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Hunting zeros of Dirichlet series by linear algebra. I

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Abstract: To be able to use the linear algebra for the study of certain Dirichlet series $D(s) = \sum_{n=1}^{\infty} d_n n^{-s}$ we consider related *Dirichlet series with independent exponents*, $D_\infty(s_1, s_2, \dots) = \sum_{n=1}^{\infty} d_n n^{-s_n}$. To calculate approximate values of the initial $s_1, \dots, s_N$ we consider many approximations of $D(s)$ by finite Dirichlet series, $D_{N,m}(s) = \sum_{n=1}^{N} d_{N,m,n} n^{-s}$, and at first solve linear system $\sum_{n=1}^{N} d_{N,m,n} x_n = 0$. If in its solution $x_n \approx n^{-z}$ for certain z, then we may hope that this z is close to a zero of $D(s)$. However, *a priori*, such z need not exist at all.

The paper presents a numerical example with the alternating zeta function $\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$ in the role of $D(s)$ and specially defined approximations $D_{N,m}(s)$. In our case such z does exist, and, moreover, it is very close to a zero of $\eta(s)$; unfortunately, this zero is not a zero of the zeta function.

Key words: alternating zeta function, finite Dirichlet series, linear system.

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1 Introduction

Finding (an approximate value of) a certain zero z of (the function defined by) a given Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} d_n n^{-s} \tag{1.1}$$

is, in general, not an easy task. This paper presents a numerical experiment of reaching this goal by tools of linear algebra.

The (crazy) key idea is very trivial: *formally, $D(s)$ is a linear function of infinitely many arguments n^{-s}.* So we consider corresponding *Dirichlet series with independent exponents*

$$D_\infty(s_1, s_2, \dots) = \sum_{n=1}^{\infty} d_n n^{-s_n}. \tag{1.2}$$

Also we introduce new unknowns, x_1, x_2, $\dots$, and consider corresponding linear function

$$L_\infty(x_1, x_2, \dots) = \sum_{n=1}^{\infty} d_n x_n. \tag{1.3}$$

Formally, every solution $s = z$ of the equation

$$D(s) = 0 \tag{1.4}$$

produces the solution

$$s_1 = \dots = s_n = \dots = z \tag{1.5}$$

of the equation

$$D_\infty(s_1, s_2, \dots) = 0, \tag{1.6}$$

and the solution

$$x_n = n^{-z}, \qquad n = 1, 2, \dots \tag{1.7}$$

of the equation

$$L_\infty(x_1, x_2, \dots) = 0. \tag{1.8}$$

An assignment of values to x_1, x_2, $\dots$ of the form (1.7) (with any z) will be called *one-dimensional*, that is, generated by a single number, and this number, z, will be called the *seed* of the assignment.

Instead of solving the exact equation (1.4) we can approximate the infinite series (1.1) via a finite one,

$$D_N(s) = \sum_{n=1}^{N} d_{N,n} n^{-s} \approx \sum_{n=1}^{\infty} d_n n^{-s} = D(s). \tag{1.9}$$

If z is a zero of $D(s)$,

$$D(z) = 0, \tag{1.10}$$

and if $D_N(s)$ is a good approximation to $D(s)$ (in the vicinity of z), then we have the approximate equality

$$D_N(z) \approx 0. \tag{1.11}$$

Instead of this we can try to find $\tilde{z}$ satifying the exact equality

$$D_N(\tilde{z}) = 0 \tag{1.12}$$

in the hope that such a $\tilde{z}$ will be close to z.

We can do similar things with $L_\infty(x_1, x_2, \dots)$. First, we cut this infinite sum and consider the linear form

$$L_N(x_1, \dots, x_N) = \sum_{n=1}^{N} d_{N,n} x_n \tag{1.13}$$

(with coefficients taken from (1.9)) and the corresponding linear equation

$$L_N(x_1, \dots, x_N) = 0. \tag{1.14}$$

Of course, this single homogeneous equation has infinitely many solutions, including the trivial one,

$$x_1 = \dots = x_N = 0. \tag{1.15}$$

The crucial idea is as follows: *while we cannot change our originally given series* (1.1)*, we can construct many different finite sums (for the same value of N) giving good approximations to (the function defined by) our infinite series, and impose many conditions of the form* (1.14)*.*

To eliminate the degenerated solution (1.15) we assume that $d_{N,1} \neq 0$ and impose the non-homogeneous equation

$$x_1 = 1 \tag{1.16}$$

(this condition complies with (1.7)).

Further, we need $N - 1$ versions of finite Dirichlet series giving good approximations to $D(s)$,

$$D_{N,m}(s) = \sum_{n=1}^{N} d_{N,m,n} n^{-s} \approx D(s), \qquad m = 0, \dots, N-2. \tag{1.17}$$

Using corresponding linear forms

$$L_{N,m}(x_1,\dots,x_N)=\sum_{n=1}^{N} d_{N,m,n}x_n, \qquad m=0,\dots,N-2, \tag{1.18}$$

we define numbers $\tilde{x}_1$, ..., $\tilde{x}_N$ as the solution of the system consisting of N linear equations, namely, of equation (1.16) and equations

$$L_{N,m}(x_1,\dots,x_N)=0, \qquad m=0,\dots,N-2. \tag{1.19}$$

We have applied no efforts to get a solution which would be approximately one-dimensional. Now we can look to what extent our solution posses such similarity.

Every one-dimensional solution is completely multiplicative, that is,

$$x_k x_l = x_{kl}. \tag{1.20}$$

Question I. *Does our solution satisfy approximate counterparts of equations* (1.20)*, that is,*

$$\tilde{x}_k\tilde{x}_l \approx \tilde{x}_{kl}? \tag{1.21}$$

From (1.7) we get that

$$\mathrm{Re}(z)=-\log_2(|x_2|)=-\log_3(|x_3|)=\dots. \tag{1.22}$$

Question II. *Is it true that*

$$-\log_2(|\tilde{x}_2|)\approx\dots\approx-\log_N(|\tilde{x}_N|)? \tag{1.23}$$

The most intriguing question is whether numbers $\tilde{x}_2,\dots,\tilde{x}_N$ carry any information about the zeros of our initial Dirichlet series $D(s)$. It is natural to search for them near numbers $\tilde{z}_2$, ..., $\tilde{z}_N$ such that

$$\tilde{x}_n = n^{-\tilde{z}_n}. \qquad n=2,\dots,N. \tag{1.24}$$

This condition defines the $\tilde{z}$'s only up to integer multiples of $2\pi/\log(n)$,

$$\tilde{z}_n=-\left(\log_n(|\tilde{x}_n|)+\mathrm{i}\left(\frac{\arg(\tilde{x}_n)}{\log(n)}+j_n\frac{2\pi}{\log(n)}\right)\right). \tag{1.25}$$

Thus the knowledge of a single $\tilde{x}_n$ is not sufficient for pinpointing a candidate for an approximation to a zero of $D(s)$. Luckily, we have a bunch of $\tilde{x}$'s, and this might help.

Question III. *Is there a way to calculate, on the basis of* $\tilde{x}_1, \dots, \tilde{x}_N$*, some integers* $j_2, \dots, j_N$ *such that all numbers* (1.25) *would be close to the same zero of the series* (1.1)*?*

Of course, the answers to the above questions depend on our choice of finite approximations (1.17). Below we present a case when the answers to Questions I and II surprisingly are positive, and indicate a way to calculate an approximation to a sought-for zero of $D(s)$.

2 A numerical example

For the role of (1.1) we take the *alternating zeta function* (known also as *Dirichlet eta function*):

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2 \times 2^{-s}) \sum_{n=1}^{\infty} n^{-s} = (1 - 2 \times 2^{-s})\zeta(s). \tag{2.1}$$

In terms of this function the celebrated *functional equation* can be written as

$$\xi(1-s) = \xi(s) \tag{2.2}$$

where

$$\xi(s) = h(s)\eta(s), \tag{2.3}$$

$$h(s) = \frac{\pi^{-\frac{s}{2}}(s-1)\Gamma(1+\frac{s}{2})}{1 - 2 \times 2^{-s}}. \tag{2.4}$$

Our definition of a finite approximation (1.9) is inspired[1] by *Hamburger theorem* [1]. Loosely speaking, this theorem tells us that among all Dirichlet series the (alternating) zeta function is (up to a constant factor) defined by the functional equation (plus some other extra condition).

Let us consider the formal counterpart of the xi function,

$$\xi_N(s) = h(s) D_N(s). \tag{2.5}$$

[1] Similar, but slightly different, method for constructing such approximations was presented in author's paper [5].

If $D_N(s)$ and $D_N(1-s)$ give good approximations to $\eta(s)$ and $\eta(1-s)$ respectively, then

$$\xi_N(1-s) \approx \xi_N(s). \tag{2.6}$$

We shall demand that in the case of our finite approximations the imaginary parts of the left and right hand sides in (2.6) should coincide, that is

$$\mathrm{Im}\big(\xi_N(1-s)\big) = \mathrm{Im}\big(\xi_N(s)\big), \tag{2.7}$$

for a specially selected finite set of values of s.

Clearly, (2.7) is trivially satified by

$$d_{N,1} = \cdots = d_{N,N} = 0. \tag{2.8}$$

To avoid such a degeneration we impose *condition of normalization*,

$$d_{N,1} = 1. \tag{2.9}$$

In order to rewrite condition (2.7) as a linear equation, we represent the coefficients of $D_N(s)$ as

$$d_{N,n} = d^{\mathrm{R}}_{N,n} + \mathrm{i}d^{\mathrm{I}}_{N,n} \tag{2.10}$$

with real

$$d^{\mathrm{R}}_{N,1},\ \ldots,\ d^{\mathrm{R}}_{N,n}, \qquad d^{\mathrm{I}}_{N,1},\ \ldots,\ d^{\mathrm{I}}_{N,n}. \tag{2.11}$$

Respectively, let

$$D^{\mathrm{R}}_N(s) = \sum_{n=1}^{N} d^{\mathrm{R}}_{N,n} n^{-s}, \qquad D^{\mathrm{I}}_N(s) = \sum_{n=1}^{N} d^{\mathrm{I}}_{N,n} n^{-s}. \tag{2.12}$$

With this notation conditions (2.9) and (2.7) can be written as

$$d^{\mathrm{R}}_{N,1} = 1, \qquad d^{\mathrm{I}}_{N,1} = 0, \tag{2.13}$$

and

$$\xi^{\mathrm{I}}_N(1-s) = \xi^{\mathrm{I}}_N(s) \tag{2.14}$$

where

$$\xi^{\mathrm{I}}_N(s) = \mathrm{Re}\big(h(s)\big)D^{\mathrm{I}}_N(s) + \mathrm{Im}\big(h(s)\big)D^{\mathrm{R}}_N(s). \tag{2.15}$$

Having fixed the value of $d^{\mathrm{R}}_{N,1}$ and $d^{\mathrm{I}}_{N,1}$ by (2.13), we still have in (2.11) other $2N-2$ coefficients at our disposal. In order to determine them we select values of two parameters, a and δ, and supplement (2.13) by the following $2N-2$ copies of (2.14):

$$\xi^{\mathrm{I}}_N(1-(a+k\delta)) = \xi^{\mathrm{I}}_N(a+k\delta), \qquad k = 0, \ldots, 2N-3. \tag{2.16}$$

Having solved this system we get an instance of finite series (1.9) which will be denoted as

$$D_{a,\delta,N}(s) \;=\; \sum_{n=1}^{N}\left(d^{\mathrm{R}}_{a,\delta,N,n} - \mathrm{i}d^{\mathrm{I}}_{a,\delta,N,n}\right)n^{-s} = \sum_{n=1}^{N} d_{a,\delta,N,n}n^{-s}. \tag{2.17}$$

Respectively, we get corresponding instance of linear form (1.13), namely,

$$L_{a,\delta,N}(x_1,\ldots,x_N) \;=\; \sum_{n=1}^{N} d_{a,\delta,N,n}x_n. \tag{2.18}$$

The $N-1$ linear forms (1.18) will be instances of $L_{a,\delta,N}(x_1,\ldots,x_N)$ with different values of a. Namely, we select a value of one more parameter, Δ, and define

$$L_{a,\Delta,\delta,N,m}(x_1,\ldots,x_N) = L_{a+m\Delta,\delta,N}(x_1,\ldots,x_N). \tag{2.19}$$

At last, we define the $\tilde{x}$'s as the solution of the system consisting of equation (1.16) and the following instance of equations (1.19):

$$L_{a,\Delta,\delta,N,m}(x_1,\ldots,x_N) = 0, \qquad m = 0,\ldots,N-1. \tag{2.20}$$

Let us consider a numerical example with the following values of the parameters:

$$a = 30\mathrm{i}, \qquad \Delta = \frac{1}{100}, \qquad \delta = \frac{1}{20000}, \qquad N = 50. \tag{2.21}$$

Table 1 presents the values of the coefficients (2.11), and Table 2 demonstrates that corresponding finite Dirichlet series produces good approximations to $\eta(s)$ for large range of s.

It can be checked that the solution $\tilde{x}_1,\ldots,\tilde{x}_N$ of the system consisting of equations (1.16) and (1.19) satisfies (1.21) and (1.23) quite well. Moreover, it can be seen from Table 3 that for all n the value of $-\log_n(|\tilde{x}_n|)$ is very close to 1. This number is equal to the real part of certain zeros of $\eta(s)$. Namely, the zeros of this function are of three kinds:

- negative even integers, known as the *trivial zeros of the zeta function*;
- other zeros of the zeta function called *non-trivial*;
- the *proper zeros the eta function*, that is the zeros of the *Euler factor* in (2.1),

  $$1 - 2\times 2^{-s}, \tag{2.22}$$

  which are of the form

  $$\mathrm{w}_k = 1 + \frac{2\pi k}{\log(2)}\mathrm{i}, \qquad k = \pm 1,\ \pm 2,\ \ldots \tag{2.23}$$

Thus the data from Table 3 gives the cue that possibly

$$\tilde{x}_n \approx n^{-\mathrm{w}_k} \tag{2.24}$$

for some k. Testing a few initial values of k reveals that indeed $\tilde{x}_n$ is very close to $n^{-\mathrm{w}_3}$ – see Table 4. Table 5 shows how well numbers (1.25) (with suitable choice of the j's) approximate w_3. We observe that in both tables the approximation for $n = 2,\ 4,\ 8,\ 16$, and 32 is much better than when n is not a power of 2.

This numerical example poses at least two enigmas:

- why the solution of our linear system is so close to a one-dimensional?
- why the seed of this one-dimensional solution is a proper zero of $\eta(s)$?

As for specifical choice of w_3, a plausible answer is as follows. For constructing our approximations $D_{a,\delta,N}(s)$ we used the value $a = 30\mathrm{i}$, and among all proper zeros of $\eta(s)$ the zero

$$\mathrm{w}_3 = 1 + \frac{6\pi}{\log(2)}\mathrm{i} = 1 + 27.194160850963162857766...\mathrm{i}. \tag{2.25}$$

is the nearest to a. But the non-trivial (z)eta zero

$$1/2 + 30.424876125859513210311897530 6...\mathrm{i} \tag{2.26}$$

is closer to a, why we haven't got this zero as the seed?

It would very interesting to get in this way a non-trivial zeta zero, but so far the author was not able to do it.

3 Determining the imaginary part of the seed

In the above consideration the seed w_3 was found by trial and error method. It is desirable to have a direct way to calculate the seeds of an arbitrary (approximately) one-dimensional solutions.

If we know two exact numbers, x_2 and x_3, such that

$$x_2 = 2^{-z} \text{ and } x_3 = 3^{-z} \tag{3.1}$$

for some (unknown to us) number z, then we can determine its imaginary part via the unique solution in integers j_2 and j_3 of the equation

$$\frac{\arg(x_2)}{\log(2)} + j_2\frac{2\pi}{\log(2)} - \frac{\arg(x_3)}{\log(3)} - j_3\frac{2\pi}{\log(3)} = 0. \tag{3.2}$$

Unfortunately, if we know only approximate values,

$$\tilde{x}_2 \approx 2^{-z} \text{ and } \tilde{x}_3 \approx 3^{-z}, \tag{3.3}$$

then, on the one hand, the left hand side of the analogous equation

$$\frac{\arg(\tilde{x}_2)}{\log(2)} + j_2 \frac{2\pi}{\log(2)} - \frac{\arg(\tilde{x}_3)}{\log(3)} - j_3 \frac{2\pi}{\log(3)} = 0. \tag{3.4}$$

may never be exact zero, and, on the other hand, it can be made arbitrary small (in absolute value) by a suitable choice of j_2 and j_3.

Luckily, the functional equations of the gamma and zeta functions allows us to get by when we know many approximate values,

$$\tilde{x}_n \approx n^{-z}, \qquad n = 2, \dots, N, \tag{3.5}$$

for certain (unown to us) $z = \sigma + \mathrm{i}t$. An approximation $\tilde{z} = \tilde{\sigma} + \mathrm{i}\tilde{t}$ can be found in the following way.

First of all, we put

$$\tilde{\sigma} = -\log_2(|\tilde{x}_2|). \tag{3.6}$$

In order to determine $\tilde{t}$, we multiply left and right hand sides of two copies of the functional equation, namely,

$$4\pi(-3+2\mathrm{i}t)\Gamma\big(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}\big)\zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big) = 4\pi^{\mathrm{i}t}(1-2\mathrm{i}t)\Gamma\big(\tfrac{7}{4}-\tfrac{\mathrm{i}t}{2}\big)\zeta\big(\tfrac{3}{2}-\mathrm{i}t\big) \tag{3.7}$$

and its congruent. We get the identity

$$\begin{aligned} 16\pi^2(9+4t^2)\Gamma\big(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}\big)\Gamma\big(\tfrac{3}{4}-\tfrac{\mathrm{i}t}{2}\big)\zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big)\zeta\big(+\tfrac{1}{2}+\mathrm{i}t\big) = \\ 16(1+4t^2)\Gamma\big(\tfrac{7}{4}-\tfrac{\mathrm{i}t}{2}\big)\Gamma\big(\tfrac{7}{4}+\tfrac{\mathrm{i}t}{2}\big)\zeta\big(\tfrac{3}{2}-\mathrm{i}t\big)\zeta\big(\tfrac{3}{2}+\mathrm{i}t\big). \end{aligned} \tag{3.8}$$

By the functional equation of the gamma function

$$(9+4t^2)\Gamma\big(\tfrac{3}{4}+\tfrac{\mathrm{i}t}{2}\big)\Gamma\big(\tfrac{3}{4}-\tfrac{\mathrm{i}t}{2}\big) = 16\Gamma\big(\tfrac{7}{4}-\tfrac{\mathrm{i}t}{2}\big)\Gamma\big(\tfrac{7}{4}+\tfrac{\mathrm{i}t}{2}\big), \tag{3.9}$$

so we can cancel these factors in (3.8) and get the identity

$$16\pi^2\zeta\big(-\tfrac{1}{2}+\mathrm{i}t\big)\zeta\big(+\tfrac{1}{2}+\mathrm{i}t\big) = (1+4t^2)\zeta\big(\tfrac{3}{2}-\mathrm{i}t\big)\zeta\big(\tfrac{3}{2}+\mathrm{i}t\big). \tag{3.10}$$

Having at our disposal the coefficients of an approximation (1.9), we can calculate approximate value of the $\zeta(s)$ when $\mathrm{Im}(s) = t$, namely, for a real ς

$$\zeta(\varsigma+\mathrm{i}t) \approx \frac{\sum_{n=1}^{N} d_{N,n}\tilde{x}_n n^{\tilde{\sigma}-\varsigma}}{1-2\tilde{x}_2 2^{\tilde{\sigma}-\varsigma}}, \tag{3.11}$$

$$\zeta(\varsigma - \mathrm{i}t) = \overline{\zeta(\varsigma + \mathrm{i}t)} \tag{3.12}$$

where $\overline{z}$, as usual, denotes the complex conjugate of z. Replacing the exact values of the zeta function in (3.10) by such approximations, we get a quadratic equation; two its solutions, $\tilde{t}_1$ and $\tilde{t}_2$, differ only be the sign and are candidates for the value of $\tilde{t}$.

To avoid this ambiguity (and to stay in the framework of linear algebra) one can rewrite (3.10) as

$$1 + 4t^2 = \frac{16\pi^2\zeta\left(-\frac{1}{2} + \mathrm{i}t\right)\zeta\left(+\frac{1}{2} + \mathrm{i}t\right)}{\zeta\left(\frac{3}{2} - \mathrm{i}t\right)\zeta\left(\frac{3}{2} + \mathrm{i}t\right)} \tag{3.13}$$

and differentiate the both sides; this gives the identity

$$t = \left(\frac{2\pi^2\zeta\left(-\frac{1}{2} + \mathrm{i}t\right)\zeta\left(+\frac{1}{2} + \mathrm{i}t\right)}{\zeta\left(\frac{3}{2} - \mathrm{i}t\right)\zeta\left(\frac{3}{2} + \mathrm{i}t\right)}\right)'. \tag{3.14}$$

Derivatives of the zeta function can be calculated similar to (3.11) and (3.12):

$$\zeta'(\varsigma + \mathrm{i}t) \approx \frac{-\sum_{n=1}^{N} d_{N,n}\tilde{x}_n \log(n) n^{\tilde{\sigma}-\varsigma}}{1 - 2\tilde{x}_2 2^{\tilde{\sigma}-\varsigma}} - \frac{2\tilde{x}_2 \ln(2) 2^{\tilde{\sigma}-\varsigma} \sum_{n=1}^{N} d_{N,n}\tilde{x}_n n^{\tilde{\sigma}-\varsigma}}{(1 - 2\tilde{x}_2 2^{\tilde{\sigma}-\varsigma})^2}, \tag{3.15}$$

$$\zeta'(\varsigma - \mathrm{i}t) = \overline{\zeta'(\varsigma + \mathrm{i}t)}. \tag{3.16}$$

Replacing exact values of the zeta function and its derivatives in (3.14) by such approximations we get the desired value of $\tilde{t}$.

Let us continue our numerical example from the previous section with parameters (2.21). For the role of $d_{N,n}$ in (3.11) and (3.15) we take $d_{a,\delta,N,n}$. This gives the following values for the imaginary part of the seed:

- by solving corresponding quadratic equation

$$\begin{aligned} \tilde{t}_{1,2} &= \pm 27.194162084133\ldots && (3.17)\\ &= \pm(\mathrm{Im}(\mathrm{w}_3) + 1.2331\ldots \cdot 10^{-6}); && (3.18) \end{aligned}$$

- by solving corresponding linear equation

$$\begin{aligned} \tilde{t} &= 27.1940973709788\ldots && (3.19)\\ &= \mathrm{Im}(\mathrm{w}_3) - 6.3479\ldots \cdot 10^{-5}. && (3.20) \end{aligned}$$

4 Conclusion

We have observed an interesting phenomenon: the solution of a system of linear equations (coefficients of which were taken from particular finite Dirichlet series approximating the alternating zeta function $\eta(s)$) is almost one-dimensional, that is, the values of all the unknowns are very close the same power of numbers 1, 2, ..., moreover, this power is a non-real zero of $\eta(s)$.

This is an example of what the author suggests to call *nearby property*[2] *of the (z)eta function*. This is a rather informal notion; in general, a nearby property of an infinite Dirichlet series is a property of akin to it certain finite Dirichlet series. Due to the finite number of summands, nearby properties typically are expressed via inaccurate, imprecise equalities.

Nearby properties can be classified as *inherited properties* and *own properties*. As an example of an inherited nearby property we can indicate the approximate equality (2.6). The phenomenon demonstrated in this paper is clearly an own nearby property because it is formulated in terms of particular approximations (2.17) and need not manifest itself for another kind of approximations.

Other examples of own nearby properties of the zeta function can be found in [2]–[7].

References

[1] H. Hamburger. Über die Riemannsche Funktionalgleichung der ζ-Funktion (Zweite Mitteilung). *Mathematische Zeitschrift*, 11:224–245, 1921.

[2] Yu. Matiyasevich. Calculation of Riemann's zeta function via interpolating determinants. Preprint of Max Planck Institute for Mathematics in Bonn, 18 (2013), 31 pages; `http://www.mpim-bonn.mpg.de/preblob/5368`, `https://logic.pdmi.ras.ru/~yumat/talks/bonn_2013/5368.pdf`.

[3] G. Beliakov and Yu. Matiyasevich. Approximation of Riemann's zeta function by finite Dirichlet series: a multiprecision numerical approach. *Exp. Math.*, 24(2):150–161, 2015. DOI 10.1080/10586458.2014.976801 (see also `http://arxiv.org/abs/1402.5295`).

[4] Ю. В. Матиясевич. Дзета-функция Римана и конечные ряды Дирихле. *Алгебра и анализ*, 27(6):174–198, 2015. `http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=aa&paperid=1472&option_lang=rus`.

[2] In Russian: *близлежащее свойство*

Translated as: Yu. V. Matiyasevich, Riemann's zeta function and finite Dirichlet series, *St. Petersburg Math. J.*, 27:6 (2016), 985–1002. DOI: 10.1090/spmj/F1431.

[5] Yuri Matiyasevich. Computational aspects of Hamburger's theorem. Pages 195—223 in: *Algorithms and Complexity in Mathematics, Epistemology, and Science*, eds. Fillion N., Corless R., and Kotsireas I., Springer, New York, NY, 2019, ISBN 978-1-4939-9051-1. DOI: 10.1007/978-1-4939-9051-1_8. See also *POMI Preprints* 01/2018, DOI: 10.13140/RG.2.2.14495.46247.

[6] Yu. V. Matiyasevich. Crop circles drawn by Riemann's zeta function and some other its nearby properties, *POMI Preprints*, 2, 2019, 74 pages. http://www.pdmi.ras.ru/preprint/2019/19-02.html, DOI: 10.13140/RG.2.2.11612.03200.

[7] Yu. Matiyasevich. WWW personal journal, https://logic.pdmi.ras.ru/~yumat/personaljournal/finitedirichlet.

n	$d^{\mathrm{R}}_{a,\delta,N,n}$	$d^{\mathrm{I}}_{a,\delta,N,n}$
1	1	0
2	-0.9999999999999999999987384...	0.0000000000000000000009610...
3	0.9999999999999999999451356...	0.0000000000000000000506117...
4	-1.0000000000000000039063388...	0.0000000000000000016192825...
5	0.9999999999999998949792440...	0.0000000000000001562155195...
6	-0.9999999999999967398113182...	0.0000000000000049491614266...
7	1.0000000000001121894356317...	-0.0000000000000700126979112...
8	-1.0000000000015136088002456...	-0.0000000000015571747562656...
9	0.9999999999885648646288825...	0.0000000000249693457260385...
10	-0.9999999997215215614914186...	-0.0000000000139179086259024...
11	0.9999999986510053535937505...	-0.0000000019251016515142514...
12	-1.0000000055057478774135872...	0.0000000159674499265889323...
13	1.0000000989562982641844183...	-0.0000000364321350166636839...
14	-1.0000005143313795505609993...	-0.0000002685595829422189918...
15	1.0000008517280681589695984...	0.0000027120159940242184644...
16	-0.9999947401670473246914846...	-0.0000113290843549299284333...
17	0.9999552617857702784785804...	0.0000211425096931070581812...
18	-0.9998265232881319931823219...	0.0000371204541551598688474...
19	0.9996073347777870598431287...	-0.0004230628859792799118234...
20	-0.9997030707767648407577070...	0.0016820558863920625076930...
21	1.0016323050916021405946775...	-0.0043064040732798217605510...
22	-1.0086248132202087942655607...	0.0073510189534798201351446...
23	1.0247854554152142629046068...	-0.0059516475070275725367104...
24	-1.0514094416060493048161708...	-0.0106208345225715736366974...
25	1.0810645787258061563371074...	0.0571124654682986230022531...
26	-1.0936755799058347276309630...	-0.1444466492432231927360503...
27	1.0604680332473785584570485...	0.2686334886716153435042459...
28	-0.9574870450804144308288388...	-0.4045896957644686705275135...
29	0.7819273229364294945603960...	0.5125945924385168361163381...
30	-0.5593173572669934203403673...	-0.5565338670449731982907508...
31	0.3343623985424523757486907...	0.5228105117227437592413758...
32	-0.1503328094610327953613906...	-0.4268247149849277954988260...
33	0.0305876048136105141176186...	0.3030140945226535498153739...
34	0.0267142245012646970476706...	-0.1864936962302580318140525...
35	-0.0403508709504571265125850...	0.0987625840338074206468000...
36	0.0326518590146981886559426...	-0.0443531600488926600075650...
37	-0.0199083151373364928284639...	0.0164136756477463206557076...
38	0.0098584308541324315918269...	-0.0046846228685254646823180...
39	-0.0040634498663572072035669...	0.0008201527216634910241769...
40	0.0014027840212830671622398...	0.0000609570347403268294484...
41	-0.0004039590774944196007334...	-0.0001178555325562601894455...
42	0.0000957942226514277520267...	0.0000551004274613413239060...
43	-0.0000182554890708354054022...	-0.0000172688961722773212123...
44	0.0000026704624612408329699...	0.0000040756411955584791797...
45	-0.0000002703581805562546025...	-0.0000007418550617686660312...
46	0.0000000126036221487498755...	0.0000001032408182278302116...
47	0.0000000011100802765469123...	-0.0000000106385991035510443...
48	-0.0000000002586638961962758...	0.0000000007628904065376235...
49	0.0000000000205604069318146...	-0.0000000000337362509724946...
50	-0.0000000000006559441873436...	0.0000000000006828003488777...

Table 1: Coefficients of $D_{a,\delta,N}(s)$ for $a = 30\mathrm{i}$, $\delta = 1/20000$, and $N = 50$

t	$\sigma=-1$	$\sigma=0$	$\sigma=1/2$	$\sigma=1$	$\sigma=2$
0	$7.88...\cdot 10^{-21}$	$7.74...\cdot 10^{-21}$	$6.57...\cdot 10^{-21}$	$5.35...\cdot 10^{-21}$	$3.30...\cdot 10^{-21}$
1	$2.98...\cdot 10^{-21}$	$2.07...\cdot 10^{-21}$	$1.77...\cdot 10^{-21}$	$1.57...\cdot 10^{-21}$	$1.18...\cdot 10^{-21}$
2	$2.56...\cdot 10^{-21}$	$4.05...\cdot 10^{-22}$	$3.72...\cdot 10^{-22}$	$5.07...\cdot 10^{-22}$	$5.05...\cdot 10^{-22}$
3	$2.93...\cdot 10^{-21}$	$6.05...\cdot 10^{-22}$	$4.12...\cdot 10^{-22}$	$3.70...\cdot 10^{-22}$	$2.78...\cdot 10^{-22}$
4	$4.44...\cdot 10^{-21}$	$1.16...\cdot 10^{-21}$	$6.44...\cdot 10^{-22}$	$3.80...\cdot 10^{-22}$	$1.63...\cdot 10^{-22}$
5	$6.69...\cdot 10^{-21}$	$1.60...\cdot 10^{-21}$	$7.62...\cdot 10^{-22}$	$3.38...\cdot 10^{-22}$	$4.75...\cdot 10^{-23}$
6	$1.04...\cdot 10^{-20}$	$2.40...\cdot 10^{-21}$	$1.09...\cdot 10^{-21}$	$4.71...\cdot 10^{-22}$	$8.55...\cdot 10^{-23}$
7	$1.75...\cdot 10^{-20}$	$4.10...\cdot 10^{-21}$	$1.94...\cdot 10^{-21}$	$9.15...\cdot 10^{-22}$	$2.23...\cdot 10^{-22}$
8	$3.14...\cdot 10^{-20}$	$7.52...\cdot 10^{-21}$	$3.60...\cdot 10^{-21}$	$1.70...\cdot 10^{-21}$	$3.75...\cdot 10^{-22}$
9	$6.03...\cdot 10^{-20}$	$1.46...\cdot 10^{-20}$	$6.98...\cdot 10^{-21}$	$3.25...\cdot 10^{-21}$	$6.48...\cdot 10^{-22}$
10	$1.24...\cdot 10^{-19}$	$3.08...\cdot 10^{-20}$	$1.47...\cdot 10^{-20}$	$6.86...\cdot 10^{-21}$	$1.36...\cdot 10^{-21}$
11	$2.80...\cdot 10^{-19}$	$7.24...\cdot 10^{-20}$	$3.50...\cdot 10^{-20}$	$1.63...\cdot 10^{-20}$	$3.21...\cdot 10^{-21}$
12	$6.96...\cdot 10^{-19}$	$1.97...\cdot 10^{-19}$	$9.70...\cdot 10^{-20}$	$4.46...\cdot 10^{-20}$	$8.12...\cdot 10^{-21}$
13	$1.89...\cdot 10^{-18}$	$6.89...\cdot 10^{-19}$	$3.61...\cdot 10^{-19}$	$1.56...\cdot 10^{-19}$	$2.22...\cdot 10^{-20}$
14	$4.87...\cdot 10^{-18}$	$3.44...\cdot 10^{-18}$	$6.38...\cdot 10^{-18}$	$7.88...\cdot 10^{-19}$	$5.82...\cdot 10^{-20}$
15	$9.26...\cdot 10^{-18}$	$3.96...\cdot 10^{-18}$	$2.21...\cdot 10^{-18}$	$9.11...\cdot 10^{-19}$	$1.12...\cdot 10^{-19}$
16	$1.56...\cdot 10^{-17}$	$4.87...\cdot 10^{-18}$	$2.45...\cdot 10^{-18}$	$1.12...\cdot 10^{-18}$	$1.94...\cdot 10^{-19}$
17	$2.90...\cdot 10^{-17}$	$8.37...\cdot 10^{-18}$	$4.15...\cdot 10^{-18}$	$1.95...\cdot 10^{-18}$	$3.67...\cdot 10^{-19}$
18	$6.23...\cdot 10^{-17}$	$1.80...\cdot 10^{-17}$	$8.99...\cdot 10^{-18}$	$4.22...\cdot 10^{-18}$	$7.99...\cdot 10^{-19}$
19	$1.54...\cdot 10^{-16}$	$4.88...\cdot 10^{-17}$	$2.47...\cdot 10^{-17}$	$1.14...\cdot 10^{-17}$	$2.01...\cdot 10^{-18}$
20	$4.35...\cdot 10^{-16}$	$1.82...\cdot 10^{-16}$	$1.00...\cdot 10^{-16}$	$4.32...\cdot 10^{-17}$	$5.78...\cdot 10^{-18}$
21	$1.17...\cdot 10^{-15}$	$9.46...\cdot 10^{-16}$	$1.06...\cdot 10^{-14}$	$2.25...\cdot 10^{-16}$	$1.58...\cdot 10^{-17}$
22	$2.42...\cdot 10^{-15}$	$1.11...\cdot 10^{-15}$	$6.24...\cdot 10^{-16}$	$2.66...\cdot 10^{-16}$	$3.33...\cdot 10^{-17}$
23	$4.84...\cdot 10^{-15}$	$1.83...\cdot 10^{-15}$	$9.63...\cdot 10^{-16}$	$4.40...\cdot 10^{-16}$	$6.73...\cdot 10^{-17}$
24	$1.09...\cdot 10^{-14}$	$5.05...\cdot 10^{-15}$	$2.83...\cdot 10^{-15}$	$1.22...\cdot 10^{-15}$	$1.54...\cdot 10^{-16}$
25	$2.44...\cdot 10^{-14}$	$2.05...\cdot 10^{-14}$	$4.73...\cdot 10^{-13}$	$4.97...\cdot 10^{-15}$	$3.48...\cdot 10^{-16}$
26	$4.24...\cdot 10^{-14}$	$1.93...\cdot 10^{-14}$	$1.08...\cdot 10^{-14}$	$4.70...\cdot 10^{-15}$	$6.11...\cdot 10^{-16}$
27	$7.20...\cdot 10^{-14}$	$2.53...\cdot 10^{-14}$	$1.32...\cdot 10^{-14}$	$6.18...\cdot 10^{-15}$	$1.04...\cdot 10^{-15}$
28	$1.44...\cdot 10^{-13}$	$4.97...\cdot 10^{-14}$	$2.57...\cdot 10^{-14}$	$1.21...\cdot 10^{-14}$	$2.11...\cdot 10^{-15}$
29	$3.52...\cdot 10^{-13}$	$1.41...\cdot 10^{-13}$	$7.59...\cdot 10^{-14}$	$3.46...\cdot 10^{-14}$	$5.17...\cdot 10^{-15}$
30	$9.54...\cdot 10^{-13}$	$6.82...\cdot 10^{-13}$	$5.36...\cdot 10^{-13}$	$1.67...\cdot 10^{-13}$	$1.40...\cdot 10^{-14}$
31	$2.18...\cdot 10^{-12}$	$1.52...\cdot 10^{-12}$	$1.04...\cdot 10^{-12}$	$3.74...\cdot 10^{-13}$	$3.21...\cdot 10^{-14}$
32	$4.31...\cdot 10^{-12}$	$2.56...\cdot 10^{-12}$	$1.53...\cdot 10^{-12}$	$6.28...\cdot 10^{-13}$	$6.31...\cdot 10^{-14}$
33	$7.86...\cdot 10^{-12}$	$7.31...\cdot 10^{-12}$	$2.90...\cdot 10^{-11}$	$1.78...\cdot 10^{-12}$	$1.14...\cdot 10^{-13}$
34	$1.16...\cdot 10^{-11}$	$5.52...\cdot 10^{-12}$	$3.09...\cdot 10^{-12}$	$1.34...\cdot 10^{-12}$	$1.67...\cdot 10^{-13}$
35	$1.79...\cdot 10^{-11}$	$6.79...\cdot 10^{-12}$	$3.56...\cdot 10^{-12}$	$1.64...\cdot 10^{-12}$	$2.56...\cdot 10^{-13}$
36	$3.41...\cdot 10^{-11}$	$1.36...\cdot 10^{-11}$	$7.24...\cdot 10^{-12}$	$3.29...\cdot 10^{-12}$	$4.81...\cdot 10^{-13}$
37	$7.63...\cdot 10^{-11}$	$4.75...\cdot 10^{-11}$	$3.14...\cdot 10^{-11}$	$1.14...\cdot 10^{-11}$	$1.06...\cdot 10^{-12}$
38	$1.55...\cdot 10^{-10}$	$1.13...\cdot 10^{-10}$	$9.00...\cdot 10^{-11}$	$2.71...\cdot 10^{-11}$	$2.12...\cdot 10^{-12}$
39	$2.74...\cdot 10^{-10}$	$1.28...\cdot 10^{-10}$	$6.98...\cdot 10^{-11}$	$3.05...\cdot 10^{-11}$	$3.71...\cdot 10^{-12}$
40	$5.47...\cdot 10^{-10}$	$3.00...\cdot 10^{-10}$	$1.73...\cdot 10^{-10}$	$7.10...\cdot 10^{-11}$	$7.27...\cdot 10^{-12}$
41	$1.11...\cdot 10^{-9}$	$1.08...\cdot 10^{-9}$	$3.42...\cdot 10^{-9}$	$2.54...\cdot 10^{-10}$	$1.45...\cdot 10^{-11}$
42	$1.90...\cdot 10^{-9}$	$1.14...\cdot 10^{-9}$	$6.64...\cdot 10^{-10}$	$2.67...\cdot 10^{-10}$	$2.44...\cdot 10^{-11}$
43	$3.13...\cdot 10^{-9}$	$2.62...\cdot 10^{-9}$	$2.43...\cdot 10^{-9}$	$6.12...\cdot 10^{-10}$	$3.95...\cdot 10^{-11}$
44	$4.40...\cdot 10^{-9}$	$2.62...\cdot 10^{-9}$	$1.63...\cdot 10^{-9}$	$6.08...\cdot 10^{-10}$	$5.46...\cdot 10^{-11}$
45	$5.96...\cdot 10^{-9}$	$2.35...\cdot 10^{-9}$	$1.22...\cdot 10^{-9}$	$5.41...\cdot 10^{-10}$	$7.27...\cdot 10^{-11}$
46	$1.00...\cdot 10^{-8}$	$3.81...\cdot 10^{-9}$	$1.96...\cdot 10^{-9}$	$8.72...\cdot 10^{-10}$	$1.20...\cdot 10^{-10}$
47	$2.16...\cdot 10^{-8}$	$1.07...\cdot 10^{-8}$	$5.99...\cdot 10^{-9}$	$2.45...\cdot 10^{-9}$	$2.54...\cdot 10^{-10}$
48	$5.04...\cdot 10^{-8}$	$5.17...\cdot 10^{-8}$	$2.54...\cdot 10^{-6}$	$1.17...\cdot 10^{-8}$	$5.84...\cdot 10^{-10}$
49	$9.69...\cdot 10^{-8}$	$7.82...\cdot 10^{-8}$	$5.06...\cdot 10^{-8}$	$1.76...\cdot 10^{-8}$	$1.10...\cdot 10^{-9}$
50	$1.53...\cdot 10^{-7}$	$1.45...\cdot 10^{-7}$	$1.77...\cdot 10^{-7}$	$3.26...\cdot 10^{-8}$	$1.72...\cdot 10^{-9}$

Table 2: Approximations of $\eta(s)$ by $D_{a,\delta,N}(s)$ for $a = 30\mathrm{i}$, $\delta = 1/20000$, $N = 50$, and $s = \sigma + \mathrm{i}t$ with different σ and t

n	$\left\| -\log_n(\|\tilde{x}_n\|) - 1 \right\|$	n	$\left\| -\log_n(\|\tilde{x}_n\|) - 1 \right\|$
		26	$1.801077\ldots \cdot 10^{-37}$
2	$1.149155\ldots \cdot 10^{-71}$	27	$1.912995\ldots \cdot 10^{-37}$
3	$2.478495\ldots \cdot 10^{-37}$	28	$1.799200\ldots \cdot 10^{-37}$
4	$3.736145\ldots \cdot 10^{-71}$	29	$1.498868\ldots \cdot 10^{-37}$
5	$4.036585\ldots \cdot 10^{-37}$	30	$1.062796\ldots \cdot 10^{-37}$
6	$1.519682\ldots \cdot 10^{-37}$	31	$5.456515\ldots \cdot 10^{-38}$
7	$3.080976\ldots \cdot 10^{-37}$	32	$7.419118\ldots \cdot 10^{-61}$
8	$4.364747\ldots \cdot 10^{-69}$	33	$5.278373\ldots \cdot 10^{-38}$
9	$2.736641\ldots \cdot 10^{-37}$	34	$1.000277\ldots \cdot 10^{-37}$
10	$2.821452\ldots \cdot 10^{-37}$	35	$1.389593\ldots \cdot 10^{-37}$
11	$9.917281\ldots \cdot 10^{-38}$	36	$1.677963\ldots \cdot 10^{-37}$
12	$1.095777\ldots \cdot 10^{-37}$	37	$1.856775\ldots \cdot 10^{-37}$
13	$2.287797\ldots \cdot 10^{-37}$	38	$1.925431\ldots \cdot 10^{-37}$
14	$2.271759\ldots \cdot 10^{-37}$	39	$1.889893\ldots \cdot 10^{-37}$
15	$1.334827\ldots \cdot 10^{-37}$	40	$1.761140\ldots \cdot 10^{-37}$
16	$2.417916\ldots \cdot 10^{-65}$	41	$1.553656\ldots \cdot 10^{-37}$
17	$1.244996\ldots \cdot 10^{-37}$	42	$1.284047\ldots \cdot 10^{-37}$
18	$2.080360\ldots \cdot 10^{-37}$	43	$9.698290\ldots \cdot 10^{-38}$
19	$2.378695\ldots \cdot 10^{-37}$	44	$6.284199\ldots \cdot 10^{-38}$
20	$2.168629\ldots \cdot 10^{-37}$	45	$2.763427\ldots \cdot 10^{-38}$
21	$1.576386\ldots \cdot 10^{-37}$	46	$7.137296\ldots \cdot 10^{-39}$
22	$7.693392\ldots \cdot 10^{-38}$	47	$4.016023\ldots \cdot 10^{-38}$
23	$8.715101\ldots \cdot 10^{-39}$	48	$7.033747\ldots \cdot 10^{-38}$
24	$8.567838\ldots \cdot 10^{-38}$	49	$9.679844\ldots \cdot 10^{-38}$
25	$1.445023\ldots \cdot 10^{-37}$	50	$1.188989\ldots \cdot 10^{-37}$

Table 3: Comparison of $-\log_n(|\tilde{x}_n|)$ with 1 for $a = 30$, $\Delta = 1/100$, $\delta = 1/20000$, and $N = 50$

n	$\left\|\frac{\tilde{x}_n}{n^{-w_3}} - 1\right\|$
2	$1.014557...\cdot 10^{-71}$
3	$1.285341...\cdot 10^{-36}$
4	$5.199667...\cdot 10^{-71}$
5	$1.129395...\cdot 10^{-36}$
6	$1.285341...\cdot 10^{-36}$
7	$7.581245...\cdot 10^{-37}$
8	$2.200497...\cdot 10^{-68}$
9	$6.780420...\cdot 10^{-37}$
10	$1.129395...\cdot 10^{-36}$
11	$1.321720...\cdot 10^{-36}$
12	$1.285341...\cdot 10^{-36}$
13	$1.076955...\cdot 10^{-36}$
14	$7.581245...\cdot 10^{-37}$
15	$3.842456...\cdot 10^{-37}$
16	$9.936331...\cdot 10^{-65}$
17	$3.615524...\cdot 10^{-37}$
18	$6.780420...\cdot 10^{-37}$
19	$9.360853...\cdot 10^{-37}$
20	$1.129395...\cdot 10^{-36}$
21	$1.257003...\cdot 10^{-36}$
22	$1.321720...\cdot 10^{-36}$
23	$1.328879...\cdot 10^{-36}$
24	$1.285341...\cdot 10^{-36}$
25	$1.198746...\cdot 10^{-36}$
26	$1.076955...\cdot 10^{-36}$
27	$9.276619...\cdot 10^{-37}$
28	$7.581245...\cdot 10^{-37}$
29	$5.749993...\cdot 10^{-37}$
30	$3.842456...\cdot 10^{-37}$
31	$1.910852...\cdot 10^{-37}$
32	$3.905632...\cdot 10^{-60}$
33	$1.852435...\cdot 10^{-37}$
34	$3.615524...\cdot 10^{-37}$
35	$5.264549...\cdot 10^{-37}$
36	$6.780420...\cdot 10^{-37}$
37	$8.149068...\cdot 10^{-37}$
38	$9.360853...\cdot 10^{-37}$
39	$1.040997...\cdot 10^{-36}$
40	$1.129395...\cdot 10^{-36}$
41	$1.201309...\cdot 10^{-36}$
42	$1.257003...\cdot 10^{-36}$
43	$1.296936...\cdot 10^{-36}$
44	$1.321720...\cdot 10^{-36}$
45	$1.332092...\cdot 10^{-36}$
46	$1.328879...\cdot 10^{-36}$
47	$1.312981...\cdot 10^{-36}$
48	$1.285341...\cdot 10^{-36}$
49	$1.246934...\cdot 10^{-36}$
50	$1.198746...\cdot 10^{-36}$

Table 4: Comparison of $\tilde{x}_n$ with $n^{-\mathrm{w}_3}$ for $a = 30$, $\Delta = 1/100$, $\delta = 1/20000$, and $N = 50$

n	j_n	$\lvert \mathrm{w}_3 - \tilde{z}_n \rvert$	n	j_n	$\lvert \mathrm{w}_3 - \tilde{z}_n \rvert$
			26	-14	$3.30547\ldots\cdot 10^{-37}$
2	-3	$1.46\ldots\cdot 10^{-71}$	27	-14	$2.81464\ldots\cdot 10^{-37}$
3	-5	$1.16\ldots\cdot 10^{-36}$	28	-14	$2.27514\ldots\cdot 10^{-37}$
4	-6	$3.75\ldots\cdot 10^{-71}$	29	-15	$1.70759\ldots\cdot 10^{-37}$
5	-7	$7.01\ldots\cdot 10^{-37}$	30	-15	$1.12973\ldots\cdot 10^{-37}$
6	-8	$7.17\ldots\cdot 10^{-37}$	31	-15	$5.56452\ldots\cdot 10^{-38}$
7	-8	$3.89\ldots\cdot 10^{-37}$	32	-15	$1.12692\ldots\cdot 10^{-60}$
8	-9	$1.05\ldots\cdot 10^{-68}$	33	-15	$5.29795\ldots\cdot 10^{-38}$
9	-10	$3.08\ldots\cdot 10^{-37}$	34	-15	$1.02528\ldots\cdot 10^{-37}$
10	-10	$4.90\ldots\cdot 10^{-37}$	35	-15	$1.48074\ldots\cdot 10^{-37}$
11	-10	$5.51\ldots\cdot 10^{-37}$	36	-16	$1.89211\ldots\cdot 10^{-37}$
12	-11	$5.17\ldots\cdot 10^{-37}$	37	-16	$2.25678\ldots\cdot 10^{-37}$
13	-11	$4.19\ldots\cdot 10^{-37}$	38	-16	$2.57336\ldots\cdot 10^{-37}$
14	-11	$2.87\ldots\cdot 10^{-37}$	39	-16	$2.84149\ldots\cdot 10^{-37}$
15	-12	$1.41\ldots\cdot 10^{-37}$	40	-16	$3.06162\ldots\cdot 10^{-37}$
16	-12	$3.58\ldots\cdot 10^{-65}$	41	-16	$3.23491\ldots\cdot 10^{-37}$
17	-12	$1.27\ldots\cdot 10^{-37}$	42	-16	$3.36306\ldots\cdot 10^{-37}$
18	-13	$2.34\ldots\cdot 10^{-37}$	43	-16	$3.44819\ldots\cdot 10^{-37}$
19	-13	$3.17\ldots\cdot 10^{-37}$	44	-16	$3.49274\ldots\cdot 10^{-37}$
20	-13	$3.77\ldots\cdot 10^{-37}$	45	-16	$3.49937\ldots\cdot 10^{-37}$
21	-13	$4.12\ldots\cdot 10^{-37}$	46	-17	$3.47089\ldots\cdot 10^{-37}$
22	-13	$4.27\ldots\cdot 10^{-37}$	47	-17	$3.41021\ldots\cdot 10^{-37}$
23	-14	$4.23\ldots\cdot 10^{-37}$	48	-17	$3.32026\ldots\cdot 10^{-37}$
24	-14	$4.04\ldots\cdot 10^{-37}$	49	-17	$3.20398\ldots\cdot 10^{-37}$
25	-14	$3.72\ldots\cdot 10^{-37}$	50	-17	$3.06426\ldots\cdot 10^{-37}$

Table 5: Approximations of w_3 (defined by (2.23)) by $\tilde{z}_n$ (defined by (1.25)) for $a = 30$, $\Delta = 1/100$, $\delta = 1/20000$, and $N = 50$