{T}$ has at least one odd edge. By definition, an odd edge is incident to every tetrahedron of $\mathcal{T}$. Thus $\mathcal{T}$ is connected. Further arguments are obvious. $\square$

Let $\mathcal{T}$ be an ideal triangulation of a connected compact 3-manifold M with boundary belonging to $\mathscr{T}_o \cup \mathscr{T}_e$. It follows that a given tetrahedron Δ in $\mathcal{T}$ falls into one of the following categories that are determined by an edge identification scheme of the corresponding model tetrahedron $\widetilde{\Delta}$.

Type 1: Δ has a single edge e, i.e. all the model edges of $\widetilde{\Delta}$ are identified to form e. In this case, the edge e must be odd in $\mathcal{T}$.

Type 2: Δ has two edges e_1, e_2 such that there are three model edges of $\widetilde{\Delta}$ with a common vertex that are identified to form e_1, and the other three model edges of $\widetilde{\Delta}$ are identified to form e_2. In this case, the edge e_1 must be even in $\mathcal{T}$, while the edge e_2 must be odd.

Type 3: Δ has two edges e_1, e_2 such that there is a pair of opposite model edges of $\widetilde{\Delta}$ that are identified to form e_2, and the other four model edges of $\widetilde{\Delta}$ are identified to form e_1. In this case, the edge e_1 must be even in $\mathcal{T}$, while the edge e_2 must be odd.

Type 4: Δ has three edges e_1, e_2, e_3 such that for each $i \in \{1, 2, 3\}$ there is pair of opposite model edges of $\widetilde{\Delta}$ that are identified to form e_i. In this case, each edge e_i must be odd in $\mathcal{T}$.

By Lemma 4.2, the set $\mathscr{T}_o$ can be divided into two pairwise disjoint subsets. One of them consists of one-edged ideal triangulations and is denoted by $\mathscr{T}_o^1$. The other consists of ideal triangulations with three edges and is denoted by $\mathscr{T}_o^2$.

By definition, any ideal triangulation $\mathcal{T}$ belonging to $\mathscr{T}_e$ has at least one even edge. It is convenient to divide even edges e of $\mathcal{T}$ into two types:

Type A: e is incident to tetrahedra of type 2 only.

Type B: e is incident to a tetrahedron of type 3.

There is a deep reason to divide the set $\mathscr{T}_e$ into two pairwise disjoint subsets $\mathscr{T}_e^1$ and $\mathscr{T}_e^2$. Denote by $\mathtt{w}(\mathcal{T})$ the number of even edges of $\mathcal{T}$ that have type B. Then $\mathcal{T}$ is in $\mathscr{T}_e^1$ if $\mathtt{w}(\mathcal{T}) = 0$. Otherwise $\mathcal{T}$ is in $\mathscr{T}_e^2$.

So we have the partition of $\mathscr{T}_o \cup \mathscr{T}_e$ into four subsets: $\mathscr{T}_o^1$, $\mathscr{T}_o^2$, $\mathscr{T}_e^1$, and $\mathscr{T}_e^2$. The following lemma summarises the information about this partition.

Lemma 4.3. *Let $\mathcal{T}$ be an ideal triangulation, then the following hold:*

- *$\mathcal{T}$ belongs to $\mathscr{T}_o^1$ if and only if all the tetrahedra in $\mathcal{T}$ are of type 1.*
- *$\mathcal{T}$ belongs to $\mathscr{T}_o^2$ if and only if all the tetrahedra in $\mathcal{T}$ are of type 4.*
- *$\mathcal{T}$ belongs to $\mathscr{T}_e^1$ if and only if all the tetrahedra in $\mathcal{T}$ are of types 1–2 and at least one of them is of type 2.*
- *$\mathcal{T}$ belongs to $\mathscr{T}_e^2$ if and only if all the tetrahedra in $\mathcal{T}$ are of types 1–3 and at least one of them is of type 3.*

4.2 Proof of Theorem 4.1

Let P be a special spine of M that is dual to $\mathcal{T}$. Recall that $\mathtt{d}(P)$ and $\mathtt{v}(P)$ denote the number of 2-components and the number of true vertices of P, respectively. Let $\alpha_0, \ldots, \alpha_{\mathtt{d}(P)-1}$ be the 2-components and $\gamma_0, \ldots, \gamma_{2\mathtt{v}(P)-1}$ be the edges of P. Consider a part of the chain complex of P with $\mathbb{Z}_2$-coefficients:

$$C_2 \xrightarrow{\partial} C_1.$$

Recall that $B_1 = \partial C_2$ is the group of 1-dimensional boundaries. We claim that the following are equivalent:

(a) $\mathcal{T}$ belongs to the union $\mathscr{T}_o \cup \mathscr{T}_e$.

(b) $\mathcal{T}$ has only odd and even edges.

(c) $\partial\alpha_i = 0$ or $\partial\alpha_i = \gamma_0 + \ldots + \gamma_{2\mathtt{v}(P)-1}$ for every $i \in \{0, \ldots, \mathtt{d}(P) - 1\}$.

(d) $\dim B_1 = 1$.

(e) $\mathtt{d}(P) = \beta_2(M, \mathbb{Z}_2) + 1$.

(f) $\mathtt{v}(P) = \beta_1(M, \mathbb{Z}_2)$.

(g) $\mathcal{T}$ consists of $\beta_1(M, \mathbb{Z}_2)$ tetrahedra.

(h) M belongs to $\mathcal{M}_h$ and $\mathcal{T}$ is minimal.

Indeed, the implication (b)$\Rightarrow$(a) is the statement of Lemma 4.2. The reverse implication (a)$\Rightarrow$(b) is clear by definition. The equivalences (b)$\Leftrightarrow$(c) and (f)$\Leftrightarrow$(g) come from the duality between P and $\mathcal{T}$. The implication (c)$\Rightarrow$(d) is evident, while the equality (2) implies the implication (d)$\Rightarrow$(c). The equivalence (d)$\Leftrightarrow$(e) is clear by (3), and (e)$\Leftrightarrow$(f) is a direct corollary of Lemma 3.2. And the final equivalence (g)$\Leftrightarrow$(h) is obtained by Theorem 3.1 and by the definition of $\mathcal{M}_h$. This completes the proof of the theorem.

5 Exact cover of $\mathcal{M}_h$ with four subclasses

The partition of $\mathscr{T}_o \cup \mathscr{T}_e$ induces a cover of $\mathcal{M}_h$ with four subsets $\mathcal{M}_o^1$, $\mathcal{M}_o^2$, $\mathcal{M}_e^1$ and $\mathcal{M}_e^2$. Where $\mathcal{M}_o^1$ (resp., $\mathcal{M}_o^2$, $\mathcal{M}_e^1$, $\mathcal{M}_e^2$) denote the set of connected compact 3-manifolds with boundary that posses an ideal triangulation from $\mathscr{T}_o^1$ (resp., $\mathscr{T}_o^2$, $\mathscr{T}_e^1$, $\mathscr{T}_e^2$). Now we show that this cover is exact.

Theorem 5.1. *The sets $\mathcal{M}_o^1$, $\mathcal{M}_o^2$, $\mathcal{M}_e^1$, and $\mathcal{M}_e^2$ are pairwise disjoint and cover $\mathcal{M}_h$.*

Let M belong to $\mathcal{M}_h$ and let $\mathcal{T}$ be a minimal ideal triangulation of M. Denote by $\mathtt{e}(\mathcal{T})$ the number of edges of $\mathcal{T}$. Now we establish the number of boundary components of M.

Lemma 5.2. *Let $\mathcal{T}$ be an ideal triangulation of a connected compact 3-manifold M with boundary. Then the following holds:*

1. *If $\mathcal{T}$ belongs to $\mathscr{T}_o$, then ∂M is connected.*
2. *If $\mathcal{T}$ belongs to $\mathscr{T}_e$, then ∂M has exactly $\mathtt{e}(\mathcal{T}) - \mathtt{w}(\mathcal{T})$ connected components.*

Proof. Note that there is a one-to-one correspondence between the connected components of ∂M and the vertices of $\mathcal{T}$. Suppose that under the assumptions of the lemma, $\mathcal{T}$ belongs to the union $\mathscr{T}_o \cup \mathscr{T}_e$.

To establish the conclusion we first prove three claims, each showing that, under certain conditions, some model vertices of a model tetrahedron are identified to the same vertex of $\mathcal{T}$.

Claim 1 If the three model edges incident to a model face σ are identified to form the same edge of $\mathcal{T}$, then the three model vertices incident to σ are identified to form the same vertex of $\mathcal{T}$.

Claim 2 If each pair of opposite model edges of a model tetrahedron $\widetilde{\Delta}$ are identified to form the same edge of $\mathcal{T}$, then all the model vertices of $\widetilde{\Delta}$ are identified to form the same vertex of $\mathcal{T}$.

Claim 3 Each tetrahedron of type 1, 3 or 4 has exactly one vertex.

Indeed, let the three model edges incident to a model face σ be identified to form an edge e of $\mathcal{T}$. Suppose that e has distinct endpoints that are denoted by u and v. It follows that each model edge incident to σ has endpoints on the pre-images of u and v, a contradiction. This proves Claim 1.

Let us prove Claim 2. Denote by Δ the tetrahedron in $\mathcal{T}$ that corresponds to $\widetilde{\Delta}$. On the contrary, suppose that there is an edge e in Δ that has distinct endpoints, say u and v. By hypothesis, there is a pair of opposite model edges in $\widetilde{\Delta}$ that are identified to form e. It follows that $\widetilde{\Delta}$ has exactly two vertices. This contradicts our hypothesis and completes the proof of Claim 2.

Finally, Claim 3 follows directly from Claim 1 and Claim 2.

Now we return to the proof of Lemma 5.2. The statement 1 follows from Lemma 4.3 and Claim 3.

Now let $\mathcal{T}$ belongs to $\mathscr{T}_e$. By definition, $\mathcal{T}$ has at least two edges, one of which, say e_0, is odd, and the others are even. By Lemma 4.3, all the tetrahedra in $\mathcal{T}$ are of types 1–3 and at least one of them is of type 2 or 3. Since all tetrahedra of types 1–3 are incident of e_0, applying Claim 1 (to a tetrahedron of type 2) or Claim 3 (to a tetrahedron of type 1 or 3) we obtain that the ends of e_0 coincide (denote this vertex by v_0. Recall that every even edge of type B is incident to a tetrahedron of type 3. Hence, by Claim 3, its ends coincide with v_0 too. Finally, one endpoint of an even edge of type A is v_0, while the other is a degree one vertex. It implies that $\mathcal{T}$ has exactly $\mathtt{e}(\mathcal{T}) - \mathtt{w}(\mathcal{T})$ vertices. This completes the proof. $\square$

Proof of Theorem 5.1. Let $\mathcal{T}_o^1$, $\mathcal{T}_o^2$, $\mathcal{T}_e^1$, and $\mathcal{T}_e^2$ be ideal triangulations belonging to $\mathscr{T}_o^1$, $\mathscr{T}_o^2$, $\mathscr{T}_e^1$, and $\mathscr{T}_e^2$, respectively. We need to prove that the corresponding manifolds M_o^1, M_o^2, M_e^1, and M_e^2 are pairwise non-homeomorphic. Since these ideal triangulations are minimal (Theorem 4.1), we may assume they consist of the same number, say n, of tetrahedra; otherwise the manifolds are non-homeomorphic.

Recall if we are given an ideal triangulation $\mathcal{T}$ of a connected compact 3-manifold M with boundary, we obtain the dual special spine P of M. As we noticed above in the proof of Lemma 3.2, $\chi(M) = \chi(P) = \mathtt{d}(P) - \mathtt{v}(P)$. Since true vertices, edges and 2-components of P correspond to tetrahedra, faces and edges of $\mathcal{T}$, we have $\chi(M) = \mathtt{e}(\mathcal{T}) - \mathtt{t}(\mathcal{T})$. Then we apply the formula to show that M_o^1 is different from M_o^2, M_e^1, and M_e^2. In fact, $\chi(M_o^1) = 1 - n$, while $\chi(M_o^2) = 3 - n$, $\chi(M_e^1) = \mathtt{e}(\mathcal{T}_e^1) - n$, and $\chi(M_e^2) = \mathtt{e}(\mathcal{T}_e^2) - n$, where by definition $\mathtt{e}(\mathcal{T}_e^1) \geq 2$ and $\mathtt{e}(\mathcal{T}_e^2) \geq 2$. Hence, $M_o^1 \cap M_o^2 = \emptyset$, $M_o^1 \cap M_e^1 = \emptyset$, and $M_o^1 \cap M_2^2 = \emptyset$.

Now there are three cases to consider, depending on the equality between the Euler characteristic.

First, if $\chi(M_o^2) = \chi(M_e^1)$, then $\mathtt{e}(\mathcal{T}_e^1) = 3$. By Lemma 5.2, ∂M_e^1 has three boundary components, while ∂M_o^2 has only one. Hence, $M_o^2 \cap M_e^1 = \emptyset$.

Second, if $\chi(M_e^1) = \chi(M_e^2)$, then $\mathtt{e}(\mathcal{T}_e^1) = \mathtt{e}(\mathcal{T}_e^2)$. By Lemma 5.2, ∂M_e^1 has more boundary components than ∂M_e^2. Hence, $M_e^1 \cap M_e^2 = \emptyset$.

Consider the last case assuming $\chi(M_o^2) = \chi(M_e^2)$. It follows that $\mathtt{e}(\mathcal{T}_e^2) = 3$. Now we switch from the ideal triangulations $\mathcal{T}_o^2$ and $\mathcal{T}_e^2$ to its dual special spines, denoted P and Q, respectively.

To prove that the manifolds M_o^2, M_e^2 are non-homeomorphic, we use the ε-invariant of Matveev – Ovchinnikov – Sokolov (see ???), which is the homologically trivial part of the order 5 Turaev–Viro invariant. We give the definition of the ε-invariant following [9]. Let R be a special spine of a connected compact 3-manifold M with boundary. Denote by $\mathcal{F}(R)$ the set of all simple subpolyhedra of R including R and the empty set. Set $\varepsilon = (1+\sqrt{5})/2$, a solution of the equation $\varepsilon^2 = \varepsilon + 1$. With each $K \in \mathcal{F}(R)$ we associate its *ε-weight* by the formula

$$w_\varepsilon(K) = (-1)^{\mathtt{v}(K)} \varepsilon^{\chi(K) - \mathtt{v}(K)},$$

where $\mathtt{v}(K)$ is the number of true vertices of K and $\chi(K)$ is its Euler characteristic. Set

$$t(M) = \sum_{K \in \mathcal{F}(R)} w_\varepsilon(K).$$

As shown in [9], $t(M)$ is an invariant of M.

To complete the proof of the Theorem we show that $t(M_o^2) \neq t(M_e^2)$. Let us calculate $t(M_o^2)$ by its special spine P. By the compactness of a simple subpolyhedron if it contains a point of a 2-component, then it contains the whole of it. Thus, to describe a simple subpolyhedron of P it is enough to indicate which 2-components of P it includes (its triple points and true vertices then will be determined uniquely). Since $\mathcal{T}_o^2$ has exactly three edges, and each of these edges is odd, P has exactly three 2-components, denoted ξ_1, ξ_2, ξ_3, and each boundary curve $\partial\xi_i$ passes exactly once along each edge of P. Hence, $\mathcal{F}(P) = \{\emptyset, P, P_1, P_2, P_3\}$, where $P_i = P \setminus \xi_i$. It is easy to see that $\mathtt{v}(P) = n$, $\chi(P) = 3 - n$, $\mathtt{v}(P_i) = 0$, and $\chi(P_i) = 2 - n$, $1 \leq i \leq 3$. Summing up the ε-weights $w_\varepsilon(\emptyset) = 1$, $w_\varepsilon(P) = (-1)^n \varepsilon^{3-2n}$, and $w_\varepsilon(P_i) = \varepsilon^{2-n}$, we get

$$t(M_o^2) = 1 + (-1)^n \varepsilon^{3-2n} + 3\varepsilon^{2-n}.$$

Now we calculate $t(M_e^2)$ by the special spine Q. Since $\mathcal{T}_e^2$ has exactly one odd edge and two even edges, Q has exactly three 2-components, denoted ξ_0, ξ_1, and ξ_2. Let ξ_0 corresponds to the odd edge, while ξ_1 and ξ_2 correspond to the even ones. We claim Q has exactly three proper simple subpolyhedra. Indeed, two of these polyhedra are connected closed surfaces, denoted Q_1, Q_2, such that each Q_i, $i = 1$ or 2, contains ξ_i and do not contain the other 2-components of Q. Therefore, $Q_1 \cap Q_2 = \emptyset$. The third polyhedron, denoted Q_3, is the union $Q_1 \cup Q_2$ (i.e. a closed surface too).

So we have $\mathcal{F}(Q) = \{\emptyset, Q, Q_1, Q_2, Q_3\}$, $\mathtt{v}(Q) = n$, $\chi(Q) = 3-n$, and $\mathtt{v}(Q_i) = 0$, $1 \leq i \leq 3$. Summing up the ε-weights we get

$$t(M_e^2) = 1 + (-1)^n \varepsilon^{3-2n} + \sum_{i=1}^{3} \varepsilon^{\chi(Q_i)}.$$

For each i, $1 \leq i \leq 3$, we claim $\chi(Q_i) > 2 - n$. Indeed, let v_i^+, e_i^+, and d_i^+ denote the number of true vertices, edges, and 2-components of P, respectively, belonging to Q_i. By construction, $d_1^+ = d_2^+ = 1$ and $d_3^+ = 2$. We set $v_i^- = \mathtt{v}(Q) - v_i^+$ and $e_i^- = \mathtt{e}(Q) - e_i^+$. Since $\mathtt{e}(Q) = 2\mathtt{v}(Q)$, we have

$$\chi(Q_i) = v_i^+ - e_i^+ + d_i^+ = (2 - v) + (e_i^- - v_i^-) + (d_i^+ - 2).$$

The inequality $e_i^- - v_i^- > 1$ can be easily proved by induction on v_i^- by using the fact that the surface Q_i does not contain all the edges of the special spine Q. It follows that $\chi(Q_i) > 2 - n$. Hence, $3\varepsilon^{2-n} < \sum_{i=1}^{3} \varepsilon^{\chi(Q_i)}$, and we have $t(M_o^2) \neq t(M_e^2)$. This then completes the proof. $\square$

6 Hyperbolicity of manifolds in $\mathcal{M}_h$

We say that a compact 3-manifolds is *hyperbolic* if, after removing the boundary tori and Klein bottles, we get a complete riemannian manifold of constant sectional curvature $K_\sigma = -1$ with finite volume and totally geodesic boundary.

Theorem 6.1. *Each M in $\mathcal{M}_h$, with a few exceptions, is a hyperbolic manifold with totally geodesic boundary components and some cusps.*

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