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Plausible ways for calculating the Riemann zeta function via the Riemann–Siegel theta function

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Abstract. This paper demonstrate on numerical examples several plausible ways of calculating values of the Riemann zeta function via the Riemann–Siegel theta function. These new methods are mainly of theoretical interest for the study of the zeta function, they are not claimed to be more efficient than other already known techniques.

Key words: Riemann zeta function, Hardy Z function, Hardy–Siegel theta function.

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1 Introduction. Syntactical and semantical approximations

Suppose that we are interested to study some function $D(s)$ defined by a certain Dirichlet series,

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}. \tag{1.1}$$

We could try to get information about $D(s)$ via its approximations by some finite Dirichlet series

$$D_N(s) = a_{N,1} \times 1^{-s} + \cdots + a_{N,N} \times N^{-s}. \tag{1.2}$$

Informally, we can classify such approximations as *syntactical* and *semantical*.

In the former case the starting point is the right-hand side in (1.1), the Dirichlet series. Namely, we define numbers $a_{N,n}$ in some way via numbers a_n, and then we examine to what extent function $D_N(s)$, defined by (1.2), is similar to $D(s)$. The simplest case is the plain truncation

$$a_{N,n} = a_n. \tag{1.3}$$

Typically, better numerical approximation is achieved for a *smoothed truncation*

$$a_{N,n} = w_{N,n} a_n \tag{1.4}$$

with some weights $w_{N,n}$ such that $w_{N,n} \approx 1$ for small n and $w_{N,n} \approx 0$ for n close to N; often there are many ways to select such weights giving good approximations.

In the case of semantical approximation the starting point is the function $D(s)$ itself, and we need not know the coefficients of its Dirichlet series. Numbers $a_{N,n}$ are defined in the framework of a certain *mode of similarity* saying that function $D_N(s)$ behaves like $D(s)$ in a *certain* respect. After that we ask: in what *other* respects are these two functions similar? Also we can scrutinize the numbers $a_{N,n}$ themselves.

In general, semantical approximations (in comparison with syntactical ones) are less amenable to analysis because of the indirect character of the definition of their coefficients.

Many results found in the literature can be viewed as studies of particular modes of similarity, and below a few sample are briefly described; two new modes are introduced in Section 3, and Sections 4–5 present corresponding numerical data.

Example 1 Having in (1.2) N parameters $a_{N,1}, \ldots, a_{N,N}$, we can select N different *testing points* $s_1, \ldots, s_N$ and demand that

$$D_N(s_m) = D(s_m). \tag{1.5}$$

A particular case of such an interpolation was considered in [7] (for prehistory see [8], for further development see [1, 9, 12]). There $D(s)$ is the Riemann zeta function, and its initial non-trivial zeros were used in the role of the testing points. Respectively, conditions (1.5) simplified to

$$D_N(s_m) = 0. \tag{1.6}$$

Clearly, this is trivially satisfied by $a_{N,1} = \cdots = a_{N,N} = 0$. In order to avoid such a degeneration one syntactical *condition of normalization* was imposed, namely,

$$a_{N,1} = 1. \tag{1.7}$$

For an odd N, $N = 2K + 1$, K pairs of the initial conjugate zeta zeroes

$$\rho_{\pm 1} = 1/2 \pm \mathrm{i}\gamma_1, \ \ldots, \rho_{\pm K} = 1/2 \pm \mathrm{i}\gamma_K \tag{1.8}$$

were used as testing points, that is, for $n > 1$ the coefficients $a_{N,n}$ were defined by solving the linear system

$$D_N(1/2 \pm \mathrm{i}\gamma_m) = 0, \qquad m = 1, \ldots, K. \tag{1.9}$$

The first numerical discovery was as follows: $D_N(s)$ also vanishes very close to a certain amount of subsequent zeroes $\rho_{\pm(K+1)}, \ldots, \rho_{\pm K'}$ (see [7, Section 2], [1, Table 2]) and very close to several initial trivial zeroes $-2, \ \ldots, -2J$ (see [1, Table 3]).

Numbers (1.8) are also zeros of Euler *alternating zeta function*

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2 \times 2^{-s})\zeta(s). \tag{1.10}$$

The next numerical discovery was as follows: $D_N(s)$ vanishes also near the zeros of $1 - 2 \times 2^{-s}$, that are small in absolute value but different from $s = 1$ (see [1, Table 2]). Moreover, the values of $D_N(s)$ approximate values of $\eta(s)$ very well. This holds for a large range of s, both inside the critical strip and to the left of it, and this is true also for the derivatives of $D_N(s)$ and $\eta(s)$ (see [7, Figure 5–8], [1, Table 5]).

The coefficients of $D_N(s)$ by themselves are very interesting numbers encoding primes in several ways (for details see [7, 1])).

Example 2 The similarity condition exploited in [4, 5, 6] is formulated via *smoothed approximate functional equations* (see [13, Sect. 3.2]). In spite of the name, they are exact equalities of the form

$$h(s)D(s) = \sum_{n=1}^{\infty} \big(f_G(s,n)n^{-s} + f_G(c-s,n)n^{c-s}\big)a_n \tag{1.11}$$

where c and $h(s)$ are respectively the constant and the function from a functional equation

$$h(s)D(s) = h(c-s)D(c-s) \tag{1.12}$$

satisfied by $D(s)$. Function f_G in (1.13) is defined via an auxiliary entire function G which should satisfy some mild conditions on its growth. For any two such functions G_1 and G_2 we have the identity

$$\sum_{n=1}^{\infty} \big(f_{G_1}(s,n)n^{-s} + f_{G,1}(c-s,n)n^{c-s}\big)a_n = \\ \sum_{n=1}^{\infty} \big(f_{G_2}(s,n)n^{-s} + f_{G_2}(c-s,n)n^{c-s}\big)a_n \tag{1.13}$$

and can demand that finite series (1.3) should satisfy similar equalities

$$\sum_{n=1}^{N} \big(f_{G_1}(s,n)n^{-s} + f_{G,1}(c-s,n)n^{c-s}\big)a_{N,n} = \\ \sum_{n=1}^{N} \big(f_{G_2}(s,n)n^{-s} + f_{G_2}(c-s,n)n^{c-s}\big)a_{N,n} \tag{1.14}$$

for a suitable set of triples $\langle s, G_1, G_2 \rangle$. It is demonstrated in [4, 5, 6] that in many cases it is possible to calculate in this way multiprecision values of the function satisfying (1.12), and to determine it.

Example 3 An essentially different way of using a functional equation for defining a mode of similarity was introduced in [10, 11]. It is based on the "genuine" functional equation (1.12) rather than on its smoothed counterpart (1.13). Namely, one demands that finite series (1.3) should satisfy the exact replica of (1.12),

$$h(s)D_N(s) = h(c-s)D_N(c-s), \tag{1.15}$$

but for a certain finite set of values of s only. Surprisingly, it turned out possible to use very large real values of s, that is, such that $c-s$ is less than the abscissa of convergency of series (1.1). This allows one to select for s special values (typically, integer or half-integer) for which the gamma factors from the both sides of (1.15) can be cancelled due to the functional equation for the gamma function; the resulting form of (1.15) is much simpler than (1.14).

In [11] such simplified version of mode of similarity (1.15) was used for getting very good approximations for $\eta(s)$ and $\zeta(s)$ and for the initial derivatives of these functions in broad range of values of the arguments inside and to the left of the critical strip.

In [10] a simplified version of mode (1.15) was used for rediscovering *Ramanujan tau numbers* which are the coefficients of the Dirichlet series for Ramanujan function $L_\tau(s)$.

Example 4 Another mode of similarity considered in [10, 11] can be viewed as as the limiting form of (1.5) when $s_k \to c/2$.

Namely, the *functional* equation (1.12) is equivalent to the infinite system of *numerical* equalities

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}s^k} h(s)D(s)\right|_{s=c/2} = 0 \tag{1.16}$$

which hold for all odd k. Respectively, a mode of similarity can consists of the validity of equalities

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}s^k} h(s)D_N(s)\right|_{s=c/2} = 0. \tag{1.17}$$

for a certain set of odd values of k.

Calculations presented in [11] demonstrate how the well-known Davenport–Heilbronn function $f(s)$ (for definition see [15, Subsection 10.25]) can be found as the solution of corresponding system of equations of the form (1.17).

Calculations presented in [11] demonstrate that Ramanujan tau numbers can also be rediscovered via (1.17).

2 Riemann–Siegel and Hardy functions

The two new modes of similarity considered in this paper are also based on a functional equation; they will be introduced here for the case of the zeta function.

This function has well-known representation

$$\zeta(1/2 + \mathrm{i}t) = \mathrm{e}^{-\mathrm{i}\theta(t)} Z(t) \tag{2.1}$$

for real t where continuous real valued functions $\theta(t)$ and $Z(t)$ are known respectively as the *Riemann–Siegel theta function* and the *Hardy Z-function.* The former function can be defined as

$$\theta(t) = \operatorname{Im}\left(\ln\Gamma\left(\mathrm{i}t/2 + 1/4\right)\right) - \ln(\pi)t/2, \quad \theta(0) = 0, \tag{2.2}$$

where lnΓ is the continuous version of the natural logarithm of the gamma function; respectively,

$$Z(t) = \mathrm{e}^{\mathrm{i}\theta(t)}\zeta(1/2 + \mathrm{i}t). \tag{2.3}$$

The significance of representation (2.1) is due to the fact that $Z(t)$ is real for real t (this is implied by the functional equation for the zeta function). Thus in (2.1) the information about $\zeta(1/2+\mathrm{i}t)$ is split into two parts: $\theta(t)$ is its argument (up to an integer multiple of π), and $Z(t)$ is its absolute value (up to the sign). However, *Riemann–Siegel formula* ([14], see also [3, Chapter 7]) allows one to calculate approximate value of $Z(t)$ via function θ. Thus, in a sense this function alone contains full information about the zeta function.

Below we present numerical data suggesting that plausibly there are several other ways for calculating the zeta function via function θ. Actually we will deal with the eta function (1.10); in its terms the reality of $Z(t)$ is expressed as

$$\operatorname{Im}\left(\frac{\mathrm{e}^{\mathrm{i}\theta(t)}\eta(1/2 + \mathrm{i}t)}{1 - 2 \times 2^{-1/2-\mathrm{i}t}}\right) = 0. \tag{2.4}$$

3 First new mode of similarity

Our first mode of similarity consists of the formal counterparts of equality (2.4):

$$\operatorname{Im}\left(\frac{\mathrm{e}^{\mathrm{i}\theta(t)}D_N(1/2 + \mathrm{i}t)}{1 - 2 \times 2^{-1/2-\mathrm{i}t}}\right) = 0. \tag{3.1}$$

We shall see how well $\eta(s)$ is approximated by $D_N(s)$ for two particular ways of selecting values of t; with some abuse of the language, they will be called testing points as well.

3.1 Classical Gram points

Well-known *Gram points* are defined via the solutions of equation

$$\theta(t) = m\pi. \tag{3.2}$$

For a non-negative integer m Gram point g_m is the unique (see Fig. 1) positive solution of this equation. For $m = -1$ there are two positive solutions, they are usually denoted as $g_{-2} = 3.436218\ldots$ and $g_{-1} = 9.666908\ldots$.

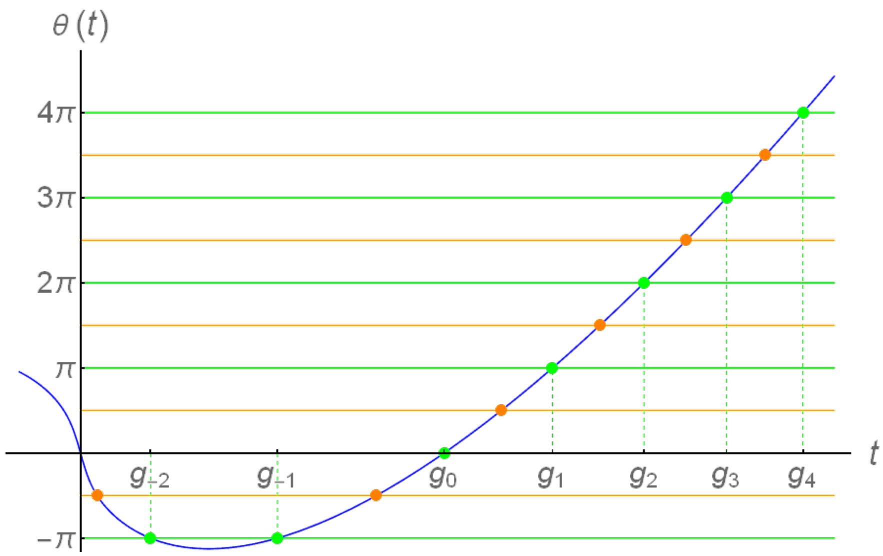

Figure 1: Plot of $\theta(t)$ and Gram points g_m

The reality of $Z(t)$ implies that

$$\operatorname{Im}(\zeta(1/2 + \mathrm{i}g_m)) = 0 \tag{3.3}$$

for any Gram point g_m.

The reason to use Gram points for testing is as follows: in this case condition (3.1) simplifies to

$$\operatorname{Im}\left(\frac{D_N(1/2 + \mathrm{i}g_m)}{1 - 2 \times 2^{-1/2-\mathrm{i}g_m}}\right) = 0. \tag{3.4}$$

In other words, function θ comes into this mode of similarity via Gram points only. However, the plain usage of Gram points does not work: clearly, $D_N(s) = 1 - 2 \times 2^{-s}$ trivially satisfies (3.4). Thus if we want to avoid such a degeneration, we need to add some extra condition.

One possible way to do it is to use, besides (1.7), yet another syntactical condition, namely,

$$a_{N,2} = -1; \tag{3.5}$$

the rest of the coefficients is still defined via semantical conditions (3.4) for $m = -2, \ldots, N - 5$.

We can write down an explicit expressions for $D_N(s)$. Consider $(N-1) \times (N-1)$ matrix

$$M_N(s) = \Big(\mu_{m,n}(s)\Big)\Big|_{m=1}^{N-1}\Big|_{n=1}^{N-1} \tag{3.6}$$

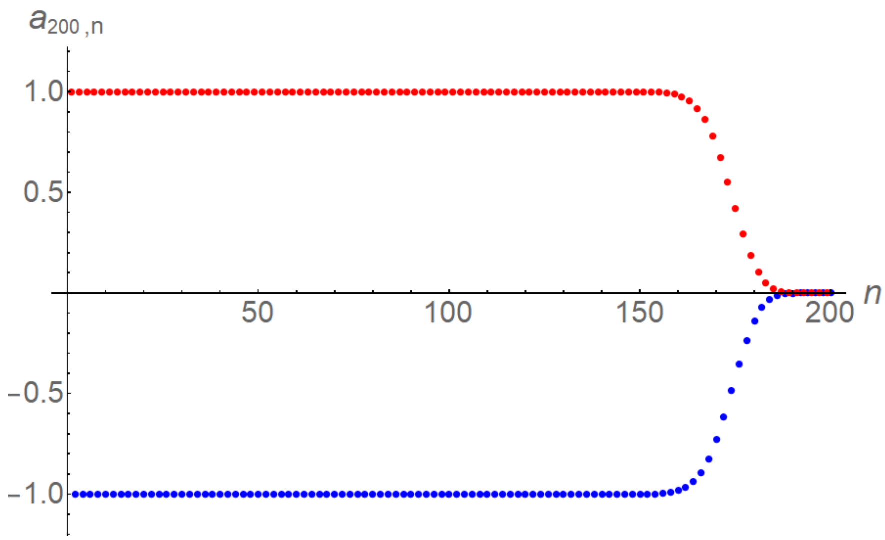

Figure 2: Coefficients $a_{200,n}$ defined by (1.7), (3.4), and (3.5); points are red for odd values of n, and blue otherwise

where

$$\mu_{m,n}(s) = \begin{cases} 1-2^{-s}, & \text{if } m=1,\ n=1; \\ (n+1)^{-s}, & \text{if } m=1,\ n>1; \\ \operatorname{Im}\left(\frac{1-2^{-1/2-\mathrm{i}g_{m-4}}}{1-2\times 2^{-1/2-\mathrm{i}g_{m-4}}}\right), & \text{if } m>1,\ n=1; \\ \operatorname{Im}\left(\frac{(n+1)^{-1/2-\mathrm{i}g_{m-4}}}{1-2\times 2^{-1/2-\mathrm{i}g_{m-4}}}\right), & \text{otherwise.} \end{cases} \tag{3.7}$$

Let L_N be the $(N-2)\times(N-2)$ matrix resulting from $M_N(s)$ by deletion of the first row and the first column. In this notation

$$D_N(s) = \frac{\det(M_N(s))}{\det(L_N)}. \tag{3.8}$$

If $N' < N$ then $M_{N'}$ is a submatrix of M_N, and so are $L_{N'}$ and L_N as well. Thanks to this, Gauss elimination can be organized in such a way that the coefficients of *all* $D_1(s), \ldots, D_N(s)$ are computed after $O(N^3)$ arithmetical operations (see [2]).

Figure 2 presents coefficients of $D_{200}(s)$ for the testing points $t_m = g_{m-3}$ in (3.4) for $m = 1, \ldots, 198$. We see that the coefficients looks as if they were defined via a smooth truncation (1.4) of the coefficients from the alternating series from (1.10). What is remarkable is the following: we haven't "invented" the weights $w_{N,n}$, they emerged naturally from the solution of the system consisting of equations (1.7), (3.4), and (3.5).

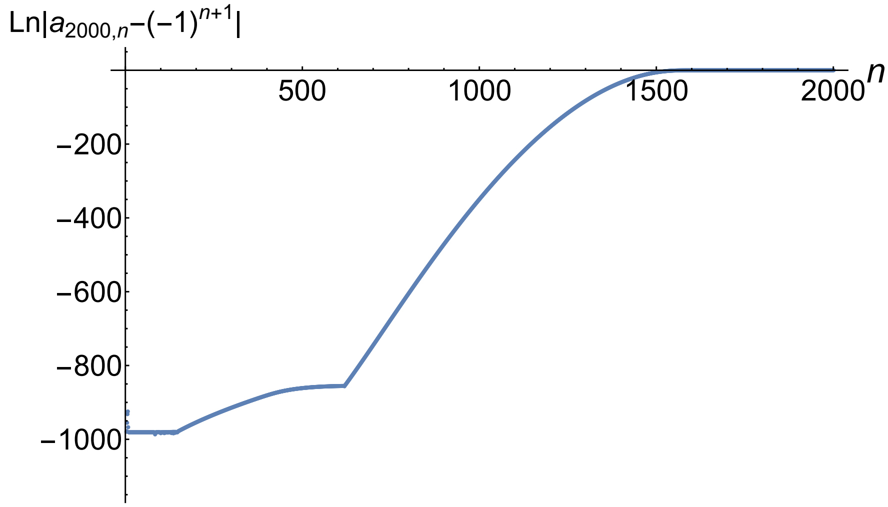

Figure 3: Absolute values of the differences between $a_{2000,n}$ (defined by (1.7), (3.4), and (3.5)) and $(-1)^{n+1}$ in logarithmic scale

Figure 2 and other calculations performed for greater value of N suggest the following surmise.

Conjecture A. *For every fixed n and $N \to \infty$*

$$a_{N,n} \to (-1)^{n+1} \tag{3.9}$$

where $a_{N,n}$ is defined by (1.7)*,* (3.4)*, and* (3.5)*.*

In order to scrutinize the way in which $a_{N,n}$ approaches $(-1)^{n+1}$ we can consider the difference of these two quantities and switch to logarithmic scale — see Figure 3. The picture looks rather similar to [7, Fig. 30] and [9, Fig. 5.5]. However, there is an essential difference: the two latter picture were produced using "complicated" non-trivial zeros of the zeta function while for Figure 3 we need only "simple" Gram points.

Table 1 shows the accuracy with which $D_{500}(s)$ approximates $\eta(s)$ for diverse arguments. This data and calculations performed for different values of s and greater values of N suggest the following surmise.

Conjecture B. *For every s such that $1 - 2 \times 2^{-s} \neq 0$,*

$$\zeta(s) = \lim_{N\to\infty} \frac{\det(M_N(s))}{(1 - 2 \times 2^{-s})\det(L_N)}. \tag{3.10}$$

s	$\left\lvert \frac{D_{500}(s)}{\eta(s)} - 1 \right\rvert$	s	$\left\lvert \frac{D_{500}(s)}{\eta(s)} - 1 \right\rvert$
$-10+900\mathrm{i}$	$4.41730\ldots\cdot 10^{-22}$	$600\mathrm{i}$	$1.28877\ldots\cdot 10^{-59}$
$-8+500\mathrm{i}$	$2.78142\ldots\cdot 10^{-56}$	$700\mathrm{i}$	$4.41919\ldots\cdot 10^{-58}$
$-6+900\mathrm{i}$	$1.53815\ldots\cdot 10^{-22}$	$800\mathrm{i}$	$1.08592\ldots\cdot 10^{-52}$
$-5+500\mathrm{i}$	$1.78207\ldots\cdot 10^{-57}$	$900\mathrm{i}$	$5.58937\ldots\cdot 10^{-23}$
$-4+900\mathrm{i}$	$9.25972\ldots\cdot 10^{-23}$	0.5	$1.71437\ldots\cdot 10^{-46}$
$-3+500\mathrm{i}$	$2.87433\ldots\cdot 10^{-58}$	$0.5+100\mathrm{i}$	$4.99083\ldots\cdot 10^{-57}$
$-2+900\mathrm{i}$	$5.79830\ldots\cdot 10^{-23}$	$0.5+200\mathrm{i}$	$4.96293\ldots\cdot 10^{-60}$
-1	$1.55777\ldots\cdot 10^{-42}$	$0.5+300\mathrm{i}$	$4.07432\ldots\cdot 10^{-59}$
$-1+500\mathrm{i}$	$4.77712\ldots\cdot 10^{-59}$	$0.5+400\mathrm{i}$	$2.96713\ldots\cdot 10^{-58}$
0	$3.15370\ldots\cdot 10^{-45}$	$0.5+500\mathrm{i}$	$1.75993\ldots\cdot 10^{-59}$
$100\mathrm{i}$	$3.24485\ldots\cdot 10^{-56}$	$0.5+600\mathrm{i}$	$6.35035\ldots\cdot 10^{-60}$
$200\mathrm{i}$	$2.47524\ldots\cdot 10^{-59}$	$1+700\mathrm{i}$	$3.86352\ldots\cdot 10^{-59}$
$300\mathrm{i}$	$2.64273\ldots\cdot 10^{-59}$	$1+800\mathrm{i}$	$9.62379\ldots\cdot 10^{-54}$
$400\mathrm{i}$	$1.99062\ldots\cdot 10^{-59}$	$2+800\mathrm{i}$	$8.44679\ldots\cdot 10^{-56}$
$500\mathrm{i}$	$2.10556\ldots\cdot 10^{-59}$	$3+800\mathrm{i}$	$6.41067\ldots\cdot 10^{-58}$

Table 1: Approximation of $\eta(s)$ by $D_{500}(s)$ defined by (1.7), (3.5) and (3.1) for $t_m = g_m$, $m = -2, \ldots, 495$

Conjectures A and B tell us that full information about the values of the zeta function and the coefficients of its Dirichlet series is contained in the modest numerical equalities (3.3) and values (1.7) and (3.5) of the first two Dirichlet coefficients of the eta function. Thus these two conjectures proffer

Conjecture C. *Riemann's zeta function is the only function $R(s)$ such that*

- *it can be defined as*

$$R(s) = \frac{1 - 2^{-s} + \sum_{n=3}^{\infty} a_n n^{-s}}{1 - 2 \times 2^{-s}} \tag{3.11}$$

  where the Dirichlet series in the numerator converges for $\mathrm{Re}(s) > 0$ *to an entire function ;*

- $R(1/2 + \mathrm{i}g_m)$ *is real for* $m = -2,\ -1,\ \ldots$

3.2 Gram points with half-integer indices

Instead of syntactical condition (3.5) we can use semantical condition (3.1) at additional testing points. In particular, we can consider Gram points with half-integer indices. They are defined by the same equation (3.2); for $m = -1/2$ there are (see Fig. 1) two positive solutions which we denote as $g_{-3/2} = 0.819545\ldots$ and $g_{-1/2} = 14.517919\ldots$ For Gram point g_m with half-integer index m condition (3.1) simplifies to

$$\mathrm{Re}\left(\frac{D_N(1/2 + \mathrm{i}g_m)}{1 - 2 \times 2^{-1/2-\mathrm{i}g_m}}\right) = 0. \tag{3.12}$$

There is no need to use all initial Gram points (with integer and half-integer indices) for testing, moreover, it seems more natural that for calculation of $\zeta(s)$ we should select Gram points in the vicinity $\mathrm{Im}(s)$.

Let us consider the following example. We wish to calculate $\zeta(1/2+239\mathrm{i})$; we put $N = 120$ and use 119 Gram points

$$g_{71} = 186.810\ldots,\ g_{71.5} = 187.735\ldots,\ \ldots,\ g_{130} = 289.038\ldots \tag{3.13}$$

in conditions (3.4) and (3.12). Calculation gives, as we might expect, quite good approximation

$$D_{120}(1/2 + 239\mathrm{i}) = 3.0796627956\ldots - 0.8238469365\ldots\mathrm{i}, \tag{3.14}$$

$$\left|\eta(1/2 + 239\mathrm{i}) - D_{120}(1/2 + 239\mathrm{i})\right| = 1.50430\ldots \times 10^{-7}. \tag{3.15}$$

But in fact this is rather surprising if one looks at the coefficients of $D_{120}(s)$ presented in Table 2. Almost one third of the initial coefficients are very close to the alternating ±1 (similar to Fig. 2), but others have very large absolute values.

4 Second new mode of similarity

If we wish to calculate $\zeta(s)$ for $s = \sigma + \mathrm{i}\tau$, then the ultimate form of selecting the testing points in the vicinity of τ is just to put $t_m = \tau$ for all m. Of course, in such a case we need to modify (3.1). Namely, (2.4) implies that

$$\mathrm{Im}\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{\mathrm{e}^{\mathrm{i}\theta(t)}\eta(1/2 + \mathrm{i}t)}{1 - 2 \times 2^{-1/2-\mathrm{i}t}}\right) = 0, \tag{4.1}$$

and we replace (3.1) by counterpart of (4.1),

$$\mathrm{Im}\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{\mathrm{e}^{\mathrm{i}\theta(t)}D_N(1/2 + \mathrm{i}t)}{1 - 2 \times 2^{-1/2-\mathrm{i}t}}\bigg|_{t=\tau}\right) = 0. \tag{4.2}$$

n	$a_{120,n}$	n	$a_{120,n}$	n	$a_{120,n}$	n	$a_{120,n}$
1	1.0...	31	.99...	61	-.44...$\cdot 10^{12}$	91	.78...$\cdot 10^{18}$
2	-1.0...	32	-.99...	62	-.17...$\cdot 10^{13}$	92	.68...$\cdot 10^{18}$
3	.99...	33	.99...	63	-.64...$\cdot 10^{13}$	93	.54...$\cdot 10^{18}$
4	-.99...	34	-.99...	64	-.22...$\cdot 10^{14}$	94	.40...$\cdot 10^{18}$
5	.99...	35	.98...	65	-.75...$\cdot 10^{14}$	95	.27...$\cdot 10^{18}$
6	-.99...	36	-.95...	66	-.23...$\cdot 10^{15}$	96	.17...$\cdot 10^{18}$
7	.99...	37	.90...	67	-.68...$\cdot 10^{15}$	97	.10...$\cdot 10^{18}$
8	-.99...	38	-.89...	68	-.19...$\cdot 10^{16}$	98	.58...$\cdot 10^{17}$
9	.99...	39	.11...$\cdot 10^{1}$	69	-.49...$\cdot 10^{16}$	99	.30...$\cdot 10^{17}$
10	-.99...	40	-.17...$\cdot 10^{1}$	70	-.11...$\cdot 10^{17}$	100	.15...$\cdot 10^{17}$
11	.99...	41	.30...$\cdot 10^{1}$	71	-.27...$\cdot 10^{17}$	101	.70...$\cdot 10^{16}$
12	-.99...	42	-.47...$\cdot 10^{1}$	72	-.57...$\cdot 10^{17}$	102	.30...$\cdot 10^{16}$
13	.99...	43	.63...$\cdot 10^{1}$	73	-.11...$\cdot 10^{18}$	103	.12...$\cdot 10^{16}$
14	-.99...	44	-.79...$\cdot 10^{1}$	74	-.21...$\cdot 10^{18}$	104	.48...$\cdot 10^{15}$
15	.99...	45	.77...$\cdot 10^{1}$	75	-.36...$\cdot 10^{18}$	105	.17...$\cdot 10^{15}$
16	-.99...	46	-.13...$\cdot 10^{2}$	76	-.59...$\cdot 10^{18}$	106	.58...$\cdot 10^{14}$
17	.99...	47	-.20...$\cdot 10^{2}$	77	-.89...$\cdot 10^{18}$	107	.18...$\cdot 10^{14}$
18	-.99...	48	-.18...$\cdot 10^{3}$	78	-.12...$\cdot 10^{19}$	108	.53...$\cdot 10^{13}$
19	.99...	49	-.11...$\cdot 10^{4}$	79	-.16...$\cdot 10^{19}$	109	.14...$\cdot 10^{13}$
20	-.99...	50	-.68...$\cdot 10^{4}$	80	-.19...$\cdot 10^{19}$	110	.35...$\cdot 10^{12}$
21	.99...	51	-.41...$\cdot 10^{5}$	81	-.22...$\cdot 10^{19}$	111	.78...$\cdot 10^{11}$
22	-.99...	52	-.24...$\cdot 10^{6}$	82	-.22...$\cdot 10^{19}$	112	.15...$\cdot 10^{11}$
23	.99...	53	-.14...$\cdot 10^{7}$	83	-.20...$\cdot 10^{19}$	113	.28...$\cdot 10^{10}$
24	-.99...	54	-.78...$\cdot 10^{7}$	84	-.17...$\cdot 10^{19}$	114	.44...$\cdot 10^{9}$
25	.99...	55	-.42...$\cdot 10^{8}$	85	-.11...$\cdot 10^{19}$	115	.58...$\cdot 10^{8}$
26	-.99...	56	-.22...$\cdot 10^{9}$	86	-.61...$\cdot 10^{18}$	116	.62...$\cdot 10^{7}$
27	.99...	57	-.11...$\cdot 10^{10}$	87	-.62...$\cdot 10^{17}$	117	.49...$\cdot 10^{6}$
28	-.99...	58	-.53...$\cdot 10^{10}$	88	.37...$\cdot 10^{18}$	118	.24...$\cdot 10^{5}$
29	.99...	59	-.24...$\cdot 10^{11}$	89	.66...$\cdot 10^{18}$	119	.25...$\cdot 10^{3}$
30	-.99...	60	-.10...$\cdot 10^{12}$	90	.79...$\cdot 10^{18}$	120	-.36...$\cdot 10^{2}$

Table 2: Coefficients of $D_{120}(s)$ defined by (3.4), (3.12), and (3.13)

s	$\left\lvert \frac{D_{50}(s)}{\eta(s)} - 1 \right\rvert$
-5	$6.70323\ldots \cdot 10^{-21}$
$-5+14\mathrm{i}$	$2.38782\ldots \cdot 10^{-26}$
$-5+28\mathrm{i}$	$2.69647\ldots \cdot 10^{-17}$
$-4+14\mathrm{i}$	$5.96096\ldots \cdot 10^{-26}$
$-4+28\mathrm{i}$	$1.10006\ldots \cdot 10^{-17}$
-3	$1.19722\ldots \cdot 10^{-22}$
$-3+14\mathrm{i}$	$1.39945\ldots \cdot 10^{-25}$
$-3+28\mathrm{i}$	$5.82625\ldots \cdot 10^{-18}$
$-2+14\mathrm{i}$	$3.14114\ldots \cdot 10^{-25}$
$-2+28\mathrm{i}$	$4.15247\ldots \cdot 10^{-18}$
-1	$4.01685\ldots \cdot 10^{-24}$
$-1+14\mathrm{i}$	$7.16153\ldots \cdot 10^{-25}$
$-1+28\mathrm{i}$	$3.85242\ldots \cdot 10^{-18}$
0	$8.10411\ldots \cdot 10^{-25}$
$7\mathrm{i}$	$1.58864\ldots \cdot 10^{-25}$

s	$\left\lvert \frac{D_{50}(s)}{\eta(s)} - 1 \right\rvert$
$14\mathrm{i}$	$2.35600\ldots \cdot 10^{-24}$
$21\mathrm{i}$	$2.86786\ldots \cdot 10^{-22}$
0.5	$4.46752\ldots \cdot 10^{-25}$
$0.5+7\mathrm{i}$	$1.54883\ldots \cdot 10^{-25}$
$0.5+14\mathrm{i}$	$8.85659\ldots \cdot 10^{-24}$
$0.5+21\mathrm{i}$	$6.92554\ldots \cdot 10^{-21}$
$0.5+28\mathrm{i}$	$3.75762\ldots \cdot 10^{-18}$
$1+7\mathrm{i}$	$1.40013\ldots \cdot 10^{-25}$
$1+14\mathrm{i}$	$2.12103\ldots \cdot 10^{-24}$
$1+21\mathrm{i}$	$2.94894\ldots \cdot 10^{-22}$
$1+28\mathrm{i}$	$2.82332\ldots \cdot 10^{-18}$
2	$1.06736\ldots \cdot 10^{-25}$
$2+28\mathrm{i}$	$5.50696\ldots \cdot 10^{-19}$
3	$4.76131\ldots \cdot 10^{-26}$
$3+28\mathrm{i}$	$7.91534\ldots \cdot 10^{-20}$

Table 3: Approximation of $\eta(s)$ by $D_{50}(s)$ defined by (1.7) and (4.2) for $\tau = 14$ and $k = 4, \ldots, 52$

In contrast to Taylor series, in order to calculate $\zeta(s)$ we need not use (4.2) with consecutive initial values of k. Table 3 shows the accuracy with which $\eta(s)$ at diverse arguments is approximated by $D_{50}(s)$ defined by (1.7) and (4.2) for $\tau = 14$ and $k = 4, \ \ldots, \ 52$.

5 Directions for further investigations

Selection of Gram points with integer and half-integer indices for the role of testing points was motivated by the simplicity of resulting equations (3.4) and (3.12). The distances between consecutive Gram points are of the same order as the average distance between the nearby zeta zeros, that is, they decrease as $2\pi/\ln(n)$. This seems to be important for getting well-behaved coefficients in finite Dirichlet series. The usage of equidistant testing points in (3.1) (such as $t_n = n$, for example) produces finite Dirichlet series giving good approximations but having more complicated and less understood structure of their coefficients.

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