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телефоны:(812)312-40-58; (812) 571-57-54

e-mail: admin@pdmi.ras.ru

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The Riemann Hypothesis as the parity of special binomial coefficients

YU. V. MATIYASEVICH

St.Petersburg Department
of V. A. Steklov Institute of Mathematics
of Russian Academy of Sciences

yumat@pdmi.ras.ru

Abstract. The Riemann Hypothesis has many equivalent reformulations. Some of them are arithmetical, that is, thewy are statements about properties of integers or natural numbers. Among them the reformulations with the simplest logical structure are those from the class Π_1^0 from the arithmetical hierachy, that is, having the form "for every $x_1, \ldots, x_m$ relation $A(x_1, \ldots, x_m)$ holds", where A is decidable. As an example one can take the reformulation of the Riemann Hypothsis as the assertion that certain Diophantine equation has no solution (such particular equation can be given explicitly).

While the logical structure of this reformulation is indeed very simple, all known methods for constructing such Diophantine equation produce equations occupying several pages. On the other hand, there are known other reformulation also belonging to class Π_1^0 but having rather short wording. As examples one can mention the criteria of the validity of the Riemann Hypothesis proposed by J.-L. Nicolas, by G. Robin, and by J. Lagarias. The shortcoming of these reformulations (as compared to Diophantine equations) consists in the usage of constants and funtions which are "more complicated" than integers and addition and multiplication sufficient for constructing Diophantine equations.

The paper presents a system of 9 conditions imposed on 9 variables. In order to state these conditions one needs only addition, multiplication, exponentiation (unary, with fixed base 2), congruences and remainders, inequalities, and binomial coefficient. The whole system can be written explicitly on a single sheet of paper. It is proved that the system is inconsistent if and only if the Riemann Hypothesis is true.

Key words: the Riemann Hypothesis, binomial coefficients.

*The following formulas were corrected: (2.4), (2.15), (2.16), (2.37), (2.41), and (2.42).

ПРЕПРИНТЫ
Санкт-Петербургского отделения
Математического института им. В. А. Стеклова
Российской академии наук

PREPRINTS
of the St.Petersburg Department
of Steklov Institute of Mathematics

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1 Introduction

The Riemann Hypothesis, like many other great problems, has very large number of equivalent reformulations. Many of them are presented in recent two-volume monograph [1, 2]. Such reformulations are given in very diverse terms, but powerful technique of *arithmetization*, developed by K. Gödel [3] allows one to transform them into statements about integers or natural numbers. In this paper we will restrict ourselves to such *arithmetical* reformulations.

A. Turing, who made made important cotributions to verification of the Riemann Hypothesis (see, for example, [1, 4, 5, 6, 7]), was also interested to know how simple, *from logical point of view*, can a reformulation of the Riemann Hypothesis be. In [8] he introduced the notion of *number-theoretical theorem*:

> By a number-theoretic theorem we shall mean a theorem of the form "$\theta(x)$ vanishes for infinitely many natural numbers x", where $\theta(x)$ is a primitive recursive function. ...An alternative form for number-theoretic theorems is "for each natural number x there exists a natural number y such that $f(x, y)$ vanishes", where $f(x, y)$ is primitive recursive.

Number-theoretical theorems of Turing are equivalent to provable formulas from class Π_2^0 of the *arithmetical hierarchy.* This class can be described as ckass of formulas of the form

$$\forall x_1 \ldots x_m \exists y_1 \ldots y_n A(x_1, \ldots, x_m, y_1, \ldots, y_n), \tag{1.1}$$

where $A(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a decidable relation among natural numbers x_1, $\ldots$, x_m, $y_1, \ldots, y_n$. Justifying his definition, Turing constructed a formula from class Π_2^0 which is equivalent to the Riemann hypothesis.

This result was improved by G. Kreisel [9] who reformulated the Riemann Hypothesis by a formula from class Π_1^0 consisting of formulas of the form

$$\forall x_1 \ldots x_m A(x_1, \ldots, x_m). \tag{1.2}$$

Such formulas can be characterized as *effectively refutable*: if a formula (1.2) is false then this can be established be exhibiting one particular set of n numbers $x_1 \ldots x_n$ not satisfying relation A. Using the decidability of this relation one can construct, saty, a Turing machine (o write a program in some programming language) which would perform the exaustive search of possible values of $x_1 \ldots x_n$ trying to find the required counterexample. Such a machine/program will work eternally if and only if formula (1.2) is true.

Thanks to this result of Kreisel it became possible to find such a machine/program for the Riemann Hypothesis, and this was actually done in a number of papers. S. Aaronson and A. Yedidia [10] constructed a particular Turing machine with two-symbol tape alphabet which, having started with the empty tape, will never halt if and only if the Rieman Hypothesis is true. In [10] the machine has 5372 states; later this was improved to 744 states (see [11]). C. Calude, E. Calude, and M. Dinneen [12, 13] and the author [14] constructed several versions of *register machines* with analogous property.

In 1970 the author made the last step in the proof of what today is often called DPRM-theorem[1]. This result allowed one to transform an arbitrary formula from class Π_1^0 into an equivalent formula from the same class having the following special form:

$$\forall x_1 \dots x_m P(x_1, \dots, x_m) \neq 0, \tag{1.3}$$

where $P(x_1, \dots, x_m)$ is a polynomial with integer coefficients. It particular, it is possible to explicitly specify a polynomial $R(x_1, \dots, x_m)$ such that the Riemann Hypothesis is equivalent to the statement that Diophantine equation

$$R(x_1, \dots, x_m) = 0 \tag{1.4}$$

has no solution. Methods for constructing such a polynomial are presented in [17, Section 2] and [16, Subsection 6.4]; more details are given in [18, 19]; see also [20].

Reformulation (1.3) of the Riemann Hypothesis certainly has very simple *structure*: it contains universal quantifiers only, and verification of the condi-tition consists just in calculation of the value of a polynomial. On the other hand, while 9 variables are sufficient for such a polynomial ([21], for details see [22]), all earlier known methods pruced polynomial occupying several pages.

There is quite a few other reformulation of of the Riemann Hypothesis having the form (1.2) in which relation A can be written very shortly but which are more difficult for verification; several such examples are given below.

Many classical results have the form resembling (1.2) but contain, for example, big O notation having hidden existential quantifier. One can get rid of it by finding explicit numerical value of the implied constant.

Diophantine equation (1.4) in [17] and Turing machine in [10] are based on the reformulation of the Riemann Hypothesis proposed by H. Shapiro (see [17, Section 2] and [1, Subsection 10.2]). It is given in terms of *Chebyshev function* $\psi(n)$ which is defined as follows:

$$\psi(n) = \ln(\mathrm{LCM}(1, \dots, n)) = \ln(2) \log_2(\mathrm{LCM}(1, \dots, n)), \tag{1.5}$$

[1] After M. Davis, H. Putnam, J. Robinson, and the author of this paper; detailed proofs of the theorem are given, for example, in [15, 16].

where LCM is the least common multiple. The Riemann Hypothesis is equivalent to the following statement:

$$\psi(n) = n + O\sqrt{n}\ln^2(n). \tag{1.6}$$

In order to avoid hidden in the big O constant, Shapiro considered summatory function

$$\psi_1(n) = \sum_{1\le m<n} \psi(m) \tag{1.7}$$

and established that the Riemann Hypothesis is equivalent to the following inequality with an explicite constant:

$$\left|\psi_1(m) - \frac{m^2}{2}\right| < 6m\sqrt{m}. \tag{1.8}$$

Later L. Schoenfeld ([23], see also [1, Theorem 4.9]) found an explicit value for the constant in (1.6), namely, he proved that the Riemann Hypothesis is equivalent to the validity of the inequality

$$|\psi(n) - n| < \frac{1}{8\pi}\sqrt{n}\ln(n)^2 \tag{1.9}$$

for $n \ge 74$. It was the usage of this criterium (insread of (1.8)) that allowed to simplify construction of polynomial (1.3) in [16] and to reduce the number of states of the Turing machine in [11].

J.-L. Nikolas ([24], see also [1, Theorem 5.31]) established that the Riemann Hypothesis is equivalent to the inequality

$$\mathrm{e}^{\gamma}\log(\log(N_n)) < \frac{N_n}{\phi(N_n)} \tag{1.10}$$

where $\mathrm{e} = 2.71828\ldots$, $\gamma = 0.577215\ldots$ is the Euler constant, N_n denotes the product of the first n prime numbers, $\phi(m)$ is Euler totient function (the quantity of number which are not greater than m and are relatively prime with this number).

G. Robin ([25], see also [1, Theorem 7.16]) proved that the the Riemann Hypothesis is equivalent to the validity, for $n \ge 5040$, of the inequality

$$\sigma(n) < \mathrm{e}^{\gamma} n \log(\log(n)), \tag{1.11}$$

where $\sigma(n)$ is the sum of all divisors of n. This necessary and sufficient condition is also known as *criterium of Ramanujan–Robin*, because S. Ramanujan established inequality (1.11) for sufficiently large n under the assumption of the validity of the Riemann Hypothesis.

J. C. Lagarias ([26], see also [1, Theorem 7.18]) replaced the right-hand side in (1.11) and got yet another condition which is both necessary and sufficient for the validity of the Riemann Hypothesis:

$$\sigma(n) < H_n + \mathrm{e}^{H_n} \log(H_n) \tag{1.12}$$

where $H_n = 1 + 1/2 + \cdots + 1/n$ and n is arbitrary.

Conditions of the type (1.8)–(1.12) are decidable and have short wording; however, they contain real constants and functions like $\psi(n)$, N_n, $\phi(n)$, $\sigma(n)$, which are more "complicated" in comparison to the integer coefficients and operations of addition and multiplication used in (1.4). The goal of this paper is to propose a "compromise" reformulation of the Riemann Hypothesis. Its advantage over the Diophantine equation is its brevity – it can be written on a sinle sheet of paper. Its disadvantage in comparison with (1.4), but advantage in comparision with (1.8)–(1.12) consists in the functions used. Besides addition and multiplication, we need only exponentiation (unary, with base 2), square root (it can be easily eliminated), $\mathrm{rem}(a, b)$ (the remainder of dividing a by b), inequalities, congruences, and binomial coefficient which plays a key role.

The binomial coefficients have surprisingly great expressive power. H. B. Mann and и D. Shanks [27] gave a criteria of primality in terms of divisibility of certain entries in the Pascal triangle. L. Hsu and P. J.-S. Shiue [28] reformulated Fermat's Last Theorem as vanishing of certain sums of products of binomial coefficients. The author [29] gave, in the form of divisibility of a single binomial coefficient, criteria for

1. number p to be prime;
2. numbers p and $p + 2$ be twin-primes;
3. number p be Fermat !! prime;
4. number p be Mersenne !! prime.

In [30] the author reformulated the Four Color Conjecture (now Theorem) as non-divisibility of a certain product of binomial coefficients. In similar style M. Margenstern and the author [31] reformulated well-known $3x + 1$-problem.

The constructions in [29, 30, 31] are based on the following properties of binomial coefficients.

Theorem (E. Kummer [32]). *Let a and b be numbers with the following p-bas positional representation where p is a prime:*

$$a = \sum_{k=0}^{m} a_k p^k,\ b = \sum_{k=0}^{m} b_k p^k,\ 0 \le a_k < p,\ 0 \le b_k < p,\ k = 0, \dots, m; \qquad (1.13)$$

then the exponent of p in the prime factorization of binomial coefficient $\binom{a+b}{a}$ is equal to the number of carries performed during adding a and b.

This result of Kummer for a long time remained little-known and was rediscovered by many authors; proofs of the theorem can be found also in [16, 33].

We shall use the following corllary of Kummer's theorem for the case $p = 2$ in (1.13). Let us say that a *masks* b (and write $a \succeq b$), if $a_k \ge b_k$ for $k = 0, \dots, m$. Rummer's theorem implies that:

$$\binom{a}{b} \equiv 1 \pmod 2 \iff a \succeq b. \qquad (1.14)$$

This can be deduced from a special case of Lukas's theotem [34, Section XXI]:

$$\binom{a}{b} \equiv \binom{a_0}{b_0} \dots \binom{a_m}{b_m} \pmod p. \qquad (1.15)$$

2 New reformulation of the Riemann Hypothesis

The inequality (1.9) will be our starting point, but we modify it in two ways, differently for necessary and for sufficient conditions:

- *the Riemann Hypothesis implies that for all $n > 1$*

$$\psi(n) > n - \sqrt{n}\log_2^2(n); \qquad (2.1)$$

- *if the Riemann Hypothesis is not true than there are infinitely many values of n for which*

$$\psi(n) < n - 20\sqrt{n}\log_2^2(n). \qquad (2.2)$$

We shall use the fact that the right-hand side in the necessary condition (2.1) is larger than the right-hand side in the sufficient condition (2.2). The inequality (2.1) for $n \ge 74$ follows from (1.9), and the remaining cases $n = 2, \dots, 73$ can be verified by numerical calculation. The sufficiency of condition (2.2) follows from $\Omega_{\pm}$-result for function $\psi(n)$, which was obtained by E. Schmidt ([35], see also [36, Theorem 32] and [1, Theorem 4.8]).

Theorem 1. *Let us cosider the following system of conditions:*

$$2^l \leq n < 2^{l+1}, \tag{2.3}$$

$$2^m \leq 2q < 2^{m+1}, \tag{2.4}$$

$$s = \frac{B^{n+1}\left(B^{(n+1)n} - n - 1\right) + n}{\left(B^{n+1} - 1\right)^2}, \tag{2.5}$$

$$t = \frac{(2^m - 1)\left(B^{n^2} - 1\right)}{B^n - 1}, \tag{2.6}$$

$$\binom{t}{r} \equiv 1 \pmod 2, \tag{2.7}$$

$$u = \operatorname{rem}\left(rs, B^{n^2-n}\right), \tag{2.8}$$

$$rs - u \equiv \frac{B^{n^2-n}\left(B^n - 1\right)}{B - 1} q \pmod{B^{n^2}}, \tag{2.9}$$

$$p = \operatorname{rem}(r, B^n + 1), \tag{2.10}$$

$$mp < nq - 15l^2 q\sqrt{n}, \tag{2.11}$$

where B *is an abridgement for* 2^{l+m+1}.

(A) If the Riemann Hypothesis is true then system (2.3)–(2.11) *has no solution in positive integers* $l, m, n, p, q, r, s, t, u$.

(B) If the Riemann Hypothesis is false then the system (2.3)–(2.11) *has infinitely many such solutions.*

Proof of the Part (A) will be given "by contradiction". Suppose that there are numbers l, m, n, p, q, r, s, t and u satisfying conditions (2.3)–(2.11).

According to (2.3),

$$n > 1 \tag{2.12}$$

and

$$l = \lfloor \log_2(n) \rfloor. \tag{2.13}$$

Clearly,

$$1 \leq l, \quad 0 \leq \log_2(n) - l < 1. \tag{2.14}$$

Similar, according to (2.4)

$$m = \lfloor \log_2(q) \rfloor + 1 \tag{2.15}$$

and

$$0 < m - \log_2(q) \leq 1. \tag{2.16}$$

Let us consider B-base notation of numbers s, t, r and rs.

It is easy to check that (2.5) implies that

$$s = \sum_{j=1}^{n} jB^{(n-j)(n+1)}. \tag{2.17}$$

This means that numbers $1, \ldots, n$ are the only non-zero digits of number s, and they are separated blocks of n zeros.

Similar, (2.6) implies that

$$t = \sum_{k=1}^{n} (2^m - 1)B^{(k-1)n}; \tag{2.18}$$

in other words, all non-zero digits of number t are equal to $2^m - 1$, and they are separated blocks of $n - 1$ zeros.

The binary notation of any number a can be obtained from its B-base notation by replacing each B-base digits by its by its binary notation prepended, if necessary, by leading zeros to the length $l + m + 1$. This implies that a masks b if and only if each B-base digit of a masks corresponding B-base digit of b.

According to (1.14), (2.7) implies that $t \succeq r$ and hence number r has the form

$$r = \sum_{k=1}^{n} r_k B^{(k-1)n}, \tag{2.19}$$

where

$$r_k \leq 2^m - 1, \quad k = 1, \ldots, n. \tag{2.20}$$

Let

$$rs = \sum_{i=0}^{2n^2} d_i B^i, \quad 0 \leq d_i < B, \quad i = 0, \ldots 2n^2. \tag{2.21}$$

According to (2.17) and (2.19)

$$rs = \sum_{j=1}^{n} \sum_{k=1}^{n} j r_k B^{(n-j)(n+1)+(k-1)n}. \tag{2.22}$$

It is easy to check that for $1 \le j \le n$, $1 \le k \le n$ all numbers $(n-j)(n+1)+(k-1)n$ are pairwise distinct. Also (2.3) and (2.19) imply that

$$jr_k \le n(2^m - 1) < 2^{l+1}(2^m - 1) < 2^{l+m+1} = B. \tag{2.23}$$

Thus, all possible products jr_k constitute all non-zero digits of number rs, more precisely,

$$d_i = \begin{cases} jr_k, & \text{if } i = (n-j)(n+1) + (k-1)n \\ 0, & \text{otherwise.} \end{cases} \tag{2.24}$$

In particular, in the case $j = k$ we have:

$$d_{n^2-k} = kr_k, \quad k = 1, \ldots, n. \tag{2.25}$$

According to (2.8) and (2.21)

$$u = \sum_{i=0}^{n^2-n-1} d_i B^i. \tag{2.26}$$

In other words, number u is the "tail" of the product rs, formed by its $n^2 - n$ least significant digits. Respectively,

$$rs - u = \sum_{i=n^2-n}^{2n^2} d_i B^i \equiv \sum_{i=n^2-n}^{n^2-1} d_i B^m \pmod{B^{n^2}}. \tag{2.27}$$

We have the identity

$$\sum_{i=n^2-n}^{n^2-1} qB^i = \frac{(B^n - 1)\, B^{n^2-n}}{B - 1} q; \tag{2.28}$$

according to it (2.9), (2.25) and (2.27) imply that

$$kr_k = d_{n^2-k} = q, \quad k = 1, \ldots, n. \tag{2.29}$$

From this we get the following values of the digits of number r:

$$r_k = \frac{q}{k}, \quad k = 1, \ldots, n. \tag{2.30}$$

According to (2.29) q is divisible by $1, \dots, n$, hence,

$$\mathrm{LCM}(1, \dots, n) \le q. \tag{2.31}$$

The evident congruence

$$B^n \equiv -1 \pmod{B^n + 1} \tag{2.32}$$

and identity (2.19) imply that

$$p \equiv \sum_{k=1}^{n} (-1)^{k-1} r_k \pmod{B^n + 1}. \tag{2.33}$$

The summands in the alternating sum (2.33) decrease in absolute value, the first summand id equal to q, hence, the sum is positive and is at most q. Thus, both left- and right-hand sides in the congruence (2.33) are positive and do not exceed its modulo, hence they are equal. Respectively,

$$\frac{p}{q} = \sum_{k=1}^{n} \frac{(-1)^{k-1} r_k}{q} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \approx \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln(2) \tag{2.34}$$

and we have elementary inequalities

$$\frac{1}{2} \le \frac{p}{q}, \quad \left| \frac{p}{q} - \ln(2) \right| < \frac{1}{2n}. \tag{2.35}$$

Now (2.11) and (2.35) imply that

$$m < \frac{n - 15 l^2 p \sqrt{n}}{p/q} \le 2n. \tag{2.36}$$

Further, according to (2.35), (2.36), (2.16), (2.31), (1.5), and (2.12), we have:

$$\begin{aligned} \frac{p}{q} m > \left(\ln(2) - \frac{1}{2n} \right) m = \ln(2) m - \frac{m}{2n} > \ln(2) \log_2(q) - 1 = \\ \ln(q) - 1 \ge \ln(\mathrm{LCM}(1, 2, \dots, n)) - 1 = \\ \psi(n) - 1 > \psi(n) - 2\sqrt{n} \log_2^2(n). \end{aligned} \tag{2.37}$$

On the other hand, according to (2.11) и (2.14)

$$\frac{p}{q} m < n - 15 l^2 \sqrt{n} < n - 3\sqrt{n} \log_2^2(n). \tag{2.38}$$

The three inequalities, (2.1), (2.37), and (2.38), give the required contradiction. Part (A) is proved.

Proof of Part (B). For the role of n we take any integer whichis greater than 1 and satisfies (2.2). From the proof of Part (A) we can see that the values of all other variables are almost uniquely determined by the value of n.

Let us take l according to (2.13), so (2.3) and (2.14) hold.

Let

$$q = \mathrm{LCM}(1, \ldots, n), \tag{2.39}$$

and select m according to (2.15), then (2.4) and (2.16) hold.

Let us select s according to (2.17), so (2.5) holds.

Let numbers r_k and r be defined according to (2.30) and (2.19), then (2.16) implies the validity of (2.20) and (2.23). The binary notation of number $2^m - 1$ consists of m units, hence (2.23) implies that

$$2^m - 1 \succeq r_k, \quad k = 1, \ldots, n. \tag{2.40}$$

Let us take t according to (2.18), then (2.6) holds. All non-zero digits of number t are equal to $2^m - 1$ and, according to (2.40), they mask corresponding digits of number r. This implies that что $t \succeq r$ and, according to (1.14), condition (2.7) is fulfilled.

Similar to the proof of Part (A), we conclude that the digits d_i in representation (2.21) are defined by equality (2.24) and шеы particular case (2.25).

Let us take u according to (2.26), then (2.8) and (2.27) hold. According to (2.25) in the second sum in (2.27) all d_i are equal to q. Due to the identity (2.28), condition (2.9) is fulfilled.

Similat to the proof of Part (A) we conclude that that inequalities (2.35) hold.

According to (2.39), (1.5), and (2.2)

$$\log_2(q) = \log_2\big(\mathrm{LCM}(1, \ldots, n)\big) = \psi(n)/\ln(2) < 2\psi(n) < 3n. \tag{2.41}$$

Using also (2.35), (2.16), (2.12), (2.2), and (2.14), we get that

$$\begin{aligned} \frac{p}{q}m < \left(\ln(2) + \frac{1}{2n}\right)(\log_2(q) + 1) = \psi(n) + \frac{\log_2(q)}{2n} + \ln(2) + \frac{1}{2n} < \\ < \psi(n) + 3\sqrt{n}\log_2^2(n) < n - 17\sqrt{n}\log_2^2(n) < n - 17\sqrt{n}l^2 \end{aligned} \tag{2.42}$$

hence, condition (2.11) is fulfilled.

Part (B) is proved. Theorem is proved.

Remark. If we allow exponentiation with arbitrary base (not only числа 2 as in (2.3)–(2.11)), then we can eliminate the binomial coefficient:

$$\binom{t}{r} \equiv 1 \pmod 2 \iff \operatorname{rem}((2^t+1)^t, 2^{rt+1}) > 2^{rt}. \tag{2.43}$$

Replacing condition (2.7) by the right-hand side in (2.43), we get a system of conditions each of which can be easily transformed into an exponential Diophantine equation at the cost of introduction of new variables. All these equations can be easily combined into a single exponential Diophantine equation the undecidability of which is equivalent to the Riemann Hypothesis. Standard technique (see, for example, [15, 16]) allows us to transform this exponential Diophantine equation into an equivalent equation with additional variables with relatively short wording.

Conclusion

We have established that the Riemann Hypothesis is equivalent to the inconsistency of the conditions (2.3)–(2.11). It seems interesting to investigate systems of conditions resulting from (2.3)–(2.11) by deletion of one of the conditions or replacing it by a weaker one. For example, can we find a transparent description of the solutions of the system resulting from (2.3)–(2.11) via replacing the binomial condition (2.7) by its corollary $r \leq t$?

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