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L_2-theory for two viscous fluids of different type: compressible and incompressible

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Abstract. We prove the stability of the rest state in the problem of evolution of two viscous fluids, compressible and incompressible, contained in a bounded vessel and separated by a free interface. The fluids are subject to mass and capillary forces. The proof of stability is based on "maximal regularity" estimates for the solution in the anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{r,r/2}$ with an exponential weight.

Key words and phrases: free boundaries, compressible and incompressible fluids, Sobolev–Slobodetskiĭ spaces

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1 Introduction

We consider free boundary problem governing the motion of two different viscous fluids, compressible and incompressible, contained in a fixed bounded domain $\Omega \subset \mathbb{R}^3$ and separated by a variable interface Γ_t, $t > 0$. It is assumed that the incompressible fluid fills a strictly interior subdomain $\Omega_t^- \subset \Omega$ and a compressible fluid fills the domain $\Omega_t^+ = \Omega \backslash \overline{\Omega}_t^-$ surrounding Ω_t^-. The boundary Σ of Ω is bounded away from Γ_t: $\Sigma \cap \Gamma_t = \varnothing$. The fluids are subject to the mass forces $\boldsymbol{f}(x,t)$, $x \in \Omega$, and to the capillary forces at the interface Γ_t. The motion of the fluids is governed by the system of equations

$$\begin{cases} \rho^-(\mathcal{D}_t \boldsymbol{v}^- + (\boldsymbol{v}^- \cdot \nabla)\boldsymbol{v}^-) - \nabla \cdot \mathbb{T}^-(\boldsymbol{v}^-) + \nabla p^- = \rho^- \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{v}^- = 0 \text{ in } \Omega_t^-, \\ \rho^+(\mathcal{D}_t \boldsymbol{v}^+ + (\boldsymbol{v}^+ \cdot \nabla)\boldsymbol{v}^+) - \nabla \cdot \mathbb{T}^+(\boldsymbol{v}^+) + \nabla p(\rho^+) = \rho^+ \boldsymbol{f}, \\ \mathcal{D}_t \rho^+ + \nabla \cdot (\rho^+ \boldsymbol{v}^+) = 0 \text{ in } \Omega_t^+, \\ \boldsymbol{v}^\pm|_{t=0} = \boldsymbol{v}_0^\pm \text{ in } \Omega_0^\pm, \quad \rho^+|_{t=0} = \rho_0^+ \text{ in } \Omega_0^+, \\ \boldsymbol{v}^+|_\Sigma = 0, \quad [\boldsymbol{v}]|_{\Gamma_t} = 0, \quad V_n = \boldsymbol{v} \cdot \boldsymbol{n}|_{\Gamma_t}, \\ (-p(\rho^+) + p^-)\boldsymbol{n} + [\mathbb{T}(\boldsymbol{u})\boldsymbol{n}] = -\sigma H \boldsymbol{n} \text{ on } \Gamma_t, \end{cases} \tag{1.1}$$

where the unknowns are the velocity vector fields of both fluids $\boldsymbol{v}^\pm(x,t)$, $x \in \Omega_t^\pm$, the density $\rho^+(x,t) > 0$ of the compressible fluid and the pressure $p^-(x,t)$ of the incompressible one. The pressure in the compressible fluid is given by a positive strictly monotone increasing function of density $p(\rho^+)$; $\rho^- > 0$ is a given constant density of the incompressible fluid. The viscous parts of the stress tensors are denoted by $\mathbb{T}^\pm(\boldsymbol{v}^\pm)$:

$$\mathbb{T}^-(\boldsymbol{v}^-) = \mu^- \mathbb{S}(\boldsymbol{v}^-), \quad \mathbb{T}^+(\boldsymbol{v}^+) = \mu^+ \mathbb{S}(\boldsymbol{v}^+) + \mu_1^+ \mathbb{I} \nabla \cdot \boldsymbol{v}^+,$$

where $\mu^\pm > 0$, $\mu_1^+ > -2\mu^+/3$ are constant viscosity coefficients,

$$\mathbb{S}(\boldsymbol{w}) = (\nabla \otimes \boldsymbol{w}) + (\nabla \otimes \boldsymbol{w})^T$$

is the doubled rate-of-strain tensor, the superscript T means transposition, $\mathbb{I}$ is the identity matrix, σ is a positive constant coefficient of the surface tension, H is the doubled mean curvature of Γ_t, V_n is the velocity of evolution of Γ_t in the direction of $\boldsymbol{n}$, the exterior normal to Γ_t with respect to Ω_t^-, $[u]|_{\Gamma_t}$ is the jump of the functions $u^\pm$ given in $\Omega_t^\pm$ on the surface Γ_t, i.e.,

$$[u]|_{\Gamma_t} = u^+|_{\Gamma_t} - u^-|_{\Gamma_t}.$$

Since one of the fluids is incompressible, the quantities $|\Omega_t^\pm| = \operatorname{mes} \Omega_t^\pm$ and the mean value of the density $\overline{\rho}^+ = M^+/|\Omega_t^+|$, where M^+ is a total mass of the compressible fluid, are independent of t. Upon setting

$$\vartheta^+ = \rho^+ - \overline{\rho}^+, \quad \vartheta^- = p^- - p(\rho^+) - \frac{2\sigma}{R_0},$$

where R_0 is the radius of the ball $B_{R_0}^-$ such that $|\Omega_t^-| = 4\pi R_0^3/3$, the jump conditions on Γ_t can be written as follows:

$$[\boldsymbol{v}] = 0, \quad -\big(p(\overline{\rho}^+ + \vartheta^+) - p(\overline{\rho}^+) - \vartheta^-\big)\boldsymbol{n} + [\mathbb{T}(\boldsymbol{v})\boldsymbol{n}] = -\sigma\Big(H + \frac{2}{R_0}\Big)\boldsymbol{n}.$$

It is clear that

$$\int\limits_{\Omega_t^+} \vartheta^+(x,t)\,\mathrm{d}x = 0.$$

We write (1.1) as a nonlinear problem in a fixed domain $\Omega_0^+ \cup \Gamma_0 \cup \Omega_0^-$ by passing to the Lagrangian coordinates $y \in \Omega_0^+ \cup \Gamma_0 \cup \Omega_0^-$ related to the Eulerian coordinates $x \in \Omega_t^+ \cup \Gamma_t \cup \Omega_t^-$ by the equation

$$x = y + \int\limits_0^t \boldsymbol{u}(y,\tau) d\tau \equiv X_{\boldsymbol{u}}(y,t), \tag{1.2}$$

where $\boldsymbol{u}(y,\tau)$ is the velocity vector field written as a function of the Lagrangian coordinates. Then problem (1.1) takes the form

$$\begin{cases} \rho^- \mathcal{D}_t \boldsymbol{u}^- - \nabla_{\boldsymbol{u}} \cdot \mathbb{T}_{\boldsymbol{u}}^-(\boldsymbol{u}^-) + \nabla_{\boldsymbol{u}} \theta^- = \rho^- \widehat{\boldsymbol{f}}, \\ \nabla_{\boldsymbol{u}} \cdot \boldsymbol{u}^- = 0 \text{ in } \Omega_0^-, \\ \widehat{\rho}^+ \mathcal{D}_t \boldsymbol{u}^+ - \nabla_{\boldsymbol{u}} \cdot \mathbb{T}_{\boldsymbol{u}}^+(\boldsymbol{u}^+) + \nabla_{\boldsymbol{u}} p(\widehat{\rho}^+) = \widehat{\rho}^+ \widehat{\boldsymbol{f}}, \\ \mathcal{D}_t \widehat{\rho}^+ + \widehat{\rho}^+ \nabla_{\boldsymbol{u}} \cdot \boldsymbol{u}^+ = 0, \quad \widehat{\rho}^+|_{t=0} = \rho_0^+ \text{ in } \Omega_0^+, \\ \boldsymbol{u}^\pm|_{t=0} = \boldsymbol{u}_0^\pm \equiv \boldsymbol{v}_0^\pm \text{ in } \Omega_0^\pm, \quad \boldsymbol{u}^+|_\Sigma = 0, \quad [\boldsymbol{u}]|_{\Gamma_0} = 0, \\ (-p(\widehat{\rho}^+) + p(\overline{\rho}^+) + \theta^-)\boldsymbol{n} + [\mathbb{T}_u(\boldsymbol{u})\boldsymbol{n}] = -\sigma(\widehat{H} + \frac{2}{R_0})\boldsymbol{n} \text{ on } \Gamma_0, \end{cases} \tag{1.3}$$

where $\widehat{\boldsymbol{f}}(y,t) = \boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t)$, $\widehat{\rho}^+ = \overline{\rho}^+ + \theta^+$, $\widehat{H} = H(X_{\boldsymbol{u}})$, $\theta^\pm = \vartheta^\pm(X_{\boldsymbol{u}},t)$, $\nabla_{\boldsymbol{u}} = (\mathbb{L}^{-1})^T \nabla_y = \mathbb{L}^{-T}\nabla_y$ is the transformed gradient ∇_x, $\mathbb{L} = \left(\frac{\partial x}{\partial y}\right)$ is the Jacobi matrix of the transformation (1.2), the subscript T means transposition, $\widehat{\mathbb{L}} = \mathbb{L}^{-T} L$, $L = \det \mathbb{L}$, $L = 1$ in Ω_t^-, $\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} \otimes \boldsymbol{u} + (\nabla_{\boldsymbol{u}} \otimes \boldsymbol{u})^T$ is the transformed doubled rate-of-strain tensor,

$$\mathbb{T}_{\boldsymbol{u}}^-(\boldsymbol{u}^-) = \mu^- \mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}^-), \quad \mathbb{T}_{\boldsymbol{u}}^+(\boldsymbol{u}^+) = \mu^+ \mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}^+) + \mu_1^+ \mathbb{I} \nabla_{\boldsymbol{u}} \cdot \boldsymbol{u}^+.$$

The elements of the transposed co-factor matrix $\widehat{\mathbb{L}}^T$ are given by

$$(\widehat{\mathbb{L}}^T)_{im} = (\nabla X_j \times \nabla X_k)_m, \tag{1.4}$$

where $X_j = (X_{\boldsymbol{u}})_j$ and (i,j,k) is a cyclic permutation of $(1,2,3)$.

The kinematic condition $V_n = \boldsymbol{u} \cdot \boldsymbol{n}$ is fulfilled automatically. The normal $\boldsymbol{n}(X_{\boldsymbol{u}})$ to Γ_t is related to the normal $\boldsymbol{n}_0$ to Γ_0 by the formula

$$\boldsymbol{n} = \frac{\widehat{\mathbb{L}}^T \boldsymbol{n}_0(y)}{|\widehat{\mathbb{L}}^T \boldsymbol{n}_0(y)|}. \tag{1.5}$$

Since $H\boldsymbol{n} = \Delta(t) X_u$, where $\Delta(t)$ is the Laplace–Beltrami operator on Γ_t, it can be shown that the corresponding linear problem has the form

$$\begin{cases} \overline{\rho}^+ \mathcal{D}_t \boldsymbol{v}^+ - \mu^+ \nabla^2 \boldsymbol{v}^+ - (\mu^+ + \mu_1^+)\nabla(\nabla \cdot \boldsymbol{v}) + p_1 \nabla \theta^+ = \boldsymbol{f}^+, \\ \mathcal{D}_t \theta^+ + \overline{\rho}^+ \nabla \cdot \boldsymbol{v}^+ = h^+ \text{ in } \Omega_0^+, \\ \rho^- \mathcal{D}_t \boldsymbol{v}^- - \mu^- \nabla^2 \boldsymbol{v}^- + \nabla \theta^- = \boldsymbol{f}^-, \quad \nabla \cdot \boldsymbol{v}^- = h^- \text{ in } \Omega_0^-, \quad \boldsymbol{v}^+|_\Sigma = 0, \\ \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0 \text{ in } \Omega_0^+ \cup \Omega_0^-, \quad \theta^+|_{t=0} = \theta_0^+ \text{ in } \Omega_0^+, \\ [\boldsymbol{v}]|_{\Gamma_0} = 0, \quad [\mu \Pi_0 \mathbb{S}(\boldsymbol{v}) \boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{b}, \\ -p_1 \theta^+ + \theta^- + [\boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{v})\boldsymbol{n}_0] + \sigma \boldsymbol{n}_0 \cdot \int\limits_0^t \Delta(0) \boldsymbol{v}(y,\tau)\,\mathrm{d}\tau|_{\Gamma_0} = b + \int\limits_0^t B\,\mathrm{d}\tau, \end{cases} \tag{1.6}$$

where $\boldsymbol{f}^{\pm}$, $h^{\pm}$, $\boldsymbol{b}$, b, B, $\boldsymbol{v}_0$, θ_0^+ are some given functions, $p_1 = p'(\overline{\rho}^+) > 0$.

In the present paper problems (1.3) and (1.6) are studied in the Sobolev–Slobodetskiĭ spaces $W_2^r(\Omega)$ and $W_2^{r,r/2}(Q_T)$ with the norms

$$\|u\|^2_{W_2^r(\Omega)} = \sum_{0\leqslant|j|\leqslant r} \|\mathcal{D}^j u\|^2_{L_2(\Omega)} \equiv \sum_{0\leqslant|j|\leqslant r} \int\limits_{\Omega} |\mathcal{D}^j u(x)|^2\,\mathrm{d}x,$$

if $r = [r]$, i.e., r is an integer, and

$$\|u\|^2_{W_2^r(\Omega)} = \|u\|^2_{W_2^{[r]}(\Omega)} + \sum_{|j|=[r]} \int\limits_{\Omega}\int\limits_{\Omega} |\mathcal{D}^j u(x) - \mathcal{D}^j u(y)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n+2\rho}},$$

if $r = [r]+\rho$, $\rho \in (0,1)$; here, $\Omega \subset \mathbb{R}^n$, $Q_T = \Omega \times (0,T)$, $r > 0$. As usual, $\mathcal{D}^j u$ denotes a (generalized) partial derivative $\frac{\partial^{|j|}u}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}}$ where $j = (j_1, j_2, \ldots j_n)$ and $|j| = j_1 + \cdots + j_n$. The anisotropic space $W_2^{r,r/2}(Q_T)$ can be defined as

$$L_2((0,T), W_2^r(\Omega)) \cap W_2^{r/2}((0,T), L_2(\Omega)) \equiv W_2^{r,0}(Q_T) \cap W_2^{0,r/2}(Q_T)$$

and supplied with the norm

$$\begin{aligned} \|u\|^2_{W_2^{r,r/2}(Q_T)} &= \int\limits_0^T \|u(\,\cdot\,,t)\|^2_{W_2^r(\Omega)}\,\mathrm{d}t + \int\limits_{\Omega} \|u(x,\,\cdot\,)\|^2_{W_2^{r/2}(0,T)}\,\mathrm{d}x \\ &\equiv \|u\|^2_{W_2^{r,0}(Q_T)} + \|u\|^2_{W_2^{0,r/2}(Q_T)}. \end{aligned} \tag{1.7}$$

In addition, we set $\|u\|_{\Omega} = \|u\|_{L_2(\Omega)}$,

$$|u|^{(r+l,l/2)}_{Q_T} = \big(\|u\|^2_{W_2^{r+l,0}(Q_T)} + \|u\|^2_{W_2^{l/2}(0,T);W_2^r(\Omega))}\big)^{1/2}, \tag{1.8}$$

and

$$\|u\|_{W_2^r(\cup\Omega^{\pm})} = \big(\|u\|^2_{W_2^r(\Omega^+)} + \|u\|^2_{W_2^r(\Omega^-)}\big)^{1/2},$$

if $\Omega = \cup\Omega^{\pm}$ and $u(x)$ can be discontinuous on $\bar{\Omega}^+ \cap \bar{\Omega}^-$. Finally,

$$\|u\|^2_{H^{r,r/2}(Q_T)} = \|u\|^2_{\widehat{W}_2^{r,r/2}(Q_T)} + \sum_{0<k<r/2-1/2} \sup_{t<T} \|\mathcal{D}_t^k u(\,\cdot\,,t)\|^2_{W_2^{r-1-2k}(\Omega)}, \tag{1.9}$$

where

$$\|u\|_{\widehat{W}_2^{r,r/2}(Q_T)} = \|u\|_{W_2^{r,r/2}(Q_T)} \tag{1.10}$$

if $r/2$ is an integer or $r/2 = [r/2]+\rho$, $1/2 < \rho < 1$,

$$\|u\|^2_{\widehat{W}_2^{r,r/2}(Q_T)} = \|u\|^2_{W_2^{r,r/2}(Q_T)} + \frac{1}{T^{2\rho}}\|\mathcal{D}_t^{[r/2]}u(\,\cdot\,,t)\|^2_{L_2(Q_T)} \tag{1.11}$$

if $\rho \in (0,1/2)$ (the case of $\rho = 1/2$ is excluded).

For arbitrary $T > 0$, the norms (1.9)–(1.11) are equivalent to the norm (1.7); they are useful in the analysis of problems (1.3) and (1.6) in a small time interval $(0,T)$ (see [1] and

§2). Functions with finite norm (1.11), where $r \in (0,1/2)$, can be extended by zero into the domain $t<0$ with preservation of regularity, and the norm (1.11) is equivalent to the norm $\|u\|_{W_2^{r,r/2}(\Omega\times(-\infty,T))}$ defined by

$$\|u\|^2_{W_2^{r,r/2}(\Omega\times(-\infty,T))} = \|u\|^2_{W_2^{r,0}(Q_T)} + \|\boldsymbol{u}\|^2_{\dot W_2^{0,r/2}(\Omega\times(-\infty,T))},$$

$$\|\boldsymbol{u}\|^2_{\dot W_2^{0,r/2}(\Omega\times(-\infty,t))} = \int\limits_0^\infty \frac{\mathrm{d}h}{h^{1+r}} \int\limits_0^T \|\Delta_t(-h)u(\,\cdot\,,t)\|^2_{L_2(\Omega)}\,\mathrm{d}t,$$

where $\Delta_t(-h)u(x,t) = u(x,t-h)-u(x,t)$ is a finite difference of u with respect to t.

Our starting point is the following theorem.

Theorem 1 (see [6,7])**.** *Suppose that $\Sigma \in W_2^{l+3/2}$, $\Gamma_0 \in W_2^{l+3/2}$, $l\in(1/2,1)$. For arbitrary $\boldsymbol{f}\in W_2^{l,l/2}(\cup Q_T^\pm)$, $h^-\in W_2^{l+1,(l+1)/2}(Q_T^-)$ such that*

$$\begin{gathered}
\mathcal{D}_t h^- = \nabla\cdot\boldsymbol{H} + H_1, \quad \boldsymbol{H}, H_1 \in W_2^{0,l/2}(Q_T^-),\\
h^+ \in W_2^{l+1,0}(Q_T^+)\cap W_2^{l/2}((0,T);W_2^1(\Omega_0^+)), \quad \boldsymbol{b}\in W_2^{l+1/2,l/2+1/4}(G_T),\\
b\in W_2^{l+1/2,0}(G_T)\cap \widehat{W}_2^{l/2}((0,T),W_2^{1/2}(\Gamma_0)), \quad B\in \widehat{W}_2^{l-1/2,l/2-1/4}(G_T),\\
\boldsymbol{v}_0\in W_2^{l+1}(\Omega_0^\pm), \quad h_0^+\in W_2^{l+1}(\Omega_0^+),
\end{gathered}$$

where $Q_T^\pm = \Omega_0^\pm\times(0,T)$, $G_T=\Gamma_0\times(0,T)$, satisfying the compatibility conditions

$$\begin{gathered}
\nabla\cdot\boldsymbol{v}_0^-(y) = h_0^-(y) \text{ in } \Omega_0^-, \quad [\mu\Pi_0\mathbb{S}(\boldsymbol{v}_0)\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{b}(y,0),\\
\boldsymbol{b}\cdot\boldsymbol{n}_0 = 0, \quad [\boldsymbol{v}_0^+]|_{\Gamma_0}=0, \quad \boldsymbol{v}_0^+|_\Sigma = 0,
\end{gathered}\tag{1.12}$$

problem (1.6) has a unique solution in an arbitrary finite time interval $(0,T)$, and we have

$$\begin{aligned}
&\|\boldsymbol{v}\|_{H^{2+l,1+l/2}(\cup Q_T^\pm)} + \|\theta^-\|_{\widehat{W}_2^{l/2}((0,T);W_2^1(\Omega_0^-))} + \|\theta^-\|_{W_2^{l+1,0}(Q_T^-)}\\
&+\|\theta^+\|_{\widehat{W}_2^{l/2}((0,T);W_2^1(\Omega_0^+))} + \|\theta^+\|_{W_2^{l+1,0}(Q_T^+)}\\
&+\|\mathcal{D}_t\theta^+\|_{\widehat{W}_2^{l/2}((0,T);W_2^1(\Omega_0^+))} + \|\mathcal{D}_t\theta^+\|_{W_2^{l+1,0}(Q_T^+)}\\
&\leqslant c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(\cup Q_T^\pm)} + \|h^-\|_{W_2^{l+1,0}(\cup Q_T^-)} + \|\boldsymbol{H}\|_{\widehat{W}_2^{0,l/2}(Q_T^-)}\\
&+\|H_1\|_{\widehat{W}_2^{0,l/2}(Q_T^-)} + \|h^+\|_{W_2^{l+1,0}(\cup Q_T^+)} + \|h^+\|_{\widehat{W}_2^{l/2}((0,T);W_2^1(\Omega_0^+))}\\
&+\|\boldsymbol{b}\|_{H^{l+1/2,l/2+1/4}(G_T)} + \|b\|_{W_2^{l+1/2,0}(G_T)} + \|b\|_{\widehat{W}_2^{1/2}((0,T);W_2^{1/2}(\Gamma_0))}\\
&+\|B\|_{\widehat{W}_2^{l-1/2,l/2-1/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\cup\Omega_0^\pm)} + \|\theta_0^+\|_{W_2^{l+1}(\Omega_0^+)}\Big),
\end{aligned}\tag{1.13}$$

where $\Pi_0\boldsymbol{g} = \boldsymbol{g}-\boldsymbol{n}_0(\boldsymbol{g}\cdot\boldsymbol{n}_0)$, $c(T)$ is a monotone nondecreasing function of T.

We recall that

$$\begin{gathered}
\|\boldsymbol{u}\|^2_{H^{2+l,1+l/2}(Q_T^\pm)} = \|\boldsymbol{u}\|^2_{W_2^{2+l,0}(Q_T^\pm)} + \|\mathcal{D}_t\boldsymbol{u}\|^2_{\widehat{W}_2^{l,l/2}(Q_T^\pm)} + \sup_{t<T}\|\boldsymbol{u}(\,\cdot\,,t)\|^2_{W_2^{l+1}(\cup\Omega_0^\pm)},\\
\|\boldsymbol{b}\|^2_{H^{l+1/2,l/2+1/4}(G_T)} = \|\boldsymbol{b}\|^2_{W_2^{l+1/2,l/2+1/4}(G_T)} + \sup_{t<T}\|\boldsymbol{b}(\,\cdot\,,t)\|^2_{W_2^{l-1/2}(\Gamma_0)}.
\end{gathered}$$

For a nonlinear problem (1.3), the existence of a unique solution in a finite time interval is established in §2. In §§3 and 5 estimates of the solution with exponential weight $e^{\beta t}$, $\beta > 0$, are obtained, and the solution is extended into the infinite time interval $(0,\infty)$ if the data of the problem satisfy some smallness conditions. It is shown that the solution tends to a rest state of the problem (1.1) as $t\to\infty$: $\boldsymbol{v}=0$, $p^-,\vartheta^\pm$ are constants in $\Omega^\pm_\infty$, Ω^-_∞ is a ball of radius R_0 centered at a point h_∞ close to the barycenter of Ω^-_0. For two incompressible fluids these results were obtained in [2–5], see also [10].

Theorem 1 and the local existence theorem for a nonlinear problem were proved earlier in [6] under some additional assumptions on the viscosity coefficients, which were lifted in [3, 7–10]. The case of $\sigma=0$ was studied in [7–9].

2 Local solution of problem (1.3).

In this section, we study problem (1.3) in a finite time interval $(0,T)$ with $T>1$ fixed later. By separating linear and nonlinear terms, we transform (1.3) into

$$\begin{cases}\rho^-\mathcal{D}_t\boldsymbol{u}^- - \nabla\cdot\mathbb{T}^-(\boldsymbol{u}^-)+\nabla\theta^- = \boldsymbol{l}_1^-(\boldsymbol{u}^-,\theta^-)+\rho^-\widehat{\boldsymbol{f}},\\ \nabla\cdot\boldsymbol{u}^- = l_2^-(\boldsymbol{u}^-) \quad \text{in } \Omega_0^-,\ \ t>0,\\ \overline{\rho}^+\mathcal{D}_t\boldsymbol{u}^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{u}^+)+p_1\nabla\theta^+ = \boldsymbol{l}_1^+(\boldsymbol{u}^+,\theta^+)+(\overline{\rho}^++\theta^+)\widehat{\boldsymbol{f}},\\ \mathcal{D}_t\theta^+ +\overline{\rho}^+\nabla\cdot\boldsymbol{u} = l_2^+(\boldsymbol{u}^+,\theta^+) \text{ in } \Omega_0^+,\ \ t>0,\\ \boldsymbol{u}^\pm|_{t=0}=\boldsymbol{u}_0^\pm \text{ in } \Omega_0^\pm, \quad \theta^+|_{t=0}=\theta_0^+=\rho_0^+-\overline{\rho}^+,\\ [\boldsymbol{u}]|_{\Gamma_0}=0,\quad [\mu\Pi_0\mathbb{S}(\boldsymbol{u})\boldsymbol{n}_0]|_{\Gamma_0}=\boldsymbol{l}_3(\boldsymbol{u})|_{\Gamma_0},\\ -p_1\theta^+ +\theta^- + [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{u})\boldsymbol{n}_0]\big|_{\Gamma_0}+\sigma\boldsymbol{n}_0\cdot\int_0^t\Delta(0)\boldsymbol{u}(y,\tau)\,d\tau\big|_{\Gamma_0}\\ = l_4(\boldsymbol{u})-\int_0^t(l_5(\boldsymbol{u})+l_6(\boldsymbol{u}))\,d\tau-\sigma(H_0+\frac{2}{R_0}),\quad \boldsymbol{u}|_\Sigma=0,\end{cases}\tag{2.1}$$

where $H_0=H|_{t=0}$,

$$\begin{gathered}\boldsymbol{l}_1^-(\boldsymbol{u},\theta)=\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^-_{\boldsymbol{u}}(\boldsymbol{u}^-)-\nabla\cdot\mathbb{T}^-(\boldsymbol{u}^-)+(\nabla-\nabla_{\boldsymbol{u}})\theta^-,\\ \boldsymbol{l}_1^+(\boldsymbol{u},\theta)=\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^+_{\boldsymbol{u}}(\boldsymbol{u}^+)-\nabla\cdot\mathbb{T}^+(\boldsymbol{u}^+)\\ +p_1(\nabla-\nabla_{\boldsymbol{u}})\theta^+-\nabla_{\boldsymbol{u}}\big(p(\overline{\rho}^++\theta^+)-p(\overline{\rho}^+)-p_1\theta^+\big)-\theta^+\mathcal{D}_t\boldsymbol{u}^+,\\ l_2^-(\boldsymbol{u})=(\nabla-\nabla_{\boldsymbol{u}})\cdot\boldsymbol{u}^-=\nabla\cdot\boldsymbol{L}(\boldsymbol{u}^-),\ \boldsymbol{L}(\boldsymbol{u}^-)=(\mathbb{I}-\mathbb{L}^{-1})\boldsymbol{u}^-=(\mathbb{I}-\widehat{\mathbb{L}})\boldsymbol{u}^-,\\ l_2^+(\boldsymbol{u},\theta)=\overline{\rho}^+(\nabla-\nabla_{\boldsymbol{u}})\cdot\boldsymbol{u}^+-\theta^+\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^+,\\ \boldsymbol{l}_3(\boldsymbol{u})=[\mu(\Pi_0(\Pi_0\mathbb{S}(\boldsymbol{u})\boldsymbol{n}_0-\Pi\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n})]\big|_{\Gamma_0},\\ l_4(\boldsymbol{u},\theta)=[\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{u})\boldsymbol{n}_0-\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}]+(p^+(\overline{\rho}^++\theta^+)-p^+(\overline{\rho}^+)-p_1\theta^+)\big|_{\Gamma_0},\\ l_5(\boldsymbol{u})=\sigma\mathcal{D}_t(\boldsymbol{n}\Delta(t))\cdot\int_0^t\boldsymbol{u}(y,\tau)\,d\tau+\sigma(\boldsymbol{n}\cdot\Delta(t)-\boldsymbol{n}_0\cdot\Delta(0))\boldsymbol{u},\\ l_6(\boldsymbol{u})=\sigma(\dot{\boldsymbol{n}}\Delta(t)+\boldsymbol{n}\dot{\Delta}(t))\cdot\boldsymbol{y}\Big|_{\Gamma_0},\quad \dot{\boldsymbol{n}}=\mathcal{D}_t\boldsymbol{n},\quad \dot{\Delta}(t)=\mathcal{D}_t\Delta(t),\\ \Pi_0\boldsymbol{g}=\boldsymbol{g}-\boldsymbol{n}_0(\boldsymbol{n}_0\cdot\boldsymbol{g}),\quad \Pi\boldsymbol{g}=\boldsymbol{g}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{g}).\end{gathered}\tag{2.2}$$

Remark. 1. Since $\widehat{\mathbb{L}}^T\nabla\cdot\boldsymbol{u}^- = \nabla\cdot\widehat{\mathbb{L}}\boldsymbol{u}^-$, the expression $l_2^-(\boldsymbol{u})$ can be written in the divergence form: $l_2^-(\boldsymbol{u}^-) = \nabla\cdot\boldsymbol{L}\boldsymbol{u}^-$, $\boldsymbol{L} = (\mathbb{I}-\widehat{\mathbb{L}})$.

2. The relation $\Pi_0[\mu\mathbb{S}(\boldsymbol{u})\boldsymbol{n}_0] = \boldsymbol{l}_3(\boldsymbol{u})$ is equivalent to $\Pi_0\Pi[\mu\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}] = 0$, which implies $\Pi[\mu\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}] = 0$, provided that $\boldsymbol{n}\cdot\boldsymbol{n}_0 > 0$. This condition will be justified later as a consequence of the smallness condition (2.10).

The operator $\Delta(t)$ is given by

$$\Delta(t) = \frac{1}{\sqrt{g}}\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial s_\alpha}g^{\alpha\beta}\sqrt{g}\frac{\partial}{\partial s_\beta}, \tag{2.3}$$

where $g = \det(g_{\alpha\beta})$,$\alpha,\beta = 1,2$, the $g_{\alpha\beta} = \frac{\partial X_{\boldsymbol{u}}}{\partial s_\alpha}\cdot\frac{\partial X_{\boldsymbol{u}}}{\partial s_\beta}$ are elements of the metric tensor on Γ_t, $g^{\alpha\beta}$ and $\widehat{g}_{\alpha\beta}$ are elements of the inverse and transposed co-factors matrices to $(g_{\alpha\beta})$, respectively. We assume that (s_1,s_2) are local Cartesian coordinates on the tangential plane to Γ_0 with the origin at the point $y_0\equiv 0$. Let $\Gamma_0'\subset\Gamma_0$ be a neighborhood of the origin defined by the equation

$$s_3 = \phi(s_1,s_2)\in W_2^{5/2+l}(K), \quad K = \{s_1^2+s_2^2\leqslant d^2\},$$

the y_3-axis being directed along $\boldsymbol{n}_0(y_0)$. Then the set $\Gamma_t' = X_{\boldsymbol{u}}\Gamma_0'\subset\Gamma_t$ is given by the equations

$$\begin{aligned} z_\gamma &= s_\gamma + \int_0^t \widetilde{u}_\gamma(s_1,s_2,\phi(s_1,s_2),\tau)\,\mathrm{d}\tau, \quad \gamma = 1,2,\\ z_3 &= \phi(s_1,s_2) + \int_0^t \widetilde{u}_3(s_1,s_2,\phi(s_1,s_2),\tau)\,\mathrm{d}\tau, \end{aligned} \tag{2.4}$$

where the $\widetilde{u}_i$ are the projections of $\boldsymbol{u}$ to the s_i-axes and

$$\begin{aligned} g_{\alpha\beta} &= \sum_{i=1}^{3}\frac{\partial z_i}{\partial s_\alpha}\frac{\partial z_i}{\partial s_\beta} = \delta_{\alpha\beta}+\phi_\alpha\phi_\beta+\phi_\alpha U_{3\beta}+\phi_\beta U_{3\alpha}+U_{\alpha\beta}+U_{\beta\alpha}+\sum_{i=1}^{3}U_{i\alpha}U_{i\beta},\\ U_{i\alpha} &= \int_0^t\Big(\frac{\partial\widetilde{u}_i}{\partial s_\alpha}+\phi_\alpha\frac{\partial\widetilde{u}_i}{\partial s_3}\Big)\,\mathrm{d}\tau, \quad \phi_\alpha = \frac{\partial\phi}{\partial s_\alpha}. \end{aligned} \tag{2.5}$$

The time derivative $\dot{\Delta}(t)$ of $\Delta(t)$ is given by

$$\dot{\Delta}(t) = -\frac{\dot{g}}{2g}\Delta(t) + \frac{1}{\sqrt{g}}\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial s_\alpha}\widetilde{g}_{\alpha\beta}\frac{\partial}{\partial s_\beta}, \tag{2.6}$$

where $\widetilde{g}_{\alpha\beta} = \mathcal{D}_t\frac{\widehat{g}_{\alpha\beta}}{\sqrt{g}}$, $\dot{g} = \mathcal{D}_t g$.

The main result of the present section is the following theorem.

Theorem 2. *Assume that* $\Gamma_0\in W_2^{l+5/2}$, $\Sigma\in W_2^{l+3/2}$, $l\in(1/2,1)$, $p(\rho^+)$ *is a* C^2*-function with Lipschitz continuous second derivatives, and the compatibility conditions* (1.12), *i.e.*,

$$\nabla\cdot\boldsymbol{v}_0^- = 0, \quad [\mu\Pi_0\mathbb{S}(\boldsymbol{v}_0)\boldsymbol{n}_0]|_{\Gamma_0} = 0, \quad [\boldsymbol{v}_0]|_{\Gamma_0} = 0, \quad \boldsymbol{v}_0|_\Sigma = 0,$$

as well as the smallness conditions

$$\|\boldsymbol{u}_0\|_{W_2^{l+1}(\cup\Omega_0^\pm)}+\|\theta_0^+\|_{W_2^{l+1}(\Omega_0^+)}+\sigma\Big\|H_0+\frac{2}{R_0}\Big\|_{W_2^{l+1/2}(\Gamma_0)}+\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}\leqslant\varepsilon\ll 1, \tag{2.7}$$

are satisfied with $\varepsilon=\varepsilon(T)$*, moreover,* $\boldsymbol{f},\nabla\boldsymbol{f}^\pm\in W_2^{l,l/2}(Q_T^\pm)$*, where* $Q_T^\pm=\Omega_0^\pm\times(0,T)$*. Then problem* (1.3) *has a unique solution* $(\boldsymbol{u}^\pm,\theta^\pm)$ *such that*

$$\boldsymbol{u}^\pm\in W_2^{2+l,1+l/2}(\cup Q_T^\pm),\quad \theta^+,\mathcal{D}_t\theta^+\in W_2^{l+1,0}(Q_T^+)\cap W_2^{l/2}((0,T);W_2^1(\Omega_0^+)),$$
$$\theta^-\in W_2^{l+1,0}(Q_T^-)\cap W_2^{l/2}((0,T);W_2^1(\Omega_0^-))$$

and we have the estimate

$$\begin{aligned}
Y(\boldsymbol{u},\theta)\equiv{}&\|\boldsymbol{u}\|_{H^{2+l,1+l/2}(\cup Q_T^\pm)}+\|\theta^-\|_{\widehat{W}_2^{l/2}((0,T);W_2^1(\Omega_0^-))}\\
&+\|\theta^-\|_{W_2^{l+1,0}(Q_T^-)}+\|\theta^+\|_{W_2^{l+1,0}(Q_T^+)}+\|\theta^+\|_{\widehat{W}_2^{l/2}((0,T);W_2^1(\Omega_0^+))}\\
&+\|\mathcal{D}_t\theta^+\|_{W_2^{l+1,0}(Q_T^+))}+\|\mathcal{D}_t\theta^+\|_{\widehat{W}_2^{l/2}((0,T);W_2^1(\Omega_0^+))}\\
\leqslant{}& c(T)\big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\cup\Omega_0^\pm)}+\sigma\Big\|H_0+\frac{2}{R_0}\Big\|_{W_2^{l+1/2}(\Gamma_0)}\\
&+\|\theta_0^+\|_{W_2^{l+1}(\Omega_0^+)}+\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T)}\big),
\end{aligned}\tag{2.8}$$

where $c(T)$ *is a monotone nondecreasing function of* T*.*

The proof is based on Theorem 1 and on the estimate of nonlinear terms (2.2).

Proposition 1. *Let*

$$\begin{aligned}
Z(\boldsymbol{u},\theta)={}&\|\boldsymbol{l}_1^\pm\|_{\widehat{W}_2^{l,l/2}(\cup Q_T^\pm)}+\|l_2^\pm\|_{W_2^{l+1,0}(\cup Q_T^\pm)}+\|\mathcal{D}_t\boldsymbol{L}(\boldsymbol{u})\|_{\widehat{W}_2^{0,l/2}(Q_T)}\\
&+\|l_4(\boldsymbol{u})\|_{\widehat{W}_2^{l/2}((0,T);W_2^{1/2}(\Gamma_0))}+\|\boldsymbol{l}_3(\boldsymbol{u})\|_{H^{l+1/2,l/2+1/4}(G_T)}\\
&+\|l_2^\pm\|_{\widehat{W}_2^{l/2}((0,T);W_2^1(\Omega_0^\pm))}+\|l_5(\boldsymbol{u})\|_{\widehat{W}_2^{l-1/2,l/2-1/4}(G_T)}.
\end{aligned}\tag{2.9}$$

If

$$\begin{aligned}
&\sup_{t<T}\|\theta^+(\,\cdot\,,t)\|_{W_2^{l+1}(\Omega_0^+)}+\sup_{t<T}\|\boldsymbol{U}(\,\cdot\,,t)\|_{W_2^{l+2}(\cup\Omega_0^\pm)}\\
&\quad\leqslant\sup_{t<T}\|\theta^+(\,\cdot\,,t)\|_{W_2^{l+1}(\Omega_0^+)}+(1+\sqrt{T})\|\boldsymbol{u}\|_{W_2^{l+2,0}(\cup Q_T^\pm)}\leqslant\delta\ll 1,
\end{aligned}\tag{2.10}$$

where $\boldsymbol{U}(\xi,t)=\int\limits_0^t\boldsymbol{u}(\xi,\tau)\,\mathrm{d}\tau$*, then*

$$Z(\boldsymbol{u},\theta)\leqslant c\sqrt{T}Y^2(\boldsymbol{u},\theta)\leqslant\delta Y(\boldsymbol{u},\theta) \tag{2.11}$$

and

$$\begin{aligned}
\|l_6(\boldsymbol{u})\|_{\widehat{W}_2^{l-1/2,l/2-1/4}(G_T)}\leqslant{}& c\big(\|\nabla\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa,0}(G_T)}\\
&+\|\nabla\boldsymbol{u}\|_{\widehat{W}_2^{l-1/2}((0,T);W_3^{3/2-l}(\Gamma_0))}\big),
\end{aligned}\tag{2.12}$$

where $\varkappa\in(0,l-1/2)$*. If* $\boldsymbol{f}\in W_2^{l,l/2}(Q_T)$ *and* $\nabla\boldsymbol{f}\in L_2(Q_T)$*, then*

$$\|\widehat{\boldsymbol{f}}\|_{W_2^{l,l/2}(\cup Q_T^\pm)}\leqslant c\big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}+\|\nabla\boldsymbol{f}\|_{L_2(Q_T)}\sup_{Q_T}|\boldsymbol{u}(y,t)|\big). \tag{2.13}$$

Proof. We cite some auxiliary inequalities (see [1,4]), namely,

$$\begin{aligned}
&\|uv\|_{W_2^l(\mathfrak{Q})} \leqslant c\|u\|_{W_2^l(\mathfrak{Q})}\|v\|_{W_2^s(\mathfrak{Q})},\\
&\|uv\|_{L_2(\mathfrak{Q})} \leqslant c\|u\|_{W_2^l(\mathfrak{Q})}\|v\|_{W_2^{n/2-l}(\mathfrak{Q})}, \text{ if } l < n/2,\\
&\|uv\|_{W_2^l(\mathfrak{Q})} \leqslant c\big(\|u\|_{W_2^l(\mathfrak{Q})}\|v\|_{W_2^s(\mathfrak{Q})} + \|v\|_{W_2^l(\mathfrak{Q})}\|u\|_{W_2^s(\mathfrak{Q})}\big), \text{ if } l \geqslant n/2,
\end{aligned} \tag{2.14}$$

where $\mathfrak{Q}$ is a bounded domain in $\mathbb{R}^n$, $n = 2,3$, $s > n/2$. If u, v depend also on $t \in (0,T)$, then (2.14) implies

$$\|uv\|_{W_2^{l,0}(\mathfrak{Q}_T)} \leqslant c\|u\|_{W_2^{l,0}(\mathfrak{Q}_T)} \sup_{t\in(0,T)} \|v(\,\cdot\,,t)\|_{W_2^{n/2+\varkappa}(\mathfrak{Q})}, \quad \mathfrak{Q}_T = \mathfrak{Q}\times(0,T), \tag{2.15}$$

where $l < n/2$, $\varkappa \in (0, l-1/2)$. In addition, from

$$\|\Delta_t(-h)uv\|_{\mathfrak{Q}} \leqslant \sup_{\mathfrak{Q}} |v(y,t)| \|\Delta_t(-h)u(\,\cdot\,,t)\|_{\mathfrak{Q}} + \|\Delta_t(-h)v\|_{L_q(\mathfrak{Q})}\|u\|_{L_p(\mathfrak{Q})}$$

it follows that

$$\begin{aligned}
\|uv\|_{\dot W_2^{0,l/2}(\mathfrak{Q}_T)} &\leqslant c \sup_{\mathfrak{Q}'_T} |v(y,t)| \|u\|_{\dot W_2^{0,l/2}(\mathfrak{Q}_T)}\\
&\quad + c\|v\|_{W_2^{l/2}((0,T);W_2^{n/2-l}(\mathfrak{Q}))} \sup_{t<T} \|u(\,\cdot\,,t)\|_{W_2^l(\mathfrak{Q}')}),
\end{aligned} \tag{2.16}$$

where $l - n/2 + n/p = 0$, $1/q = 1/2 - 1/p$, $l < n/2$. If $l > n/2$, then

$$\|uv\|_{\dot W_2^{0,l/2}(\mathfrak{Q}_T)} \leqslant c\big(\sup_{\mathfrak{Q}'_T} |u(y,t)| \|v\|_{\dot W_2^{0,l/2}(\mathfrak{Q}_T)} + \sup_{\mathfrak{Q}_T} |v(y,t)| \|u\|_{\dot W_2^{0,l/2}(\mathfrak{Q}_T)}\big). \tag{2.17}$$

We pass to the estimates of expressions l_i on the right-hand side of (2.1). Inequality (2.10) implies

$$\begin{aligned}
&\|\widehat{\mathbb{L}} - \mathbb{I}\|_{W_2^{1+l}(\cup\Omega_0^\pm)} + \|\boldsymbol{n} - \boldsymbol{n}_0\|_{W_2^{l+1/2}(\Gamma_0)} \leqslant c\sqrt{T}\|\nabla \boldsymbol{u}\|_{W_2^{l+1}(\cup Q_T^\pm)} \leqslant c\delta,\\
&\|\mathcal{D}_t\widehat{\mathbb{L}}\|_{W_2^{l+1}(\cup\Omega_0^\pm)} \leqslant c\|\nabla \boldsymbol{u}\|_{W_2^{l+1}(\cup\Omega_0^\pm)},
\end{aligned} \tag{2.18}$$

hence the expressions $\boldsymbol{l}_1^\pm(\boldsymbol{u},\theta^\pm)$, $l_2^\pm$, $\nabla_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{u}}^\pm(\boldsymbol{u}^\pm) - \nabla\mathbb{T}^\pm(\boldsymbol{u}^\pm)$, $\theta^+\mathcal{D}_t\boldsymbol{u}^+$, as well as $\boldsymbol{l}_3$, $[\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{u})\boldsymbol{n}_0 - \boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n})]_{\Gamma_0}$ are estimated by the same arguments as in [1] (see also calculations in §4), by $c\delta Y$, i.e., the norms of all these expressions satisfy (2.11). The $W_2^{l,l/2}(Q_T)$-norm of $\widehat{\boldsymbol{f}}$ is estimated as in [4], i.e., by passing to the Eulerian coordinates under the integral sign and by using the relation

$$\boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t) - \boldsymbol{f}(X_{\boldsymbol{u}}(y,t-\tau),t) = -\int_0^1 \nabla\boldsymbol{f}(X_{\boldsymbol{u}}(y,t-\lambda\tau),t)\boldsymbol{u}(y,t-\lambda\tau)\tau\,\mathrm{d}\lambda.$$

Indeed, we have

$$\|\boldsymbol{f}(X_{\boldsymbol{u}},t)\|^2_{L_2(Q_T^\pm)} = \int_0^T \int_{\Omega_0^\pm} |\boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t)|^2\,\mathrm{d}y\,\mathrm{d}t$$

$$= \int_0^T \mathrm{d}t \int_{\Omega_t^\pm} |\boldsymbol{f}(x,t)|^2 L^{-1}\,\mathrm{d}x \leqslant c\|\boldsymbol{f}\|^2_{L_2(Q_T)},$$

$$\int_0^T \mathrm{d}t \int_{\Omega_0^\pm}\int_{\Omega^\pm} |\boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t) - \boldsymbol{f}(X_{\boldsymbol{u}}(z,t),t)|^2 \frac{\mathrm{d}y\,\mathrm{d}z}{|y-z|^{3+2l}}$$

$$\leqslant c\int_0^T \mathrm{d}t \int_{\Omega_t^\pm}\int_{\Omega_t^\pm} |\boldsymbol{f}(x,t) - \boldsymbol{f}(x',t)|^2 \frac{\mathrm{d}x\,\mathrm{d}x'}{|x-x'|^{3+2l}}$$

$$\leqslant c\int_0^T \mathrm{d}t \int_{\Omega}\int_{\Omega} |\boldsymbol{f}(x,t) - \boldsymbol{f}(x',t)|^2 \frac{\mathrm{d}x\,\mathrm{d}x'}{|x-x'|^{3+2l}},$$

$$\int_0^T\int_0^T\int_{\Omega_0^\pm} |\boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t) - \boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t')|^2 \frac{\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}t'}{|t-t'|^{1+l}}$$

$$\leqslant c\int_0^T\int_0^T\int_{\Omega} \boldsymbol{f}(x,t) - \boldsymbol{f}(x,t)|^2 \frac{\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t'}{|t-t'|^{1+l}},$$

$$\int_0^T \mathrm{d}t \int_0^t \frac{\mathrm{d}\tau}{\tau^{1+l}} \int_{\Omega_0^\pm} |\boldsymbol{f}(X_{\boldsymbol{u}}(y,t),t) - \boldsymbol{f}(X_{\boldsymbol{u}}(y,t-\tau),t)|^2\,\mathrm{d}y$$

$$\leqslant \int_0^1 \mathrm{d}\lambda \int_0^T \mathrm{d}t \int_0^t \frac{\mathrm{d}\tau}{\tau^{1+l}} \int_{\Omega_0^\pm} \big|\nabla\boldsymbol{f}(X_{\boldsymbol{u}}(y,t-\lambda\tau),t)\big|^2 |\boldsymbol{u}(y,t-\lambda\tau)|^2\tau^2\,\mathrm{d}y$$

$$\leqslant cT^{2-l}\|\nabla\boldsymbol{f}\|^2_{L_2(Q_T)} \sup_{Q_T^\pm} |\boldsymbol{u}(y,t)|^2.$$

Hence

$$\|\widehat{\boldsymbol{f}}\|_{\widehat{W}_2^{l,l/2}(\Omega_0^\pm)} \leqslant c\big(\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(\Omega)} + T^{1-l/2}\|\nabla\boldsymbol{f}\|_{L_2(Q_T)} \sup_{Q_T^\pm}|\boldsymbol{u}(y,t)|\big).$$

Let us consider the term

$$\boldsymbol{P} \equiv \nabla_{\boldsymbol{u}}(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+) - p_1\theta^+) = \nabla_{\boldsymbol{u}} \int_0^1 (p'(\overline{\rho}^+ + s\theta^+) - p'(\overline{\rho}^+))\,\mathrm{d}s\theta^+.$$

Since $p \in C^{2+1}(\min\rho^+, \max\rho^+)$, we have

$$\|\boldsymbol{P}\|_{W_2^{l,0}(Q_T^+)} \leqslant c\|\nabla\theta^+\|_{W_2^l(Q_T^+)} \sup_{t<T}\|\theta^+\|_{W_2^{l+1}(\Omega_0^+)} \leqslant c\delta\|\nabla\theta^+\|_{W_2^l(Q_T^+)},$$

$$
\begin{aligned}
&\|\Delta_t(-h)\boldsymbol{P}\|_{L_2(\Omega_0^+)} \\
&\leqslant c(\|\Delta_t(-h)\nabla\theta^+\|_{L_2(\Omega_0^+)}\sup_{Q_T^+}|\theta^+(y,t)| + \|\Delta_t(-h)\theta^+\|_{L_6(\Omega_0^+)}\|\nabla\theta^+\|_{L_3(\Omega_0^+)} \\
&\leqslant c\|\Delta_t(-h)\theta^+\|_{W_2^1(\Omega_0^+)}\|\theta^+\|_{W_2^{1+l}(\Omega_0^+)} \\
&\leqslant c\int\limits_0^h \|\mathcal{D}_t\theta^+(\,\cdot\,,t-\tau)\|_{W_2^1(\Omega_0^+)}\,\mathrm{d}t\|\theta^+\|_{W_2^{1+l}(\Omega_0^+)}, \\
&\frac{1}{T^l}\int\limits_0^T \|\boldsymbol{P}\|^2_{L_2(\Omega_0^+)}\,\mathrm{d}t \leqslant cT^{1-l}\sup_{t<T}\|\nabla\theta^+(\,\cdot\,,t)\|^2_{L_2(\Omega_0^+)}\sup_{Q_T^+}|\theta^+(y,t)|^2,
\end{aligned}
$$

which implies

$$
\|\boldsymbol{P}\|_{\widehat{W}_2^{l,l/2}(Q_T^+)} \leqslant c(T)\delta\big(\|\mathcal{D}_t\theta^+\|_{W_2^{l+1,0}(Q_T^+)} + \|\theta_0^+\|_{W_2^{l+1}(\Omega_0)}\big),
$$

because

$$
\|\theta^+\|_{W_2^{l+1}(\Omega_0)} \leqslant \|\theta_0^+\|_{W_2^{l+1}(\Omega_0)} + \int\limits_0^t \|\mathcal{D}_t\theta^+\|_{W_2^{l+1,0}(Q_T^+)}\,\mathrm{d}\tau.
$$

The term in l_4 containing the expression $(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+) - p_1\theta^+)\big|_{\Gamma_0}$ is estimated in a similar way.

We proceed with the estimates of $l_5(\boldsymbol{u})$ and $l_6(\boldsymbol{u})$. From formulas (2.3)–(2.6) it follows that the the coefficients $g_{\alpha\beta}$ in $\Delta(t)$ are uniformly bounded and coefficients $\dot{g}_{\alpha\beta}$ in $\dot{\Delta}(t)$ are controlled by $\sup|\nabla\boldsymbol{u}|$. By (2.2), l_5 is equal to the sum $l_5 = l_{51} + l_{52}$ with $l_{51} = \sigma\mathcal{D}_t(\boldsymbol{n}\Delta(t))\cdot\int_0^t \boldsymbol{u}(y,\tau)\,\mathrm{d}\tau$, whence

$$
\|l_{51}\|_{W_2^{l-1/2}(\Gamma_0)} \leqslant c\|\nabla\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\Big\|\int\limits_0^t \boldsymbol{u}\,\mathrm{d}\tau\Big\|_{W_2^{l+3/2}(\Gamma_0)} \leqslant c\delta\|\nabla\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)},
$$

$$
\begin{aligned}
\|\Delta_t(-h)l_{51}\|_{L_2(\Gamma_0)} \leqslant{}& c\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{W_2^{3/2-l}(\Gamma_0)}\int\limits_0^t \|\boldsymbol{u}(\,\cdot\,,\tau)\,\mathrm{d}\tau\|_{W_2^{3/2+l}(\Gamma_0)} \\
&+ \|\nabla\boldsymbol{u}\|_{W_2^{3/2-l}(\Gamma_0)}\int\limits_0^h \|\boldsymbol{u}(\,\cdot\,,t-\tau)\|_{W_2^{3/2+l}(\Gamma_0)}\,\mathrm{d}\tau,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{T^{l-1/2}}\int\limits_0^T \|l_{51}\|^2_{L_2(\Gamma_0)}\,\mathrm{d}t \leqslant{}& \frac{c}{T^{l-1/2}}\int\limits_0^T \|\nabla\boldsymbol{u}\|^2_{W_2^{3/2-l}(\Gamma_0)}\,\mathrm{d}t\Big(\int\limits_0^t \|\boldsymbol{u}\|_{W_2^{l+3/2}(\Gamma_0)}\,\mathrm{d}\tau\Big)^2 \\
&\leqslant \frac{c\delta^2T}{T^{l-1/2}}\int\limits_0^t \|\nabla\boldsymbol{u}\|^2_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\,\mathrm{d}\tau, \quad \varkappa \leqslant l - 1/2,
\end{aligned}
$$

which implies

$$\|l_{51}\|_{\widehat{W}_2^{l-1/2,l/2-1/4}(G_T)} \leqslant c(T)\delta\big(\|\nabla \boldsymbol{u}\|_{\widehat{W}_2^{l/2-1/4}((0,T);W_2^{1-\varkappa}(\Gamma_0))} + \|\boldsymbol{u}\|_{W_2^{l+3/2-\varkappa,0}(G_T)}\big). \tag{2.19}$$

The expression $l_{52} = \sigma(\int\limits_0^t \dot{\boldsymbol{n}}\,\mathrm{d}\tau \cdot \Delta(t)\boldsymbol{u} + \boldsymbol{n}_0 \cdot \int\limits_0^t \dot{\Delta}(\tau)\,\mathrm{d}\tau \boldsymbol{u})$ is estimated in the same way.

It remains to estimate $l_6(\boldsymbol{u})$. We have

$$\begin{aligned}\|l_6\|_{W_2^{l-1/2}(\Gamma_0)} &\leqslant c\big(\|\dot{\boldsymbol{n}}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)} + \|\nabla \boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\big)\|\boldsymbol{y}\|_{W_2^{l+3/2}(\Gamma_0)}\\ &\leqslant c\|\nabla \boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)},\end{aligned}$$

$$\frac{1}{T^{l-1/2}}\int\limits_0^T \|l_6\|^2_{L_2(\Gamma_0)}\,\mathrm{d}t \leqslant cT^{3/2-l}\sup_{t<T}\|\nabla \boldsymbol{u}\|^2_{W_2^{3/2-l}(\Gamma_0)},$$

$$\begin{aligned}\|\triangle_t(-h)l_6\|_{L_2(\Gamma_0)} \leqslant c\big(&\|\triangle_t(-h)\nabla \boldsymbol{u}\|_{W_2^{3/2-l}(\Gamma_0)}\\ &+\|\nabla \boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\sqrt{h}\|\nabla \boldsymbol{u}\|_{W_2^{l+1/2-\varkappa,0}(G_{t-h,t})}\big)\|\boldsymbol{y}\|_{W_2^{l+3/2}(\Gamma_0)},\end{aligned}$$

hence

$$\|l_6\|_{\widehat{W}_2^{l-1/2,l/2-1/4}(G_T)} \leqslant c\big(\|\nabla \boldsymbol{u}\|_{\widehat{W}_2^{l/2-1/4}((0,T);W_2^{1-\varkappa}(\Gamma_0))} + \sup_{t<T}\|\nabla \boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\big).$$

In view of Propositions 1 and 2 in [1], the estimates of the expressions (2.2) obtained above imply inequalities (2.11), (2.12) with the constants bounded for small T.

We continue the estimate of l_6 by applying the interpolation inequality with the constants bounded for small T. According to Proposition 1 in [1], the vector field $\boldsymbol{u}^{\pm}$ admits an extension $\boldsymbol{U}^{\pm}(x,t)$ into the domain $t<0$ such that the norms $\|\boldsymbol{U}^{\pm}\|_{W_2^{2+l,1+l/2}(\Omega_0^{\pm}\times(-\infty,T))}$ and $\|\boldsymbol{u}^{\pm}\|_{H^{2+l,1+l/2}(Q_T^{\pm}))}$ are equivalent and

$$\|\boldsymbol{U}^{\pm}\|_{W_2^{2+l,1+l/2}(\Omega_0^{\pm}\times(-\infty,0))} \leqslant c\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_0^{\pm})}.$$

The inequality

$$\begin{aligned}&\|\nabla \boldsymbol{u}\|_{\widehat{W}_2^{l/2-1/4}((0,T);W_2^{l+1/2-\varkappa}(\Gamma_0))} + \sup_{t<T}\|\nabla \boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\\ &\leqslant \epsilon_1\|\boldsymbol{U}\|_{W_2^{2+l,1+l/2}(\cup\Omega_0^{\pm}\times(-\infty,T))} + c(\epsilon_1)\|\boldsymbol{U}\|_{L_2(\cup\Omega_0^{\pm}\times(-\infty,T))}\\ &\leqslant c(\epsilon_1)\|\boldsymbol{u}\|_{H^{2+l,1+l/2}(\cup Q_T^{\pm})} + c(\epsilon_1)(\|\boldsymbol{u}\|_{L_2(Q_T^{\pm})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Gamma_0)})\end{aligned}$$

yields the desired result:

$$\|l_6\|_{\widehat{W}_2^{l-1/2,1/2-1/4}(G_T)} \leqslant c\epsilon_1\|\boldsymbol{u}\|_{H^{2+l,1+l/2}(G_T^{\pm})} + c(\epsilon_1)\big(\|\boldsymbol{u}\|_{L_2(Q_T^{\pm})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Gamma_0)}\big). \tag{2.20}$$

Similar inequalities hold for l_5 and $\boldsymbol{P}$. This completes the proof of Proposition 1. $\square$

Proof of Theorem 2. We seek a solution of (1.3) in the form $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{w}$, $\theta = \theta_1 + \theta_2$, where $\boldsymbol{u}_1, \theta_1$ and $\boldsymbol{w}, \theta_2$ are defined as the solutions of

$$\begin{cases} \rho^- \mathcal{D}_t \boldsymbol{u}_1^- - \nabla \cdot \mathbb{T}^-(\boldsymbol{u}_1^-) + \nabla \theta_1^- = 0, \quad \nabla \cdot \boldsymbol{u}_1^- = 0 \text{ in } \Omega_0^-, \\ \overline{\rho}^+ \mathcal{D}_t \boldsymbol{u}_1^+ - \nabla \cdot \mathbb{T}^+(\boldsymbol{u}_1^+) + p_1 \nabla \theta_1^+ = 0, \\ \mathcal{D}_t \theta_1^+ + \overline{\rho}^+ \nabla \cdot \boldsymbol{u}_1^+ = 0 \quad \text{in } \Omega_0^+, \ t > 0, \\ \boldsymbol{u}_1^+|_\Sigma = 0, \quad \boldsymbol{u}_1^\pm(y,0) = \boldsymbol{u}_0^\pm(y) \text{ in } \Omega_0^\pm, \quad \theta_1^+(y,0) = \theta_0^+(y) \text{ in } \Omega_0^+, \\ [\boldsymbol{u}_1]|_{\Gamma_0} = 0, \quad [\mu \Pi_0 \mathbb{S}(\boldsymbol{u}_1)\boldsymbol{n}_0]|_{\Gamma_0} = 0, \\ -p_1 \theta_1^+ + \theta_1^- + [\boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{u}_1)\boldsymbol{n}_0] + \sigma \boldsymbol{n}_0 \cdot \Delta(0) \int\limits_0^t \boldsymbol{u}_1(\xi,\tau)\,d\tau|_{\Gamma_0} \\ \qquad = -\sigma(H|_{t=0} + \frac{2}{R_0}), \end{cases} \tag{2.21}$$

$$\begin{cases} \rho^- \mathcal{D}_t \boldsymbol{w}^- - \nabla \cdot \mathbb{T}^-(\boldsymbol{w}^-) + \nabla \theta_2^- = \boldsymbol{l}_1^-(\boldsymbol{u},\theta) + \rho^- \widehat{\boldsymbol{f}}^-, \\ \qquad \nabla \cdot \boldsymbol{w}^- = l_2^-(\boldsymbol{u}) \text{ in } \Omega_0^-, \\ \overline{\rho}^+ \mathcal{D}_t \boldsymbol{w}^+ - \nabla \cdot \mathbb{T}^+(\boldsymbol{w}^+) + p_1 \nabla \theta_2^+ = \boldsymbol{l}_1^+(\boldsymbol{u},\theta) + (\overline{\rho}^+ + \theta^+)\widehat{\boldsymbol{f}}^+, \\ \mathcal{D}_t \theta_2^+ + \overline{\rho}^+ \nabla \cdot \boldsymbol{w}^+ = l_2^+(\boldsymbol{u},\theta), \quad \theta_2^+(y,0) = 0 \text{ in } \Omega_0^+, \\ \boldsymbol{w}^+|_\Sigma = 0, \quad \boldsymbol{w}^\pm(y,0) = 0, \\ [\boldsymbol{w}]|_{\Gamma_0} = 0, \quad [\mu \Pi_0 \mathbb{S}(\boldsymbol{w})\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{l}_3(\boldsymbol{u}), \\ -p_1 \theta_2^+ + \theta_2^- + [\boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{w})\boldsymbol{n}_0] + \sigma \boldsymbol{n}_0 \cdot \Delta(0) \int\limits_0^t \boldsymbol{w}(y,\tau) d\tau|_{\Gamma_0} \\ = l_4(\boldsymbol{u}) - \int\limits_0^t (l_5(\boldsymbol{u}) + l_6(\boldsymbol{u}))) d\tau \end{cases} \tag{2.22}$$

By Theorem 1, problem (2.21) is uniquely solvable and the solution satisfies the inequality

$$\begin{aligned} Y(\boldsymbol{u}_1, \theta_1) \leqslant c(T)\Big(&\|\boldsymbol{u}_0\|_{W_2^{1+l}(\cup \Omega_0^\pm)} + \|\theta_0^+\|_{W_2^{l+1}(\Omega_0^+)} \\ &+ \sigma \Big\| H_0 + \frac{2}{R_0} \Big\|_{W_2^{l+1/2}(\Gamma_0)} \Big) \leqslant c(T)\epsilon. \end{aligned} \tag{2.23}$$

The solution of (2.22) can be constructed by iteration in accordance with the following scheme:

$$\begin{cases} \rho^- \mathcal{D}_t \boldsymbol{w}_{m+1}^- - \nabla \cdot \mathbb{T}^-(\boldsymbol{w}_{m+1}^-) + \nabla \theta_{2,m+1}^- = \boldsymbol{l}_1^-(\boldsymbol{u}_m, \theta_m) + \rho^- \widehat{\boldsymbol{f}}_m^-, \\ \nabla \cdot \boldsymbol{w}_{m+1}^- = l_2^-(\boldsymbol{u}_m) \text{ in } \Omega_0^-, \\ \overline{\rho}^+ \mathcal{D}_t \boldsymbol{w}_{m+1}^+ - \nabla \cdot \mathbb{T}^+(\boldsymbol{w}_{m+1}^+) + p_1 \nabla \theta_{2,m+1}^+ \\ \qquad = \boldsymbol{l}_1^+(\boldsymbol{u}_m, \theta_m) + (\overline{\rho}^+ + \theta_{2,m}^+)\widehat{\boldsymbol{f}}_m^+, \\ \mathcal{D}_t \theta_{2,m+1}^+ + \overline{\rho}^+ \nabla \cdot \boldsymbol{w}_{m+1}^+ = l_2^+(\boldsymbol{u}_m, \theta_m), \\ \boldsymbol{w}_{m+1}^-|_\Sigma = 0, \quad \boldsymbol{w}_{m+1}(y,0) = 0, \quad \theta_{2,m+1}^+(y,0) = 0 \text{ in } \Omega_0^+, \\ [\boldsymbol{w}_{m+1}]|_{\Gamma_0} = 0, \quad [\mu \Pi_0 \mathbb{S}(\boldsymbol{w}_{m+1})\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{l}_3(\boldsymbol{u}_m), \\ -p_1 \theta_2^+ + \theta_2^- + [\boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{w}_{m+1})\boldsymbol{n}_0]|_{\Gamma_0} + \sigma \boldsymbol{n}_0 \cdot \Delta(0) \int\limits_0^t \boldsymbol{w}_{m+1}(y,\tau)\,d\tau|_{\Gamma_0} \\ = l_4(\boldsymbol{u}_m) - \int\limits_0^t (l_5(\boldsymbol{u}_m) + l_6(\boldsymbol{u}_m))\,d\tau, \end{cases} \tag{2.24}$$

where $m = 1, 2, \dots$, $\widehat{\boldsymbol{f}}_m = \boldsymbol{f}(X_{\boldsymbol{u}_m}, t)$, $\boldsymbol{u}_m = \boldsymbol{w}_m + \boldsymbol{u}_1$, $\theta_m = \theta_1 + \theta_{2,m}$; we also set $\boldsymbol{w}_1 = 0$, $\theta_{2,1} = 0$. In view of Theorem 1 and Proposition 1, Problem (2.24) with given $\boldsymbol{w}_m \in H^{2+l,1+l/2}(Q_T^{\pm})$, $\nabla\theta_{2,m}^{+} \in \widehat{W}_2^{l,l/2}(Q_T^{\pm})$, $\theta_{2,m} \in \widehat{W}_2^{0,l/2}(G_T)$ satisfying (2.10) is uniquely solvable and the solution satisfies the inequality

$$\begin{aligned} Y(\boldsymbol{w}_{m+1}, \theta_{2,m}^{+}) \leqslant c(T)\Big(& Z(\boldsymbol{u}_m, \theta_m^{+}) + \|l_4(\boldsymbol{u}_m, \theta_m^{+})\|_{W_2^{l/2}((0,T),W_2^{1/2}(\Gamma_0))} \\ &+ \|l_5(\boldsymbol{u}_m)\|_{W_2^{l-1/2,l/2-1/4}(G_T)} + \|l_6(\boldsymbol{u}_m)\|_{W_2^{l-1/2,l/2-1/4}(G_T)} \\ &+ \|\widehat{\boldsymbol{f}}_m\|_{W_2^{l,l/2}(\cup Q^{\pm})} + \|\theta_m^{+}\widehat{\boldsymbol{f}}_m^{+}\|_{W_2^{l,l/2}(Q_T^{+})}\Big). \end{aligned}$$

In view of Proposition 1, we have

$$\|\widehat{\boldsymbol{f}}_m\|_{W_2^{l,l/2}(Q_T^{\pm})} \leqslant c\big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \sup_{t<T}\|\boldsymbol{u}_m\|_{W_2^{l+1-\varkappa}(\cup\Omega_0^{\pm})}\big),$$

$$\begin{aligned} \|\theta_m^{+}\widehat{\boldsymbol{f}}_m\|_{W_2^{l,l/2}(Q_T^{+})} \leqslant c\Big(&\big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \sup_{t<T}\|\boldsymbol{u}_m\|_{W_2^{l+1-\varkappa}(\cup\Omega_0^{\pm})}\big) \\ &\times \sup_{t<T}\|\theta_m^{+}(\,\cdot\,,t)\|_{W_2^{l+1}(\Omega_0^{+})} + \|\boldsymbol{f}\|_{W_2^{l,0}(Q_T)}\|\mathcal{D}_t\theta_m^{+}\|_{W_2^{1,0}(Q_T^{+})}\Big) \\ \leqslant c\big(&\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \delta\big)Y_m(T). \end{aligned}$$

Assume that

$$(1+\sqrt{T})Y_m(T) \leqslant \delta, \tag{2.25}$$

where $Y_m(T) = Y(\boldsymbol{u}_m, \theta_m)$ (this condition implies (2.10), because

$$\begin{aligned} \|\theta^{+}\|_{W_2^{l+1}(\Omega_0^{+})} &\leqslant \|\theta_0^{+}\|_{W_2^{l+1}(\Omega_0^{+})} + \int_0^t \|\mathcal{D}_t\theta^{+}\|_{W_2^{l+1}(\Omega_0^{+})}\,\mathrm{d}\tau \\ &\leqslant \|\theta_0^{+}\|_{W_2^{l+1}(\Omega_0^{+})} + \sqrt{t}\|\mathcal{D}_t\theta^{+}\|_{W_2^{l+1,0}(Q_t^{+})}). \end{aligned}$$

Then, collecting the above estimate, we arrive at

$$\begin{aligned} Y_{m+1}(T) &\leqslant Y(\boldsymbol{u}_1, \theta_1) + Y(\boldsymbol{w}_m, \theta_{2,m}) \\ &\leqslant c(T)\delta_1 Y_m(T) + c(\delta_1)\|\boldsymbol{u}_m\|_{L_2(Q_T)} + cF(T), \end{aligned} \tag{2.26}$$

where $\delta_1 = \delta + \epsilon$ and

$$F(T) = \|\boldsymbol{u}_0\|_{W_2^{1+l}(\cup\Omega_0^{\pm})} + \|\theta_0^{+}\|_{W_2^{l+1}(\Omega_0^{+})} + \|H_0 + \frac{2}{R_0}\|_{W_2^{l+1/2}(\Gamma_0)} + \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}.$$

Inequality (2.26) is valid for arbitrary $t < T$, and it implies

$$Y_{m+1}^2(t) \leqslant \delta_2 Y_m^2(t) + c_1\int_0^t Y_m^2(\tau)\,\mathrm{d}\tau + c_2F^2(t), \quad \delta_2 = 2c^2(T)\delta_1^2, \tag{2.27}$$

because

$$\|\boldsymbol{u}_m(\,\cdot\,,t)\|^2_{\Omega_0} \leqslant \|\boldsymbol{u}_0\|^2_{\Omega_0} + 2\int\limits_0^t \|\boldsymbol{u}_m\|_{\Omega_0}\|\mathcal{D}_t\boldsymbol{u}_m\|_{\Omega_0}\,\mathrm{d}\tau \leqslant \|\boldsymbol{u}_0\|^2_{\Omega_0} + 2Y^2_m(t)$$

and

$$\int\limits_0^t \|\boldsymbol{u}_m\|^2_{\Omega_0}\,\mathrm{d}\tau \leqslant t\|\boldsymbol{u}_0\|^2_{\Omega_0} + 2\int\limits_0^t Y^2_m(\tau)\,\mathrm{d}\tau.$$

If (2.25) is true for all $Y_j(T)$, $j = 1,\dots,m$, then (2.27) yields

$$\begin{aligned}Y^2_{m+1}(t) &\leqslant c_2F^2(t) + \mathcal{A}(Y^2_m + c_2F^2(t))\\ &\leqslant \mathcal{A}^{m+1}Y^2_0(t) + c_2(F^2 + \mathcal{A}F^2 + \cdots + \mathcal{A}^mF^2(t)),\end{aligned}$$

where $\mathcal{A}f(t) = \delta_2 f(t) + c_1\int\limits_0^t f(\tau)\,\mathrm{d}\tau$, hence $\mathcal{A}^jf(t) \leqslant \sum_{j=0}^m C^j_m\delta_2^{m-j}\frac{(c_1t)^j}{j!}f(t)$. From the formula

$$\sum_{m=0}^\infty\sum_{j=0}^m C^j_m\delta_2^{m-j}\frac{(c_1t)^j}{j!} = \frac{1}{1-\delta}e^{\frac{c_1t}{1-\delta}} \tag{2.28}$$

which is due to N. D. Filonov, it follows that

$$Y^2_{m+1}(t) \leqslant c(T)\frac{1}{1-\delta_2}e^{\frac{c_1t}{1-\delta_2}}|F^2(t)$$

with a constant independent of m. For ϵ sufficiently small, the right-hand side is smaller than $\delta/(1+\sqrt{T})$, hence (2.25) is fulfilled for $Y_{m+1}(T)$ and consequently for all $Y_j(T)$, $j = 1,2,\dots$.

The convergence of the sequence $(\boldsymbol{u}_m,\theta_m)$ to the solution of (1.3) is established by estimating

$$Y^2(\boldsymbol{u}_{m+1}-\boldsymbol{u}_m,\theta_{m+1}-\theta_m) = Y^2(\boldsymbol{w}_{m+1}-\boldsymbol{w}_m,\theta_{2,m+1}-\theta_{2,m}) \equiv y_{m+1}(t)$$

by the differences $l(\boldsymbol{u}_m,\theta_m) - l(\boldsymbol{u}_{m-1},\theta_{m-1})$ of the expressions (2.2) and by

$$\widehat{\boldsymbol{f}}_m - \widehat{\boldsymbol{f}}_{m-1} = \int\limits_0^1\int\limits_0^1 \nabla\boldsymbol{f}(X_{m-1} + \lambda(X_m - X_{m-1})\,\mathrm{d}\lambda(X_m - X_{m-1}), \quad X_m = X_{\boldsymbol{u}_m}.$$

The details of a lengthy proof are omitted; in particular, the conditions $\nabla\boldsymbol{f} \in W^{l,l/2}_2(Q_T)$ and $\mathcal{D}^2_y\boldsymbol{f} \in L_2(Q_T)$ should be used. As a result, we obtain

$$y_{m+1}(t) \leqslant \delta y_m + c\int\limits_0^t y_m(\tau)\,\mathrm{d}\tau \leqslant \mathcal{A}y_m \tag{2.29}$$

and, as a consequence, $y_{m+1}(t) \leqslant \mathcal{A}^my_1(t)$, which guarantees the convergence of the sequence $(\boldsymbol{u}_m,\theta_m)$.

The uniqueness of the solution of Problem (1.3) follows from the same estimate (2.29) applied to the difference of two possible solutions of (1.6). Theorem 2 is proved. □

In view of (1.4), (1.5), and (2.10), we have $\boldsymbol{n}\cdot\boldsymbol{n}_0=\frac{\boldsymbol{n}_0\cdot\widehat{\mathbb{L}}^T\boldsymbol{n}_0}{|\widehat{\mathbb{L}}^T\boldsymbol{n}_0|}>0$.

We proceed with establishing some additional properties of the solution of problem (1.3) that are necessary for the construction of the solution in the infinite time interval. We notice that the boundedness of the norm $Y(\boldsymbol{u},\theta)$ in (2.8) implies

$$H=\boldsymbol{n}\cdot\Delta(t)X_{\boldsymbol{u}}\in W_2^{l-1/2,0}(G_T),\quad \mathcal{D}_tH\in W_2^{l-1/2,0}(G_T).$$

We assume that Γ_0 is close to $S_{R_0}=\{|y|=R_0\}$ and can be defined by the equation

$$y=\eta+\boldsymbol{N}(\eta)r_0(\eta), \tag{2.30}$$

where $\boldsymbol{N}(\eta)=\boldsymbol{\eta}/|\eta|$, $\eta\in S_{R_0}$, and r_0 is a given small function belonging to $W_2^{l+5/2}(S_{R_0})$. Without loss of generality we may assume that the origin is the barycenter of Ω_0^-. For $t>0$, the barycenter of Ω_t^- is located at the point with the coordinates $h_i=|\Omega_0^-|^{-1}\int_{\Omega_t^-}x_i\,\mathrm{d}x$, $i=1,2,3$, hence

$$\frac{\mathrm{d}h_i(t)}{\mathrm{d}t}=\frac{1}{|\Omega_0^-|}\int\limits_{\Omega_t^-}\nabla\cdot x_i\boldsymbol{v}^-(x,t)\,\mathrm{d}x=\frac{1}{|\Omega_0^-|}\int\limits_{\Omega_0^-}v_i^-(x,t)\,\mathrm{d}x=\frac{1}{|\Omega_0^-|}\int\limits_{\Omega_0^-}u_i^-(y,t)\,\mathrm{d}y,$$

$$h_i(t)=\frac{1}{|\Omega_0^-|}\int\limits_0^t\mathrm{d}\tau\int\limits_{\Omega_0^-}u_i^-(y,\tau)\,\mathrm{d}y.$$

The surface Γ_t can be defined by an equation similar to (2.30) on the sphere of radius R_0 with center at the point $h(t)$. This is equivalent to the fact that the shifted surface $\Gamma_{t,h}=\{x=X_{\boldsymbol{u}}(y,t)-h(t)\equiv X_{\boldsymbol{u},h}(y,t),\ \ y\in\Gamma_0,\}$ is given by

$$x=\eta+\boldsymbol{N}(\eta)r(\eta,t),\quad \eta\in S_{R_0}. \tag{2.31}$$

It is clear that η is the point of S_{R_0} closest to $\Gamma_{\boldsymbol{u},h}$:

$$\eta=\overline{x}=\overline{X}_{\boldsymbol{u},h}(y,t)=R_0\frac{X_{\boldsymbol{u},h}}{|X_{\boldsymbol{u},h}|}\equiv\mathcal{X}(y,t), \tag{2.32}$$

whereas $r(\eta,t)=|X_{\boldsymbol{u},h}|-R_0\equiv r'(y,t)$ is the signed distance of $X_{\boldsymbol{u},h}\Gamma_0$ to S_{R_0}. For small δ and ϵ, equation $x=X_{\boldsymbol{u}}(y,t)$ establishes one-to-one correspondence between Ω_0^+ and Ω_t^+, as well as between Γ_0 and Γ_t, and (2.32) maps Γ_0 onto S_{R_0}. It follows that

$$c_1\|f\|_{W_2^\mu(\Gamma_0)}\leqslant\|f_1\|_{W_2^\mu(S_{R_0})}\leqslant c_2\|f\|_{W_2^\mu(\Gamma_0)}, \tag{2.33}$$

where $f_1(\eta)=f(\overline{X}_{\boldsymbol{u},h})$ and $\mu\leqslant l+3/2$, in particular, we have

$$\begin{aligned}&\|r(\,\cdot\,,t)\|_{W_2^{l+3/2}(S_{R_0})}\\&\leqslant c\|r'(\,\cdot\,,t)\|_{W_2^{l+3/2}(\Gamma_0)}\leqslant c\||X_{\boldsymbol{u},h}|-R_0\|_{W_2^{l+3/2}(\Gamma_0)}\\&\leqslant c(\||\boldsymbol{y}|-R_0|\|_{W_2^{l+3/2}(\Gamma_0)}+\int\limits_0^t\|\frac{\mathrm{d}}{\mathrm{d}\tau}X_{\boldsymbol{u},h}(\,\cdot\,,\tau)\|_{W_2^{l+3/2}(\Gamma_0)}\,\mathrm{d}\tau)\\&\leqslant c(\|\rho_0\|_{W_2^{l+3/2}(S_{R_0})}+\sqrt{t}\|\boldsymbol{u}\|_{W_2^{l+3/2,0}(\Gamma_0)})\leqslant c(\delta+\epsilon),\end{aligned} \tag{2.34}$$

where $r'(y,t) = |X_{\boldsymbol{u},h}| - R_0$. The relation $|\Omega_t^-| = 4\pi R_0^3/3$ and the fact that the origin is a barycenter of the shifted domain Ω_t^- can be expressed in terms of $r(\eta,t)$ as follows:

$$
\begin{aligned}
&\int\limits_{S_{R_0}} \big((R_0 + r(\eta,t))^3 - R_0^3\big)\,\mathrm{d}S = 0,\\
&\int\limits_{S_{R_0}} \eta_i\big((R_0 + r(\eta,t))^4 - R_0^4\big)\,\mathrm{d}S = 0, \quad i = 1,2,3.
\end{aligned} \tag{2.35}
$$

In the variables $\eta \in S_{R_0}$, the equation $-(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+)) + \theta^- + [\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}] = -\sigma(H + \frac{2}{R_0})$ has the form

$$
\begin{aligned}
&-(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+)) + \theta^- + [\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}]|_{y=\mathcal{X}^{-1}(\eta,t)}\\
&= -\sigma\Big(\frac{R_0^2}{R_0 + r}\nabla_{S_{R_0}}\cdot\frac{\nabla_{S_{R_0}} r}{\sqrt{g}} - \frac{2}{\sqrt{g}} + \frac{2}{R_0}\Big),
\end{aligned} \tag{2.36}
$$

where $g = (R_0 + r)^2 + |R_0\nabla_{S_{R_0}} r|^2$; it can be viewed as a nonlinear elliptic equation on S_{R_0} with respect to r with $-(p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+)) + \theta^- + [\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}\boldsymbol{n}] \in W_2^{l+1/2,0}(\mathcal{S}_T) \cap W_2^{l/2}((0,T); W_2^{1/2}(S_{R_0}))$, $\mathcal{S}_T = S_{R_0}\times(0,T)$, the solution of which satisfies inequality (2.34). Hence one can conclude from the regularity theorem for elliptic equations that $r \in W_2^{l+5/2,0}(\mathcal{S}_T)$ and

$$
\begin{aligned}
\|r\|_{W_2^{5/2+l,0}(\mathcal{S}_T)} &\leqslant c\sum_{\pm}\big(\|\theta^\pm\|_{W_2^{l+1/2,0}(\mathcal{S}_T)} + \|\nabla\boldsymbol{u}^\pm\|_{W_2^{l+1/2,0}(\mathcal{S}_T)}\big)\\
&\leqslant c\big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\cup\Omega_0^\pm)} + \|\theta_0^+\|_{W_2^{l+1}(\Omega_0^+)}\\
&\quad + \sigma\|H_0 + \frac{2}{R_0}\|_{W_2^{l+1/2}(\Gamma_0)} + \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}\big).
\end{aligned} \tag{2.37}
$$

We also estimate the time derivative of $r(\eta,t)$. Let $S'_{R_0} \subset S_{R_0}$ and let (φ_1,φ_2) be local coordinates on S'_{R_0}; they can be regarded also as local coordinates on $\Gamma'_{t,h} \subset \Gamma_{t,h} = \{x = \eta + N(\eta)r(\eta,t), \eta \in S_{R_0}\}$. Since

$$
\begin{aligned}
&\mathcal{D}_t r'(y,t) = \mathcal{D}_t r(\eta,t) + \sum_{\alpha=1}^{2}\frac{\partial r(\eta,t)}{\partial\varphi_\alpha}\frac{\partial\varphi_\alpha(y,t)}{\partial t}\Big|_{\eta=\mathcal{X}^{-1}(y,t)}, \quad y \in S'_{R_0},\\
&\|\mathcal{D}_t r'\|_{W_2^{l+3/2,0}(G_T)} \leqslant c\|\boldsymbol{u}\|_{W_2^{l+3/2,0}(G_T)},
\end{aligned}
$$

we show, in view of (2.32) and

$$
\begin{aligned}
\frac{\partial\boldsymbol{\eta}(y,t)}{\partial t} &= R_0\frac{\partial}{\partial t}\frac{X_{\boldsymbol{u},h}}{|X_{\boldsymbol{u},h}|}\\
&= R_0\Big(\Big(\boldsymbol{u}(y,t) - \frac{\mathrm{d}\boldsymbol{h}(t)}{\mathrm{d}t}\Big)\frac{1}{|X_{\boldsymbol{u},h}|} - \frac{X_{\boldsymbol{u},h}}{|X_{\boldsymbol{u},h}|^3}\Big(\Big(\boldsymbol{u} - \frac{\mathrm{d}\boldsymbol{h}(t)}{\mathrm{d}t}\Big)\cdot X_{\boldsymbol{u},h}\Big)\Big),
\end{aligned}
$$

that the inequalities

$$\begin{aligned}
&\|r\|_{W_2^{l+5/2,0}(\mathcal{S}_T)} + \|\mathcal{D}_t r\|_{W_2^{l+3/2,0}(\mathcal{S}_T)} \\
&\leqslant c\Big(\sum_{\pm} \big(\|\theta^{\pm}\|_{W_2^{l+1/2,0}(\mathcal{S}_T)} + \|\nabla \boldsymbol{u}^{\pm}\|_{W_2^{l+1/2,0}(\mathcal{S}_T)}\big) + \|\boldsymbol{u}\|_{W_2^{l+1/2,0}(\mathcal{S}_T)}\Big) \qquad (2.38)\\
&\leqslant c(T)\Big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\cup\Omega_0^{\pm})} + \|\theta_0^{+}\|_{W_2^{l+1}(\Omega_0^{+})} + \sigma\Big\|H_0 + \frac{2}{R_0}\Big\|_{W_2^{l+1/2}(\Gamma_0)} + \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}\Big)
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&\sup_{t<T} \|r\|_{W_2^{l+2}(S_{R_0})} \\
&\leqslant c(T)\Big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\cup\Omega_0^{\pm})} + \|\theta_0^{+}\|_{W_2^{l+1}(\Omega_0^{+})} + \|r_0\|_{W_2^{l+5/2}(S_{R_0})} + \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}\Big), \qquad (2.39)
\end{aligned}$$

are satisfied (see [1]).

Thus, we have proved that under the assumptions of Theorem 2 problem (1.3) is solvable and the solution satisfies (2.8), (2.38), and (2.39). Inequality (2.10) is fulfilled with $\delta = c(T)\epsilon$, which can be made small by the choice of ϵ.

In §4 it will be shown that $r(y,t) \in W_2^{l+5/2}$ if $t > 0$ and $p, \boldsymbol{f}$ satisfy some additional assumptions.

3 Estimate of the solution in the norms with exponential weight.

In this section, we obtain estimates of the solution of problem (1.3) that are necessary for its extension into an infinite time interval. We notice that inequalities (2.11) and (2.38) extend to the weighted Sobolev–Slobodetskiĭ spaces with the exponential weight $e^{\beta t}$, $\beta > 0$. For technical reasons, we fix $T > 2$.

Proposition 1'. *If* (2.10) *is fulfilled, then the solution of* (1.3) *satisfies the inequalities*

$$\begin{aligned}
&Z(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) \leqslant c\delta Y(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta),\\
&\|e^{\beta t} l_6\|_{W_2^{l-1/2,l/2-1/4}(G_T)} \leqslant c\big(\|e^{\beta t}\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa,0}(G_T)} \\
&\qquad\qquad + \|e^{\beta t}\nabla_0 \boldsymbol{u}\|_{W_2^{l/2-1/4}((0,T);W_2^{3/2-l}(\Gamma_0))}\big),\\
&\|e^{\beta t} r\|_{W_2^{l+5/2,0}(\mathcal{S}_T)} + \|e^{\beta t}\mathcal{D}_t r\|_{W_2^{l+3/2,0}(\mathcal{S}_T)} \\
&\qquad\qquad \leqslant c\Big(\sum_{\pm}\big(\|e^{\beta t}\theta^{\pm}\|_{W_2^{l+1/2,0}(\mathcal{S}_T)} + \|e^{\beta t}\nabla \boldsymbol{u}^{\pm}\|_{W_2^{l+1/2,0}(\mathcal{S}_T)}\big)\Big),\\
&\|e^{\beta t}\widehat{\boldsymbol{f}}\|_{W_2^{l,l/2}(Q_T)} \leqslant c\big(\|e^{\beta t}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \|\nabla \boldsymbol{f}\|_{L_2(Q_T)} \sup_{Q_T} e^{\beta t}|\boldsymbol{u}(y,t)|\big).
\end{aligned}$$

The proof is the same as that of Proposition 1 and inequality (2.38).

We pass to the estimates of the solution of (1.3) in weighted norms.

Theorem 3. *The solution of problem* (1.1) *constructed above satisfies the inequality*

$$
\begin{aligned}
&e^{2\beta t}\big(\|\boldsymbol{v}(\,\cdot\,,t)\|^2_{\Omega}+\|\vartheta^+\|^2_{\Omega_t^+}+\|r(\,\cdot\,,t)\|^2_{W_2^1(S_{R_0})}\big)\\
&+\int\limits_0^t e^{2\beta\tau}\big(\|\boldsymbol{v}(\,\cdot\,,\tau)\|^2_{\Omega}+\|\vartheta^+(\,\cdot\,,\tau)\|^2_{\Omega_t^+}+\|r(\,\cdot\,,\tau)\|^2_{W_2^1(S_{R_0})}\big)\,\mathrm{d}\tau\\
&\leqslant c\Big(\|\boldsymbol{v}_0\|^2_{\Omega}+\|r_0\|^2_{W_2^1(S_{R_0})}+\|\theta_0^+\|^2_{\Omega_0^+}+\int\limits_0^t e^{2\beta\tau}\|\boldsymbol{f}(\,\cdot\,,\tau)\|^2_{\Omega}d\tau\Big),
\end{aligned}
\tag{3.1}
$$

with a certain $\beta>0$ *and with a constant* c *independent of* $t\in(0,T)$.

Proof. We make use of the energy relation for the solution of (1.1):

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\sqrt{\rho^{\pm}}\boldsymbol{v}\big\|^2_{\Omega}+\int\limits_{\Omega}\mathbb{T}(\boldsymbol{v}):\nabla\boldsymbol{v}\,\mathrm{d}x-\int\limits_{\Omega_t^+}p(\rho^+)\nabla\cdot\boldsymbol{v}^+\,\mathrm{d}x&\\
-\sigma\int\limits_{\Gamma_t}H\boldsymbol{n}\cdot\boldsymbol{v}\,\mathrm{d}S=\int\limits_{\Omega}\rho\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x.&
\end{aligned}
\tag{3.2}
$$

Since $\int_{\Omega_t^+}\nabla\cdot\boldsymbol{v}\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_t^+|=0$, $\int_{\Gamma_t}H\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S=-\frac{\mathrm{d}}{\mathrm{d}t}|\Gamma_t|$, we have

$$
\int\limits_{\Omega_t^+}p(\rho^+)\nabla\cdot\boldsymbol{v}\,\mathrm{d}x=p_1\int\limits_{\Omega_t^+}\vartheta^+\nabla\cdot\boldsymbol{v}\,\mathrm{d}x+\int\limits_{\Omega_t^+}\big(p(\rho^+)-p(\overline{\rho}^+)-p_1\vartheta^+\big)\nabla\cdot\boldsymbol{v}\,\mathrm{d}x
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t^+}\vartheta^{+2}\,\mathrm{d}x&=\int\limits_{\Omega_t^+}\big(2\vartheta^+\mathcal{D}_t\vartheta^++\nabla\cdot(\boldsymbol{v}^+\vartheta^{+2})\big)\,\mathrm{d}x\\
&=\int\limits_{\Omega_t^+}\big(-2\vartheta^+\nabla\cdot(\overline{\rho}^++\vartheta^+)\boldsymbol{v}^++\nabla\cdot(\boldsymbol{v}^+\vartheta^{+2})\big)\,\mathrm{d}x\\
&=-2\overline{\rho}^+\int\limits_{\Omega_t^+}\vartheta^+\nabla\cdot\boldsymbol{v}^+\,\mathrm{d}x+\int\limits_{\Omega_t^+}\vartheta^{+2}\nabla\cdot\boldsymbol{v}^+\,\mathrm{d}x,
\end{aligned}
$$

where $\vartheta(x,t)=\rho^+-\overline{\rho}^+=\theta^+(y,t)$. Moreover, (3.2) implies

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big\|\sqrt{\rho}\boldsymbol{v}\big\|^2_{\Omega}+\frac{p_1}{\overline{\rho}^+}\|\vartheta^+\|^2_{\Omega_t^+}+2\sigma\big(|\Gamma_t|-4\pi R_0^2\big)\Big)+\int\limits_{\Omega_t^+}\mathbb{T}(\boldsymbol{v}):\nabla\boldsymbol{v}\,\mathrm{d}x&\\
=\int\limits_{\Omega}\rho^{\pm}\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x+K_1,&
\end{aligned}
\tag{3.3}
$$

where

$$
K_1=\int\limits_{\Omega_t^+}\big(p(\rho^+)-p(\overline{\rho}^+)-p_1\vartheta^+\big)\nabla\cdot\boldsymbol{v}\,\mathrm{d}x+\frac{p_1}{2\overline{\rho}^+}\int\limits_{\Omega_t^+}\vartheta^{+2}\nabla\cdot\boldsymbol{v}^+\,\mathrm{d}x.
$$

Next, we construct an auxiliary vector field $\boldsymbol{W} \in W_2^1(\Omega)$ such that

$$
\begin{gathered}
-\nabla_x \cdot \boldsymbol{W}^+(x,t) = \vartheta^+(x,t) \text{ in } \Omega_t^+, \quad -\nabla_x \cdot \boldsymbol{W}^-(x,t) = 0 \text{ in } \Omega_t^-, \\
\boldsymbol{W}\big|_{\Gamma_t} = \boldsymbol{n}\frac{\widetilde{r}}{|\widehat{\mathbb{L}}^T\boldsymbol{n}_0|}\Big|_{y=X_{\boldsymbol{u}}^{-1}(x,t)}, \quad \boldsymbol{W}|_\Sigma = 0,
\end{gathered}
\tag{3.4}
$$

where $\widetilde{r}(y,t) = r'(y,t) - \frac{1}{|\Gamma_0|}\int_{\Gamma_0} r'(y,t)\,\mathrm{d}S_y$ and $r'(y,t) = |X_{\boldsymbol{u},h}| - R_0$. In the Lagrangian coordinates, (3.4) takes the form

$$
\begin{gathered}
-\nabla_y \cdot \boldsymbol{w}^+(y,t) = L\theta^+(y,t) \text{ in } \Omega_0^+, \quad -\nabla_y \cdot \boldsymbol{w}^-(y,t) = 0 \text{ in } \Omega_0^-, \\
\boldsymbol{w}\big|_{\Gamma_0} = \widetilde{r}\boldsymbol{n}_0, \quad \boldsymbol{w}|_\Sigma = 0,
\end{gathered}
\tag{3.5}
$$

where $\boldsymbol{w}(y,t) = \widehat{\mathbb{L}}\boldsymbol{W}(X_{\boldsymbol{u}},t)$. Since the compatibility condition

$$
\int\limits_{\Omega_0^+} L\theta^+(y,t)\,\mathrm{d}y = \int\limits_{\Gamma_0} \widetilde{r}\,\mathrm{d}S_y = 0
$$

is satisfied, the vector field $\boldsymbol{w}$ belongs to $W_2^1(\Omega)$ and satisfies the inequality

$$
\|\boldsymbol{w}\|_{W_2^1(\Omega)} \leqslant c\big(\|\theta^+\|_{L_2(\Omega_0^+)} + \|r'\|_{W_2^{1/2}(\Gamma_0)}\big).
$$

Moreover, $\boldsymbol{W} \in W_2^1(\Omega)$, because $[\widehat{\mathbb{L}}^T\boldsymbol{n}_0]|_{\Gamma_0} = 0$ (by virtue of (1.4)) and

$$
\|\boldsymbol{W}\|_{W_2^1(\Omega)} \leqslant c\big(\|\vartheta^+\|_{L_2(\Omega_t^+)} + \|r'\|_{W_2^{1/2}(\Gamma_0)}\big).
$$

By differentiating (3.5) with respect to time, we obtain a problem for $\mathcal{D}_t\boldsymbol{w}$ the solution of which is subject to the inequality

$$
\|\mathcal{D}_t\boldsymbol{w}\|_{L_2(\Omega)} \leqslant c(\|\mathcal{D}_t\theta^+\|_{L_2(\Omega_0^+)} + \|\mathcal{D}_t r'\|_{L_2(\Gamma_0)}) \leqslant c\big(\|\mathcal{D}_t r'\|_{L_2(\Gamma_0)} + \|\nabla\boldsymbol{u}^+\|_{L_2(\Omega_0^+)}\big).
$$

Now, from

$$
\begin{gathered}
\mathcal{D}_t\boldsymbol{W}(x,t) = \mathcal{D}_t\boldsymbol{W}(X_{\boldsymbol{u}},t) - \nabla_x\boldsymbol{W}(x,t)\boldsymbol{u}|_{y=X_{\boldsymbol{u}}^{-1}(x,t)}, \\
\|\mathcal{D}_t r'\|_{\Gamma_0} \leqslant c\Big\|\mathcal{D}_t\frac{X_{\boldsymbol{u},h}}{|X_{\boldsymbol{u},h}|}\Big\|_{\Gamma_0} \leqslant c\|\boldsymbol{u} - \mathcal{D}_t\boldsymbol{h}(t)\|_{\Gamma_0}
\end{gathered}
$$

we conclude, taking account of the estimates obtained in §2, that

$$
\begin{aligned}
\|\mathcal{D}_t\boldsymbol{W}\|_{L_2(\Omega)} &\leqslant \|\mathcal{D}_t(\mathbb{L}\boldsymbol{w}(\,\cdot\,,t))\|_{L_2(\Omega)} + \sup_\Omega|\boldsymbol{u}(y,t)|\|\nabla_x\boldsymbol{W}(\,\cdot\,,t)\|_{L_2(\Omega)} \\
&\leqslant c\big(\|\mathcal{D}_t r'\|_{L_2(\Gamma_0)} + \|\nabla\boldsymbol{u}\|_{L_3(\Omega)}\|\boldsymbol{w}\|_{L_6(\Omega)} + \sup_\Omega|\boldsymbol{u}(y,t)|\|\nabla_x\boldsymbol{W}(\,\cdot\,,t)\|_{L_2(\Omega)}\big) \\
&\leqslant c\big(\|\boldsymbol{u}\|_{W_2^1(\cup\Omega_0^\pm)} + \|\theta^+\|_{L_2(\Omega_0^+)} + \|r'\|_{W_2^{1/2}(\Gamma_0)}\big).
\end{aligned}
$$

Multiplying the first and the third equations in (1.1) by $\boldsymbol{W}$, and integrating we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \rho\boldsymbol{v}\cdot\boldsymbol{W}\,\mathrm{d}x - \int\limits_\Omega \rho\boldsymbol{v}\cdot(\partial_t\boldsymbol{W} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{W})\,\mathrm{d}x + \int\limits_\Omega \mathbb{T}(\boldsymbol{v}) : \nabla\boldsymbol{W}\,\mathrm{d}x \\
&- \int\limits_{\Omega_t^+} p(\overline{\rho}^+ + \vartheta^+)\nabla\cdot\boldsymbol{W}\,\mathrm{d}x - \sigma\int\limits_{\Gamma_t}\Big(H + \frac{2}{R_0}\Big)\frac{\widetilde{r}}{|\widehat{\mathbb{L}}^T\boldsymbol{n}_0|}\Big|_{y=X_{\boldsymbol{u}}^{-1}(x,t)}\mathrm{d}S \\
&= \int\limits_\Omega \rho\boldsymbol{f}\cdot\boldsymbol{W}\,\mathrm{d}x,
\end{aligned}
\tag{3.6}
$$

because $\int_{\Gamma_t} \frac{\widetilde{r}}{|\widehat{\mathbb{L}}^T \boldsymbol{n}_0|}\Big|_{y=X_{\boldsymbol{u}}^{-1}} \mathrm{d}S_x = \int_{\Gamma_0} \widetilde{r}(y,t)\,\mathrm{d}S_y = 0$. Since $\int_{\Omega_t^+} \nabla\cdot\boldsymbol{W}\,\mathrm{d}x = -\int_{\Omega_t^+} \vartheta^+(x,t)\,\mathrm{d}x = 0$, we have

$$-\int\limits_{\Omega_t^+} p(\overline{\rho}^+ + \vartheta^+)\nabla\cdot\boldsymbol{W}\,\mathrm{d}x = \int\limits_{\Omega_t^+} \big(p(\overline{\rho}^+ + \vartheta^+) - p(\overline{\rho}^+) - p_1\vartheta^+\big)\vartheta^+\,\mathrm{d}x + p_1\int\limits_{\Omega_t^+} \vartheta^{+2}\,\mathrm{d}x,$$

moreover, the surface integral in (3.6) can be written as

$$J = -\int\limits_{S_{R_0}} \Big(H + \frac{2}{R_0}\Big)\widetilde{r}\frac{|\widehat{\mathfrak{L}}^T\boldsymbol{N}(\eta)|}{|\widehat{\mathbb{L}}^T\boldsymbol{n}_0(y)|}\Bigg|_{y=\mathcal{X}^{-1}(\eta,t)}\mathrm{d}S_\eta,$$

where $\mathfrak{L} = \big(\frac{\partial}{\partial\eta}(\boldsymbol{\eta} + \boldsymbol{N}(\eta)r(\eta,t)\big)$ (see [13]) and

$$H = \frac{R_0^2}{R_0 + r}\nabla_{S_{R_0}}\cdot\frac{\nabla_{S_{R_0}} r}{\sqrt{g}} - \frac{2}{\sqrt{g}}, \quad g = (R_0+r)^2 + |R_0\nabla_{S_{R_0}} r|^2. \tag{3.7}$$

As shown in [2],

$$-\int\limits_{S_{R_0}} \Big(H + \frac{2}{R_0}\Big)\widetilde{r}\,\mathrm{d}S_y = \int\limits_{S_{R_0}} \Big(|\nabla_{S_{R_0}} r|^2 - \frac{2}{R_0^2}r^2\Big)\,\mathrm{d}S_y + K_2,$$

where $|K_2| \leqslant \delta\|r\|^2_{W_2^1(S_{R_0})}$, and the same inequality is satisfied by

$$K_3 = \int\limits_{S_{R_0}} \Big(H + \frac{2}{R_0}\Big)\widetilde{r}\bigg(\frac{|\widehat{\mathfrak{L}}^T\boldsymbol{N}(\eta)|}{|\widehat{\mathbb{L}}^T\boldsymbol{n}_0(y)|} - 1\bigg)\bigg|_{y=\mathcal{X}^{-1}(\eta,t)}\mathrm{d}S_\eta.$$

Now, we add (3.2) and (3.6) multiplied by a small positive γ_0, which leads to

$$\begin{aligned}\frac{\mathrm{d}E_0}{\mathrm{d}t} &+ \sigma\gamma_0\int\limits_{S_{R_0}} \Big(|\nabla_{S_{R_0}} r|^2 - \frac{2}{R_0^2}r^2\Big)\,\mathrm{d}S_\eta + \int\limits_{\Omega} \mathbb{T}(\boldsymbol{v}) : \nabla\boldsymbol{v}\,\mathrm{d}x \\ &+ \frac{\gamma_0 p_1}{\overline{\rho}^+}\int\limits_{\Omega_t^+} \vartheta^{+2}\,\mathrm{d}x + K = \int\limits_{\Omega} \boldsymbol{f}\cdot(\boldsymbol{v} + \gamma_0\boldsymbol{W})\,\mathrm{d}x,\end{aligned}$$

where

$$E_0 = \frac{1}{2}\|\sqrt{\rho}\boldsymbol{v}\|^2_{L_2(\Omega)} + \frac{p_1}{\overline{\rho}^+}\|\vartheta^+\|^2_{L_2(\Omega_t^+)} + \sigma(|\Gamma_t| - 4\pi R_0) + \gamma_0\int\limits_{\Omega} \rho^{\pm}\boldsymbol{v}\cdot\boldsymbol{W}\,\mathrm{d}x$$

and $K = K_1 + \gamma_0(K_2 + K_3)$. If r is a small function satisfying (2.35), then

$$c_1\|r\|^2_{W_2^1(S_{R_0})} \leqslant (|\Gamma_t| - 4\pi R_0) + \int\limits_{S_{R_0}} \Big(|\nabla_{S_{R_0}} r|^2 - \frac{2}{R_0^2}r^2\Big)\,\mathrm{d}S_\eta \leqslant c_2\|r\|^2_{W_2^1(S_{R_0})},$$

which implies

$$c_2\big(\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \|\vartheta^+\|^2_{L_2(\Omega_t^+)} + \|r\|^2_{W_2^1(S_{R_0})}\big) \\ \leqslant E_0(t) \leqslant c_4\big(\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \|\vartheta^+\|^2_{L_2(\Omega_t^+)} + \|r\|^2_{W_2^1(S_{R_0})}\big).$$

By estimating the positive form $\int_\Omega \mathbb{T}(\boldsymbol{v}) : \nabla\boldsymbol{v}\,\mathrm{d}x$ from below with the help of the Korn inequality, we prove that the expression

$$E_1(t) = \sigma\gamma_0 \int\limits_{S_{R_0}} (|\nabla_{S_{R_0}} r|^2 - \frac{2}{R_0^2}r^2)\,\mathrm{d}S_\eta + \int\limits_\Omega \mathbb{T}(\boldsymbol{v}) : \nabla\boldsymbol{v}\,\mathrm{d}x + \frac{\gamma_0 p_1}{\rho_m^+}\int\limits_{\Omega_t^+} \vartheta^{2+}\,\mathrm{d}x + K$$

satisfies

$$E_1(t) \geqslant c(\|\boldsymbol{v}\|^2_{W_2^1(\Omega)} + \|\vartheta^+\|^2_{L_2(\Omega_t^+)} + \|r\|^2_{W_2^1(S_{R_0})}) \geqslant 2aE_0(t),$$

if δ and γ_0 are small. Thus, we have $\frac{\mathrm{d}E_0(t)}{\mathrm{d}t} + E_1(t) \leqslant |\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}+\gamma_0\boldsymbol{W})\,\mathrm{d}x|$, which implies (3.1) (with $\beta < a$) and completes the proof of the theorem. □

The idea and method of estimating "the generalized energy" E_0 used above are due to M. Padula [15, 16].

From (3.1) it follows that

$$\begin{aligned} &e^{2\beta t}\Big(\|\boldsymbol{u}(\,\cdot\,,t)\|^2_\Omega + \|\theta^+\|^2_{\Omega_0^+} + \|r(\,\cdot\,,t)\|^2_{W_2^1(S_{R_0})}\Big) \\ &+\int\limits_0^t e^{2\beta\tau}\Big(\|\boldsymbol{u}(\,\cdot\,,\tau)\|^2_\Omega + \|\theta^+(\,\cdot\,,\tau)\|^2_{\Omega_0^+} + \|r(\,\cdot\,,\tau)\|^2_{W_2^1(S_{R_0})}\Big)\,\mathrm{d}\tau \\ &\leqslant c\Big(\|\boldsymbol{v}_0\|^2_\Omega + \|r_0\|^2_{W_2^1(S_{R_0})} + \|\theta_0^+\|^2_{\Omega_0^+} + \int\limits_0^t e^{2\beta\tau}\|\boldsymbol{f}(\,\cdot\,,\tau)\|^2_\Omega d\tau\Big), \quad t\in T. \end{aligned} \tag{3.8}$$

We proceed with the estimate of the norm $\|e^{\beta t}\theta^-\|_{W_2^{0,l/2}(Q_T^-)}$.

Theorem 4. *The function θ^- satisfies the inequalities*

$$\|e^{\beta t}\theta^-\|_{Q_T^-} \leqslant c\Big(\|e^{\beta t}g\|_{G_T} + \|e^{\beta t}\nabla\boldsymbol{u}\|_{Q_T^-} + \|e^{\beta t}\boldsymbol{f}\|_{Q_T^-}\Big) \tag{3.9}$$

and

$$\begin{aligned} \|e^{\beta t}\theta^-\|_{W_2^{0,l/2}(Q_T^-)} \leqslant c\Big(&\|e^{\beta t}\theta^-\|_{Q_T^-} + \|e^{\beta t}g\|_{W_2^{0,l/2}(G_T)} \\ &+ \|e^{\beta t}\nabla\boldsymbol{u}\|_{W_2^{0,l/2}(Q_T^-)} + \|e^{\beta t}\boldsymbol{f}\|_{W_2^{0,l/2}(Q_T^-)}\Big), \end{aligned} \tag{3.10}$$

where

$$g = \theta^-\big|_{\Gamma_0} = p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+) - [\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}] - \sigma\Big(H + \frac{2}{R_0}\Big). \tag{3.11}$$

Proof. We start with the proof of (3.9) by using the relations

$$\mathcal{D}_t \boldsymbol{u}^- - \nu^- \nabla^2_{\boldsymbol{u}} \boldsymbol{u}^- + \frac{1}{\rho^-} \nabla_u \theta^- = \widehat{\boldsymbol{f}}^-, \quad \theta^-\big|_{\Gamma_0} = g.$$

Let $\psi(y,t)$ be a solution of the Dirichlet problem

$$\nabla^2_{\boldsymbol{u}} \psi = \theta^- \text{ in } \Omega_0^-, \quad \psi|_{\Gamma_0} = 0.$$

If δ in (2.10) is small, then

$$\|\psi\|_{W_2^2(\Omega_0^-)} \leqslant c\|\theta^-\|_{\Omega_0^-}.$$

Indeed, since

$$\nabla^2_{\boldsymbol{u}} \psi = \mathbb{L}^{-T} \nabla \cdot \mathbb{L}^{-T} \nabla \psi = \nabla \cdot \mathbb{L}^{-1} \mathbb{L}^{-T} \nabla \psi = \nabla \cdot (\mathbb{L}^{-1} \mathbb{L}^{-T} - \mathbb{I}) \nabla \psi + \nabla^2 \psi,$$

$$\begin{aligned}
|(\mathbb{L}^{-1}\mathbb{L}^{-T} - \mathbb{I}| &\leqslant c \sup_{\cup \Omega_0^\pm} |\nabla \boldsymbol{U}| \leqslant c\delta, \\
\|\nabla(\mathbb{L}^{-1}\mathbb{L}^{-T}) \cdot \nabla \psi\|_{\Omega_0^-} &\leqslant c \|\nabla \mathbb{L}^{-1}\mathbb{L}^{-T}\|_{L_3(\Omega_0^-)} \|\nabla \psi\|_{L_6(\Omega_0^-)} \\
&\leqslant c\|\boldsymbol{U}\|_{W_2^{5/2}(\cup \Omega_0^\pm)} \|\psi\|_{W_2^2(\Omega_0^-)},
\end{aligned}$$

we obtain

$$\|\psi\|_{W_2^2(\Omega_0^-)} \leqslant c\|\Delta \psi\|_{L_2(\Omega_0^-)} \leqslant c\big(\|\theta^-\|_{\Omega_0^-} + c\delta \|\psi\|_{W_2^2(\Omega_0^-)}\big),$$

from which the desired inequality follows.

Now, we make use of the relation

$$\int_{\Omega_0^-} \Big(\mathcal{D}_t \boldsymbol{u}^- - \nu^- \nabla^2_{\boldsymbol{u}} \boldsymbol{u}^- + \frac{1}{\rho^-} \nabla_{\boldsymbol{u}} \theta^- \Big) \cdot \nabla_{\boldsymbol{u}} \psi \, \mathrm{d}y = \int_{\Omega_0^-} \widehat{\boldsymbol{f}} \cdot \nabla_{\boldsymbol{u}} \psi \, \mathrm{d}y. \tag{3.12}$$

By passing to the Eulerian coordinates under the integral sign, we prove that $\int_{\Omega_0^-} \nabla^2_{\boldsymbol{u}} \boldsymbol{u} \cdot \nabla_{\boldsymbol{u}} \psi \, \mathrm{d}y = 0$, because $\nabla_{\boldsymbol{u}} \cdot \boldsymbol{u} = 0$ and $L|_{\Omega_0^-} = 1$. We also have

$$\begin{aligned}
\int_{\Omega_0^-} \mathcal{D}_t \boldsymbol{u}^- \cdot \nabla_{\boldsymbol{u}} \psi \, \mathrm{d}y &= -\int_{\Omega_0^-} (\nabla \cdot \mathbb{L}^{-1} \mathcal{D}_t \boldsymbol{u}^-) \psi \, \mathrm{d}y = \int_{\Omega_0^-} (\mathcal{D}_t \mathbb{L}^{-T}) \nabla \cdot \boldsymbol{u}^- \psi \, \mathrm{d}y, \\
\int_{\Omega_0^-} \nabla_{\boldsymbol{u}} \theta^- \cdot \nabla_{\boldsymbol{u}} \psi \, \mathrm{d}y &= -\int_{\Omega_0^-} \theta^{-2} \, \mathrm{d}y + \int_{\Gamma_0} g \boldsymbol{n}_0 \cdot \mathbb{L}^{-1} \nabla_{\boldsymbol{u}} \psi \, \mathrm{d}S,
\end{aligned}$$

hence (3.12) implies

$$\begin{aligned}
\|\theta^-\|^2_{\Omega_0^-} &\leqslant c\Big(\|g\|_{\Gamma_0} \|\nabla \psi\|_{\Gamma_0} + \||\nabla \boldsymbol{u}|^2\|_{\Omega_0^-} \sup_{\Omega_0^-} |\psi| + \|\boldsymbol{f}\|_{\Omega_0^-} \|\nabla \psi\|_{\Omega_0^-}\Big) \\
&\leqslant c\|\theta^-\|_{\Omega_0^-} \Big(\|g\|_{\Gamma_0} + \|\nabla \boldsymbol{u}^-\|_{\Omega_0^-} + \|\boldsymbol{f}\|_{\Omega_0^-}\Big),
\end{aligned}$$

from which (3.9) follows. Next, we evaluate the norm

$$\Big(\int\limits_0^T \frac{\mathrm{d}\tau}{\tau^{1+l}} \int\limits_0^T e^{2\beta t}\|\theta^-(\,\cdot\,,t)-\theta^-(\,\cdot\,,t-\tau)\|^2_{\Omega_0^-}\,\mathrm{d}t\Big)^{1/2} \equiv \|e^{\beta t}\theta^-\|_{\dot W_2^{0,l/2}(Q_T^-)},$$

assuming that $\theta = 0$ for $t<0$. We make use of the equation

$$\begin{aligned} &\int\limits_{\Omega_0^-} \Big(\mathcal{D}_t \boldsymbol{u}^- - \nu^- \nabla^2_{\boldsymbol{u}} \boldsymbol{u}^- - \frac{1}{\rho^-}\nabla_{\boldsymbol{u}}\theta^- - \widehat{\boldsymbol{f}}\Big)\cdot \nabla_{\boldsymbol{u}}(\psi-\psi')\,\mathrm{d}y \\ -&\int\limits_{\Omega_0^-} \Big(\mathcal{D}_t \boldsymbol{u}'^- - \nu^- \nabla^2_{\boldsymbol{u}} \boldsymbol{u}'^- - \frac{1}{\rho^-}\nabla_{\boldsymbol{u}'}\theta'^- - \widehat{\boldsymbol{f}}'\Big)\cdot \nabla_{\boldsymbol{u}'}(\psi-\psi')\,\mathrm{d}y = 0, \end{aligned} \tag{3.13}$$

where $u'(y,t) = u(y,t-h)$; we set $\boldsymbol{u},\theta = 0$ for $t<0$. The terms containing $\nabla^2_{\boldsymbol{u}}\boldsymbol{u}$ and $\nabla^2_{\boldsymbol{u}'}\boldsymbol{u}'$ vanish and other terms can be calculated as follows:

$$\begin{aligned} &\int\limits_{\Omega_0^-} (\nabla_{\boldsymbol{u}}\theta^- \cdot \nabla_{\boldsymbol{u}}(\psi-\psi') - \nabla_{\boldsymbol{u}'}\theta'^- \nabla_{\boldsymbol{u}'}(\psi-\psi'))\,\mathrm{d}y \\ &= -\int\limits_{\Omega_0^-} (\theta^-\nabla^2_{\boldsymbol{u}}(\psi-\psi') - \theta'^-\nabla^2_{\boldsymbol{u}'}(\psi-\psi'))\,\mathrm{d}y \\ &+ \int\limits_{\Gamma_0} \mathbb{L}^{-1}\boldsymbol{n}_0 g\cdot\nabla_{\boldsymbol{u}}(\psi-\psi') - \mathbb{L}'^{-1}\boldsymbol{n}_0 g'\cdot\nabla_{\boldsymbol{u}'}(\psi-\psi'))\,\mathrm{d}S \\ &= -\int\limits_{\Gamma_0} \big((\theta^- - \theta'^-)\nabla^2_{\boldsymbol{u}}(\psi-\psi') - \theta'^-(\nabla^2_{\boldsymbol{u}} - \nabla^2_{\boldsymbol{u}'})(\psi-\psi')\big)\,\mathrm{d}y \\ &+ \int\limits_{\Gamma_0} (\mathbb{L}^{-1}g - \mathbb{L}'^{-1}g')\boldsymbol{n}_0\nabla_{\boldsymbol{u}}(\psi-\psi')\,\mathrm{d}S + \int\limits_{\Gamma_0} \mathbb{L}'^{-1}g'\boldsymbol{n}_0\cdot(\nabla_{\boldsymbol{u}} - \nabla_{\boldsymbol{u}'})(\psi-\psi')\,\mathrm{d}S \\ &= -\int\limits_{\Omega_0^-} (\theta^- - \theta'^-)^2\,\mathrm{d}y - \int\limits_{\Omega_0^-} (\theta^- - \theta'^-)(\nabla^2_{\boldsymbol{u}} - \nabla^2_{\boldsymbol{u}'})\psi'\,\mathrm{d}y \\ &- \int\limits_{\Omega_0^-} \theta'^-(\nabla^2_{\boldsymbol{u}} - \nabla^2_{\boldsymbol{u}'})(\psi-\psi')\,\mathrm{d}y + \int\limits_{\Gamma_0} (\mathbb{L}^{-1}g - \mathbb{L}'^{-1}g')\boldsymbol{n}_0\nabla_{\boldsymbol{u}}(\psi-\psi')\,\mathrm{d}S \\ &+ \int\limits_{\Gamma_0} \mathbb{L}'^{-1}g'\boldsymbol{n}_0\cdot(\nabla_{\boldsymbol{u}} - \nabla_{\boldsymbol{u}'})(\psi-\psi')\,\mathrm{d}S = \sum_{k=1}^5 J_k, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\int\limits_{\Omega_0^-} (\widehat{\boldsymbol{f}}\nabla_{\boldsymbol{u}}(\psi-\psi') - \widehat{\boldsymbol{f}}'\nabla_{\boldsymbol{u}'}(\psi-\psi'))\,\mathrm{d}y \\ &\qquad = \int\limits_{\Omega_0^-} ((\widehat{\boldsymbol{f}} - \widehat{\boldsymbol{f}}')\nabla_{\boldsymbol{u}}(\psi-\psi') + \widehat{\boldsymbol{f}}'(\nabla_{\boldsymbol{u}} - \nabla_{\boldsymbol{u}'})(\psi-\psi'))\,\mathrm{d}y = J_6, \end{aligned} \tag{3.15}$$

$$\int\limits_{\Omega_0^-}(\mathcal{D}_t\boldsymbol{u}^-\cdot\nabla_{\boldsymbol{u}}(\psi-\psi')-\mathcal{D}_t\boldsymbol{u}'^-\cdot\nabla_{\boldsymbol{u}'}(\psi-\psi'))\,\mathrm{d}y$$

$$=-\int\limits_{\Omega_0^-}((\mathcal{D}_t\mathbb{L}^{-1}):\nabla\boldsymbol{u}^- -(\mathcal{D}_t\mathbb{L}'^{-1}):\nabla\boldsymbol{u}'^-)(\psi-\psi'))\,\mathrm{d}y=J_7, \quad (3.16)$$

because $\mathcal{D}_t\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}=0$. Thus, (3.14) is equivalent to $\sum_{r=1}^7 J_k=0$.

We proceed with the estimate of the expression

$$(\nabla_{\boldsymbol{u}}^2-\nabla_{\boldsymbol{u}'}^2)\psi=\nabla\cdot\mathbb{M}\nabla\psi=\mathbb{M}:\nabla\nabla\psi+\nabla\mathbb{M}\cdot\nabla\psi,$$

where $\mathbb{M}=\mathbb{L}^{-1}\mathbb{L}^{-T}-\mathbb{L}'^{-1}\mathbb{L}'^{-T}$. We have

$$\begin{aligned}\|(\nabla_{\boldsymbol{u}}^2-\nabla_{\boldsymbol{u}'}^2)\psi\|_{\Omega_0^-}&\leqslant c\Big(\sup_{\Omega_0^-}|\mathbb{M}(\,\cdot\,,t)|\|\psi\|_{W_2^2(\Omega_0^-)}+\|\nabla\mathbb{M}\|_{L_3(\Omega_0^-)}\|\nabla\psi\|_{L_6(\Omega)}\Big)\\
&\leqslant c\Big(\int\limits_0^h\sup_{\Omega_0^-}|\nabla\boldsymbol{u}^-(\,\cdot\,,t-\tau)|\,\mathrm{d}\tau\sup_{\Omega_0^-}|\boldsymbol{U}(\,\cdot\,,t)|\\
&\quad+\int\limits_0^h\|\boldsymbol{u}^-(\,\cdot\,,t-\tau)\|_{W_2^2(\Omega_0^-)}\,\mathrm{d}\tau\sup_{\Omega_0^-}|\nabla\boldsymbol{U}|\Big)\|\psi\|_{W_2^2(\Omega_0^-)}\\
&\leqslant c\sqrt{h}\delta\|\psi\|_{W_2^2(\Omega_0^-)}\end{aligned}$$

and, similarly,

$$\|(\nabla_{\boldsymbol{u}}^2-\nabla_{\boldsymbol{u}'}^2)(\psi-\psi')\|_{\Omega_0^-}\leqslant c\sqrt{h}\delta\|\psi-\psi'\|_{W_2^2(\Omega_0)};$$

moreover, since $\psi-\psi'$ is a solution of the Dirichlet problem

$$\nabla_{\boldsymbol{u}}^2(\psi-\psi')=\theta^- -\theta'^- -(\nabla_{\boldsymbol{u}}^2-\nabla_{\boldsymbol{u}'}^2)\psi',\quad \psi-\psi'|_{\Gamma_0}=0,$$

we obtain

$$\|\psi-\psi'\|_{W_2^2(\Omega_0^-)}\leqslant c\big(\|\theta^- -\theta'-\|_{\Omega_0^-}+\delta\sqrt{h}\|\theta'^-\|_{\Omega_0^-}\big).$$

From the above inequalities it follows that

$$\begin{aligned}\int\limits_0^T\frac{\mathrm{d}h}{h^{1+l}}\int\limits_0^T e^{2\beta t}|J_2|\,\mathrm{d}t&\leqslant\Big(\int\limits_0^T\frac{\mathrm{d}h}{h^{1+l}}\int\limits_0^T e^{2\beta t}\|\theta^- -\theta'^-\|_{\Omega_0^-}^2\,\mathrm{d}t\Big)^{1/2}\\
&\quad\times\Big(\int\limits_0^T\frac{\mathrm{d}h}{h^{1+l}}\int\limits_0^T e^{2\beta t}\|(\nabla_{\boldsymbol{u}}^2-\nabla_{\boldsymbol{u}'}^2)\psi'\|_{\Omega_0^-}^2\,\mathrm{d}t\Big)^{1/2}\\
&\leqslant c\delta\|e^{\beta t}\theta^-\|_{\dot{W}_2^{0,l/2}(Q_T^-)}\|e^{\beta t}\theta^-\|_{Q_T^-},\end{aligned}$$

$$\int\limits_0^T\frac{\mathrm{d}h}{h^{1+l}}\int\limits_0^T e^{2\beta t}|J_3|\,\mathrm{d}t\leqslant c\delta\Big(\int\limits_0^T\frac{h\,\mathrm{d}h}{h^{1+l}}\Big)^{1/2}\|e^{\beta t}\theta^-\|_{Q_T^-}\|e^{\beta t}\theta^-\|_{\dot{W}_2^{0,l/2}(Q_T^-)},$$

$$\int\limits_0^T \frac{\mathrm{d}h}{h^{1+l}} \int\limits_0^T e^{2\beta t}(|J_4| + |J_5|)\,\mathrm{d}t \leqslant c\|e^{\beta t} g\|^2_{W_2^{0,l/2}(G_T)} \|e^{\beta t}\theta^-\|_{\dot{W}_2^{0,l/2}(Q_T^-)},$$

$$\int\limits_0^T \frac{\mathrm{d}h}{h^{1+l}} \int\limits_0^T e^{2\beta t}|J_6|\,\mathrm{d}t \leqslant c\|e^{\beta t}\boldsymbol{f}\|_{W_2^{0,l/2}(Q_T^-)} \|e^{\beta t}\theta^-\|_{\dot{W}_2^{0,l/2}(Q_T^-)},$$

$$\int\limits_0^T \frac{\mathrm{d}h}{h^{1+l}} \int\limits_0^T e^{2\beta t}|J_7|\,\mathrm{d}t \leqslant c\|e^{\beta t}\nabla\boldsymbol{u}\|_{W_2^{0,l/2}(Q_T^-)} \|e^{\beta t}\theta^-\|_{W_2^{0,l/2}(Q_T^-)}.$$

Collecting the above estimates we arrive at (3.10) after easy calculations. The theorem is proved.

By (3.9) and (3.10), we have

$$\|e^{\beta t}\theta^-\|_{W_2^{0,l/2}(Q_T^-)} \leqslant c\big(\mathbf{Y}'_T + \|e^{\beta t}\boldsymbol{f}\|_{W_2^{0,l/2}(Q_T^-)}\big), \tag{3.17}$$

where

$$\begin{aligned} \mathbf{Y'}_T &= \|e^{\beta t}\theta^+\|_{W_2^{0,l/2}(G_T)} + \sum_{\pm} \|e^{\beta t}\nabla\boldsymbol{u}^{\pm}\|_{W_2^{0,l/2}(G_T)} \\ &\quad + \|e^{\beta t}\nabla\boldsymbol{u}^-\|_{W_2^{0,l/2}(Q_T^-)} + \|e^{\beta t}r\|_{W_2^{l/2}((0,T);W_2^2(\Gamma_0))} \\ &\leqslant \epsilon_1 \mathbf{Y}_T^{(+)} + c(\epsilon_1)\big(\|e^{\beta t}\boldsymbol{u}\|_{Q_T} + \|e^{\beta t}\theta^+\|_{G_T} + \|e^{\beta t}r\|_{G_T}\big), \end{aligned} \tag{3.18}$$

$\epsilon_1 \ll 1$, and

$$\begin{aligned} \mathbf{Y}_T^{(+)} &= \|e^{\beta t}\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(\cup Q_T^{\pm})} + |e^{\beta t}\theta^+|^{(l+1,l/2)}_{\Omega_0^+} + |e^{\beta t}\mathcal{D}_t\theta^+|^{(l+1,l/2)}_{\Omega_0^+} \\ &\quad + \|e^{\beta t}r\|_{W_2^{l+5/2,0}(\mathcal{S}_T)} + \|e^{\beta t}\mathcal{D}_t r\|_{W_2^{l+3/2,0}(\mathcal{S}_T)}. \end{aligned} \tag{3.19}$$

We pass to the estimate of higher order weighted Sobolev norms of the solution of (1.3). We make use of the localization method and estimate the solution in the neighborhood of the surfaces Σ, Γ_0 and in the interior of $\Omega_0^{\pm}$. We start with the interior estimates and consider two model problems

$$\begin{cases} \rho^-\mathcal{D}_t\boldsymbol{u}^- - \nabla\cdot\mathbb{T}^-(\boldsymbol{u}^-) + \nabla\sigma^- = \boldsymbol{f}^-, \quad \nabla\cdot\boldsymbol{u}^- = h^- \\ \text{in } \mathfrak{Q} = \{|z_\alpha| \leqslant d_0, \quad \alpha = 1,2, \quad 0 < z_3 < 2d_0\}, \quad \boldsymbol{u}^-|_{t=0} = 0 \end{cases} \tag{3.20}$$

and

$$\begin{cases} \overline{\rho}^+\mathcal{D}_t\boldsymbol{u}^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{u}^+) + p_1\nabla\cdot\sigma^+ = \boldsymbol{f}^+, \quad \mathcal{D}_t\sigma^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{u}^+ = h^+ \text{ in } \mathfrak{Q}, \\ \boldsymbol{u}^+|_{t=0} = 0, \quad \sigma^+|_{t=0} = 0. \end{cases} \tag{3.21}$$

Concerning $(\boldsymbol{u}^-, \sigma^-)$ and $(\boldsymbol{u}^+, \sigma^+)$, we assume that these functions are compactly supported in $\mathfrak{Q}$ and are defined for $t \in (-\infty, \infty)$ and vanished for $t < 0$. Applying the Fourier–Laplace transform, we convert (3.20) into

$$(s + \nu^-|\xi'|^2)\widetilde{\boldsymbol{u}}^- - \nu^-\mathcal{D}_{y_3}^2\widetilde{\boldsymbol{u}}^- + \frac{1}{\rho^-}\widetilde{\nabla}\widetilde{\theta}^- = \frac{1}{\rho^-}\widetilde{\boldsymbol{f}}, \quad \widetilde{\nabla}\cdot\widetilde{\boldsymbol{u}}^- = \widetilde{\boldsymbol{h}}^-, \tag{3.22}$$

where $\xi' = \frac{\pi \boldsymbol{k}'}{d_0}$, $\widetilde{\nabla} = (i\xi_1, i\xi_2, \mathcal{D}_{y_3})$. The parameter $\operatorname{Re} s$ can take small negative values (such that $|s + \nu^-|\xi'|^2| \geqslant c(|s| + |\xi'|^2)$). If $|\boldsymbol{k}'| > 0$, then the solution of (3.22) is sought in the form $(\widetilde{\boldsymbol{u}}^- = \widetilde{\boldsymbol{u}}_1^- + \widetilde{\boldsymbol{u}}_2^-, \sigma^-)$, where $\widetilde{\boldsymbol{u}}_1^- = \widetilde{\nabla}\widetilde{\Phi}$, $\widetilde{\Phi}$ solves the problem

$$\widetilde{\nabla}^2\widetilde{\Phi} = \widetilde{h}^-, \quad \widetilde{\Phi}|_{y_3=0} = 0, \quad \widetilde{\Phi} \to 0 \text{ as } y_3 \to \infty$$

and is given by

$$\widetilde{\boldsymbol{u}}_1^- = \frac{1}{2|\xi'|}\widetilde{\nabla}\Big(\int\limits_0^\infty e^{-|\xi'|(y_3+z_3)}\widetilde{h}^-(z_3)\,\mathrm{d}z_3 - \int\limits_0^\infty e^{-|\xi'||y_3-z_3|}\widetilde{h}^-(z_3)\,\mathrm{d}z_3\Big).$$

It is easily verified that $\nabla \cdot \boldsymbol{u}_1^- = h^-$ and

$$\begin{aligned}\|e^{\beta t}\boldsymbol{u}_1^-\|_{W_2^{2+l,0}(Q_T)} &\leqslant c\|e^{\beta t}h^-\|_{W_2^{1+l,0}(Q_T)},\\ \|e^{\beta t}\mathcal{D}_t u_1^-\|_{W_2^{0,l/2}(Q_T)} &\leqslant c\big(\|e^{\beta t}\boldsymbol{H}\|_{W_2^{0,l/2}(Q_T)} + \|e^{\beta t}H_1\|_{W_2^{0,l/2}(Q_T)}\big),\end{aligned}$$

provided that $\mathcal{D}_t h^- = \nabla \cdot \boldsymbol{H} + H_1$; here, $Q_T = Q \times (0,T)$, $Q = \mathfrak{Q}' \times (0,\infty)$, $\beta = -\operatorname{Re} s$. It follows that

$$\begin{aligned}\|e^{\beta t}\boldsymbol{u}_1^-\|_{W_2^{2+l,1+l/2}(Q_T)} \leqslant c\Big(&\|e^{\beta t}\boldsymbol{h}^-\|_{W_2^{l+1,0}(Q_T)}\\ &+ \|e^{\beta t}\boldsymbol{H}\|_{W_2^{0,l/2}(Q_T)} + \|e^{\beta t}H_1\|_{W_2^{0,l/2}(Q_T)}\Big).\end{aligned} \tag{3.23}$$

Now, we estimate $(\boldsymbol{u}_2^-, \sigma^-)$, assuming again that $|\boldsymbol{k}'| > 0$. These functions satisfy the Stokes problem

$$\mathcal{D}_t\boldsymbol{u}_2^- - \nu^-\nabla^2\boldsymbol{u}_2^- + \frac{1}{\rho^-}\nabla\sigma^- = \boldsymbol{f}_1, \quad \nabla \cdot \boldsymbol{u}_2^- = 0, \quad \boldsymbol{u}_2|_{y_3=0} = 0, \tag{3.24}$$

where $\boldsymbol{f}_1 = \frac{1}{\rho^-}\boldsymbol{f} - \mathcal{D}_t\boldsymbol{u}_1^- + \nu^-\nabla^2\boldsymbol{u}_1^-$. Taking the Fourier–Laplace transforms, we obtain

$$(s + \nu^-|\xi'|^2)\widetilde{\boldsymbol{u}}_2^- - \nu^-\mathcal{D}_{y_3}^2\widetilde{\boldsymbol{u}}_2^- + \frac{1}{\rho^-}\widetilde{\nabla}\widetilde{\sigma}^- = \widetilde{\boldsymbol{f}}_1, \quad \widetilde{\nabla} \cdot \widetilde{\boldsymbol{u}}_2^- = 0, \quad \widetilde{\boldsymbol{u}}_2^-|_{y_3=0} = 0.$$

By using the energy relation we obtain

$$\begin{aligned}(|s| + |\xi'|^2)\|\widetilde{\boldsymbol{u}}_2^-\|^2 + \|\mathcal{D}_{y_3}\widetilde{\boldsymbol{u}}_2^-\|^2 &\leqslant c\|\widetilde{\boldsymbol{f}}_1\|^2,\\ (|s|^2 + |\xi'|^2\operatorname{Re}\bar{s})\|\widetilde{\boldsymbol{u}}_2^-\|^2 + \operatorname{Re}\bar{s}\|\mathcal{D}_{y_3}\widetilde{\boldsymbol{u}}_2^-\|^2 &\leqslant c|s|\|\widetilde{\boldsymbol{f}}_1\|\|\widetilde{\boldsymbol{u}}_2^-\|,\\ (|s| + |\xi'|^2)|\xi'|^2\|\widetilde{\boldsymbol{u}}_2^-\|^2 + |\xi|^2\|\mathcal{D}_{y_3}\widetilde{\boldsymbol{u}}_2^-\|^2 &\leqslant c|\xi'|^2\|\widetilde{\boldsymbol{u}}_2^-\|\|\widetilde{\boldsymbol{f}}_1\|,\end{aligned}$$

where all the norms are in $L_2(\mathbb{R}_+)$. For small negative $\operatorname{Re} s$, these estimates yield the inequality

$$\|e^{\beta t}\boldsymbol{u}_2^-\|_{W_{2,tan}^{2,1}(Q_T)} \leqslant c\|e^{\beta t}\boldsymbol{f}_1\|_{L_2(Q_T)},$$

where "tan" means that only the tangential derivative of $\boldsymbol{u}_2^-$ enters into the norm. Moreover, the relations

$$\nabla^2\sigma^- = \rho^-\nabla \cdot \boldsymbol{f}_1, \quad \sigma|_{y_3=0} = 0$$

yield the estimate of $\nabla\sigma^-$ and, as a consequence, of $\mathcal{D}^2_{y_3}\boldsymbol{u}^-_2$. Thus, we have

$$\|e^{\beta t}\boldsymbol{u}^-_2\|_{W^{2,1}_2(Q_T)} + \|e^{\beta t}\nabla\sigma^-\|_{L_2(Q_T)} \leqslant c\|e^{\beta t}\boldsymbol{f}_1\|_{L_2(Q_T)}.$$

It follows that

$$\begin{aligned}\|e^{\beta t}\mathcal{D}_t\boldsymbol{u}^-_2\|_{W^{l,l/2}_{2,tan}(Q_T)} + \sum_{j=1}^3 \|e^{\beta t}\mathcal{D}^2_{y_j}\boldsymbol{u}^-_2\|_{W^{l,l/2}_{2,tan}(Q_T)} + \|e^{\beta t}\nabla\sigma^-\|_{W^{l,l/2}_{2,tan}(Q_T)} \\ \leqslant c\|\boldsymbol{f}_1\|_{W^{l,l/2}_{2,\tan}(Q_T)}.\end{aligned}$$

Finally, the missing norms $\|e^{\beta t}\mathcal{D}^2_{y_j}\boldsymbol{u}^-_2\|_{W^{l,0}_{2,y_3}(Q_T)}$ are estimated by using equations (3.24) and interpolation inequalities for the mixed derivatives (see [18]). As a result, we obtain

$$\|e^{\beta t}\boldsymbol{u}^-_2\|_{W^{2+l,1+l/2}_2(Q_T)} + \|e^{\beta t}\nabla\sigma^-\|_{W^{l,l/2}_2(Q_T)} \leqslant c\|e^{\beta t}\boldsymbol{f}_1\|_{W^{l,l/2}_2(Q_T)}. \tag{3.25}$$

Together with (3.23), this yields the desired estimate of $\boldsymbol{u}^-, \sigma^-$.

If $\boldsymbol{k}' = 0$, then $\widetilde{\boldsymbol{u}}^-$, $\widetilde{\sigma}^-$ satisfy the relations

$$\begin{cases} s\widetilde{u}^-_\alpha - \nu^-\mathcal{D}^2_{y_3}\widetilde{u}^-_\alpha = \widetilde{f}_{1\alpha}, \quad \widetilde{u}^-_\alpha|_{y_3=0,2d_0} = 0, \quad \alpha = 1,2, \\ s\widetilde{u}^-_3 - \nu^-\mathcal{D}^2_{y_3}\widetilde{u}^-_3 + \dfrac{1}{\rho^-}\mathcal{D}_{y_3}\widetilde{\sigma}^- = \widetilde{f}_{13}, \quad \mathcal{D}_{y_3}\widetilde{u}^-_3 = \widetilde{h}^-, \quad \widetilde{u}^-_3|_{y_3=0,2d_0} = 0.\end{cases}$$

We expand $\widetilde{u}^-_\alpha$ and $\widetilde{f}_{1\alpha}$ in a Fourier series in $\sin\frac{k_3\pi y_3}{2d_0}$, $k_3 = 1,\dots$, in the interval $(0,2d_0)$. For the Fourier coefficients $\check{u}^-_\alpha$ we obtain the relation

$$(s + \nu^-|\xi_3|^2)\check{u}^-_\alpha = \check{f}_{1\alpha}, \quad \xi_3 = \frac{k_3\pi}{2d_0},$$

hence

$$\|e^{\beta t}u^-_\alpha\|_{W^{2+l,1+l/2}_2(\mathfrak{I}_T)} \leqslant c\|e^{\beta t}\boldsymbol{f}_1\|_{W^{l,l/2}_2(\mathfrak{I}_T)},$$

where $\mathfrak{I}_T = I\times(0,T)$. $I = (0,2d_0)$. In addition, we have

$$\widetilde{u}^-_3 = \int_0^{y_3}\widetilde{h}^-(z_3,s)\,\mathrm{d}z_3, \quad s\widetilde{u}^-_3 = \widetilde{H_3}(y_3,s) + \int_0^{y_3,s}\widetilde{H}_1(z_3,s)\,\mathrm{d}z_3$$

if $\mathcal{D}_t h^- = \nabla\cdot\boldsymbol{H} + H_1$. Hence

$$\begin{aligned}&\|e^{\beta t}\boldsymbol{u}^-_2\|_{W^{2+l,1+l/2}_2(\mathfrak{I}_T)} + \|e^{\beta t}\mathcal{D}_{y_3}\sigma^-\|_{W^{l,l/2}_2(\mathfrak{I}_T)} \\ &\leqslant c\big(\|\mathrm{e}^{\beta t}\boldsymbol{f}_1\|_{W^{l,l/2}_2(\mathfrak{I}_T)} + \|h^-\|_{W^{l+1,0}_2(\mathfrak{I}_T)} + \|H_3\|_{W^{0,l/2}_2(\mathfrak{I}_T)} + \|H_1\|_{W^{0,l/2}_2(\mathfrak{I}_T)}\big).\end{aligned}$$

Collecting the estimates of $\boldsymbol{u}^-_1, \boldsymbol{u}^-_2, \sigma^-$, (both for $|\boldsymbol{k}'| > 0$ and $\boldsymbol{k}' = 0$) we obtain

$$\begin{aligned}\|e^{\beta t}\boldsymbol{u}^-\|_{W^{2+l,1+l/2}_2(\mathfrak{Q}_T)} + \|e^{\beta t}\nabla\sigma^-\|_{W^{l,l/2}_2(\mathfrak{Q}_T)} \leqslant c\big(\|e^{\beta t}\boldsymbol{f}^-\|_{W^{l,l/2}_2(\mathfrak{Q}_T)} \\ + \|e^{\beta t}\boldsymbol{h}^-\|_{W^{l+1,0}_2(\mathfrak{Q}_T)} + \|e^{\beta t}\boldsymbol{H}\|_{W^{0,l/2}_2(\mathfrak{Q}_T)} + \|e^{\beta t}H_1\|_{W^{0,l/2}_2(\mathfrak{Q}_T)}\big),\end{aligned} \tag{3.26}$$

where $\mathfrak{Q}_T = \mathfrak{Q}(0,T)$.

As for Problem (3.21), similar problems were studied in [14] (see, for instance, (2.43), (2.47)). A final estimate of the solution is as in [14], (2.54):

$$\begin{aligned} &\|e^{\beta t}\boldsymbol{u}^+\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + |e^{\beta t}\sigma^+|^{(l+1,l/2)}_{\mathfrak{Q}_T} + |e^{\beta t}\mathcal{D}_t\sigma^+|^{(l+1,l/2)}_{\mathfrak{Q}_T} \\ &\leqslant c(\|\boldsymbol{f}^+\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + |e^{\beta t}h^+|^{(l+1,l/2)}_{\mathfrak{Q}_T}), \quad \mathfrak{Q}_T = \mathfrak{Q}\times(0,T). \end{aligned} \tag{3.27}$$

We apply (3.26), (3.27) to the problems arising in the estimates of the solution of (1.3) inside Ω_0^- and Ω_0^+. Let the cube $\mathfrak{Q}^- = \{|y_j - y_{j0}^-| \leqslant d_0\}$, $j=1,2,3$, be included in Ω_0^-. We introduce smooth cut-off functions $\zeta(t)$ and $\varphi(z)$ such that $0 \leqslant \zeta(t), \varphi^-(z) \leqslant 1$, $\zeta(t) = 0$ for $t \leqslant 1/2$, $\zeta(t) = 1$ for $t \geqslant 1$, $\varphi(z) = 1$ for $|z| \leqslant d_0/2$, $\varphi(z) = 0$ for $|z| \geqslant d_0$. The functions $\boldsymbol{w}^-(y,t) = \boldsymbol{u}^-(y,t)\gamma^-(y,t)$ and $\chi^-(y,t) = \theta^-(y,t)\gamma^-(y,t)$, where $\gamma^-(y,t) = \zeta(t)\varphi^-(y - y_0^-)$, satisfy the equations

$$\begin{aligned} &\rho^-\mathcal{D}_t\boldsymbol{w}^- - \nabla\cdot\mathbb{T}^-(\boldsymbol{w}^-) + \nabla\chi^- = \boldsymbol{l}_1^-(\boldsymbol{w}^-,\chi^-;\boldsymbol{u}^-) + \boldsymbol{m}_1^-(\boldsymbol{u}^-,\theta^-) + \rho^-\widehat{\boldsymbol{f}}\gamma^-, \\ &\nabla\cdot\boldsymbol{w}^- = l_2^-(\boldsymbol{w}^-;\boldsymbol{u}^-) + m_2^-(\boldsymbol{u}^-) \text{ in } \mathfrak{Q}^-, \quad \boldsymbol{w}^-|_{t=0} = 0, \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} \boldsymbol{l}_1^-(\boldsymbol{w}^-,\chi^-,\boldsymbol{u}^-,\theta^-) &= (\nabla_{\boldsymbol{u}} - \nabla)\cdot\mathbb{T}^-(\boldsymbol{w}^-) + (\nabla - \nabla_{\boldsymbol{u}})\chi^-, \\ \boldsymbol{m}_1^- &= -\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^-(\gamma^-\boldsymbol{u}^-) + \gamma^-\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^-_{\boldsymbol{u}}(\boldsymbol{u}^-)) + \theta^-\nabla_{\boldsymbol{u}}\gamma^- + \rho^-\boldsymbol{u}^-\cdot\mathcal{D}_t\gamma^-, \\ l_2^-(\boldsymbol{w}^-;\boldsymbol{u}) &= (\nabla - \nabla_{\boldsymbol{u}})\boldsymbol{w}^- = (\mathbb{I} - \mathbb{L}^{-T})\nabla\cdot\boldsymbol{w}^- = \nabla\cdot(\mathbb{I} - \mathbb{L}^{-1})\boldsymbol{w}, \\ m_2^-(\boldsymbol{u}) &= \boldsymbol{u}^-\cdot\nabla_{\boldsymbol{u}}\gamma^-. \end{aligned}$$

Similarly, if $\mathfrak{Q}^+ = \{|y_j - y_{j0}^+| \leqslant d_0\}$, $j=1,2,3$, is included in Ω_0^+, then the functions $\boldsymbol{w}^+(y,t) = \boldsymbol{u}^+(u,t)\gamma^+(y,t)$, $\chi^+(y,t) = \theta^+(y,t)\gamma^+(y,t)$, where $\gamma^+(y,t) = \zeta(t)\varphi^+(y - y_0^+)$, satisfy the equations

$$\begin{cases} \overline{\rho}^+\mathcal{D}_t\boldsymbol{w}^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{w}^+) + \nabla\chi^+ \\ \qquad = \boldsymbol{l}_1^+(\boldsymbol{w}^+,\chi^+;\boldsymbol{u}^+) + \boldsymbol{m}_1^+(\boldsymbol{u}^+,\theta^+) + (\overline{\rho}^+ + \theta^+)\widehat{\boldsymbol{f}}^+\gamma^+, \\ \mathcal{D}_t\chi^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{w}^+ = l_2^+(\boldsymbol{w}^+;\boldsymbol{u}^+) + m_2^+(\boldsymbol{u}^+) \text{ in } \mathfrak{Q}^+, \\ \qquad \boldsymbol{w}^+|_{t=0} = 0, \quad \chi^+|_{t=0} = 0, \end{cases} \tag{3.29}$$

where

$$\begin{aligned} \boldsymbol{l}_1^+(\boldsymbol{w}^+,\chi^+;\boldsymbol{u}^+,\theta^+) &= (\nabla_u - \nabla)\cdot\mathbb{T}^+(\boldsymbol{w}^+) + p_1(\nabla - \nabla_{\boldsymbol{u}})\chi^+ \\ &\quad - (p'(\overline{\rho}^+ + \theta^+) - p'(\overline{\rho}^+))\nabla\chi^+ - \theta^+\mathcal{D}_t\boldsymbol{w}^+, \\ \boldsymbol{m}_1^+ &= -\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^+(\gamma^+\boldsymbol{u}^+) + \gamma^+\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^+_{\boldsymbol{u}}(\boldsymbol{u}^+)) \\ &\quad + p_1\theta^+\nabla_{\boldsymbol{u}}\gamma^+ + (\overline{\rho}^+ + \theta^+)\boldsymbol{u}^+\mathcal{D}_t\gamma^+, \\ l_2^+(\boldsymbol{w}^+,\chi^+;\boldsymbol{u}^+,\theta^+) &= \overline{\rho}^+(\nabla - \nabla_{\boldsymbol{u}})\cdot\boldsymbol{w}^+ - \theta^+\nabla_{\boldsymbol{u}}\cdot\boldsymbol{w}^+, \\ m_2^+(\boldsymbol{u}^+,\theta^+) &= (\overline{\rho}^+ + \theta^+)\boldsymbol{u}^+\cdot\nabla_{\boldsymbol{u}}\gamma^+ + \theta^+\mathcal{D}_t\gamma^+. \end{aligned}$$

We proceed with the estimates of $\boldsymbol{w}^-, \chi^-$. Let $\mathfrak{Q}_T^\pm = \mathfrak{Q}^\pm\times(0,T)$,

$$Y_T(e^{\beta t}\boldsymbol{w}^-, e^{\beta t}\chi^-) = \|e^{\beta t}\boldsymbol{w}^-\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T^-)} + \|e^{\beta t}\nabla\chi^-\|_{W_2^{l,l/2}(\mathfrak{Q}_T^-)}.$$

We have

$$\|e^{\beta t}\boldsymbol{l}_1^-(\boldsymbol{w}^-,\chi^-;\boldsymbol{u}^-,\theta^-)\|_{W_2^{l,l/2}(\mathfrak{Q}_T^-)} \leqslant c(\delta+\epsilon)Y_T(e^{\beta t}\boldsymbol{w}^-,e^{\beta t}\chi^-),$$
$$\|e^{\beta t}\boldsymbol{m}_1^-\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} \leqslant c\big(\|e^{\beta t}\boldsymbol{u}^-\|_{W_2^{l+1.l/2+1/2}(\mathfrak{Q}_T)} + \|e^{\beta t}\theta^-\|_{W_2^{l,l/2}(\mathfrak{Q}_T)}\big),$$

and similar inequalities are true for the norms of l_2^+, m_2^+. Moreover, $\boldsymbol{h}^- \equiv l_2^- + m_2^-$ satisfies $\mathcal{D}_t h^- = \nabla\cdot\boldsymbol{H} + H_1$, where

$$\boldsymbol{H} = \mathcal{D}_t(\mathbb{I}-\mathbb{L}^{-1})\boldsymbol{w}^- + (\mathbb{L}^{-1}\mathbb{T}_u(\boldsymbol{u}^-) - \mathbb{L}^{-1}\chi^-)\nabla_{\boldsymbol{u}}\gamma^-,$$
$$H_1 = -(\mathbb{L}^{-T}\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}^-) - \mathbb{L}^{-T}\theta^-)\nabla\cdot\nabla_{\boldsymbol{u}}\gamma + \widehat{\boldsymbol{f}}\cdot\nabla_{\boldsymbol{u}}\gamma^-,$$

hence in the case of small δ and ϵ we have

$$Y_T(e^{\beta t}\boldsymbol{w}^-,e^{\beta t}\chi^-) \leqslant c\big(Y_T'(e^{\beta t}\boldsymbol{u}^-,e^{\beta t}\theta^-) + \|e^{\beta t}\boldsymbol{f}\gamma\|_{W_2^{l,l/2}(\mathfrak{Q}_T^-)}\big), \tag{3.30}$$

where $Y_T'(\boldsymbol{u}^-,\chi^-)$ is the sum of lower order norms of $\boldsymbol{u}^-$ and θ^- (in comparison with Y_T), which can be estimated by the interpolation inequality

$$Y_T' \leqslant \epsilon_1 Y_T + c(\epsilon_1)\big(\|e^{\beta t}\boldsymbol{u}^-\|_{L_2(\mathfrak{Q}_T^-)} + \|e^{\beta t}\theta^-\|_{W_2^{0,l/2}(\mathfrak{Q}_T^-)}\big). \tag{3.31}$$

For the functions $(\boldsymbol{w}^+,\chi^+)$ satisfying (3.29), we have the inequality

$$Y_T(e^{\beta t}\boldsymbol{w}^+,e^{\beta t}\chi^+) \leqslant c\big(Y_T'(e^{\beta t}\boldsymbol{u}^+,e^{\beta t}\theta^+) + \|e^{\beta t}\boldsymbol{f}\gamma\|_{W_2^{l,l/2}(\mathfrak{Q}_T^+)}\big) \tag{3.32}$$

similar to (3.30) where

$$\begin{aligned} Y_T(e^{\beta t}\boldsymbol{w}^+,e^{\beta t}\chi^+) &= \|e^{\beta t}\boldsymbol{w}^+\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T^+)} \\ &\quad + |e^{\beta t}\chi^+|^{(1+l,l/2)}_{\mathfrak{Q}_T^+} + |e^{\beta t}\mathcal{D}_t\chi^+|^{(1+l,l/2)}_{\mathfrak{Q}_T^+}; \\ Y_T'(e^{\beta t}\boldsymbol{u}^+,e^{\beta t}\theta^+) &\leqslant \epsilon_2 Y_T\big(e^{\beta t}\boldsymbol{u}^+,e^{\beta t}\theta^+\big) \\ &\quad + c(\epsilon_2)\big(\|e^{\beta t}\boldsymbol{u}^+\|_{L_2(\mathfrak{Q}_T^+)} + \|e^{\beta t}\theta^+\|_{L_2(\mathfrak{Q}_T^+)}\big). \end{aligned} \tag{3.33}$$

Our main attention is paid to the most complicated estimates of solution of (1.3) near the interface Γ_0. We pass to local Cartesian coordinates in a neighborhood of an arbitrary point $y_0\in\Gamma_0$. Without loss of generality, it may be assumed that $y_0=0$ and the y_3-axis is directed along $\boldsymbol{n}_0(0)$. Let

$$y_3 = \phi(y')$$

be an equation of Γ_0 near the origin. The coordinate transformation

$$z = \mathcal{F}y:\;\; z'=y',\quad z_3 = y_3-\phi(y') \tag{3.34}$$

establishes one-to-one correspondence between $\mathfrak{Q}' = \{|z_\alpha|\leqslant d_0\;\alpha=1,2\}$ and a subset $\Gamma_0' = \{y'\in\mathfrak{Q}',\;\; y_3=\phi(y')\}$ near zero if d_0 is small. We set $\mathfrak{Q}'\times(-d_0,d_0)\equiv\mathfrak{Q}(2d_0)$. Since $\phi\in W_2^{l+5/2}(\mathfrak{Q}')$ and $\phi(0),\mathcal{D}_{z_3}\phi(0)=0$, we have $|\phi|\leqslant cd_0^2$, $|\nabla\phi|\leqslant cd_0$ in $\mathfrak{Q}(2d_0)$. The Jacobi matrix of the transformation $\mathcal{F}$ is given by

$$\mathcal{J} = \Big(\frac{\partial z}{\partial y}\Big) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -\phi_1 & -\phi_2 & 1\end{pmatrix},\quad \phi_\alpha = \frac{\partial\phi}{\partial y_\alpha}.$$

As above, we introduce two functions $\zeta(t)$ and $\varphi(z)$ such that $\zeta(t) = 0$ for $0 \leqslant t \leqslant 1/2$, $\zeta(t) = 1$ for $t \geqslant 1$, $\varphi(z) = 1$ for $z \in \mathfrak{Q}(d_0)$, $\varphi(y') = 0$ outside $\mathfrak{Q}(2d_0)$, $0 \leqslant \zeta(t), \varphi(z) \leqslant 1$, and we set $\gamma(z,t) = \varphi(z)\zeta(t)$, $\boldsymbol{w}^{\pm}(z,t) = \boldsymbol{u}^{\pm}\gamma$ and $\chi^{\pm}(z,t) = \theta^{\pm}\gamma$. From (1.3) it follows that

$$\begin{cases} \rho^-\mathcal{D}_t\boldsymbol{w}^- - \nabla\cdot\mathbb{T}^-(\boldsymbol{w}^-) + \nabla\chi^- \\ \quad = \boldsymbol{l}_1^-(\boldsymbol{w}^-,\chi^-;\boldsymbol{u}) + \boldsymbol{m}_1^-(\boldsymbol{u},\theta) + \boldsymbol{\lambda}_1^-(\boldsymbol{w}^-,\chi^-) + \rho^-\widehat{\boldsymbol{f}}^-\gamma, \\ \nabla\cdot\boldsymbol{w}^- = \lambda_2^- + l_2^-(\boldsymbol{w}^-;\boldsymbol{u}) + m_2^-(\boldsymbol{u}) \text{ in } \mathfrak{Q}^-, \\ \overline{\rho}^+\mathcal{D}_t\boldsymbol{w}^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{w}^+) + p_1\nabla\chi^+ \\ \quad = \boldsymbol{l}_1^+(\boldsymbol{w}^+,\chi^+;\boldsymbol{u},\theta) + \boldsymbol{m}_1^+(\boldsymbol{u},\theta) + \boldsymbol{\lambda}_1^+(\boldsymbol{w}^+,\chi^+) + (\overline{\rho}^+ + \theta^+)\gamma\widehat{\boldsymbol{f}}^+, \\ \mathcal{D}_t\chi^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{w}^+ \\ \quad = l_2^+(\boldsymbol{w}^+,\chi^+;\boldsymbol{u}^+,\theta^+) + m_2^+(\boldsymbol{u}^+,\theta^+) + \lambda_2^+(\boldsymbol{w}^+,\chi^+) \text{ in } \mathfrak{Q}^+, \\ \boldsymbol{w}|_{t=0} = 0, \quad \chi^+|_{t=0} = 0, \quad [\boldsymbol{w}]|_{z_3=0} = 0, \\ [\mu(\mathcal{D}_{z_3}w_\alpha + \mathcal{D}_{z_\alpha}w_3)]|_{z_3=0} = l_{3\alpha} + m_{3\alpha} + \lambda_{3\alpha}(\boldsymbol{w}), \quad \alpha = 1,2, \\ -p_1\chi^+ + \chi^- + [\mathbb{T}_{33}]|_{z_3=0} = l_4 + m_4 + \lambda_4(\boldsymbol{w}) - \sigma\gamma(H + \frac{2}{R_0}), \end{cases} \tag{3.35}$$

where l_i, m_i, and λ_i are defined by

$$\begin{aligned} &\boldsymbol{l}_1^-(\boldsymbol{w}^-,\chi^-,\boldsymbol{u}^-,\theta^-) = (\nabla_{\boldsymbol{u}} - \nabla)\cdot\mathbb{T}^-(\boldsymbol{w}^-) + (\nabla - \nabla_{\boldsymbol{u}})\chi^-|_{y=\mathcal{F}^{-1}z}, \\ &\boldsymbol{m}_1^- = -\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^-(\gamma\boldsymbol{u}^-) + \gamma\nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^-(\boldsymbol{u}^-)) + \theta^-\nabla_{\boldsymbol{u}}\gamma + \rho^-\boldsymbol{u}^-\cdot\mathcal{D}_t\gamma|_{y=\mathcal{F}^{-1}z}, \\ &\boldsymbol{l}_1^+(\boldsymbol{w}^+,\chi^+;\boldsymbol{u}^+,\theta^+) = (\nabla_u - \nabla)\cdot\mathbb{T}^+(\boldsymbol{w}^+) + p_1(\nabla - \nabla_{\boldsymbol{u}})\chi^+ \\ &\qquad\qquad - (p'(\overline{\rho}^+ + \theta^+) - p'(\overline{\rho}^+))\nabla\chi^+ - \theta^+\mathcal{D}_t\boldsymbol{w}^+|_{y=\mathcal{F}^{-1}z}, \\ &\boldsymbol{m}_1^+ = -\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^+(\gamma\boldsymbol{u}) + \gamma\nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^+(\boldsymbol{u})) + p_1\theta^+\nabla_{\boldsymbol{u}}\gamma + (\overline{\rho}^+ + \theta^+)\boldsymbol{u}^+\mathcal{D}_t\gamma|_{y=\mathcal{F}^{-1}z}, \\ &l_2^-(\boldsymbol{w}^-;\boldsymbol{u}) = (\nabla - \nabla_{\boldsymbol{u}})\boldsymbol{w}^- = (\mathbb{I} - \mathbb{L}^{-T})\nabla\cdot\boldsymbol{w}^- = \nabla\cdot(\mathbb{I} - \mathbb{L}^{-1})\boldsymbol{w}^-|_{y=\mathcal{F}^{-1}z}, \\ &m_2^-(\boldsymbol{u}) = \boldsymbol{u}^-\cdot\nabla_{\boldsymbol{u}}\gamma|_{y=\mathcal{F}^{-1}z}, \\ &l_2^+(\boldsymbol{w}^+,\chi^+;\boldsymbol{u},\theta) = \overline{\rho}^+(\nabla - \nabla_{\boldsymbol{u}})\cdot\boldsymbol{w}^+ - \theta^+\nabla_{\boldsymbol{u}}\cdot\cdot\boldsymbol{w}^+|_{y=\mathcal{F}^{-1}z}, \\ &m_2^+(\boldsymbol{u}^+,\theta^+) = (\overline{\rho}^+ + \theta^+)\boldsymbol{u}^+\cdot\nabla_{\boldsymbol{u}}\gamma + \theta^+\mathcal{D}_t\gamma|_{y=\mathcal{F}^{-1}z}, \\ &\boldsymbol{l}_3(\boldsymbol{w};\boldsymbol{u}) = [\mu(\Pi_0^2\mathbb{S}(\boldsymbol{w})\boldsymbol{n}_0 - \Pi_0\Pi\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w}))\boldsymbol{n}]|_{y=\mathcal{F}^{-1}z}, \\ &\boldsymbol{m}_3 = [\mu(\Pi_0\Pi(\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}\gamma) - \gamma\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}))\boldsymbol{n}]|_{y=\mathcal{F}^{-1}z}, \\ &l_4(\boldsymbol{w},\chi;\boldsymbol{u},\theta) = [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{w})\boldsymbol{n}_0] - [\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{w})\boldsymbol{n}] + \int_0^1 (p'(\overline{\rho}^+ + s\theta^+) - p'(\overline{\rho}^+))\,\mathrm{d}s\chi^+|_{z_3=0} \\ &m_4(\boldsymbol{u}) = [\boldsymbol{n}\cdot(\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}\gamma)\boldsymbol{n} - \gamma\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n})]|_{y=\mathcal{F}^{-1}z} \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} \lambda_1^-(\boldsymbol{w}) &= \mu^-\nabla\cdot(\mathcal{J}\mathcal{J}^T - \mathbb{I})\nabla\boldsymbol{w} + (\mathbb{I} - \mathcal{J}^T)\nabla\chi^-, \\ \lambda_2^+(\boldsymbol{w}) &= \overline{\rho}^+(\mathbb{I} - \mathcal{J}^T)\nabla\cdot\boldsymbol{w} = \overline{\rho}^+\nabla\cdot(\mathbb{I} - \mathcal{J})\boldsymbol{w}, \\ \lambda_2^-(\boldsymbol{w}) &= (\nabla - \mathcal{J}^T\nabla)\boldsymbol{w} = \nabla\cdot(\mathbb{I} - \mathcal{J})\boldsymbol{w}, \\ \lambda_1^+(\boldsymbol{w}^+,\chi^+) &= \mu^+\nabla\cdot(\mathcal{J}\mathcal{J}^T - \mathbb{I})\nabla\boldsymbol{w} \\ &\quad + (\mu^+ + \mu_1^+)\big(\nabla\mathcal{J}(\mathcal{J}^T : \nabla\boldsymbol{w}) - \nabla(\nabla\cdot\boldsymbol{w})\big) + p_1(\mathbb{I} - \mathcal{J}^T)\nabla\chi^-, \\ \lambda_{3\alpha}(\boldsymbol{w}) &= [\mu(\mathcal{D}_{z_3}w_\alpha + \mathcal{D}_{z_\alpha}w_3 - (\Pi_0\mathbb{S}(\boldsymbol{w})\boldsymbol{n}_0)_\alpha)]|_{z_3=0}, \end{aligned}$$

$$\lambda_4(\boldsymbol{w}) = [\mathbb{T}_{33}(\boldsymbol{w}) - \boldsymbol{n}_0 \cdot \mathbb{T}(\boldsymbol{w})\boldsymbol{n}_0)]_{z_3=0}. \tag{3.37}$$

To get estimates of $\boldsymbol{w}^\pm$ and $\chi^\pm$ satisfying (3.35), we take the Fourier transform

$$\widetilde{u}(\xi', z_3, t) = \int\limits_{\mathfrak{Q}'_{2d_0}} e^{-i\xi'\cdot z'} u(z,t)\,\mathrm{d}z',$$

where $\xi' = (\frac{\pi}{d_0}k_1, \frac{\pi}{d_0}k_2)$, $k_1, k_2 = 0, \pm 1, \dots$. We treat differently the transformed problems with $\boldsymbol{k}' = 0$ and $|\boldsymbol{k}'| > 0$. In the first case we have

$$\widetilde{u} = \int\limits_{\mathfrak{Q}'} u(z,t)\,\mathrm{d}z'$$

and (3.35) is converted into

$$\begin{cases} \mathcal{D}_t\widetilde{w}^\pm_\alpha - \nu^+\mathcal{D}^2_{z_3}\widetilde{w}^\pm_\alpha = \frac{1}{\overline{\rho}^+}(\widetilde{l}^+_{1\alpha} + \widetilde{m}^+_{1\alpha} + \widetilde{\lambda}^+_{1\alpha}) + \widetilde{\gamma f_\alpha} \text{ in } I^\pm_{2d_0}, \\ \qquad\qquad\qquad \widetilde{w}_\alpha|_{t=0} = 0, \\ [\widetilde{w}_\alpha]|_{z_3=0} = 0, \quad \widetilde{w}^\pm_\alpha|_{z_3=\pm 2d_0} = 0, \\ \Big[\mu\mathcal{D}_{y_3}\widetilde{w}_\alpha\Big]\Big|_{z_3=0} = \widetilde{\lambda}_{3\alpha}(\boldsymbol{w}) + \widetilde{l}_{3\alpha}(\boldsymbol{w}) + \widetilde{m}_{3\alpha}(\boldsymbol{u}), \quad \alpha = 1,2, \end{cases} \tag{3.38}$$

$$\begin{cases} \mathcal{D}_t\widetilde{w}^-_3 - \nu^-\mathcal{D}^2_{z_3}\widetilde{w}^-_3 + \frac{1}{\rho^-}\mathcal{D}_{z_3}\widetilde{\chi}^- = \widetilde{l}^-_{13} + \widetilde{m}^-_{13} + \widetilde{\lambda}^-_{13} + \widetilde{\gamma f}_3, \\ \mathcal{D}_{z_3}\widetilde{w}^-_3 = \widetilde{l}^-_2 + \widetilde{m}^-_2 \text{ in } I^-_{2d_0} = (-2d_0, 0), \\ \mathcal{D}_t\widetilde{w}^+_3 - (2\nu^+ + \nu^+_1)\mathcal{D}^2_{z_3}\widetilde{w}^+_3 + \frac{p_1}{\overline{\rho}^+}\mathcal{D}_{z_3}\widetilde{\chi}^+ = \frac{1}{\overline{\rho}^+}(\widetilde{l}^+_{13} + \widetilde{m}^+_{13} + \widetilde{\lambda}^+_{13}) + \widetilde{\gamma f_3}, \\ \mathcal{D}_t\widetilde{\chi}^+ + \overline{\rho}^+\mathcal{D}_{z_3}\widetilde{w}^+_3 = \widetilde{l}^+_2 + \widetilde{m}^+_2 \text{ in } I^+_{2d_0} = (0, 2d_0), \\ [\widetilde{w}_3]|_{z_3=0} = 0, \quad \widetilde{w}_3|_{z_3=\pm 2d_0} = 0, \quad \widetilde{w}_3|_{t=0} = \widetilde{\chi}^+|_{t=0} = 0, \\ -p_1\widetilde{\chi}^+ + \widetilde{\chi}^- + (2\mu^+ + \mu^+_1)\mathcal{D}_{z_3}\widetilde{w}^+_3 - \mu^-\mathcal{D}_{z_3}\widetilde{w}^-_3|_{z_3=0} \\ = \widetilde{\lambda}_4 + \widetilde{l}_4 + \widetilde{m}_4 - \sigma \int\limits_{\mathfrak{Q}'_{2d_0}} \gamma(H + \frac{2}{R_0})\,\mathrm{d}z'. \end{cases} \tag{3.39}$$

In the case where $|\boldsymbol{k}'| > 0$, we transform the jump condition once again. We set $\gamma' = \varphi\mathcal{D}_t\zeta$, $\boldsymbol{w}' = \boldsymbol{u}\varphi\mathcal{D}_t\zeta$, $\chi' = \theta\varphi\mathcal{D}_t\zeta$,

$$l_4(\boldsymbol{w}', \chi'; \boldsymbol{u}, \theta^+) = [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{w}')\boldsymbol{n}_0 - \boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{w}')\boldsymbol{n}] + \int\limits_0^1 (p'(\overline{\rho}^+ + s\theta^+) - p'(\overline{\rho}^+))\,\mathrm{d}s\chi',$$

$$m'_4(\boldsymbol{u}) = [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{u}\gamma')\boldsymbol{n}_0 - \boldsymbol{n}\gamma'\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}], \quad \lambda'_4 = [\mathbb{T}_{33}(\boldsymbol{w}') - \boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{w}')\boldsymbol{n}_0)]$$

and write the normal jump condition in (3.35) as follows:

$$\begin{aligned} &-p_1\chi^+ + \chi^- + [T_{33}(\boldsymbol{w})] - \int\limits_0^t ([T_{33}(\boldsymbol{w}')] - p_1{\chi'}^+ + {\chi'}^-)\,\mathrm{d}\tau \\ &= \lambda_4 + l_4 + m_4 - \int\limits_0^t (l'_4 + m'_4 + \lambda'_4)\,\mathrm{d}\tau - \sigma\gamma\Big(H + \frac{2}{R_0}\Big) + \sigma\int\limits_0^t \gamma'\Big(H + \frac{2}{R_0}\Big)\,\mathrm{d}\tau. \end{aligned} \tag{3.40}$$

Since $H = \boldsymbol{n} \cdot \Delta(t) X_{\boldsymbol{u}}$, we have

$$\gamma(H + \frac{2}{R_0}) - \int\limits_0^t \mathcal{D}_\tau \gamma(H + \frac{2}{R_0})\,\mathrm{d}\tau = \int\limits_0^t \gamma \mathcal{D}_\tau(\boldsymbol{n} \cdot \Delta(t) X_{\boldsymbol{u}})\,\mathrm{d}\tau$$

$$= \int\limits_0^t \boldsymbol{n} \cdot \Delta(\tau)\boldsymbol{w}\,\mathrm{d}\tau + \int\limits_0^t \gamma(\dot{\boldsymbol{n}} \cdot \Delta(\tau) + \boldsymbol{n} \cdot \dot{\Delta}(\tau)) X_{\boldsymbol{u}}(y,\tau)\,\mathrm{d}\tau$$

$$= \int\limits_0^t \Delta' w_3\,\mathrm{d}\tau + \int\limits_0^t (\boldsymbol{n} \cdot \Delta(t) - \boldsymbol{e}_3 \cdot \Delta')\boldsymbol{w}\,\mathrm{d}\tau + \int\limits_0^t \gamma\big(\dot{\boldsymbol{n}} \cdot \Delta(\tau) + \boldsymbol{n} \cdot \dot{\Delta}(\tau)\big) X_{\boldsymbol{u}}\,\mathrm{d}\tau.$$

Hence (3.40) implies

$$-p_1\chi^+ + \chi^- + [T_{33}(\boldsymbol{w})] + \sigma \int\limits_0^t \Delta' w_3\,\mathrm{d}\tau = l_4 + m_4 + \lambda_4 - \int\limits_0^t (l_4' + m_4' + \lambda_4')\,\mathrm{d}\tau$$

$$+ \int\limits_0^t ([T_{33}(\boldsymbol{w}')] - p_1\chi'^+ + \chi'^-)\,\mathrm{d}\tau + \int\limits_0^t (l_5(\boldsymbol{w};\boldsymbol{u}) + l_6(\boldsymbol{w};\boldsymbol{u})\,\mathrm{d}\tau$$

$$= b + \sigma \int\limits_0^t B(z,\tau)\,\mathrm{d}\tau, \tag{3.41}$$

where $l_5(\boldsymbol{w};\boldsymbol{u}) = \sigma((\boldsymbol{n} \cdot \Delta(t) - \boldsymbol{e}_3 \cdot \Delta')\boldsymbol{w})$, $l_6(\boldsymbol{w};\boldsymbol{u}) = \sigma\gamma(\dot{\boldsymbol{n}} \cdot \Delta(\tau) + \boldsymbol{n} \cdot \dot{\Delta}(\tau))X_{\boldsymbol{u}})$,

$$b = l_4 + m_4 + \lambda_4 - \int\limits_0^t (l_4' + m_4' + \lambda_4')\,\mathrm{d}\tau + \int\limits_0^t [T_{33}(\boldsymbol{w}',\chi')]\,\mathrm{d}\tau, \quad B = l_5(\boldsymbol{w};\boldsymbol{u}) + l_6(\boldsymbol{w};\boldsymbol{u}).$$

Now, we estimate the functions (3.36) assuming that $|\boldsymbol{k}'| > 0$, i.e.,

$$\int\limits_{\mathfrak{Q}'} \boldsymbol{w}^\pm\,\mathrm{d}z' = 0, \quad \int\limits_{\mathfrak{Q}'} \chi^\pm\,\mathrm{d}z' = 0.$$

We extend $\boldsymbol{w}, \chi$ by zero into the domain $|z_3| > d_0$ and make use of Theorem 1 in [14]. Let $Q_T^\pm = \mathfrak{Q}^\pm \times (0,T)$, $\mathfrak{Q}_T' = \mathfrak{Q}' \times (0,T)$. Since $\boldsymbol{l}_1(\boldsymbol{w}^-, \chi^-; \boldsymbol{u}, \theta)$ is a linear differential expression with respect to the first pair of arguments $\boldsymbol{w}^-, \chi^-$ with small coefficients dependent of $\boldsymbol{u}^-, \theta^-$, one has the following estimate (in view of Proposition 1'):

$$\|e^{\beta t}\boldsymbol{l}_1(\boldsymbol{w}^-, \chi^-; \boldsymbol{u}, \theta)\|_{W_2^{l,l/2}(Q_T^-)} \leqslant c(\delta + \epsilon) Y_T(e^{\beta t}\boldsymbol{w}^-, e^{\beta t}\chi^-).$$

The expressions $\boldsymbol{l}_1^+, l_2^+, \boldsymbol{l}_3, l_4$ have similar structure and satisfy similar inequalities:

$$\begin{aligned} &\|e^{\beta t}\boldsymbol{l}_1(\boldsymbol{w}^+, \chi^+; \boldsymbol{u}, \theta)\|_{W_2^{l,l/2}(\cup Q_T^+)} + |e^{\beta t} l_2(\boldsymbol{w}^+, \chi^+; \boldsymbol{u}, \theta)|_{Q_T^+}^{(1+l,l/2)} \\ &+ \|e^{\beta t}\boldsymbol{l}_3(\boldsymbol{w};\boldsymbol{u})\|_{W_2^{l+1,2,l/2+1/4}(\mathfrak{Q}_T')} + |e^{\beta t} l_4(\boldsymbol{w}, \chi; \boldsymbol{u}, \theta)|_{\mathfrak{Q}_T'}^{(l+1/2,l/2)} \\ &\leqslant \delta_1 Y_T(e^{\beta t}\boldsymbol{w}, e^{\beta t}\chi), \end{aligned} \tag{3.42}$$

where $\delta_1 \leqslant c(\delta+\epsilon)$, $Q^{\pm} = \mathfrak{Q}' \times \mathbb{R}^{\pm}$.

We also need to compute the time derivative of $l_2^- + m_2^- + \lambda_2^-$. We have

$$\mathcal{D}_t(l_2^- + m_2^- + \lambda_2^-) = \nabla\cdot\boldsymbol{H} + H_0 = \mathcal{D}_t\nabla\cdot\boldsymbol{w}^-,$$

where

$$\boldsymbol{H} = \mathcal{D}_t\big((\mathbb{I}-\mathbb{L}^{-1})\boldsymbol{w}\big) + (\mathbb{I}-\mathcal{J})\mathcal{D}_t\boldsymbol{w} + \frac{1}{\rho^-}\big(\mathbb{L}^{-1}\mathbb{T}_u(\boldsymbol{u}) - \mathbb{L}^{-1}\theta^-\big)\nabla_{\boldsymbol{u}}\gamma\Big|_{y=\mathcal{J}^{-1}z},$$
$$H_0 = \boldsymbol{u}^-\cdot\mathcal{D}_t\nabla_{\boldsymbol{u}}\gamma - \frac{1}{\rho^-}\big(\mathbb{T}_u\colon\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\gamma - \theta^-\nabla^2_{\boldsymbol{u}}\gamma\big) + \widehat{f}\nabla_{\boldsymbol{u}}\gamma\Big|_{y=\mathcal{J}^{-1}z}$$

These functions satisfy the inequalities

$$\begin{aligned}
&\|e^{\beta t}\boldsymbol{H}\|_{W_2^{0,l/2}(Q_T^-)} + \|e^{\beta t}H_0\|_{W_2^{0,l/2}(Q_T^-)}\\
&\leqslant c(\delta+\epsilon+d_0)\|e^{\beta t}\mathcal{D}_t\boldsymbol{w}^-\|_{W_2^{0,l/2}(Q_T^-)}\\
&+ c(\delta_1)\big(\|e^{\beta t}\nabla\boldsymbol{u}^-\|_{W_2^{0,l/2}(Q_T^-)} + \|e^{\beta t}\theta^-\|_{W_2^{0,l/2}(Q_T^-)} + \|e^{\beta t}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}\big).
\end{aligned}\tag{3.43}$$

The expressions m_i in (3.36) contain some lower order derivatives of $\boldsymbol{u}$ and θ in comparison with the corresponding l_i, hence

$$\begin{aligned}
&\|e^{\beta t}\boldsymbol{m}_1(\boldsymbol{u},\theta)\|_{W_2^{l,l/2}(\cup Q_T^{\pm})} + |e^{\beta t}m_2|^{(1+l,l/2)}_{Q_T^{\pm}}\\
&+ \|e^{\beta t}\boldsymbol{m}_3(\boldsymbol{u})\|_{W_2^{l+1,l/2+1/4}(Q'_T)} + |e^{\beta t}m_4(\boldsymbol{u},\theta)|^{(l+1/2,l/2)}_{Q'_T}\\
&\leqslant c\big(\|e^{\beta t}\boldsymbol{u}\|_{W_2^{1+l,1/2+l/2}(\cup Q_T^{\pm})} + |e^{\beta t}\theta^+|^{(1+l,l/2)}_{Q_1^+} + \|e^{\beta t}\theta^-\|_{W_2^{l,l/2}(Q_T^-)}\big).
\end{aligned}\tag{3.44}$$

We proceed with the estimates of $\lambda_i^{\pm}$, λ_3, λ_4. Since the elements of $\mathbb{I}-\mathcal{J}$, i.e., the derivatives $\mathcal{D}_{z_\alpha}\phi$, $\alpha=1,2$, satisfy the inequalities

$$|\nabla_{z'}\phi| \leqslant cd_0, \quad |\mathcal{D}^2_{z'}\phi| \leqslant c, \quad \|\mathcal{D}^3_{z'}\phi(\,\cdot\,+\eta) - \mathcal{D}^3_{z'}\phi(\,\cdot\,)\|_{L_2(\mathfrak{Q}')} \leqslant c|\eta|^{l/2-1/2}$$

in $\mathfrak{Q}'$, and inequalities of the same kind are fulfilled also for $n_{\alpha 0}, 1-n_{30}$, it is not hard to show that

$$\begin{aligned}
&\|e^{\beta t}\boldsymbol{\lambda}_1(\boldsymbol{w},\chi;\boldsymbol{u},\theta)\|_{W_2^{l,l/2}(\cup Q_T^{\pm})} + |e^{\beta t}\lambda_2(\boldsymbol{w},\chi;\boldsymbol{u},\theta)|^{(1+l,l/2)}_{Q_T^+}\\
&+ \|e^{\beta t}\boldsymbol{\lambda}_3(\boldsymbol{w};\boldsymbol{u})\|_{W_2^{l+1,2,l/2+1/4}(\mathfrak{Q}'_T)} + |e^{\beta t}l_4(\boldsymbol{w},\chi;\boldsymbol{u},\theta)|^{(l+1/2,l/2)}_{\mathfrak{Q}'_T}\\
&\leqslant c\delta_2 Y_T(e^{\beta t}\boldsymbol{w}, e^{\beta t}\chi),
\end{aligned}\tag{3.45}$$

$\delta_2 = d_0 + \epsilon$.

To complete the analysis in the case where $|\boldsymbol{k}|>0$, we need to estimate the $W_2^{l-1/2,l/2-1/4}(G_T)$-norms of

$$\begin{aligned}
l_5(\boldsymbol{w};\boldsymbol{u}) &= \big((\boldsymbol{n}-\boldsymbol{n}_0)\cdot\Delta(t) + \boldsymbol{n}_0(\Delta(t)-\Delta')\big)\boldsymbol{w}\\
&= \int_0^t \dot{\boldsymbol{n}}(y,\tau)\,\mathrm{d}\tau\,\Delta(t)\boldsymbol{w}(y,t) + \boldsymbol{n}_0\cdot\int_0^t \dot{\Delta}(\tau)\,\mathrm{d}\tau\,\boldsymbol{w}(y,t)\\
&\qquad + \boldsymbol{n}_0\cdot(\Delta(0)-\Delta')\boldsymbol{w}(y,t),\\
l_6 &= (\dot{\boldsymbol{n}}\Delta + \boldsymbol{n}\dot{\Delta})X_u(y,\tau),\\
l_7 &= -(l_4' + m_4' + \lambda_4') + \big([T_{33}(\boldsymbol{w}')] - p_1{\chi'}^{+} + {\chi'}^{-}\big).
\end{aligned}\tag{3.46}$$

In view of (1.4), (2.3), and (2.6), we have

$$\|l_5\|_{W_2^{l-1/2}(\Gamma_0)} \leqslant c\int\limits_0^t \|\nabla \boldsymbol{u}(\,\cdot\,,\tau)\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\,\mathrm{d}\tau \|\boldsymbol{w}(\,\cdot\,,t)\|_{W_2^{3/2+l}(\Gamma_0)} + d_0\|\boldsymbol{w}(\,\cdot\,,t)\|_{W_2^{l+3/2}(\Gamma_0)},$$

$$\|\Delta_t(-h)l_5\|_{L_2(\Gamma_0)} \leqslant c\sqrt{h}\Big(\int\limits_0^h \|\dot{\boldsymbol{n}}(\,\cdot\,,\tau)\|^2_{W_2^{3/2-l}(\Gamma_0)}\,\mathrm{d}\tau\Big)^{1/2}\|\boldsymbol{w}\|_{W_2^{l+3/2}(\Gamma_0)},$$

$$\|l_6\|_{W_2^{l-1/2}(\Gamma_0)} \leqslant c\|\nabla\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\|X_u\|_{W_2^{l+3/2}(\Gamma_0)},$$
$$\|\Delta_t(-h)l_6\|_{L_2(\Gamma_0)} \leqslant c\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{W_2^{3/2-l}(\Gamma_0)}\|X_u\|_{W_2^{l+3/2}(\Gamma_0)},$$

which implies

$$\|e^{\beta t}l_5\|_{W_2^{l-1/2,0}(G_T)} \leqslant c(\delta+d_0)\|e^{\beta t}\boldsymbol{w}\|_{W_2^{l+1/2-\varkappa}(G_T)},$$
$$\|e^{\beta t}l_6\|_{W_2^{0,l/2-1/4}(G_T)} \leqslant c\Big(\|e^{\beta t}\nabla\boldsymbol{u}\|_{W_2^{l/2-1/4}((0,t);W_2^{3/2-l}(\Gamma_0))} + \|\mathrm{e}^{\beta t}\nabla\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\|\boldsymbol{u}\|_{W_2^{2,0}(G_T)}\Big).$$

Finally, since l_7 vanishes for $t>1$, we have

$$\|e^{\beta t}l_7\|_{W_2^{l-1/2,l/2-1/4}(G_1)} \leqslant c\Big(\sum_{\pm}\|e^{\beta t}\nabla\boldsymbol{u}^{\pm}\|_{W_2^{l-1/2,l/2-1/4}(G_1)} + \|e^{\beta t}\theta^{\pm}\|_{W_2^{l-1/2,l/2-1/4}(G_1)}\Big).$$

Collecting the estimates of the nonlinear terms in (3.36), (3.37) and making use of Theorem 1 in [14], we show that in the case of $|\boldsymbol{k}|>0$ we have

$$Y_T(e^{\beta t}\boldsymbol{w}, e^{\beta t}\chi) \leqslant c(Y'_T(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) + Y_1(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) + \|e^{\beta t}\boldsymbol{f}\gamma\|_{W_2^{l,l/2}(Q_T)}), \tag{3.47}$$

where

$$Y_T(e^{\beta t}\boldsymbol{w}, e^{\beta t}\chi) = \|e^{\beta t}\boldsymbol{w}\|_{W_2^{2+l,1+l/2}(\cup Q_T^{\pm})} + |e^{\beta t}\chi^-|^{(1+l,l/2)}_{Q_T^-} + |e^{\beta t}\chi^+|^{(1+l,l/2)}_{Q_T^+} + |e^{\beta t}\mathcal{D}_t\chi^+|^{(1+l,l/2)}_{Q_T^+},$$

$Y'_T(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta)$ is the sum of lower order norms admitting the estimate

$$Y'_T \leqslant \epsilon_3 Y_T + c(\epsilon_3)(\|e^{\beta t}\boldsymbol{u}\|_{L_2(Q_T)} + \|e^{\beta t}\theta^+\|_{L_2(Q_T^+)} + \|e^{\beta t}\theta^-\|_{W_2^{0,l/2}(Q_T^-)}), \quad \epsilon_3 \ll 1, \tag{3.48}$$

finally,

$$
\begin{aligned}
&Y_1(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) = \|e^{\beta t}\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(\cup Q_1^{\pm})} + |e^{\beta t}\theta^-|^{(1+l,l/2)}_{Q_1^-} \\
&+ |e^{\beta t}\theta^+|^{(1+l,l/2)}_{Q_1^+} + |e^{\beta t}\mathcal{D}_t\theta^+|^{(1+l,l/2)}_{Q_1^+}.
\end{aligned}
$$

In the case where $\boldsymbol{k}' = 0$, we need to estimate functions the $\widetilde{\boldsymbol{w}}, \widetilde{\chi}$ satisfying (3.38), (3.39). It is easily seen that $\boldsymbol{w}, \chi$ satisfy the same inequalities as in the case of $|\boldsymbol{k}'| > 0$ with the additional norm of $d_0^{-1}J$ on the right-hand side, where

$$
J = \sigma \int_{\mathfrak{Q}'} \gamma\Big(H + \frac{2}{R_0}\Big)\,\mathrm{d}z' = \sigma \int_{S'_{R_0}} \Big(H + \frac{2}{R_0}\Big)\frac{\gamma}{\sqrt{(1+|\nabla\phi|^2)}}\frac{|\widehat{\mathfrak{L}}^T\boldsymbol{N}(\eta)|}{|\widehat{\mathbb{L}}^T\boldsymbol{n}_0|}\bigg|_{y=\mathcal{X}^{-1}(\eta,t)}\mathrm{d}S_\eta,
$$

$S'_{R_0} = \{\eta \in S_{R_0} : z \in \mathfrak{Q}'\}$, and H is given by (3.7).

Since $\widehat{\mathfrak{L}}^T\boldsymbol{N}(\eta)$ and $\widehat{\mathbb{L}}^T\boldsymbol{n}_0(y)$ are bounded functions with time derivatives controlled by $\mathcal{D}_t r$ and $\nabla\boldsymbol{u}$, respectively, we have

$$
\begin{aligned}
\|e^{\beta t}J\|_{L_2(0,T)} &\leqslant cd_0\|e^{\beta t}r\|_{W_2^{2,0}(\mathcal{S}'_T)},\\
\|e^{\beta t}J\|_{W_2^{l/2}(0,T)} &\leqslant cd_0\Big(\|e^{\beta t}r\|_{W_2^{l/2}((0,T);W_2^2(S'_{R_0}))}\\
&\qquad + \|e^{\beta t}r\|_{W_2^{2+l,0}(\mathcal{S}'_T)}\sup_{t<T}\Big\|\mathcal{D}_t\frac{|\widehat{\mathfrak{L}^T\boldsymbol{N}}|}{|\widehat{\mathbb{L}}^T\boldsymbol{n}_0|}\Big\|_{W_2^{1-l}(S'_{R_0})}\Big)\\
&\leqslant cd_0\Big(\|e^{\beta t}r\|_{W_2^{l/2}((0,T);W_2^2(S'_{R_0}))}\\
&\qquad + \|e^{\beta t}r\|_{W_2^{2+l,0}(\mathcal{S}'_T)}\sup_{t<T}\big(\|\nabla\boldsymbol{u}\|_{W_2^{1-l}(G'_0)}\\
&\qquad + \|\mathcal{D}_t r\|_{W_2^{1-l}(\mathcal{S}'_0)}\big)\Big) \leqslant cd_0|e^{\beta t}r|^{(l/2,2+l)}_{\mathcal{S}'_T}.
\end{aligned}
$$

Putting together the cases of $|\boldsymbol{k}| > 0$ and $\boldsymbol{k} = 0$, we see that the solution of (3.35) satisfies the inequality

$$
\begin{aligned}
&Y_T(e^{\beta t}\boldsymbol{w}, e^{\beta t}\chi) \leqslant c\big(Y'_T(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta)\\
&+ Y_1(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) + \|e^{\beta t}\boldsymbol{f}\gamma\|_{W_2^{l,l/2}(Q_T)} + \delta_1|e^{\beta t}r|^{(l/2,2+l)}_{\mathcal{S}'_T}\big).
\end{aligned}
\tag{3.49}
$$

Now we give an outline of estimates of the solution of our problem near the exterior boundary Σ. Assume that $y_0 = 0 \in \Sigma$, the y_3-axis is directed along the interior normal $\boldsymbol{n}(y_0)$ to Σ, $z_3 = \psi(z') \in W_2^{l+3/2}(\mathfrak{Q}')$ is an equation of Σ near x_0. Upon introducing the functions $\gamma(z,t) = \varphi(z)\zeta(t)$, $\boldsymbol{w}^+ = \gamma\boldsymbol{u}^+$, $\chi^+ = \gamma\theta^+$ and passing to the coordinates $z \in \mathfrak{Q}^+$, as above, we arrive at the problem

$$
\begin{cases}
\overline{\rho}^+\mathcal{D}_t\boldsymbol{w}^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{w}) + p_1\nabla\chi^+\\
= \boldsymbol{l}_1^+(\boldsymbol{w}^+, \chi^+; \boldsymbol{u}, \theta) + \boldsymbol{m}_1^+(\boldsymbol{u}^+, \theta^+) + \boldsymbol{\lambda}_1^+ + \widehat{\boldsymbol{f}}\gamma \equiv \boldsymbol{F}^+,\\
\mathcal{D}_t\chi^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{w}^+\\
= l_2^+(\boldsymbol{w}^+, \chi^+; \boldsymbol{u}, \theta^+) + m_2^+(\boldsymbol{u}^+, \theta^+) + \lambda_2^+(\boldsymbol{w}, \chi^+) \equiv H^+ \text{ in } \mathfrak{Q}^+,\\
\boldsymbol{w}|_{t=0} = 0, \quad \chi^+|_{t=0} = 0, \quad \boldsymbol{w}|_{z_3=0} = 0,
\end{cases}
\tag{3.50}
$$

where l_i, m_i, λ_i are defined as above, i.e., by (3.36), (3.37) with

$$\mathcal{J} = \Big(\frac{\partial z}{\partial y}\Big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\psi_1 & -\psi_2 & 1 \end{pmatrix}.$$

The problems of this type was considered in [14]; see (2.43), Subsection 2.3. In the case of $|\boldsymbol{k}| > 0$ it was shown that

$$\begin{aligned} &\|\boldsymbol{w}\|_{W_2^{2+l,1+l/2}(Q_T)} + |\chi^+|^{(1_l,l/2)}_{Q_T^+} + |\mathcal{D}_t\chi^+|^{(1_l,l/2)}_{Q_T^+} \\ &\leqslant c(\|\boldsymbol{F}^+\|_{W_2^{l,l/2}(Q_T^+)} + |H^+|^{(1+l,l/2)}_{Q_T^+}), \end{aligned} \tag{3.51}$$

and in the case of $\boldsymbol{k} = 0$ the problem was reduced to the system

$$\begin{cases} \mathcal{D}_t\widetilde{w}_\alpha - \nu^+\mathcal{D}^2_{z_3}\widetilde{w}_\alpha = F^+_\alpha \text{ in } I^+ = (0, 2d_0), \\ \widetilde{w}_\alpha|_{t=0} = 0, \quad \widetilde{w}_\alpha|_{z_3=0,2d_0} = 0, \quad \alpha = 1,2, \end{cases} \tag{3.52}$$

$$\begin{cases} \mathcal{D}_t\widetilde{w}_3 - (2\nu^+ + \nu_1^+)\mathcal{D}^2_{z_3}\widetilde{w}_3 + \dfrac{p_1}{\overline{\rho}^+}\mathcal{D}_{z_3}\chi^+ = F_3^+ \\ \mathcal{D}_t\widetilde{\chi}^+ + \overline{\rho}^+\mathcal{D}_{z_3}\widetilde{w}_3 = H^+ \text{ in } I^+, \\ \widetilde{w}_3|_{t=0} = 0, \quad \widetilde{w}_3|_{z_3=0,d_0} = 0, \quad \alpha = 1,2 \end{cases} \tag{3.53}$$

(cf. (2.58), (2.59), Subsection 2.3 in [14]), and estimate (3.51) was obtained as well. By taking the Laplace transform $\check{w} = \int_0^\infty e^{-st}\widetilde{w}\,dt$, problems (3.52) and (3.53) were reduced to

$$s\check{w}_\alpha - \nu^+\mathcal{D}^2_{z_3}\check{w}_\alpha = \check{F}^+_\alpha, \quad \alpha = 1,2,$$

$$s\check{w}_3 - (2\nu^+ + \nu_1^+)\mathcal{D}^2_{z_3}\check{w}_3 + \frac{p_1}{\overline{\rho}^+}\mathcal{D}_{z_3}\check{\chi} = \check{F}_3^+, \quad s\check{\chi} + \overline{\rho}^+\mathcal{D}_{z_3}\check{w}_3 = \check{H}^+$$

with $\nu^+ = \mu/\overline{\rho}^+$, $\nu_1^+ = \mu^+/\overline{\rho}^+$. Upon eliminating $\check{\chi}$, the last system for $\check{w}_3$, $\check{\chi}$ was converted into

$$\big(R(s) - \mathcal{D}^2_{z_3}\big)\check{w}_3^2 = \frac{s}{as + p_1}\Big(\check{F}_3^+ - \frac{p_1}{\overline{\rho}^+ s}\check{H}^+\Big) = \check{G},$$

where $R(s) = \frac{s^2}{as+p_1}$, $a = 2\nu^+ + \nu_1^+ > 0$. By expanding $\check{w}$ in a Fourier series in $\sin\frac{\pi z_3}{d_0}k_3$, $k_3 = 1, \dots,$ one obtains

$$\widetilde{\check{w}}_\alpha = \frac{\widetilde{\check{F}}^+_\alpha}{s + \nu^+|\xi_3|^2}, \quad \alpha = 1,2, \quad \widetilde{\check{w}}_3 = \frac{\widetilde{\check{G}}}{R(s) + |\xi_3|^2}, \quad \xi_3 = \frac{k\pi}{d_0},$$

where the $\widetilde{\check{w}}$ are the Fourier coefficients of $\check{w}$. Since $|\xi_3| > c$, it follows that

$$\|e^{\beta t}\boldsymbol{w}_\alpha\|_{W_2^{2+l,1+l/2}(Q_\infty^+)} + |e^{\beta t}\chi^+|^{(1+l,l/2)}_{\mathfrak{Q}'_\infty} + |e^{\beta t}\mathcal{D}_t\chi^+|^{(1+l,l/2)}_{\mathfrak{Q}'_\infty} \leqslant c\|e^{\beta t}F^+_\alpha\|_{W_2^{l,l/2}(Q_\infty^+)},$$

where $\alpha = 1,2$,

$$\|e^{\beta t}\boldsymbol{w}_3\|_{W_2^{2+l,1+l/2}(Q_\infty^+)} \leqslant c(\|e^{\beta t}F_3^+\|_{W_2^{l,l/2}(Q_\infty^+(2d_0))} + |e^{\beta t}H^+|^{(1+l,l/2)}_{Q_\infty^+(2d_0)}).$$

We recall that $\boldsymbol{F}^+ = \boldsymbol{l}_1 + \boldsymbol{m}_1 + \boldsymbol{\lambda}_1 + \omega\widehat{\boldsymbol{f}}$, $\quad H^+ = l_2 + m_2 + \lambda_2$ and conclude that $\boldsymbol{w}^+$ and χ^+ satisfy inequality (3.32), where

$$Y_T\big(e^{\beta t}\boldsymbol{w}^+, e^{\beta t}\chi^+\big) = \|e^{\beta t}\boldsymbol{w}^+\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T^+)} + |e^{\beta t}\chi^+|^{(1+l,l/2)}_{\mathfrak{Q}_T^+} + |e^{\beta t}\mathcal{D}_t\chi^+|^{(1+l,l/2)}_{\mathfrak{Q}_T^+}$$

and

$$Y_T' \leqslant \epsilon_1 Y_T + c(\epsilon_1)\Big(\|e^{\beta t}\boldsymbol{u}^+\|_{L_2(\mathfrak{Q}_T)} + \|e^{\beta t}\theta^+\|_{L_2(\mathfrak{Q}_T^+)}\Big). \qquad \square$$

Now, we go back to Problem (1.3) and prove the main result of this section.

Theorem 5. *Let $(\boldsymbol{u}, \theta)$ be the solution of Problem* (1.3) *given for $y \in \Omega_0^\pm$, $t \leqslant T$, $T > 2$, fixed, and possessing finite norm*

$$\begin{aligned}\mathbf{Y}_T &= \|e^{\beta t}\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(\cup Q_T^\pm)} + \|e^{\beta t}\nabla\theta^-\|_{W_2^{l,l/2}(Q_T^-)} + \|e^{\beta t}\theta^-\|_{W_2^{0,l/2}(Q_T^-)} \\ &+ |e^{\beta t}\theta^+|^{(1+l,l/2)}_{Q_T^+} + |e^{\beta t}\mathcal{D}_t\theta^+|^{(1+l,l/2)}_{Q_T^+} + |e^{\beta t}r|^{(5/2+l,l/2)}_{\mathcal{S}_T} + \|e^{\beta t}\mathcal{D}_t r\|_{W_2^{3/2+l,0}(\mathcal{S}_T)},\end{aligned}$$

Assume that the data $(e^{\beta t}\boldsymbol{f}, \boldsymbol{u}_0, \theta_0^+, r_0)$ satisfy the smallness condition in Theorem 2. *Then*

$$\mathbf{Y}_T \leqslant c\mathbf{F}_T, \tag{3.54}$$

where

$$\mathbf{F}_T = \|e^{\beta t}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\cup\Omega_0^\pm)} + \|\theta_0^+\|_{W_2^{l+1}(\Omega_0^+)} + \|r_0\|_{W_2^{l+2}(S_{R_0})},$$
$$Q_T^\pm = \Omega_0^\pm \times (0,T), \quad \mathcal{S}_T = S_{R_0} \times (0,T).$$

Proof. Let ω_k be the covering of Ω with the sets

$$\begin{aligned}&\omega_k = \{|y-y_k| \leqslant d_0, \quad y_k \in \Sigma\} \quad \text{for} \quad k = 1,\dots,m_1,\\ &\omega_k = \{|y-y_k| \leqslant d_0, \quad y_k \in \Omega_0^+\} \quad \text{for} \quad k = 1+m_1,\dots,m_2,\\ &\omega_k = \{|y-y_k| \leqslant d_0, \quad y_k \in \Gamma_0\} \quad \text{for} \quad k = 1+m_2,\dots,m_3,\\ &\omega_k = \{|y-y_k| \leqslant d_0, \quad y_k \in \Omega_0^-\} \quad \text{for} \quad k = 1+m_3,\dots,m_4.\end{aligned}$$

We assume that the multiplicity of this covering is finite and ω_k, $k = m_1+1,\dots m_2$, $k = m_3+1,\dots,m_4$ are strictly interior subdomains of Ω_0^+ and Ω_0^-, respectively. Clearly, ω_k, $k = 1,\dots,m_1$ and $k = m_2+1,\dots m_3$, are sufficiently dense coverings of Σ and Γ_0 of finite multiplicity; we assume that it is independent of d_0 and d_0 is small. We introduce a smooth and monotone function $\zeta(t)$ equal to one for $t > 1$ and vanishing for $t < 1/2$, and we set $\gamma_k(y,t) = \varphi_k(y)\zeta(t)$, where the $\varphi_k(y)$ are smooth functions equal to one for $|y-y_k| \leqslant d_0$ and to zero for $|y-y_k| \geqslant 2d_0$, moreover,

$$c_1 > \sum_{j=1}^{m_4}\varphi_j(y) \geqslant c \geqslant 1$$

in Ω. It is clear that the functions $(\gamma_k\boldsymbol{u}, \gamma_k\theta) \equiv (\boldsymbol{w}_k, \chi_k)$ satisfy the relations

$$\begin{cases}\overline{\rho}^+\mathcal{D}_t\boldsymbol{w}_k^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{w}_k^+) + p_1\nabla\chi_k^+ = \boldsymbol{l}_1^+(\boldsymbol{w}_k^+,\chi_k^+;\boldsymbol{u}) + m_1^+(\boldsymbol{u},\theta)\\ \qquad +(\overline{\rho}^+ + \theta^+)\widehat{\boldsymbol{f}}\gamma_k,\\ \mathcal{D}_t\chi_k^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{w}_k^+ = l_2^+(\boldsymbol{w}_k;\boldsymbol{u}) + m_2^+(\boldsymbol{u}) \text{ in } \omega_k^+\cap\Omega_0^+,\\ \boldsymbol{w}_k^+|_{\Sigma_k'} = 0, \quad \boldsymbol{w}_k^+|_{t=0} = 0, \quad \chi_k^+|_{t=0} = 0 \quad \textit{for} \quad k = 1,\dots,m_1,\end{cases} \tag{3.55}$$

$$\begin{cases} \overline{\rho}^+\mathcal{D}_t\boldsymbol{w}_k^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{w}_k^+) + p_1\nabla\chi_k^+ \\ \qquad = \boldsymbol{l}_1^+(\boldsymbol{w}_k^+,\chi_k^+;\boldsymbol{u}) + m_1^+(\boldsymbol{u},\theta) + (\overline{\rho}^+ + \chi_k^+)\widehat{\boldsymbol{f}}\gamma_k, \\ \mathcal{D}_t\chi_k^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{w}_k^+ \\ \qquad = l_2^+(\boldsymbol{w}_k;\boldsymbol{u}) + m_2^-(\boldsymbol{u}) \text{ in } \omega_k^+, \quad \omega_k^+\cap(\Gamma_0\cup\Sigma) = \varnothing, \\ \boldsymbol{w}_k^+|_{t=0} = 0, \quad \chi_k^+|_{t=0} = 0 \quad for \quad k = m_1+1,\dots,+m_2, \end{cases} \tag{3.56}$$

$$\begin{cases} \overline{\rho}^+\mathcal{D}_t\boldsymbol{w}_k^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{w}_k^+) + p_1\nabla\chi_k^+ = \boldsymbol{l}_1^+(\boldsymbol{w}_k^+,\chi_k^+;\boldsymbol{u}) + m_1^+(\boldsymbol{u},\theta) \\ \qquad +(\overline{\rho}^+ + \theta^+)\widehat{\boldsymbol{f}}\gamma_k, \\ \mathcal{D}_t\chi_k^+ + \overline{\rho}^+\nabla\cdot\boldsymbol{w}_k^+ = l_2^+(\boldsymbol{w}_k;\boldsymbol{u}_k) + m_2^-(\boldsymbol{u}_k) \text{ in } \omega_k^+\cap\Omega_0^+, \\ \rho^-\mathcal{D}_t\boldsymbol{w}_k^- - \nabla\cdot\mathbb{T}^-(\boldsymbol{w}_k^-) + \nabla\chi_k^- \\ \qquad = \boldsymbol{l}_1^-(\boldsymbol{w}_k^-,\chi_k^-;\boldsymbol{u}) + m_1^-(\boldsymbol{u},\theta) + \rho^-\widehat{\boldsymbol{f}}\gamma_k, \\ \nabla\cdot\boldsymbol{w}_k^- = l_2^-(\boldsymbol{w}_k;\boldsymbol{u}) + m_2^-(\boldsymbol{u}) \text{ in } \omega_k^-\cap\Omega_0^-, \\ \boldsymbol{w}_k|_{t=0} = 0, \quad \chi_k^+|_{t=0} = 0, \\ [\boldsymbol{w}_k]|_{\Gamma'_{0k}} = 0, \quad \boldsymbol{w}_k^+|_{t=0} = 0, \quad \chi_k^+|_{t=0} = 0, \\ [\mu\Pi_0\mathbb{S}(\boldsymbol{w}_k)\boldsymbol{n}_0]_{\Gamma'_{0k}} = \boldsymbol{l}_3(\boldsymbol{w}_k;\boldsymbol{u}) + \boldsymbol{m}_3(\boldsymbol{u}), \\ -p_1\chi_k^+ + \chi_k^- + [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{w}_k)\boldsymbol{n}_0]_{\Gamma'_{0k}} = l_4(\boldsymbol{w}_k,\chi_k;\boldsymbol{u},\theta) + m_4(\boldsymbol{u},\chi) \\ +\sigma\gamma_k(H + \frac{2}{R_0}), \quad \text{for} \quad k = m_2+1,\dots,m_3, \end{cases} \tag{3.57}$$

$$\begin{cases} \rho^-\mathcal{D}_t\boldsymbol{w}_k^- - \nabla\cdot\mathbb{T}^-(\boldsymbol{w}_k^-) + \nabla\chi_k^- \\ \qquad = \boldsymbol{l}_1^-(\boldsymbol{w}_k^-,\chi_k^-;\boldsymbol{u}) + m_1^-(\boldsymbol{u},\theta) + \rho^-\widehat{\boldsymbol{f}}\gamma_k, \\ \nabla\cdot\boldsymbol{w}_k^- = l_2^-(\boldsymbol{w}_k;\boldsymbol{u}) + m_2^-(\boldsymbol{u}) \text{ in } \omega_k^-, \\ \boldsymbol{w}_k|_{t=0} = 0, \quad \chi_k^-|_{t=0} = 0 \quad \text{for} \quad k = m_3+1,\dots,m_4, \end{cases} \tag{3.58}$$

where

$$\begin{aligned} &\boldsymbol{l}_1^-(\boldsymbol{w}_k^-,\chi_k^-,\boldsymbol{u}^-,\theta^-) = (\nabla_u - \nabla)\cdot\mathbb{T}^-(\boldsymbol{w}_k^-) + (\nabla - \nabla_{\boldsymbol{u}})\chi_k^-, \\ &m_1^- = -\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^-(\gamma_k\boldsymbol{u}^-) + \gamma_k\nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^-(\boldsymbol{u}^-)) + \theta^-\nabla_{\boldsymbol{u}}\gamma_k + \rho^-\boldsymbol{u}^-\cdot\mathcal{D}_t\gamma_k, \\ &\boldsymbol{l}_1^+(\boldsymbol{w}_k^+,\chi_k^+;\boldsymbol{u}^+,\theta^+) = (\nabla_u - \nabla)\cdot\mathbb{T}^+(\boldsymbol{w}_k^+) + p_1(\nabla - \nabla_{\boldsymbol{u}})\chi_k^+ - \theta^+\mathcal{D}_t\gamma_k^+, \\ &m_1^+ = -\nabla_{\boldsymbol{u}}\cdot\mathbb{T}^+(\gamma_k\boldsymbol{u}) + \gamma_k\nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^+(\boldsymbol{u})) + p_1\theta^+\nabla_{\boldsymbol{u}}\gamma_k + (\overline{\rho}^+ + \theta^+)\boldsymbol{u}_k^+\mathcal{D}_t\gamma_k, \\ &l_2^-(\boldsymbol{w}_k^-;\boldsymbol{u}) = (\nabla - \nabla_{\boldsymbol{u}})\boldsymbol{w}_k^- = (\mathbb{I} - \mathbb{L}^{-T}\mathcal{J}^{-T})\nabla\cdot\boldsymbol{w}_k^- = \nabla\cdot(\mathbb{I} - \mathcal{J}^{-1}\mathbb{L}^{-1})\boldsymbol{w}_k, \\ &m_2^-(\boldsymbol{u}) = \boldsymbol{u}^-\cdot\nabla_{\boldsymbol{u}}\gamma_k, \\ &l_2^+(\boldsymbol{w}_k,\chi_k;\boldsymbol{u},\theta) = \overline{\rho}^+(\nabla - \nabla_{\boldsymbol{u}})\cdot\boldsymbol{w}_k^+ - \theta^+\nabla_{\boldsymbol{u}}\cdot\boldsymbol{w}_k, \\ &m_2^+(\boldsymbol{u}^+,\theta^+) = (\overline{\rho}^+ + \theta^+)\boldsymbol{u}\cdot\nabla_{\boldsymbol{u}}\gamma_k + \theta^+\mathcal{D}_t\gamma_k, \\ &\boldsymbol{l}_3(\boldsymbol{w}_k;\boldsymbol{u}) = [\mu\Pi_0^2\mathbb{S}(\boldsymbol{w}_k)\boldsymbol{n}_0 - \Pi_0\Pi\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w}_k)\boldsymbol{n}], \\ &\boldsymbol{m}_3(\boldsymbol{u}) = [\mu\Pi_0\Pi(\boldsymbol{u}\otimes\nabla_{\boldsymbol{u}}\gamma_k + \nabla_{\boldsymbol{u}}\gamma_k\otimes\boldsymbol{u})\boldsymbol{n}], \\ &l_4(\boldsymbol{w}_k^+,\chi_k^+;\boldsymbol{u},\theta) = [\boldsymbol{n}_0\cdot\mathbb{T}^+(\boldsymbol{w}_k)\boldsymbol{n}_0] - [\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}^+(\boldsymbol{w}_k^+)\boldsymbol{n}] \\ &\ - (p(\overline{\rho}^+ + \theta^+) - p(\overline{\rho}^+) - p_1\theta^+))\chi_k^+, \\ &m_4 = [\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}\gamma)\boldsymbol{n} - \gamma\boldsymbol{n}\cdot\mathbb{T}(\boldsymbol{u})\boldsymbol{n}]. \end{aligned} \tag{3.59}$$

In each problem (3.55), (3.57), we pass to local Cartesian coordinates $z \in \mathfrak{Q}^-$ or $z \in \mathfrak{Q}^\pm$ (in (3.56), (3.58) a change of variables is not required). As shown above, the solutions of these problems satisfy certain inequalities of the form (3.49), (3.26), (3.32). Taking the squares of these inequalities and summing up with respect to k, we arrive, in the case of small $\delta, \epsilon, \epsilon_i$, at an estimate equivalent to

$$\begin{aligned} Y_T^2(e^{\beta t}\boldsymbol{w}, e^{\beta t}\chi) &= \|e^{\beta t}\boldsymbol{w}\|^2_{W_2^{2+l,1+l/2}(\cup Q_T^\pm)} \\ &+ |e^{\beta t}\chi^-|^{(1+l,l/2)}_{Q_T^-} + |e^{\beta t}\chi^+|^{(1+l,l/2)}_{Q_T^+} + |e^{\beta t}\mathcal{D}_t\chi^+|^{(1+l,l/2)}_{Q_T^+} \\ &\leqslant c\big(Y_T'^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) + Y_1^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) \\ &+ \|\zeta\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_T)} + |e^{\beta t}r|^{2(2+l,l/2)}_{\mathcal{S}_T}\big), \end{aligned} \tag{3.60}$$

where $\boldsymbol{w} = \sum_k \boldsymbol{w}_k$, $\chi = \sum_k \chi_k$, and Y_T' is the sum of lower order norms of $e^{\beta t}\boldsymbol{u}$ and $e^{\beta t}\theta$ admitting an interpolation estimate of type (3.33).

By adding $Y_1^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta)$ to (3.60) and taking account of (2.8) (with $T = 1$), we obtain

$$\begin{aligned} Y_T^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) &\leqslant c(Y_1^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) + Y_T^2(e^{\beta t}\boldsymbol{w}, e^{\beta t}\chi)) \\ &\leqslant c\Big(Y_T'^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) + \mathbf{F}_T^2 + |e^{\beta t}r|^{2(2+l,l/2)}_{\mathcal{S}_T}\Big). \end{aligned}$$

Now we make use of the inequality

$$\begin{aligned} \|e^{\beta t}r\|^2_{W_2^{l+5/2,0}(\mathcal{S}_T))} &+ \|e^{\beta t}\mathcal{D}_t r\|^2_{W_2^{l+3/2,0}(\mathcal{S}_T)} + \|e^{\beta t}r\|^2_{W_2^{l/2}((0,1);W_2^{5/2}(S_{R_0}))} \\ &\leqslant c\sum_{\pm}\Big(|e^{\beta t}\nabla\boldsymbol{u}^\pm|^{(l+1/2,l/2)}_{G_T} + |e^{\beta t}\theta^\pm|^{(l+1/2,l/2)}_{G_T}\Big) \leqslant c\mathbf{F}_T^2, \end{aligned}$$

that is proved in the same way as (2.38); this leads to

$$\begin{aligned} Y_T^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta, e^{\beta t}r) &\equiv \|e^{\beta t}r\|^2_{W_2^{l+5/2,0}(\mathcal{S}_T)} + \|e^{\beta t}\mathcal{D}_t r\|^2_{W_2^{l+3/2,0}(\mathcal{S}_T)} + Y_T^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) \\ &\leqslant c\big(\mathbf{F}_T^2 + Y_T'^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) + |e^{\beta t}r|^{2(2+l,l/2)}_{\mathcal{S}_T}\big). \end{aligned} \tag{3.61}$$

The sum of lower order norms on the right-hand side can be estimated by the interpolation inequality

$$\begin{aligned} Y_T'^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta) &+ |e^{\beta t}r|^{2(2+l,l/2)}_{\mathcal{S}_T} \\ &\leqslant \epsilon_3 Y_T(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta, e^{\beta t}r) + c(\epsilon_3)\mathcal{Y}_0^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta, e^{\beta t}r), \end{aligned} \tag{3.62}$$

where $\epsilon_3 \ll 1$ and

$$\mathcal{Y}_0^2 = \sum_{\pm}\|e^{\beta t}\boldsymbol{u}\|^2_{L_2(Q_T^\pm)} + \|e^{\beta t}\theta^+\|^2_{L_2(Q_T^+)} + \|e^{\beta t}\theta^-\|^2_{W_2^{0,l/2}(Q_T^-)} + \|e^{\beta t}r\|^2_{L_2(\mathcal{S}_T)}.$$

The norm of θ^- satisfies inequality (3.10); the norm of $\boldsymbol{f}$ on the right-hand side of (3.10) can be included in $\mathbf{F}_T$, whereas other terms satisfy (3.18) and (3.19). Hence if ϵ_3 is sufficientrly small, then we arrive at

$$Y_T^2(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta, e^{\beta t}r) \leqslant c\big(\mathbf{F}_T^2 + \|e^{\beta t}\boldsymbol{u}\|^2_{L_2(Q_T)} + \|e^{\beta t}\theta^+\|^2_{L_2(Q_T^+)} + \|e^{\beta t}r\|^2_{L_2(G_T)}\big).$$

The L_2-norms of $\boldsymbol{u}, \theta^+$, and r can be estimated by (3.1), which leads to (3.54).

4 On the smoothness of the free boundary Γ_t

In the present section it is shown that under some additional assumptions on $p(\rho)$ and $\boldsymbol{f}$ the function $r(\,\cdot\,,t)$, $t>0$, belongs to $W_2^{l+5/2}(S_{R_0})$. The proof is based on the following theorem.

Theorem 6. *Assume that $p\in C^{3+1}(\min\rho,\max\rho)$ and $\boldsymbol{f}$ satisfies the following additional restrictions: $\boldsymbol{f}\in W_2^{\alpha_1}((t_0,T);W_2^l(\Omega))\cap W_2^{0,\alpha_1+l/2}(Q_{t_0,T})$ with $\alpha_1\in(1/2,1-l/2)$. Then $\boldsymbol{u}^{(s)}(y,t)=\boldsymbol{u}(y,t)-\boldsymbol{u}(y,t-s)$ and $\theta^{(s)}(y,t)=\theta(y,t)-\theta(y,t-s)$ satisfy the inequality*

$$\begin{aligned}\boldsymbol{Y}(t_0,t_1)\equiv\|e^{\beta t}\boldsymbol{u}^{(s)}\|_{W_2^{2+l,1+l/2}(\cup Q^{\pm}_{t_2,t_1})}+|e^{\beta t}\theta^{(s)-}|^{(l+1,l/2)}_{Q^-_{t_2,t_1}}&\\ +|e^{\beta t}\theta^{(s)+}|^{(l+1,l/2)}_{Q^+_{t_2,t_1}}+|e^{\beta t}\mathcal{D}_t\theta^{(s)+}|^{(l+1,l/2)}_{Q^+_{t_2,t_1}}\leqslant C(\boldsymbol{u},\theta,r)s^a,&\end{aligned}\tag{4.1}$$

where $a>1/2$, $0<t_0<t_1<T$, $t_2=(t_1-(t_1-t_0)/4)$, $0<s<\min(t_1-t_2,t_0)$, $Q^{\pm}_{t_2,t_1}=\Omega^{\pm}_0\times(t_2,t_1)$, and C is a constant independent of the norms of the solution of (1.3).

The theorem is proved in several steps. First we do some auxiliary constructions. Let $\lambda\in(0,(t_1-t_0)/4)$ and let $\zeta_\lambda(t)$ be a smooth monotone function of t equal to one for $t>t_0+\lambda$, to zero for $t<t_0+\lambda/2$, and satisfying the inequality $|\mathcal{D}_t^k\zeta_\lambda|\leqslant c\lambda^{-k}$, $k=1,2,3$. It can be shown that $\boldsymbol{u}^{(s)}_\lambda(y,t)=\zeta_\lambda(t)(\boldsymbol{u}(y,t)-\boldsymbol{u}(y,t-s))$, $\theta^{(s)}_\lambda(y,t)=\zeta_\lambda(\theta(y,t)-\theta(y,t-s))$ satisfy the equations

$$\begin{cases}\rho^-\mathcal{D}_t\boldsymbol{u}^{(s)-}_\lambda-\mu^-\nabla^2_{\boldsymbol{u}}\boldsymbol{u}^{(s)-}_\lambda+\nabla_{\boldsymbol{u}}\theta^{(s)-}_\lambda=\boldsymbol{F}_1^-,\quad \nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^{(s)-}_\lambda=F_2^-,\\ (\overline{\rho}^++\theta^+)\mathcal{D}_t\boldsymbol{u}^{(s)+}_\lambda-\mu^+\nabla^2_{\boldsymbol{u}}\boldsymbol{u}^{(s)+}_\lambda-(\mu^++\mu_1^+)\nabla_{\boldsymbol{u}}(\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^{(s)+}_\lambda)\\ \qquad+p'(\overline{\rho}^++\theta^+)\nabla_{\boldsymbol{u}}\theta^{(s)+}=\boldsymbol{F}_1^+,\\ \mathcal{D}_t\theta^{(s)+}_\lambda+(\overline{\rho}^++\theta^+)\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^{(s)+}_\lambda=F_2^+,\\ \boldsymbol{u}^{(s)+}_\lambda|_{t=0}=0,\quad \theta^{(s)+}_\lambda|_{t_0=0}=0,\quad \boldsymbol{u}^{(s)+}_\lambda|_\Sigma=0,\\ [\boldsymbol{u}^{(s)}_\lambda]|_{\Gamma_0}=0,\quad [\mu\Pi_{\boldsymbol{u}}\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u}^{(s)}_\lambda)\boldsymbol{n}]|_{\Gamma_0}=\boldsymbol{F}_3\end{cases}\tag{4.2}$$

for $t\leqslant t_1$. The last jump condition for $(\boldsymbol{u}^{(s)}_\lambda,\theta^{(s)}_\lambda)$ is obtained by a calculation similar to that carried out in §3 (see (3.41)). We start with the equation

$$[\boldsymbol{n}\cdot\mathbb{T}_u(\boldsymbol{u},\theta)\boldsymbol{n}]+\sigma\boldsymbol{n}\cdot\Delta(t)X_{\boldsymbol{u}}=\mathfrak{p}-\frac{2\sigma}{R_0},\tag{4.3}$$

where

$$\begin{aligned}&\mathbb{T}^+_{\boldsymbol{u}}(\boldsymbol{u}^+,\theta^+)=\mathbb{T}^+_u(\boldsymbol{u}^+)-p_1\theta^+\mathbb{I},\quad \mathbb{T}^-_{\boldsymbol{u}}(\boldsymbol{u}^-,\theta^-)=\mathbb{T}^-_u(\boldsymbol{u}^-)-\theta^-\mathbb{I},\\ &\mathfrak{p}=p(\overline{\rho}^++\theta^+)-p(\overline{\rho}^+)-p_1\theta^+.\end{aligned}$$

It implies

$$\begin{aligned}&\zeta_\lambda(t)\big([\boldsymbol{n}\cdot\mathbb{T}_u(\boldsymbol{u},\theta)\boldsymbol{n}]+\sigma\boldsymbol{n}\cdot\Delta(t)X_{\boldsymbol{u}}-\mathfrak{p}\big)\\ &=\int_0^t\mathcal{D}_t\zeta_\lambda(\tau)\big([\boldsymbol{n}\cdot\mathbb{T}_u(\boldsymbol{u},\theta)\boldsymbol{n}]+\sigma\boldsymbol{n}\cdot\Delta(t)X_{\boldsymbol{u}}-\mathfrak{p}\big)\,\mathrm{d}\tau.\end{aligned}\tag{4.4}$$

By subtracting from (4.4) a similar equation written for the time instant $t-s$, i.e.,

$$\zeta_\lambda(t)([\boldsymbol{n}'\cdot\mathbb{T}_{\boldsymbol{u}'}(\boldsymbol{u}',\theta')\boldsymbol{n}']+\sigma\boldsymbol{n}'\cdot\Delta'(t)X'_{\boldsymbol{u}}-\mathfrak{p}')$$
$$=\int\limits_0^{t-s}\mathcal{D}_t\zeta_\lambda(\tau+s))([\boldsymbol{n}\cdot\mathbb{T}_u(\boldsymbol{u},\theta)\boldsymbol{n}]+\sigma\boldsymbol{n}\cdot\Delta(t)X_{\boldsymbol{u}}-\mathfrak{p})\,\mathrm{d}\tau,$$

where $v'(t)=v(t-s)$, $v_\lambda=\zeta_\lambda(t)v(t)$, after simple calculations (integration by parts) we obtain the relation

$$\big[\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}^{(s)},\theta^{(s)})\boldsymbol{n}\big]+\sigma\int\limits_0^t\boldsymbol{n}\cdot\Delta(\tau)\boldsymbol{u}^{(s)}\,\mathrm{d}\tau|_{\Gamma_0}=F_4+\int\limits_0^t\sum_{j=5}^8F_j\,\mathrm{d}\tau+F_9. \tag{4.5}$$

The nonlinear terms in (4.2) and (4.5) are given by

$$\begin{aligned}
&\boldsymbol{F}_1^-=-\mu^-(\nabla^2_{\boldsymbol{u}}-\nabla^2_{\boldsymbol{u}'})\boldsymbol{u}_\lambda^{-'}+(\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\theta_\lambda^{-'}-\rho^-\mathcal{D}_t\zeta_\lambda\boldsymbol{u}^{(s)}+\rho^-\zeta_\lambda\widehat{\boldsymbol{f}}^{(s)},\\
&F_2^-=(\nabla_{\boldsymbol{u}'}-\nabla_{\boldsymbol{u}})\boldsymbol{u}_\lambda^{-'}=\nabla\cdot\mathcal{F}_2,\quad \mathcal{F}_2=(\widehat{\mathbb{L}}-\widehat{\mathbb{L}}')\boldsymbol{u}_\lambda^{-'},\\
&\boldsymbol{F}_1^+=-\mu^+(\nabla^2_{\boldsymbol{u}}-\nabla^2_{\boldsymbol{u}'})\boldsymbol{u}_\lambda^{+'}-(\mu^++\mu_1^+)(\nabla_{\boldsymbol{u}}\otimes\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'}\otimes\nabla_{\boldsymbol{u}'})\boldsymbol{u}_\lambda^{+'}\\
&\quad+p'\big(\overline{\rho}^++\theta^{+'}\big)(\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\theta_\lambda^{+'}+(\overline{\rho}^++\theta^+)\mathcal{D}_t\zeta_\lambda\boldsymbol{u}^{+(s)}\\
&\quad+\big(p'(\overline{\rho}^++\theta^+)-p'(\overline{\rho}^++\theta^{+'})\big)\nabla_{\boldsymbol{u}}\theta_\lambda^{+'}+(\overline{\rho}^++\theta^+)\widehat{\boldsymbol{f}}^{(s)}\zeta_\lambda+\theta^{+(s)}\zeta_\lambda\widehat{\boldsymbol{f}}',\\
&F_2^+=-(\overline{\rho}^++\theta^+)(\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\boldsymbol{u}_\lambda^{+'}-\theta^{+(s)}\nabla_{\boldsymbol{u}'}\boldsymbol{u}_\lambda^{+'},\\
&\boldsymbol{F}_3=-\big[\mu\Pi(\Pi\mathbb{S}_{\boldsymbol{u}'}(\boldsymbol{u}'_\lambda)\boldsymbol{n}-\Pi'\mathbb{S}_{\boldsymbol{u}'}(\boldsymbol{u}'_\lambda)\boldsymbol{n}'\big],\\
&F_4=\big[\boldsymbol{n}'\cdot\mathbb{T}_{\boldsymbol{u}'}(\boldsymbol{u}'_\lambda,\theta'_\lambda)\boldsymbol{n}'\big]-\big[\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}'_\lambda,\theta'_\lambda)\boldsymbol{n})\big],\\
&F_5=-\mathcal{D}_t\zeta_\lambda\big(\big[\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}',\theta')\boldsymbol{n}\big]-[\boldsymbol{n}'\cdot\mathbb{T}_{\boldsymbol{u}'}(\boldsymbol{u}',\theta')\boldsymbol{n}']\big)-\mathcal{D}_t\zeta_\lambda\big[\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u}^{(s)},\theta^{(s)})\boldsymbol{n}\big],\\
&F_6=\sigma\big(\boldsymbol{n}\Delta(\tau)-\boldsymbol{n}'\Delta'(\tau)\big)\cdot\boldsymbol{u}'_\lambda(y,\tau),\\
&F_7=\sigma\zeta_\lambda\mathcal{D}_t(\boldsymbol{n}\Delta-\boldsymbol{n}'\Delta')X_{\boldsymbol{u}}=\sigma\zeta_\lambda(t)\big((\dot{\boldsymbol{n}}\Delta-\dot{\boldsymbol{n}}'\Delta)+(\boldsymbol{n}\dot\Delta-\boldsymbol{n}'\dot\Delta')\big)X_{\boldsymbol{u}},\\
&F_8=\sigma\zeta_\lambda\mathcal{D}_t(\boldsymbol{n}'\Delta')\cdot\int\limits_{t-s}^t\boldsymbol{u}(y,\tau)\,\mathrm{d}\tau)=\sigma\zeta_\lambda(t)(\dot{\boldsymbol{n}}'\Delta'+\boldsymbol{n}'\dot\Delta')\int\limits_0^s\boldsymbol{u}(y,t-\tau)\,\mathrm{d}\tau,\\
&F_9=\mathfrak{p}_\lambda^{(s)}-\int\limits_0^t\mathcal{D}_t\zeta_\lambda\mathfrak{p}^{(s)}\,\mathrm{d}\tau.
\end{aligned}\tag{4.6}$$

We view (4.2), (4.5) as an initial-boundary value problem where the unknowns are $\boldsymbol{u}_\lambda^{(s)},\theta_\lambda^{(s)}$ and the coefficients in the system and in the boundary conditions are close to constants corresponding to $\boldsymbol{u}=0$, $\theta=0$ (in view of (2.7), (2.10)), while $\boldsymbol{n}$ is close to $\boldsymbol{n}_0$. It follows that an estimate similar to (3.54) holds, i.e.,

$$\begin{aligned}
&\|e^{\beta t}\boldsymbol{u}_\lambda^{(s)}\|_{W_2^{2+l,1+l/2}(\cup Q^\pm_{t_0,t_1})}+|e^{\beta t}\theta_\lambda^{(s)-}|^{(l+1,l/2)}_{Q^-_{t_0,t_1}}+|e^{\beta t}\theta_\lambda^{(s)+}|^{(l+1,l/2)}_{Q^+_{t_0,t_1}}\\
&+|e^{\beta t}\mathcal{D}_t\theta_\lambda^{(s)+}|^{(l+1,l/2)}_{Q^+_{t_0,t_1}}\leqslant c\Big(\|e^{\beta t}\boldsymbol{F}_1^-\|_{W_2^{l,l/2}(Q^-_{t_0,t_1})}+\|e^{\beta t}F_2^-\|_{W_2^{1+l,0}(Q^-_{t_0,t_1})}\\
&+\|e^{\beta t}\mathcal{D}_t\mathcal{F}_2\|_{W_2^{0,l/2}(Q^-_{t_0,t_1})}+\|e^{\beta t}\boldsymbol{F}_1^+\|_{W_2^{l,l/2}(Q^+_{t_0,t_1})}+|e^{\beta t}F_2^+|^{(1+l,l/2)}_{Q_{t_0,t_1}}
\end{aligned}$$

$$
\begin{aligned}
&+\|e^{\beta t}\boldsymbol{F}_3\|_{W_2^{l+1/2,l/2+1/4}(G_{t_0,t_1})}+|e^{\beta t}F_4|^{(l+1/2,l/2)}_{G_{t_0,t_1}}\\
&+\sum_{j=5}^{8}\|e^{\beta t}F_j\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})}\\
&+\|e^{\beta t}\zeta_\lambda\mathfrak{p}^{(s)}\|_{W_2^{l+1/2,l/2+1/4}(G_{t_0,t_1})}+\|e^{\beta t}\mathcal{D}_t\zeta_\lambda\mathfrak{p}^{(s)}\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})}\Big),
\end{aligned}\tag{4.7}
$$

where $G_{t_0,t_1}=\Gamma_0\times(t_0,t_1)$. This can be justified by using inequality (3.42) and conditions (2.10). We outline the estimates of the functions F_j. They involve many terms, so we can give detailed estimates only of several typical ones. We make use of the relation

$$
\begin{aligned}
\int_0^s\|u(\,\cdot\,,t-\tau)\|_{W_2^{l_1}(\Omega^\pm)}\,\mathrm{d}\tau&\leqslant c\int_0^s\|u(\,\cdot\,,t-\tau)\|^{\alpha}_{W_2^{l}(\Omega^\pm)}\|u(\,\cdot\,,t-\tau)\|^{1-\alpha}_{L_2(\Omega^\pm)}\,\mathrm{d}\tau\\
&\leqslant cs^{1-\alpha/2}\Big(\int_0^s\|u\|^2_{W_2^l(\Omega^\pm)}\,\mathrm{d}\tau\Big)^{\alpha/2}\sup_{\tau<t}\|u\|^{1-\alpha/2}_{L_2(\Omega^\pm)},
\end{aligned}\tag{4.8}
$$

where $\alpha=l_1/l<1$, hence $1-\alpha/2>1/2$. In addition, in view of (1.4) and (2.18), we have

$$
\begin{aligned}
&\|\mathcal{D}_t\widehat{\mathbb{L}}\|_{W_2^r(\Omega_0^\pm)}\leqslant c\|\nabla\boldsymbol{u}\|_{W_2^r(\Omega_0^\pm)},\quad r\leqslant l+1,\\
&\qquad\|\Delta_t(-h)\mathcal{D}_t\widehat{\mathbb{L}}\|_{W_2^{3/2-l}(\Omega_0^\pm)}\leqslant c\big(\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{W_2^{3/2-l}(\Omega_0^\pm)}\\
&\qquad\qquad+\|\nabla\boldsymbol{u}\|_{W_2^{3/2-l}(\Omega_0^\pm)}\int_0^h\|\nabla\boldsymbol{u}(\cdot,t-\tau)\|_{W_2^{l+1-\varkappa}(\Omega_0^\pm)}\,\mathrm{d}\tau\big)\\
&\qquad\qquad\leqslant c(\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{W_2^{3/2-l}(\Omega_0^\pm)}\\
&\qquad\qquad+\sqrt{h}\|\nabla\boldsymbol{u}\|_{W_2^{3/2-l,0}(\Omega_0^\pm)}\Big(\int_0^h\|\nabla\boldsymbol{u}\|^2_{W_2^{l+1-\varkappa}(\Omega_0^\pm)}\,\mathrm{d}\tau\Big)^{1/2}
\end{aligned}\tag{4.9}
$$

We proceed with the estimate of $(\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\nabla\theta^{\pm'}_\lambda$. By (4.8) and (4.9), we have

$$
\begin{aligned}
&\|(\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\theta'_\lambda\|_{W_2^l(\Omega_0^\pm)}\leqslant c\|\nabla\theta'_\lambda\|_{W_2^l(\Omega_0^\pm)}\int_0^s\|\mathcal{D}_t\widehat{\mathbb{L}}(\,\cdot\,,t-\tau)\|_{W_2^{l+1-\varkappa}(\Omega_0^\pm)}\,\mathrm{d}\tau,\\
&\qquad\|e^{\beta t}(\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\theta'_\lambda\|_{W_2^{l,0}(Q^\pm_{t_0,t_1})}\leqslant Cs^{1-\alpha/2}\|e^{\beta t}\nabla\theta'_\lambda\|_{W_2^{l,0}(Q^\pm_{t_0,t_1})},
\end{aligned}\tag{4.10}
$$

where $\alpha=(l+2-\varkappa)/(l+2)$, $\varkappa\in(0,l-1/2)$.

The $W_2^{0,l/2}$-norm of the same function is estimated as follows:

$$
\begin{aligned}
&\|\Delta_t(-h)((\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\theta'_\lambda)\|_{L_2(\Omega_0^\pm)}\\
&\leqslant\|\Delta_t(-h)\nabla\theta'_\lambda\|_{L_2(\Omega_0^\pm)}\int_0^s\|\mathcal{D}_t\widehat{\mathbb{L}}(\,\cdot\,,t-\tau)\|_{W_2^{l+1-\varkappa}(\Omega_0^\pm)}\,\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int\limits_0^s \|\Delta_t(-h)\mathcal{D}_t\widehat{\mathbb{L}}(\,\cdot\,,t-\tau)\|_{W_2^{3/2-l}(\Omega_0^\pm)}\,\mathrm{d}\tau\|\nabla\theta'_\lambda\|_{W_2^l(\Omega_0^\pm)},\\
&\|e^{\beta t}(\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\theta'_\lambda\|^2_{\dot W_2^{0,l/2}(Q^\pm_{t_0,t_1})}\\
&\leqslant c\Big(\sup_{t\in(t_0,t_1)}\Big(\int\limits_0^s\|\nabla\boldsymbol{u}(\,\cdot\,,t-\tau)\|_{W_2^{l+1-\varkappa,0}(\Omega_0^\pm)}\,\mathrm{d}\tau\Big)^2\|e^{\beta t}\nabla\theta'_\lambda\|^2_{\dot W_2^{0,l/2}(Q^\pm_{t_0,t_1})}\\
&+\|e^{\beta t}\nabla\theta'_\lambda\|^2_{W_2^{l,0}(Q^\pm_{t_0,t_1})}s\int\limits_0^s\mathrm{d}\tau\Big(\int\limits_0^{t_1-t}\|\Delta_t(-h)\nabla\boldsymbol{u}(\,\cdot\,,t-\tau)\|^2_{W_2^1(\Omega_0^\pm)}\frac{\mathrm{d}h}{h^{1+l}}\Big)^\alpha\\
&\times\Big(\int\limits_0^{t_1-t}\|\Delta_t(-h)\boldsymbol{u}(\,\cdot\,,t-\tau)\|^2_{L_2(\Omega^\pm)}\frac{\mathrm{d}h}{h^{1+l}}\Big)^{1-\alpha}\mathrm{d}\tau+\Big(\int\limits_0^s\|\nabla\boldsymbol{u}\|_{W_2^{3/2-l}(\Omega_0^\pm)}\,\mathrm{d}\tau\Big)^2\Big)\\
&\leqslant Cs^{2-\alpha}\|e^{\beta t}\nabla\theta'_\lambda\|^2_{W_2^{l,l/2}(Q^\pm_{t_0,t_1})}.
\end{aligned}\tag{4.11}
$$

The numbers $1-\alpha/2\in(1/2,1)$ in (4.10), (4.11) etc. are all different but we shall always use this generic symbol taking the minimal of the corresponding α.

The expression

$$
\mu^\pm(\nabla^2_{\boldsymbol{u}}-\nabla^2_{\boldsymbol{u}'})\boldsymbol{u}'_\lambda=\mu^\pm\Big((\widehat{\mathbb{L}}^T-\widehat{\mathbb{L}}^{T'})\nabla\cdot\widehat{\mathbb{L}}^T\nabla+\widehat{\mathbb{L}}^{T'}\nabla\cdot(\widehat{\mathbb{L}}^T-\widehat{\mathbb{L}}^{T'})\Big)\nabla\boldsymbol{u}'_\lambda \tag{4.12}
$$

has only slightly more complicated form than $(\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{u}'})\theta'_\lambda$, however it involves a term that can be estimated only by $Cs^{1/2}$. Indeed, we have

$$
\begin{aligned}
&\|e^{\beta t}\widehat{\mathbb{L}}^{T'}\nabla\cdot(\widehat{\mathbb{L}}^T-\widehat{\mathbb{L}}'^T)\nabla\boldsymbol{u}'_\lambda\|_{W_2^{l,0}(Q^\pm_{t_0,t_1})}\\
&\qquad\leqslant c\|e^{\beta t}(\widehat{\mathbb{L}}^T-\widehat{\mathbb{L}}'^T)\nabla\boldsymbol{u}'_\lambda\|_{W_2^{l+1,0}(Q^\pm_{t_0,t_1})}\\
&\qquad\leqslant c\sup_{t\in(t_0,t_1)}\int\limits_0^s\|\nabla\boldsymbol{u}\|_{W_2^{l+1,0}(\Omega_0^\pm)}\,\mathrm{d}\tau\|e^{\beta t}\nabla\boldsymbol{u}'_\lambda\|_{W_2^{l+1,0}(Q^\pm_{t_0,t_1})}\\
&\qquad\leqslant c\sqrt{s}\sup_{t\in(t_0,t_1)}\|\nabla\boldsymbol{u}\|_{W_2^{l+1,0}(Q^\pm_{t-s,t})}\|e^{\beta t}\nabla\boldsymbol{u}'_\lambda\|_{W_2^{l+1,0}(Q^\pm_{t_0,t_1})}.
\end{aligned}
$$

On the other hand, the norm of the first term in (4.12) is controlled by $Cs^{1-\alpha/2}$:

$$
\begin{aligned}
&\|e^{\beta t}(\widehat{\mathbb{L}}^T-\widehat{\mathbb{L}}^{T'})\nabla\cdot\widehat{\mathbb{L}}^T\nabla\boldsymbol{u}'_\lambda\|_{W_2^{l,0}(Q^\pm_{t_0,t_1})}\\
&\leqslant c\sup_{t\in(t_0,t_1)}\int\limits_0^s\|\nabla\boldsymbol{u}(\,\cdot\,,t-\tau)\|_{W_2^{l+1-\varkappa}(\Omega_0^\pm)}\,\mathrm{d}\tau\|e^{\beta t}\nabla\cdot\widehat{\mathbb{L}}^T\nabla\boldsymbol{u}'_\lambda\|_{W_2^{l,0}(Q^\pm_{t_0,t_1})}\\
&\leqslant Cs^{1-\alpha/2}\|e^{\beta t}\boldsymbol{u}_\lambda\|_{W_2^{3/2+l,0}(Q^\pm_{t_0,t_1})},
\end{aligned}
$$

because $\varkappa>0$. A similar estimate holds true for the $W_2^{0,l/2}(Q^\pm_{t_0,t_1})$-norm of (4.12).

The same kind of terms arises in the estimate of the norm of $F_2^{\pm}$:

$$\begin{aligned}
&\|e^{\beta t}(\nabla_{\boldsymbol{u}} - \nabla_{\boldsymbol{u}'})\boldsymbol{u}'_\lambda\|_{W_2^{l+1,0}(Q^{\pm}_{t_0,t_1})} \\
&\qquad \leqslant c \sup_{t\in(t_0,t_1)} \int\limits_0^s \|\nabla \boldsymbol{u}(\,\cdot\,, t-\tau)\|_{W_2^{l+1}(\Omega_0^{\pm})}\,\mathrm{d}\tau \|\nabla \boldsymbol{u}'_\lambda\|_{W_2^{l+1,0}(Q^{\pm}_{t_0,t_1})} \\
&\qquad \leqslant c\sqrt{s} \sup_{t\in(t_0,t_1)} \|\nabla \boldsymbol{u}\|_{W_2^{l+1,0}(Q^{\pm}_{t-s,t})} \|e^{\beta t}\nabla \boldsymbol{u}_\lambda\|_{W_2^{l+1,0}(Q^{\pm}_{t_0,t_1})}
\end{aligned}$$

and of other terms where the equation

$$\boldsymbol{n} - \boldsymbol{n}' = \frac{\widehat{\mathbb{L}}^T \boldsymbol{n}_0}{|\widehat{\mathbb{L}}^T \boldsymbol{n}_0|} - \frac{\widehat{\mathbb{L}}^{T'} \boldsymbol{n}_0}{|\widehat{\mathbb{L}}^{T'} \boldsymbol{n}_0|}$$

is used, in particular, in $\boldsymbol{F}_3$ and F_4.

We also treat the expressions containing $\boldsymbol{u}^{(s)}$ and $\theta^{(s)}$, for instance, the term

$$\boldsymbol{F}' = \big(p'(\overline{\rho}^+ + \theta^+) - p'(\overline{\rho}^+ + \theta^{+'})\big)\nabla_{\boldsymbol{u}}\theta^{+'}_\lambda = \int\limits_0^1 p''\big(\overline{\rho}^+ + \theta^{+'} + \lambda\theta^{(s)+}\big)\,\mathrm{d}\lambda\,\theta^{(s)+}_\lambda \nabla_{\boldsymbol{u}}\theta^{+'}$$

in $\boldsymbol{F}_1^+$. Since p is sufficiently smooth, we have

$$\begin{aligned}
&\|e^{\beta t}\boldsymbol{F}'\|_{W_2^{l,0}(Q^+_{t_0,t_1})} \\
&\qquad \leqslant c\|\nabla_{\boldsymbol{u}}\theta^{+'}\|_{W_2^{l,0}(Q^+_{t_0,t_1})} \sup_{t\in(t_0,t_1)} \|e^{\beta t}\theta^{(s)+}_\lambda\|_{W_2^{l+1-\varkappa}(\Omega_0^+)}, \\
&\|e^{\beta t}\boldsymbol{F}'\|_{W_2^{0,l/2}(Q^+_{t_0,t_1})} \\
&\qquad \leqslant c \sup_{t\in(t_0,t_1)} \|e^{\beta t}\theta^{(s)+}_\lambda\|_{W_2^{3/2-l}(\Omega_0^+))} \|\nabla_{\boldsymbol{u}}\theta^{+'}\|_{W_2^{l/2}((t_0+\lambda/2,t_1),W_2^l(\Omega_0^+))}.
\end{aligned} \tag{4.13}$$

The other terms in $\boldsymbol{F}_1^+$ are treated as above.

As to F_2^+, we have

$$\begin{aligned}
\|F_2^+\|_{W_2^{l+1}(\Omega_0^+)} \leqslant c \int\limits_0^s \|\nabla \boldsymbol{u}^+(\,\cdot\,, t-\tau)\|_{W_2^{l+1}(\Omega_0^+)}\,\mathrm{d}\tau \|\nabla \boldsymbol{u}^+_\lambda(\,\cdot\,, t)\|_{W_2^{l+1}(\Omega_0^+)} \\
+ \|\theta^{(s)+}(\,\cdot\,, t)\|_{W_2^{l+1}(\Omega_0^+)} \|\nabla \boldsymbol{u}^{+'}_\lambda(\,\cdot\,, t)\|_{W_2^{l+1}(\Omega_0^+)},
\end{aligned}$$

$$\begin{aligned}
\|\Delta_t(-h)F_2^+\|_{W_2^1(\Omega_0^+)} \leqslant c\Big(&\int\limits_0^s \|\Delta_t(-h)\nabla \boldsymbol{u}^+(\,\cdot\,, t-\tau)\|_{W_2^1(\Omega_0^+)}\,\mathrm{d}\tau \|\nabla \boldsymbol{u}^+_\lambda\|_{W_2^{l+1-\varkappa}(\Omega_0^+)} \\
&+ \int\limits_0^s \|\nabla \boldsymbol{u}^+\|_{W_2^{l+1-\varkappa}(\Omega_0^+)}\,\mathrm{d}\tau \|\Delta_t(-h)\nabla \boldsymbol{u}^+_\lambda\|_{W_2^1(\Omega_0^+)} \\
&+ \|\Delta_t(-h)\theta^{(s)+}_\lambda\|_{W_2^1(\Omega_0^+)} \|\nabla \boldsymbol{u}'^+\|_{W_2^{l+1-\varkappa}(\Omega_0^+)}
\end{aligned}$$

$$+\|\theta_\lambda^{(s)+}\|_{W_2^{l+1-\varkappa}(\Omega_0^+)}\|\Delta_t(-h)\nabla \boldsymbol{u}^+\|_{W_2^1(\Omega_0^+)}\Big)$$

and, as a consequence,

$$\begin{aligned}
&\|e^{\beta t}F_2^+\|_{W_2^{l+1,0}(Q^+_{t_0,t_1})}\\
&\leqslant c\Big(\sup_{t\in(t_0,t_1)}\sqrt{s}\Big(\int\limits_0^s\|\nabla \boldsymbol{u}^+(\,\cdot\,,t-\tau)\|^2_{W_2^{l+1}(\Omega_0^+)}\,\mathrm{d}\tau\Big)^{1/2}\|e^{\beta t}\nabla \boldsymbol{u}_\lambda^+\|_{W_2^{l+1,0}(Q^+_{t_0,t_1})}\\
&\qquad+\sup_{t\in(t_0,t_1)}\|e^{\beta t}\theta^{(s)+}\|_{W_2^{l+1-\varkappa}(\Omega_0^+)}\|\nabla \boldsymbol{u}_\lambda'^+\|_{W_2^{l+1,0}(Q^+_{t_0,t_1})}\Big),\\
&\|e^{\beta t}F_2^+\|_{W_2^{l/2}((t_0,t_1);W_2^1(\Omega_0^+))}\\
&\leqslant c(\|e^{\beta t}\mathcal{D}_t\theta_\lambda^{(s)+}\|_{W_2^{1,0}(Q^+_{t_0,t_1})}\|\nabla \boldsymbol{u}^+\|_{W_2^{l+1-\varkappa,0}(G_{t_0,t_1})}\\
&\quad+\sup_{t\in(t_0,t_1)}\|e^{\beta t}\theta^{(s)+}\|_{W_2^{l+1-\varkappa}(\Omega_0^+)}\|\nabla \boldsymbol{u}_\lambda^+\|_{W_2^{l/2}((t_0,t_1);W_2^1(\Omega_0^+))}\\
&\quad+\sqrt{s}\sup_{t\in(t_0,t_1)}\Big(\int\limits_{t-s}^{t}\mathrm{d}\xi\int\limits_0^{t_1-t}\|\Delta_t(-h)\nabla \boldsymbol{u}^+\|^2_{W_2^1(\Omega_0^+)}\frac{\mathrm{d}h}{h^{1+l}}\Big)^{1/2}\|\nabla \boldsymbol{u}_\lambda'^+\|_{W_2^{l+1-\varkappa,0}(Q^+_{t_0,t_1})}\\
&\quad+\sup_{t\in(t_0,t_1)}\int\limits_0^s\|e^{\beta t}\nabla \boldsymbol{u}_\lambda^+\|_{W_2^{l+1-\varkappa}(\Omega_0^+)}\,\mathrm{d}\tau\|e^{\beta t}\nabla \boldsymbol{u}_\lambda^+\|_{W_2^{l/2}((t_0,t_1);W_2^1(\Omega_0^+))}.
\end{aligned}\tag{4.14}$$

The last term is controlled by $Cs^{1-\alpha/2}$, $\alpha\in(0,1)$.

We turn to the expression F_4. The estimate of F_4 reduces to the estimate of $(\boldsymbol{n}-\boldsymbol{n}')\cdot\nabla \boldsymbol{u}'_\lambda$ and $(\boldsymbol{n}-\boldsymbol{n}')\theta'_\lambda$. We have

$$\begin{aligned}
&\|e^{\beta t}(\boldsymbol{n}-\boldsymbol{n}')\nabla \boldsymbol{u}_\lambda^\pm\|_{W_2^{l+1/2,0}(G_{t_0,t_1})}\\
&\quad\leqslant c\sup_{t\in(t_0,t_1)}\int\limits_0^s\|\nabla \boldsymbol{u}^\pm(\,\cdot\,,t-\tau)\|_{W_2^{l+1/2}(\Gamma_0)}\,\mathrm{d}\tau\|e^{\beta t}\nabla \boldsymbol{u}_\lambda^\pm\|_{W_2^{l+1/2,0}(G_{t_0,t_1})}\\
&\quad\leqslant c\sqrt{s}(\sup_{t\in(t_0,t_1)}\int\limits_0^s\|\nabla \boldsymbol{u}^\pm\|^2_{W_2^{l+1/2}(\Gamma_0)}\,\mathrm{d}\tau)^{1/2}\|e^{\beta t}\nabla \boldsymbol{u}_\lambda^\pm\|_{W_2^{l+1/2,0}(G_{t_0,t_1})},\\
&\|\Delta_t(-h)((\boldsymbol{n}-\boldsymbol{n}')\nabla \boldsymbol{u}_\lambda^\pm)\|_{W_2^{1/2}(\Gamma_0)}\\
&\quad\leqslant c\int\limits_0^s\|\Delta_t(-h)\nabla \boldsymbol{u}^\pm\|_{W_2^{1/2}(\Gamma_0)}\,\mathrm{d}\tau\|\nabla \boldsymbol{u}_\lambda^\pm\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\\
&\qquad+\int\limits_0^s\|\nabla \boldsymbol{u}^\pm(\,\cdot\,,t-\tau)\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\,\mathrm{d}\tau\|\Delta_t(-h)\nabla \boldsymbol{u}_\lambda^\pm\|_{W_2^{1/2}(\Gamma_0)},
\end{aligned}$$

hence

$$
\begin{aligned}
|e^{\beta t}F_4|^{2(l+1/2,l/2)}_{G_{t_0,t_1}} \leqslant c\sqrt{s}\sum_{\pm}\Big(\sup_{t\in(t_0,t_1)}\Big(\int\limits_0^s \|\nabla \boldsymbol{u}^{\pm}(\,\cdot\,,t-\tau)\|^2_{W_2^{l+1/2}(\Gamma_0)}\,\mathrm{d}\tau\Big)^{1/2} \\
+\sup_{t\in(t_0,t_1)}\Big(\int\limits_{t-s}^t \mathrm{d}\xi\int\limits_0^{t_1-t}\|\Delta_t(-h)\nabla \boldsymbol{u}^{\pm}(\,\cdot\,,\xi)\|_{W_2^{1/2}(\Gamma_0)}\frac{\mathrm{d}h}{h^{1+l}}\Big)^{1/2}\Big) \\
\times\big(\|e^{\beta t}\nabla \boldsymbol{u}^{\pm}_{\lambda}\|_{W_2^{l+1/2,0}(G_{t_0,t_1})}+\|e^{\beta t}\theta^{\pm}_{\lambda}\|_{W_2^{l+1/2,0}(G_{t_0,t_1})}\big) \\
+Cs^{1-\alpha/2}\big(|e^{\beta t}\nabla \boldsymbol{u}^{\pm}_{\lambda}|^{(l+1/2,l/2)}_{G_{t_0,t_1}}+|e^{\beta t}\theta^{\pm}_{\lambda}|^{(l+1/2,l/2)}_{G_{t_0,t_1}}\big). \quad (4.15)
\end{aligned}
$$

The expression $\boldsymbol{F}_3$ satisfies similar inequalities.

Now we estimate $F_5 = F_5' + F_5''$. By repeating the calculation in (4.11) and taking the behavior of $\mathcal{D}_t\zeta_\lambda$ into account, we obtain

$$
\begin{aligned}
\|e^{\beta t}F_5'\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})} \leqslant C\lambda^{-2}s^{1-\alpha/2}\sum_{\pm}\big(\|e^{\beta t}\nabla \boldsymbol{u}^{\pm}_{\lambda}\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})} \\
+\|e^{\beta t}\theta^{\pm}_{\lambda}\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})}\big), \\
\|e^{\beta t}F_5''\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})} \leqslant c\lambda^{-2}\sum_{\pm}\big(\|e^{\beta t}\nabla \boldsymbol{u}^{(s)\pm}_{\lambda}\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})} \\
+\|e^{\beta t}\theta^{(s)\pm}_{\lambda}\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})}\big). \quad (4.16)
\end{aligned}
$$

We proceed with the analysis of the terms involving the Laplace–Beltrami operator $\Delta(t)$ and its time derivative. We consider the expression

$$
\begin{aligned}
F_6 &= \sigma\big(\boldsymbol{n}\Delta(t)-\boldsymbol{n}'\Delta(t-s)\big)\cdot \boldsymbol{u}'_{\lambda} \\
&= \sigma\int\limits_0^s\big(\dot{\boldsymbol{n}}(y,t-\tau)\Delta(t-\tau)+\boldsymbol{n}(y,t-\tau)\dot{\Delta}(t-\tau)\big)\,\mathrm{d}\tau\cdot \boldsymbol{u}'_{\lambda}(y,t).
\end{aligned}
$$

It suffices to estimate the first term F_6' on the right-hand side (the second one is treated in the same way). Since $\|\Delta(t)\boldsymbol{u}\|_{W_2^{l-1/2}(\Gamma_0)} \leqslant c\|\boldsymbol{u}\|_{W_2^{3/2+l}(\Gamma_0)}$, we have

$$
\begin{aligned}
\Big\|\int\limits_0^s \dot{\boldsymbol{n}}(\,\cdot\,,t-\tau)\cdot\Delta(t-\tau)\,\mathrm{d}\tau \boldsymbol{u}'_{\lambda}(\,\cdot\,,t)\Big\|_{W_2^{l-1/2}(\Gamma_0)} \\
\leqslant \int\limits_0^s\|\nabla \boldsymbol{u}(\,\cdot\,,t-\tau)\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\,\mathrm{d}\tau\|\boldsymbol{u}'_{\lambda}\|_{W_2^{3/2+l}(\Gamma_0)}, \\
\|\Delta_t(-h)F_6\|_{L_2(\Gamma_0)} \leqslant c\Big(\int\limits_0^s\|\Delta_t(-h)\nabla \boldsymbol{u}\|_{W_2^{3/2-l}(\Gamma_0)}\,\mathrm{d}\tau\|\boldsymbol{u}'_{\lambda}\|_{W_2^{l+3/2}(\Gamma_0)} \\
+\int\limits_0^s\sup_{\Gamma_0}|\nabla \boldsymbol{u}(y,t-\tau)|\,\mathrm{d}\tau\|\Delta_t(-h)\boldsymbol{u}'_{\lambda}\|_{W_2^2(\Gamma_0)}\Big),
\end{aligned}
$$

which implies

$$\|e^{\beta t}F_6'\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})} \leqslant Cs^{1-\alpha/2}|e^{\beta t}\boldsymbol{u}_\lambda|^{(2,l/2-1/4)}_{G_{t_0,t_1}}. \tag{4.17}$$

Now we pass to the estimate of $F_7 = F_7' + F_7''$, where

$$F_7' = \sigma\zeta_\lambda((\dot{\boldsymbol{n}} - \dot{\boldsymbol{n}}')\Delta + \dot{\boldsymbol{n}}'(\Delta(t) - \Delta'(t))X_{\boldsymbol{u}},$$

$$F_7'' = \sigma\zeta_\lambda((\boldsymbol{n} - \boldsymbol{n}')\dot{\Delta} + \boldsymbol{n}'(\dot{\Delta} - \dot{\Delta}'))X_{\boldsymbol{u}}.$$

We restrict ourselves to the estimate of F_7'. Since

$$\|X_{\boldsymbol{u}}(\,\cdot\,,t)\|_{W_2^{l+3/2}(\Gamma_0)} \leqslant c,$$

$$\|(\Delta(t) - \Delta(t-s))X_{\boldsymbol{u}}\|_{W_2^{l-1/2}(\Gamma_0)} \leqslant c\int_0^s \|\nabla\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\,\mathrm{d}\tau\|X_{\boldsymbol{u}}\|_{W_2^{l+3/2}(\Gamma_0)},$$

for arbitrary $t \in (t_1, t_0)$ we have:

$$\begin{aligned}
\|F_7'\|_{W_2^{l-1/2}(\Gamma_0)} &\leqslant c\big(\|\nabla\boldsymbol{u}_\lambda^{(s)}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)} \\
&\quad + \|\nabla\boldsymbol{u}_\lambda\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\int_0^s \|\nabla\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\,\mathrm{d}\tau\big)\|X_{\boldsymbol{u}}\|_{W_2^{3/2+l}(\Gamma_0)}, \\
\|\Delta_t(-h)F_7'\|_{L_2(\Gamma_0)} &\leqslant c\big(\|\Delta_t(-h)\nabla\boldsymbol{u}_\lambda^{(s)}\|_{W_2^{3/2-l}(\Gamma_0)} \\
&\quad + \|\Delta_t(-h)\nabla\boldsymbol{u}_\lambda\|_{W_2^{3/2-l}(\Gamma_0)}\int_0^s \|\nabla\boldsymbol{u}\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\,\mathrm{d}\tau \\
&\quad + \|\nabla\boldsymbol{u}_\lambda\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\int_0^s \|\Delta_t(-h)\nabla\boldsymbol{u}\|_{W_2^{3/2-l}(\Gamma_0)}\,\mathrm{d}\tau\big)\|X_{\boldsymbol{u}}\|_{W_2^{3/2+l}(\Gamma_0)},
\end{aligned}$$

which implies

$$\begin{aligned}
\|e^{\beta t}F_7'\|_{W_2^{l-1/2,0}(G_{t_0,t_1})} &\leqslant c\|e^{\beta t}\nabla\boldsymbol{u}_\lambda^{(s)}\|_{W_2^{l+1/2-\varkappa,0}(G_{t_0,t_1})} \\
&\quad + Cs^{1-\alpha/2}\|e^{\beta t}\nabla\boldsymbol{u}_\lambda\|_{W_2^{l+1/2-\varkappa}(G_{t_0,t_1})}, \\
\|e^{\beta t}F_7'\|_{W_2^{0,l/2-1/4}(G_{t_0,t_1})} &\leqslant c\|e^{\beta t}\nabla\boldsymbol{u}_\lambda^{(s)}\|_{W_2^{l/2-1/4}((t_0,t_1);W_2^{3/2-l}(\Gamma_0))} \\
&\quad + Cs^{1-\alpha/2}(\|\nabla\boldsymbol{u}_\lambda\|_{W_2^{l+1/2-\varkappa}(G_{t_0,t_1})} \\
&\quad + \|\nabla\boldsymbol{u}\|_{W_2^{l/2-1/4}((t_0,t_1);W_2^{3/2-l}(\Gamma_0))}\big).
\end{aligned} \tag{4.18}$$

Slightly more complicated calculations lead to similar inequalities for F_7''.

We turn to the expression F_8:

$$\|F_8\|_{W_2^{l-1/2}(\Gamma_0)} \leqslant c\|\nabla\boldsymbol{u}_\lambda\|_{W_2^{l+1/2-\varkappa}(\Gamma_0)}\int_0^s \|\boldsymbol{u}(\,\cdot\,,t-\tau)\|_{W_2^{3/2+l}(\Gamma_0)}\,\mathrm{d}\tau,$$

$$\|e^{\beta t}F_8\|_{W_2^{l-1/2,0}(G_{t_0,t_1})} \leqslant c\|e^{\beta t}\nabla \boldsymbol{u}_\lambda\|_{W_2^{3/2-l,0}(G_{t_0,t_1})}\sqrt{s}\sup_{t\in(t_0,t_1)}\|\boldsymbol{u}\|_{W_2^{3/2+l}(G_{t-s,t})},$$

$$\|\Delta_t(-h)F_8\|_{L_2(\Gamma_0)} \leqslant c\|\Delta_t(-h)\nabla \boldsymbol{u}_\lambda\|_{W_2^{3/2-l}(\Gamma_0)}\int_0^s \|\boldsymbol{u}(\,\cdot\,,t)\|_{W_2^{3/2+l}(\Gamma_0)}\,\mathrm{d}\tau$$

$$+\sup_{\Gamma_0}|\nabla \boldsymbol{u}_\lambda(y,t)|\int_0^s \|\Delta_t(-h)\Delta(t)\boldsymbol{u}\|_{L_2(\Gamma_0)}\,\mathrm{d}\tau,$$

$$\|e^{\beta t}F_8\|_{\dot W_2^{0,l/2-1/4}(G_{t_0,t_1})}$$

$$\leqslant c\|e^{\beta t}\nabla \boldsymbol{u}_\lambda\|_{\dot W_2^{l/2-1/4}((t_0,t_1);W_2^{3/2-l}(\Gamma_0))}\sqrt{s}\|\boldsymbol{u}\|_{W_2^{3/2+l,0}(G_{t-s,t})}$$

$$+\|e^{\beta t}\nabla \boldsymbol{u}_\lambda\|_{W_2^{l+1/2-\varkappa,0}(G_{t_0,t_1})}\sqrt{s}\sup_{t\in(t_0,t_1)}\|\boldsymbol{u}\|_{W_2^{l/2-1/4}((t-s,t);W_2^2(\Gamma_0))}, \tag{4.19}$$

To evaluate the contribution of $\widehat{\boldsymbol{f}}$, we use the equations

$$\boldsymbol{f}(X_{\boldsymbol{u}},t)-\boldsymbol{f}(X_{\boldsymbol{u}'},t-s)=(\boldsymbol{f}(X_{\boldsymbol{u}},t)-\boldsymbol{f}(X_{\boldsymbol{u}'},t))+(\boldsymbol{f}(X_{\boldsymbol{u}'},t)-\boldsymbol{f}(X_{\boldsymbol{u}'},t-s)),$$

$$\boldsymbol{f}(X_{\boldsymbol{u}},t)-\boldsymbol{f}(X_{\boldsymbol{u}'},t)=\int_0^1 \nabla\boldsymbol{f}(X_{\boldsymbol{u}'}+\mu X_{\boldsymbol{u}}^{(s)},t)\,\mathrm{d}\mu\int_0^s \boldsymbol{u}(y,t-\tau)\,\mathrm{d}\tau,$$

whence

$$\|e^{\beta t}\widehat{\boldsymbol{f}}^{(s)}\|_{W_2^{l,l/2}(Q_{t_0,t_1})} \leqslant c\Big(\|e^{\beta t}\nabla\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{t_0,t_1})}\sup_{t\in(t_0,t_1)}\int_0^s \|\boldsymbol{u}\|_{W_2^{l+1-\varkappa}(\cup Q^{\pm}_{t-\tau,t})}\,\mathrm{d}\tau$$

$$+\|e^{\beta t}\nabla\boldsymbol{f}\|_{W_2^{l,0}(Q_{t_0,t_1})}\sup_{t\in(t_0,t_1)}\int_0^s \|\boldsymbol{u}(\,\cdot\,,t-\tau)\|_{W_2^{l/2}((t-s,t);W_2^{3/2-l}(\Gamma_0))}\,\mathrm{d}\tau$$

$$+\|e^{\beta t}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{t_1,t_0})}|s|^{\alpha_1}\Big) \leqslant C_{\boldsymbol{f}}|s|^{\alpha},$$

in view of assumptions of Theorem 6. The function $\boldsymbol{f}_\lambda=\zeta_\lambda(t)\boldsymbol{f}$ is estimated in a similar way.

In conclusion, we estimate the expression F_9 in (4.6). We have

$$\begin{aligned}\zeta_\lambda\mathfrak{p}^{(s)} &= \zeta_\lambda\big((p(\overline{\rho}^+ +\theta^+)-p(\overline{\rho}^+)-p_1\theta^+)-(p(\overline{\rho}^+ +\theta^{+'})-p(\overline{\rho}^+)-p_1\theta^{+'})\big)\\ &=\int_0^1 (p'(\overline{\rho}^+ +\theta^{+'}+\mu\theta^{(s)+})-p_1)\,\mathrm{d}\mu\theta_\lambda^{(s)+}\\ &=\int_0^1 \mathrm{d}\mu_1\int_0^1 p^{''}(\overline{\rho}^+ +\mu(\theta^{+'}+\mu_1\theta^{(s)+}))\,\mathrm{d}\mu(\theta^{+'}+\mu_1\theta^{(s)+})\theta_\lambda^{(s)+},\end{aligned}$$

$$\|\zeta_\lambda\mathfrak{p}^{(s)}\|_{W_2^{l+1/2}(\Gamma_0)} \leqslant c\|\theta^+\|_{W_2^{l+1/2}(\Gamma_0)}\int_0^s \|\mathcal{D}_t\theta_\lambda^+(\,\cdot\,,t-\tau)\|_{W_2^{l+1/2}(\Gamma_0)}\,\mathrm{d}\tau,$$

$$\|\Delta_t(-h)\zeta_\lambda \mathfrak{p}^{(s)}\|_{L_2(\Gamma_0)}$$
$$\leqslant c(\|\Delta_t(-h)\theta^+\|_{W_2^{3/2-l}(\Gamma_0)} \int_0^s \|\mathcal{D}_t\theta_\lambda^+(\,\cdot\,,t-\tau)\|_{W_2^{l-1/2}(\Gamma_0)}\,\mathrm{d}\tau$$
$$+\|\theta^+\|_{W_2^{0,l-1/2}(\Gamma_0)} \int_0^s \|\mathcal{D}_t\theta_\lambda^+(\,\cdot\,,t-\tau)\|_{W_2^{3/2-l}(\Gamma_0)}\,\mathrm{d}\tau,$$

which implies

$$\|e^{\beta t}\zeta\mathfrak{p}\|_{W_2^{l+1/2,l/2+1/4}(G_{t_0,t_1})}$$
$$\leqslant c\Big(\|e^{\beta t}\theta^+\|_{W_2^{l+1/2,0}(G_{t_0,t_1})}\sqrt{s}\Big(\int_0^s \|\mathcal{D}_t\theta_\lambda^+\|^2_{W_2^{l+1/2}(\Gamma_0)}\,\mathrm{d}\tau\Big)^{1/2} + C|s|^\alpha\Big). \tag{4.20}$$

Finally,

$$\|e^{\beta t}\mathfrak{p}^{(s)}\mathcal{D}_t\zeta_\lambda(t)\|_{W_2^{l-1/2,l/2-1/4}(G_{t_0,t_1})}$$
$$\leqslant c\lambda^{-2}\Big(\int_0^s \|\mathcal{D}_t\theta^+\|_{W_2^{l-1/2}(\Gamma_0)}\,\mathrm{d}\tau \|e^{\beta t}\theta^+\|_{W_2^{3/2-l,0}(G_{t_0,t_1})}$$
$$+\|e^{\beta t}\mathcal{D}_t\theta^+\|_{W_2^{l-1/2,0}(G_{t_0,t_1})} \int_0^s \|\nabla\theta^+\|_{W_2^{3/2-l,0}(\Gamma)}\,\mathrm{d}\tau\Big) \leqslant C\lambda^{-2}s^\alpha. \tag{4.21}$$

From the estimates obtained above, it follows that the sum of the norms on the right-hand side of (4.7) is controlled by a sum of terms proportional to $Cs^{1-\alpha/2}$, $\alpha \in (0,1)$, or to

$$\sqrt{s}\Big(\int_0^s (\|\boldsymbol{u}^\pm(\,\cdot\,,t-\tau)\|^2 + \|\theta^+\|^2)\,\mathrm{d}\tau\Big)^{1/2}$$
$$= \sqrt{s}\Big(\int_{t-s}^t (\|\boldsymbol{u}^\pm(\,\cdot\,,\xi)\|^2 + \|\theta^+\|^2)\,\mathrm{d}\xi\Big)^{1/2}, \tag{4.22}$$

or to the norms of $e^{\beta t}\boldsymbol{u}^{(s)}$ and $e^{\beta t}\theta^{(s)}$ of a lower order in comparison to the norms in $Y(e^{\beta t}\boldsymbol{u}, e^{\beta t}\theta)$ possibly multiplied by λ^{-2}. We set

$$Y^2(t_0+\lambda,t) = \|e^{\beta t}\boldsymbol{u}^{(s)}\|^2_{W_2^{2+l,1+l/2}(\cup Q_{t_0+\lambda,t_1})} + |e^{\beta t}\theta^{(s)-}|^{2(1+l,l/2)}_{Q^-_{t_0+\lambda,t_1}}$$
$$+ |e^{\beta t}\theta^{(s)+}|^{2(1+l,l/2)}_{Q^+_{t_0+\lambda,t_1}} + |e^{\beta t}\mathcal{D}_t\theta^{(s)+}|^{2(1+l,l/2)}_{Q^+_{t_0+\lambda,t_1}}$$

and we denote by $Y'(t_0+\lambda/2,t_1)$ the sum of certain weaker norms of $\boldsymbol{u}^{(s)}$ and $\theta^{(s)\pm}$ in $\Omega_0 \times (t_0+\lambda/2,t_0)$. As shown above,

$$Y(t_0+\lambda,t_1) \leqslant c_1\lambda^{-2}Y'(t_0+\lambda/2,t_1) + F(s), \tag{4.23}$$

where $F(s)$ is the sum of terms controlled by powers of s.

We estimate each term in $Y'(t_0+\lambda/2,t_1)$ by an interpolation inequality of type (3.33); the norm $\|e^{\beta t}\theta^{(s)-}\|^2_{W_2^{0,l/2-1/4}(G_{t_0+\lambda,t_1})}$, which is a part of Y' (in view of (4.16)), estimated as follows:

$$\begin{aligned}\|e^{\beta t}\theta^{(s)-}\|^2_{W_2^{0,l/2-1/4}(G_{t_0+\lambda/2,t_1})} &\leqslant \epsilon_4\|e^{\beta t}\theta^{(s)-}\|^2_{W_2^{0,l/2}(G_{t_0+\lambda/2,t_1})}\\ &+ c\epsilon_4^{-m_1}\|e^{\beta t}\theta^{(s)-}\|^2_{L_2(G_{t_0+\lambda/2,t_1})},\end{aligned}$$

$m_1>0$, $\epsilon_4\ll 1$. Thus,

$$Y'(t_0+\lambda/2,t)\leqslant \epsilon_1 Y(t_0+\lambda/2,t_1)+c\epsilon_4^{-m_1}Y_0,$$

where $m_1>0$,

$$Y_0=\|e^{\beta t}\boldsymbol{u}^{(s)}\|^2_{L_2(\cup Q^{\pm}_{t_0,t_1})}+\|e^{\beta t}\boldsymbol{\theta}^{(s)+}\|^2_{L_2(Q^{+}_{t_0,t_1})}+\|e^{\beta t}\boldsymbol{\theta}^{(s)-}\|^2_{L_2(G_{t_0,t_1})},$$

and in view of (4.23),

$$Y(t_0+\lambda,t_1)\leqslant c_1\epsilon_4\lambda^{-2}Y(t_0+\lambda/2,t_1)+c_2\epsilon_4^{-m_1}\lambda^{-2}Y_0+c_3F(s).$$

This implies

$$f(\lambda)\leqslant c\delta f(\lambda/2)+c_2\delta^{-m_1}Y_0+c_3\lambda^{2m_1+2}F(s),\tag{4.24}$$

where $\delta=\epsilon_4\lambda^{-2}2^{m_1+2}$, $f(\lambda)=\lambda^{2+m_1}Y(t_0+\lambda,t_1)$, $c=c_1 2^{m_1+2}$. We fix δ such that $c\delta\leqslant 1/2$ and, iterating (4.24), arrive at $f(\lambda)\leqslant 2(c\delta^{-m_1}Y_0+c_4\lambda^{2m_1+2}F(s))$. The norms in Y_0 can be estimated by the inequalities

$$\begin{aligned}\|e^{\beta t}\boldsymbol{u}^{(s)}\|_{L_2(Q_{t_0,t_1})}&\leqslant cs\|e^{\beta t}\mathcal{D}_t\boldsymbol{u}\|_{L_2(Q_{t_0,t_1})},\\ \|e^{\beta t}\theta^{(s)+}\|_{L_2(Q_{t_0,t_1})}&\leqslant cs\|e^{\beta t}\mathcal{D}_t\theta^{+}\|_{L_2(Q_{t_0,t_1})};\end{aligned}$$

finally, since

$$\theta^{(s)-}|_{\Gamma_0}=p(\overline{\rho}^{+}+\theta^{+})-p(\overline{\rho}^{+}+\theta^{+'})+[\boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}]-[\boldsymbol{n}'\cdot\mathbb{T}_{\boldsymbol{u}'}(\boldsymbol{u}')\boldsymbol{n}']+\sigma H^{(s)}$$

(see (3.11)), the inequality

$$\begin{aligned}\|e^{\beta t}\theta^{(s)-}\|_{L_2(G_{t_0,t_1})}\leqslant c\Big(&\|e^{\beta t}\theta^{(s)+}\|_{L_2(G_{t_0,t_1})}\\ &+\sum_{\pm}\|e^{\beta t}\nabla\boldsymbol{u}^{(s)\pm}\|_{L_2(G_{t_0,t_1})}+\|e^{\beta t}r^{(s)}\|_{W_2^2(S_{R_0}\times(t_0,t_1))}\Big)\leqslant cs^a\end{aligned}$$

holds true with $a>1/2$, due to

$$\begin{aligned}\|e^{\beta t}\theta^{(s)+}\|_{L_2(G_{t_0,t_1})}&\leqslant cs\|e^{\beta t}\mathcal{D}_t\theta^{+}\|_{L_2(G_{t_0,t_1})},\\ \|e^{\beta t}\nabla\boldsymbol{u}^{(s)\pm}\|_{L_2(G_{t_0,t_1})}&\leqslant Cs^{l/2+1/4},\\ \|e^{\beta t}r^{(s)}\|_{W_2^2(G_{t_0,t_1})}&\leqslant cs\|e^{\beta t}\mathcal{D}_t r\|_{W_2^2(G_{t_0,t_1})}.\end{aligned}$$

Hence Y_0 can be included into $F(s)$; by setting $\lambda = \lambda_0 = (t_1 - t_0)/4$, we obtain $f(\lambda_0) \leqslant cF(s)$ or

$$Y(t_0 + \lambda_0, t_1) \leqslant c\delta^{-m_1}\lambda_0^{-2-m_1} F(s) \leqslant Cs^{1/2}.$$

This implies the boundedness of the norms

$$\begin{aligned}
&\|e^{\beta t}\boldsymbol{u}\|_{W_2^{\alpha_2/2}((t_2,t_1);W_2^{2+l}(\cup\Omega_0^\pm))}, \quad \|e^{\beta t}\boldsymbol{u}\|_{W_2^{(\alpha_2+l)/2,0}(G_{t_2,t_1}))}, \\
&\|e^{\beta t}\theta^\pm\|_{W_2^{(\alpha_2+l)/2}((t_2,t_1);W_2^1(\Omega_0^\pm))} + \|e^{\beta t}\theta^\pm\|_{W_2^{\alpha_2/2}((t_2,t_1);W_2^{l+1}(\Omega_0^\pm)}, \quad \alpha_2 \in (0,1/2).
\end{aligned}$$

We separate out the terms of type sI in $F^2(s)$. The sum of the integrals I does not exceed

$$\begin{aligned}
&c\Big(\int\limits_{t-s}^{t} (\|\boldsymbol{u}(\,\cdot\,,\xi)\|^2_{W_2^{2+l}(\cup\Omega_0^\pm)} + \|\mathcal{D}_t\theta^+\|^2_{W_2^{l+1/2}(\Gamma_0)})\,\mathrm{d}\xi \\
&+\int\limits_{t-s}^{t}\mathrm{d}\xi \int\limits_0^{t_1-t} \|\Delta_t(-h)\nabla\boldsymbol{u}(\,\cdot\,,\xi)\|^2_{W_2^1(\cup\Omega^\pm)}\frac{\mathrm{d}h}{h^{1+l}} \\
&+\int\limits_{t-s}^{t}\mathrm{d}\xi \int\limits_0^{t_1-t} \|\Delta_t(-h)\nabla\boldsymbol{u}(\,\cdot\,,\xi)\|^2_{W_2^{1/2}(\Gamma_0)}\frac{\mathrm{d}h}{h^{1/2+l}}\Big) = c(I_1+I_2+I_3), \quad t\in(t_2,t_1).
\end{aligned}$$

We have

$$I_1 \leqslant s^{\alpha_2}\int\limits_{t-s}^{t} \big(\|\boldsymbol{u}(\,\cdot\,,\xi)\|^2_{W_2^{2+l}(\cup\Omega_0^\pm)} + \|\mathcal{D}_t\theta^+\|^2_{W_2^{l+1/2}(\Gamma_0)}\big)\frac{\mathrm{d}\xi}{(t-\xi)^{\alpha_2}}.$$

The last integral is controlled by a finite norm

$$\|\boldsymbol{u}\|^2_{W_2^{\alpha_2/2}((t_2,t_0);W_2^{2+l}(\cup\Omega_0^\pm))} + \|\mathcal{D}_t\theta^+\|^2_{W_2^{\alpha_2/2}((t_2,t_0);W_2^{l+1/2}(\Gamma_0))};$$

moreover,

$$I_2 \leqslant s^{\alpha_2}\int\limits_{t-s}^{t}\frac{\mathrm{d}\xi}{(t-\xi)^{\alpha_2}}\int\limits_0^{t_1-t}\|\Delta_t(-h)\nabla\boldsymbol{u}\|^2_{W_2^1(\cup\Omega_0^\pm)}\frac{\mathrm{d}h}{h^{1+l}};$$

we assume that $l/2 + \alpha_2/2 < 1$. By analyzing two cases: $t - \xi < h$ and $t - \xi > h$, it is not hard to show that

$$I_2 \leqslant cs^{\alpha_2}\|\boldsymbol{u}\|_{W_2^{l/2+\alpha_2/2}((t_2,t_1);W_2^2(\cup\Omega_0^\pm))}.$$

The third integral is estimated in the same way. Thus, $s(I_1 + I_2 + I_3) \leqslant Cs^{1+\alpha_2}$, which completes the proof of (4.1).

It follows that

$$\begin{aligned}
&\|\boldsymbol{u}(\,\cdot\,,t)\|_{W_2^{2+l}(\cup\Omega_0^\pm)} + \|\mathcal{D}_t\boldsymbol{u}(\,\cdot\,,t)\|_{W_2^l(\cup\Omega_0^\pm)} + \sum_\pm \|\theta^\pm(\,\cdot\,,t)\|_{W_2^{l+1}(\Omega_0^\pm)} \qquad (4.25)\\
&+\|\mathcal{D}_t\theta^+(\,\cdot\,,t)\|_{W_2^{l+1}(\Omega_0\pm)} + \|r(\,\cdot\,,t)\|_{W_2^{l+5/2}(S_{R_0})} \leqslant C, \quad t\in(t_2,t_1). \qquad \square
\end{aligned}$$

5 Construction of a solution in the infinite time interval

We describe the procedure of extension of the solution of Problem (1.3) from the interval $(0,T)$ to the infinite interval $t>0$. It is done step by step: first the solution is defined for $t\in(T,2T)$, then for $t\in(2T,3T)$ and so forth. We have proved that the solution satisfies (3.54) in Q_T. By the trace theorem for the Sobolev spaces, it also satisfies the inequalities

$$\begin{aligned}
&e^{2\beta T}\Big(\|\boldsymbol{u}(\,\cdot\,,T)\|^2_{W_2^{l+1}(\cup\Omega_0^\pm)}+\|\theta^+(\,\cdot\,,T)\|^2_{W_2^{l+1}(\Omega_0^+)}+\|r(\,\cdot\,,T)\|^2_{W_2^{l+2}(S_{R_0})}\Big)\\
&\leqslant cY_T^2\big(e^{\beta t}\boldsymbol{u}^\pm,e^{\beta t}\theta^\pm,e^{\beta t}r\big)\\
&\leqslant c\Big(\|\boldsymbol{u}_0\|^2_{W_2^{l+1}(\cup\Omega_0^\pm)}+\|\theta_0^+\|^2_{W_2^{l+1}(\Omega_0^+)}+\|r_0\|^2_{W_2^{l+2}(S_{R_0})}+\|e^{\beta t}\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_T)}\Big).
\end{aligned}\tag{5.1}$$

By passing to Eulerian coordinates, we obtain

$$\|\boldsymbol{v}(\,\cdot\,,T)\|^2_{W_2^{l+1}(\cup\Omega_T^\pm)}+\|\vartheta^+\|^2_{W_2^{l+1}(\Omega_T^+)}+\|r(\,\cdot\,,T)\|^2_{W_2^{l+2}(S_{R_0})}\leqslant c_1e^{-2\beta T}\epsilon^2.\tag{5.2}$$

We define $(\boldsymbol{u}^{(1)},\theta^{(1)})$ for $t\in(T,2T)$ as a solution of problem (1.3) in the domain $\Omega_T^+\cup\Gamma_T\cup\Omega_T^-$ with the initial data $\boldsymbol{u}^{(1)}(z,T)=\boldsymbol{v}(z,T)$, $z\in\Omega_T^\pm$, $\theta^{+(1)}(z,T)=\vartheta^+(z,T)$, $z\in\Omega_T^+$. Since $\Gamma_T\in W_2^{l+5/2}$, this problem has a unique solution if ϵ is chosen sufficiently small. A condition of the type (2.20), i.e.,

$$\begin{aligned}
&\sup_{t\in(T,2T)}\|\theta^{+(1)}(\,\cdot\,,t)\|_{W_2^{l+1}(\Omega_T^+)}+\sup_{t\in(T,2T)}\|\boldsymbol{U}^{(1)}(\,\cdot\,,t)\|_{W_2^{l+2}(\cup\Omega_T^\pm)}\\
&\qquad\leqslant c\Big(\sup_{t\in(T,2T)}\|\theta^{+(1)}(\,\cdot\,,t)\|_{W_2^{l+1}(\Omega_0^+)}+\sqrt{T}\|\boldsymbol{u}^{(1)}\|_{W_2^{l+2,0}(\cup Q_{T,2T}^\pm)}\Big)\leqslant\delta\ll 1,
\end{aligned}$$

is established as above in Theorem 2, i.e., by iterations. The function $r(\eta,t)$, $t\in(T,2T)$, can be defined as above in §2 (see (2.38) and Proposition 1'), hence

$$\begin{aligned}
&\|e^{\beta(t-T)}\boldsymbol{u}^{(1)}\|^2_{W_2^{2+l,1+l/2}(\cup Q_{T,2T}^\pm)}\\
&\quad+\|e^{\beta(t-T)}\nabla\theta^{-(1)}\|^2_{W_2^{l,l/2}(\cup Q_{T,2T}^\pm)}+\|e^{\beta(t-T)}\theta^{-(1)}\|^2_{W_2^{0,l/2}(Q_{T,2T}^\pm)}\\
&\quad+\|e^{\beta(t-T)}r\|^2_{W_2^{l+5/2,0}(\mathcal{S}_{T,2T})}+\|e^{\beta(t-T)}\mathcal{D}_tr\|^2_{W_2^{l+3/2,0}(\mathcal{S}_{T,2T})}\\
&\quad+|e^{\beta(t-T)}\theta^{+(1)}|^{2(1+l,l/2)}_{Q_{T,2T}^+}+|e^{\beta(t-T)}\mathcal{D}_t\theta^{+(1)}|^{2(1+l,l/2)}_{Q_{T,2T}^+})\\
&\quad\leqslant c\big((\|\boldsymbol{v}(\,\cdot\,,T)\|^2_{W_2^{l+1}(\cup\Omega_T^\pm)}+\|r(\,\cdot\,,T)\|^2_{W_2^{l+2}(S_{R_0})}\\
&\quad+\|\vartheta^+(\,\cdot\,,T)\|^2_{W_2^{l+1}(\Omega_T^+)})+\|e^{\beta(t-T)}\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{T,2T})}\big),
\end{aligned}$$

where $Q^{\pm}_{T,2T} = \Omega^{\pm}_T \times (T, 2T)$, $\mathcal{S}_{T,2T} = S_{R_0} \times (T, 2T)$. Multiplying this inequality by $e^{2\beta T}$ and taking (5.1) into account, we obtain

$$\begin{aligned}
&\|e^{\beta t}\boldsymbol{u}^{(1)}\|^2_{W_2^{2+l,1+l/2}(\cup Q^{\pm}_{T,2T})} + \|e^{\beta t}\nabla\theta^{-(1)}\|^2_{W_2^{l,l/2}(\cup Q^{-(1)}_{T,2T})} \\
&+ |e^{\beta t}\theta^{+(1)}|^{2(1+l,l/2)}_{Q^{+}_{T,2T}} + \|e^{\beta t} r\|^2_{W_2^{l+5/2,0}(\mathcal{S}_{T,2T})} \\
&+ \|e^{\beta t}\mathcal{D}_t r\|^2_{W_2^{l+3/2,0}(\mathcal{S}_{T,2T})} + |e^{\beta t}\mathcal{D}_t\theta^{+(1)}|^{2(1+l,l/2)}_{Q^{+}_{T,2T}} \\
&\leqslant c(\|\boldsymbol{v}_0\|^2_{W_2^{l+1}(\cup\Omega^{\pm}_0)} + \|\theta_0\|^2_{W_2^{l+1}(\cup\Omega^{\pm}_0)} \\
&+ \|r_0\|^2_{W_2^{l+2}(S_{R_0})} + \|e^{\beta t}\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{0,2T})}).
\end{aligned} \tag{5.3}$$

Now we go back to the Eulerian coordinates

$$x = z + \int_T^t \boldsymbol{u}^{(1)}(z,\tau)\,\mathrm{d}\tau \equiv X^{(1)}_{\boldsymbol{u}}(z,t) \in \Omega_t, \quad t \in (T, 2T),$$

and obtain an extension

$$\boldsymbol{v}(x,t) = \boldsymbol{u}^{(1)}\Big(\big(X^{(1)}_{\boldsymbol{u}}\big)^{-1}x, t\Big), \quad \vartheta(x,t) = \theta^{(1)}\Big(\big(X^{(1)}_{\boldsymbol{u}}\big)^{-1}x, t\Big),$$

of the solution of (1.3) to the interval $(T, 2T)$. Applying Theorems 3 and 5 for $t \in (0, 2T)$, we obtain

$$\mathbf{Y}^2_{2T}(e^{\beta t}\boldsymbol{u}^{\pm}, e^{\beta t}\theta^{\pm}, e^{\beta t} r) \leqslant c\mathbf{F}^2_{2T}.$$

We continue by defining $(\boldsymbol{u}^{(2)}, \theta^{(2)})$ for $t \in (2T, 3T)$ as a solution of the problem (1.3) in the domain Ω_{2T} with the initial data $\boldsymbol{u}^{(2\pm)}(z, 2T) = \boldsymbol{v}^{\pm}(z, 2T)$, $z \in \Omega^{\pm}_{2T}$, $\theta^{(2+)}(z, 2T) = \vartheta^{+}(z, 2T)$, $z \in \Omega^{+}_{2T}$. This solution expressed in the Eulerian coordinates yields an extension of $(\boldsymbol{v}(x,t), \vartheta(x,t))$ to the interval $t \in (2T, 3T)$.

We go on further in the same way, and we notice that on the kth step inequality (5.2) takes the form

$$\|\boldsymbol{v}(\,\cdot\,, kT)\|^2_{W_2^{l+1}(\cup\Omega^{\pm}_{kT})} + \|\vartheta^{+}\|^2_{W_2^{l+1}(\Omega^{+}_{kT})} + \|r(\,\cdot\,, kT)\|^2_{W_2^{l+2}(S_{R_0})} \leqslant c_1 e^{-2\beta kT}\epsilon^2,$$

hence after a final number of steps we shall have $c_1 e^{2\beta kT} < 1$, and ϵ need not be changed any more. Condition (2.10) for $t \leqslant kT$ follows from the inequality

$$\mathbf{Y}_{kT}(e^{\beta t}\boldsymbol{u}^{\pm}, e^{\beta t}\theta^{\pm}, e^{\beta t} r) \leqslant c\mathbf{F}^2_{kT} \leqslant c\epsilon^2.$$

Finally, we notice that the displacement of the barycenter of Ω^{+}_{kT} with respect to the origin (the barycenter of Ω^{+}_0) equals

$$|\boldsymbol{h}(kT)| = \frac{3}{4\pi R_0^3}\left|\int_0^{kT}\int_{\Omega^{+}_0} \boldsymbol{v}(y,t)\,\mathrm{d}S\right| \leqslant c\mathbf{Y}_{kT} \leqslant c\epsilon \ll 1,$$

so Ω^{-}_{kT} has no points of contact with Σ.

We summarize the results of the present paper.

Theorem 7. *Assume that the data of Problem* (1.3) *possess finite norm* $\boldsymbol{F}$ (3.54) *on the infinite time interval* $(0,\infty)$, *moreover, the compatibility and smallness conditions of Theorem* 2 *are satisfied, as well as additional assumptions of Theorem* 6 : $p \in C^{3+1}(\min\rho(y,t),\max\rho(y,t))$ *and*

$$\boldsymbol{f} \in W_2^{\alpha_1}\big((0,\infty); W_2^l(\Omega)\big) \cap W_2^{0,l/2+\alpha_1}(Q_\infty),$$
$$\mathcal{D}_y^j \boldsymbol{f} \in L_2(Q_{t,t+T}), \quad |j| = 1,2, \quad \forall t > 0$$

with $\alpha_1 \in (1/2, 1-l/2)$, $\|e^{\beta t}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_\infty} \leqslant \epsilon$. *Then problem* (1.3) *is uniquely solvable in the infinite time interval* $(0,\infty)$ *in the class of functions with finite norm* $\boldsymbol{Y}$, *the solution satisfies inequality* (3.54) *with* $T = \infty$, *and* $\Gamma_t \in W_2^{l+5/2}$ *for positive values of* t. *As* $t \to \infty$, *the functions* $\boldsymbol{u}, \theta^{\pm}, r$ *tend to zero and* Ω_t^- *tends to the sphere of radius* R_0 *centered at the point* $h(\infty) = \lim_{t\to\infty} h(t)$ *close to the origin, i.e., to the barycenter of* Ω_0^-.

These results extend to the case of several incompressible fluids contained in nonintersecting domains $\Omega_{k,t}$, as in [3,5], and to the case where compressible fluid is surrounded with incompressible one, as in [2,6,7]. Analysis of the problem in $W_p^{2,1}$, as in [3,5], is also possible.

In conclusion, we consider briefly the case where the compressible fluid occupies the domain Ω_t^- and the incompressible one fills Ω_t^+. Problem (1.1) takes the form

$$\begin{cases} \rho^+\mathcal{D}_t\boldsymbol{v}^+ + (\boldsymbol{v}^+\cdot\nabla)v^+) - \nabla\cdot\mathbb{T}^+(\boldsymbol{v}^+) + \nabla p^+ = \rho^+\boldsymbol{f}, \\ \nabla\cdot\boldsymbol{v}^+ = 0 \text{ in } \Omega_t^+, \\ \rho^-(\mathcal{D}_t\boldsymbol{v}^- + (\boldsymbol{v}^-\cdot\nabla)\boldsymbol{v}^-) - \nabla\cdot\mathbb{T}^-(\boldsymbol{v}^-) + \nabla p(\rho^-) = \rho^-\boldsymbol{f}, \\ \mathcal{D}_t\rho^- + \nabla\cdot(\rho^-\boldsymbol{v}^-) = 0, \quad \rho^-|_{t=0} = \rho_0^- \text{ in } \Omega_t^-, \\ \boldsymbol{v}^\pm|_{t=0} = \boldsymbol{v}_0^\pm \text{ in } \Omega_0^\pm, \quad \boldsymbol{v}^+|_\Sigma = 0, \quad [\boldsymbol{v}]|_{\Gamma_t} = 0, \\ (-p(\rho^-) + p^+)\boldsymbol{n} + [\mathbb{T}(\boldsymbol{u})\boldsymbol{n}] = -\sigma H\boldsymbol{n} \text{ on } \Gamma_t, \end{cases} \tag{5.4}$$

where

$$\mathbb{T}^+(\boldsymbol{v}^+) = \mu^+\mathbb{S}(\boldsymbol{v}^+), \quad \mathbb{T}^-(\boldsymbol{v}^-) = \mu^-\mathbb{S}(\boldsymbol{v}^-) + \mu_1^-\mathbb{I}\nabla\cdot\boldsymbol{v}^-.$$

For the sake of convenience, we assume that $[u]|_{\Gamma_t} = u^- - u^+$. By setting

$$\vartheta^- = \rho^- - \overline{\rho}^-, \quad \vartheta^+ = p^+ - p(\overline{\rho}^-) - \frac{2\sigma}{R_0},$$

where $\overline{\rho}^- = M^-/|\Omega_0^-|$, we convert (5.4) into

$$\begin{cases} \rho^+\mathcal{D}_t\boldsymbol{v}^+ + (\boldsymbol{v}^+\cdot\nabla)\boldsymbol{v}^+) - \nabla\cdot\mathbb{T}^+(\boldsymbol{v}^+) + \nabla\vartheta^+ = \rho^+\boldsymbol{f}, \\ \nabla\cdot\boldsymbol{v}^+ = 0 \text{ in } \Omega_t^+, \\ (\overline{\rho}^- + \vartheta^-)(\mathcal{D}_t\boldsymbol{v}^- + (\boldsymbol{v}^-\cdot\nabla)\boldsymbol{v}^-) - \nabla\cdot\mathbb{T}^-(\boldsymbol{v}^-) + \nabla p(\overline{\rho}^- + \vartheta^-) \\ = (\overline{\rho}^- + \vartheta^-)\boldsymbol{f}, \\ \mathcal{D}_t\vartheta^- + \nabla\cdot((\overline{\rho}^- + \vartheta^-)\boldsymbol{v}^-) = 0, \quad \vartheta^-|_{t=0} = \vartheta_0^- \text{ in } \Omega_t^-, \\ \boldsymbol{v}^\pm|_{t=0} = \boldsymbol{v}_0^\pm \text{ in } \Omega_0^\pm, \quad \boldsymbol{v}^-|_\Sigma = 0, \quad [\boldsymbol{v}]|_{\Gamma_t} = 0, \\ (-p(\overline{\rho}^- + \vartheta^-) + p(\overline{\rho}^-) + \vartheta^+)\boldsymbol{n} + [\mathbb{T}(\boldsymbol{u})\boldsymbol{n}] = -\sigma(H + \frac{2}{R_0})\boldsymbol{n} \text{ on } \Gamma_t, \end{cases} \tag{5.5}$$

We pass to Lagrangian coordinates by formula (1.2) and convert Problem (5.5) into

$$\begin{cases} \rho^+\mathcal{D}_t\boldsymbol{u}^+ - \nabla\cdot\mathbb{T}^+(\boldsymbol{u}^+) + \nabla\theta^+ = \boldsymbol{l}_1^+(\boldsymbol{u}^+,\theta^+) + \rho^+\widehat{\boldsymbol{f}}, \\ \nabla\cdot\boldsymbol{u}^+ = l_2^+(\boldsymbol{u}^+) \quad \text{in } \Omega_0^+, \quad t>0, \\ \overline{\rho}^-\mathcal{D}_t\boldsymbol{u}^- - \nabla\cdot\mathbb{T}^-(\boldsymbol{u}^-) + p_1\nabla\theta^- = \boldsymbol{l}_1^-(\boldsymbol{u}^-,\theta^-) + (\overline{\rho}^- + \theta^-)\widehat{\boldsymbol{f}}, \\ \mathcal{D}_t\theta^- + \overline{\rho}^-\nabla\cdot\boldsymbol{u}^- = l_2^-(\boldsymbol{u}^-,\theta^-) \quad \text{in } \Omega_0^-, \quad t>0, \\ \boldsymbol{u}^\pm|_{t=0} = \boldsymbol{u}_0^\pm \text{ in } \Omega_0^\pm, \quad \theta^-|_{t=0} = \theta_0^- = \rho_0^- - \overline{\rho}^-, \\ [\boldsymbol{u}]|_{\Gamma_0} = 0, \quad [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{u})\boldsymbol{n}_0]|_{\Gamma_0} = \boldsymbol{l}_3(\boldsymbol{u})|_{\Gamma_0}, \\ -p_1\theta^- + \theta^+ + [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{u})\boldsymbol{n}_0]\big|_{\Gamma_0} + \sigma\boldsymbol{n}_0\cdot\int\limits_0^t\Delta(0)\boldsymbol{u}(y,\tau)\,\mathrm{d}\tau\big|_{\Gamma_0} \\ = l_4(\boldsymbol{u}) - \int\limits_0^t(l_5(\boldsymbol{u}) + l_6(\boldsymbol{u}))\,\mathrm{d}\tau - \sigma(H_0 + \frac{2}{R_0}), \quad \boldsymbol{u}^+|_\Sigma = 0, \end{cases} \tag{5.6}$$

where

$$\begin{aligned} \boldsymbol{l}_1^+(\boldsymbol{u},\theta) &= \nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^+(\boldsymbol{u}^+) - \nabla\cdot\mathbb{T}^+(\boldsymbol{u}^+) + (\nabla - \nabla_{\boldsymbol{u}})\theta^+, \\ \boldsymbol{l}_1^-(\boldsymbol{u},\theta) &= \nabla_{\boldsymbol{u}}\cdot\mathbb{T}_{\boldsymbol{u}}^-(\boldsymbol{u}^-) - \nabla\cdot\mathbb{T}^-(\boldsymbol{u}^-) \\ &\quad + p_1(\nabla - \nabla_{\boldsymbol{u}})\theta^- - \nabla_{\boldsymbol{u}}\big(p(\overline{\rho}^- + \theta^-) - p(\overline{\rho}^-) - p_1\theta^-\big) - \theta^-\mathcal{D}_t\boldsymbol{u}^-, \\ l_2^+(\boldsymbol{u}) &= (\nabla - \nabla_{\boldsymbol{u}})\cdot\boldsymbol{u}^+ = \nabla\cdot\boldsymbol{L}(\boldsymbol{u}^+), \quad \boldsymbol{L}(\boldsymbol{u}^+) = (\mathbb{I} - \mathbb{L}^{-1})\boldsymbol{u}^+ = (\mathbb{I} - \widehat{\mathbb{L}})\boldsymbol{u}^+, \\ l_2^-(\boldsymbol{u},\theta) &= \overline{\rho}^-(\nabla - \nabla_{\boldsymbol{u}})\cdot\boldsymbol{u}^- - \theta^-\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}^-, \\ \boldsymbol{l}_3(\boldsymbol{u}) &= [\mu^\pm\Pi_0(\Pi_0\mathbb{S}(\boldsymbol{u})\boldsymbol{n}_0 - \Pi\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n})]\big|_{\Gamma_0}, \\ l_4(\boldsymbol{u}) &= [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{u})\boldsymbol{n}_0 - \boldsymbol{n}\cdot\mathbb{T}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}] - (p(\overline{\rho}^- + \theta^-) - p(\overline{\rho}^-) - p_1\theta^-)\big|_{\Gamma_0}, \\ l_5(\boldsymbol{u}) &= \sigma\mathcal{D}_t(\boldsymbol{n}\Delta(t))\cdot\int_0^t\boldsymbol{u}(y,\tau)\,\mathrm{d}\tau + \sigma(\boldsymbol{n}\cdot\Delta(t) - \boldsymbol{n}_0\cdot\Delta(0))\boldsymbol{u}|_{\Gamma_0}, \\ l_6(\boldsymbol{u}) &= \sigma(\dot{\boldsymbol{n}}\Delta(t) + \boldsymbol{n}\dot{\Delta}(t))\cdot\boldsymbol{y}\Big|_{\Gamma_0}, \quad \dot{\boldsymbol{n}} = \mathcal{D}_t\boldsymbol{n}, \quad \dot{\Delta}(t) = \mathcal{D}_t\Delta(t), \end{aligned} \tag{5.7}$$

A local solution of Problem (5.6)–(5.7) is constructed as above.

The corresponding linear problem has the form

$$\begin{cases} \overline{\rho}^-\mathcal{D}_t\boldsymbol{v}^- - \mu^-\nabla^2\boldsymbol{v}^- - (\mu^- + \mu_1^-)\nabla(\nabla\cdot\boldsymbol{v}^-) + p_1\nabla\theta^- = \boldsymbol{f}^-, \\ \mathcal{D}_t\theta^- + \overline{\rho}^-\nabla\cdot\boldsymbol{v}^- = h^- \text{ in } \Omega_0^+, \\ \rho^+\mathcal{D}_t\boldsymbol{v}^+ - \mu^+\nabla^2\boldsymbol{v}^+ + \nabla\theta^+ = \boldsymbol{f}^+, \quad \nabla\cdot\boldsymbol{v}^+ = h^+ \text{ in } \Omega_0^+, \\ \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0 \text{ in } \Omega_0^+\cup\Omega_0^-, \quad \theta^-|_{t=0} = \theta_0^- \text{ in } \Omega_0^-, \\ [\boldsymbol{v}]|_{\Gamma_0} = 0, \quad [\mu\Pi_0\mathbb{S}(\boldsymbol{v})\boldsymbol{n}_0]_{\Gamma_0} = \boldsymbol{b}, \\ -p_1\theta^- + \theta^+ + [\boldsymbol{n}_0\cdot\mathbb{T}(\boldsymbol{v})\boldsymbol{n}_0] + \sigma\boldsymbol{n}_0\cdot\int\limits_0^t\Delta(0)\boldsymbol{v}(y,\tau)\,\mathrm{d}\tau|_{\Gamma_0} = b + \sigma\int\limits_{\Gamma_0}B\,\mathrm{d}\tau. \end{cases} \tag{5.8}$$

The solution of this problem satisfies inequality (2.8), but with the roles of θ^+ and θ^- interchanged.

We pass to the definition of r. As above, we introduce the sphere with center at the origin and with radius R_0 such that $|\Omega_t^-| = \frac{4\pi R_0^3}{3}$. We assume that the barycenter of Ω_t^- is the point $y = 0$. Let Ω_t^+ be given by the relation $R(\omega,t) \leqslant |y| \leqslant \widetilde{R}(\omega,t)$, $\omega\in S_1$, and let $B^+ = \Omega\setminus\bar{B^-}$

be defined by $R_0 \leqslant |y| \leqslant \widetilde{R}(\omega,t)$. The condition that the barycenters of these two domains coincide is

$$\int\limits_{S_1} \omega_i \Big(\frac{\widetilde{R}^4}{4} - \frac{R^4}{4}\Big)\,\mathrm{d}S_\omega = \int\limits_{S_1} \omega_i \Big(\frac{\widetilde{R}^4}{4} - \frac{R_0^4}{4}\Big)\,\mathrm{d}S_\omega, \quad i=1,2,3,$$

i.e.,

$$\int\limits_{S_1} \omega_i R^4\,\mathrm{d}S_\omega = \int\limits_{S_1} \omega_i R_0^4\,\mathrm{d}S_\omega = 0.$$

This is exactly the second equation in the relations (2.35). Also the first equation $\int_{S_1}(\frac{R^3}{3} - \frac{R_0^3}{3})\,\mathrm{d}S_\omega = 0$ holds true, in view of $|\Omega_t^-| = \frac{4\pi R_0^3}{3}$.

This shows that it is possible to use the same representation of Γ_t as above (see (2.30), (2.31)), i.e., as if Ω_t^- was filled with the incompressible fluid, and carry out all the estimates of the solution of Problem (5.6), (5.7) (inequality (2.39), Theorems 2, 3, 4, 5 etc.) exactly as above; details are omitted.

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