

ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

РЕДКОЛЛЕГИЯ

**В.М.Бабич, Н.А.Вавилов, А.М.Вершик, М.А.Всемирнов, А.И.Генералов, И.А.Ибрагимов,
Л.Ю.Колотилина, Б.Б.Лурье, Ю.В.Матиясевич, Н.Ю.Нецветаев, С.И.Репин, Г.А.Серегин**

**Учредитель: Федеральное государственное бюджетное учреждение науки
Санкт-Петербургское отделение Математического института
им. В. А. Стеклова Российской академии наук**

**Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16
октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых
коммуникаций**

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

телефоны: (812)312-40-58; (812) 571-57-54

e-mail: admin@pdmi.ras.ru

<http://www.pdmi.ras.ru/preprint/>

Заведующая информационно-издательским сектором Симонова В.Н

PDMI Preprint 06/2018

Pressure drop matrix for a bifurcation of an artery with defects

Vladimir Kozlov^a, Sergei Nazarov^{a,b} and German Zavorokhin^c

June 19, 2018

^a*Department of Mathematics, Linköping University,
S-581 83 Linköping, Sweden E-mail: vlkoz@mai.liu.se*

^b*St.Petersburg State University, Universitetsky pr., 28, Peterhof, St.
Petersburg, 198504, Russia*

*Institute of Problems of Mechanical Engineering RAS
laboratory "Mathematical Methods in Mechanics of Materials"
V.O., Bolshoj pr., 61, St. Petersburg, 199178, Russia
E-mail: srgnazarov@yahoo.co.uk*

^c*St.Petersburg Department of the Steklov Mathematical Institute, Fontanka,
27, 191023, St.Petersburg, Russia E-mail: zavorokhin@pdm.ras.ru*

Abstract. We consider a bifurcation of an artery. The influence of defects of the vessel's wall near the bifurcation point on the pressure drop matrix is analyzed. The elements of this matrix are included in the modified Kirchhoff transmission conditions, which were introduced earlier in [1], [2], and which describe adequately the total pressure loss at the bifurcation point of the flow passed through it.

Keywords and phrases: Stokes' flow, bifurcation of a blood vessel, modified Kirchhoff conditions, pressure drop matrix, total pressure loss.

ПРЕПРИНТЫ
Санкт-Петербургского отделения
Математического института В. А. Стеклова
Российской академии наук

PREPRINTS
of St.Petersburg Department
of Steklov Mathematical Institute

ГЛАВНЫЙ РЕДАКТОР
С. В. Кисляков

РЕДКОЛЛЕГИЯ

В. М. Бабич, Н. А. Вавилов, А. М. Вершик,
М. А. Всемиров, А. И. Генералов, И. А. Ибрагимов,
Л. Ю. Колотилина, Б. Б. Лурье, Ю. В. Матиясевич,
Н. Ю. Нецветаев, С. И. Репин, Г. А. Серегин

1 Introduction

The main objective of this paper is to study the influence of defects in the vessel walls near the bifurcation point on *the pressure drop matrix* Q [3]. We calculate the material derivative in the case of oblong plaques or aneurysms (see Fig.1, a and b) and the topological derivative in the case of localized ones (see Fig.1, c and d). The pressure drop matrix was introduced in [3]

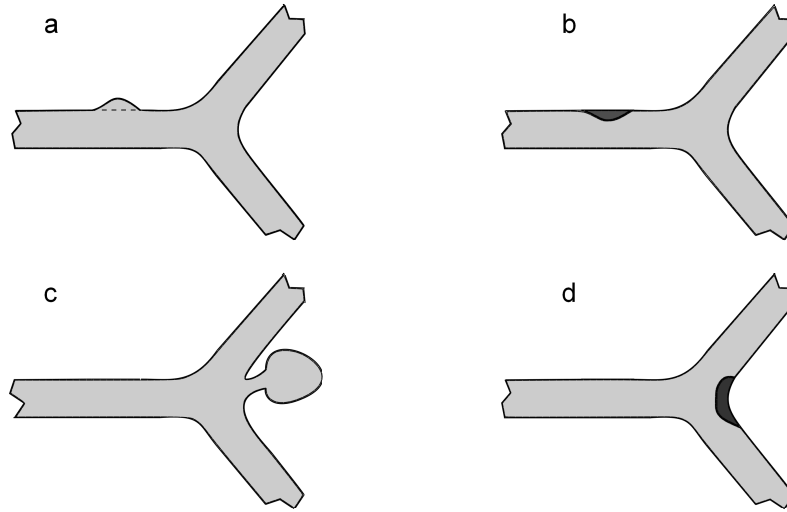


Figure 1: Variations in the shape of a bifurcation node: oblong (a) and sacular (c) aneurysms, oblong parietal (b) and localized nodular (d) cholesterol plaques.

as an integral characteristic of a junction of several pipes with absolutely rigid walls. It appears that the elements of this matrix are included in the modified Kirchhoff transmission conditions, which describe more adequately the total pressure loss at the bifurcation point of the flow passed through the corresponding junction of the pipes, see [1, 2, 4].

In the paper [2] a one-dimensional model of a fluid flow at a junction of thin vessels with rigid walls was developed. In particular, a new transmission condition at the bifurcation point was derived, which can be considered as a modification of the classical Kirchhoff condition. Clearly, the total flux at the bifurcation point is zero but continuity of the pressure is not so obvious. In fluid mechanics, one uses the total pressure loss in the flow passing the bifurcation point, see [4]. An appropriate object to describe this pressure

loss is the pressure drop matrix, elements of which are involved in the modified Kirchhoff conditions. This modification improves the model in several directions. First, the discrepancy of the approximation of three-dimensional model by the one-dimensional one is $O(e^{-\frac{\rho}{h}})$, where h is the thickness of the vessel and ρ is a positive constant. We remind that the application of the classical Kirchhoff conditions brings the discrepancy $O(h^3)$ for the velocities and $O(h)$ for the pressure. This difference is essential if we deal with a large system with many bifurcations. Second, the modified transmission conditions depend on the geometry of the bifurcation region.

The pressure drop matrix Q is the symmetric (2×2) matrix. So it has three parameters (the diagonal elements Q_{++} , Q_{--} and the off-diagonal ones $Q_{+-} = Q_{-+}$). The influence of Q on the transmission conditions can be taken into account also by the small variations in the lengths of the edges incident to the bifurcation point and by introducing *effective lengths* $L_\alpha(h)$, $\alpha = 0, \pm$, of one-dimensional images of blood vessels whilst keeping the classic Kirchhoff transmission conditions and exponential small approximation errors, see [2]. Since the number of channels is also three the effective lengths $L_\alpha(h)$ can be isomorphically determined by the entries of Q . By [2], the increments of lengths hl_α , $\alpha = 0, \pm$,

$$l_0 = -B_0 Q_{+-} = -B_0 Q_{-+}, \quad l_\pm = B_\pm (Q_{\pm\mp} - Q_{\pm\pm}), \quad B_\alpha = \frac{\pi r_\alpha^4}{8\nu}, \quad (1)$$

where ν is the viscosity of the fluid and r_α is the radius of the vessel, we introduce perturbed edges with the effective lengths

$$L_\alpha(h) = L_\alpha + hl_\alpha, \quad (2)$$

where L_α are initial lengths of the edges. The effective lengths (2) are the attributes of the vessels themselves and preserve when you change the direction of blood flow through the node.

Our aim with this article is to calculate asymptotics of the pressure drop matrix and hence the total increments $h \sum_\alpha l_\alpha$, namely,

$$\begin{aligned} h \sum_\alpha l_\alpha &= \sum_\alpha L_\alpha(h) - \sum_\alpha L_\alpha \\ &= h(Q_{+-}(B_+ + B_- - B_0) - Q_{++}B_+ - Q_{--}B_-), \end{aligned} \quad (3)$$

of the effective lengths of the vessels taking into account the influence of perturbations (e.g., plaques, aneurysms) arising near the bifurcation node

of the artery in the three-dimensional problem. As a result, we calculate the total increments of the effective lengths, and even determine their signs. Changes in the effective lengths of the vessels correspond to the presence of some defects in the vessel walls. So we can localize them by examining the process of blood flow through a bifurcation node.

In Sect. 2 we consider the Stokes system in an unbounded domain with cylindrical outlets to infinity (see, e.g., [5, 6, 7, 8, 9]) and prove the unique solvability of the problem. For obtaining the asymptotic behavior of the solution we exploit special homogeneous solutions to the Stokes problem with non-zero flux and with a linear growth in the pressure at infinity (see [3]). As a consequence, we obtain a definition of the symmetric pressure drop matrix Q , which plays a crucial role in the functioning of the bifurcation node.

Sect. 3 is the main part of this work. We analyze the influence of certain formations in the bifurcation node and close to it on the matrix Q . Using asymptotic analysis of elliptic boundary value problems in regularly (or singularly) [10, 11] perturbed domains we find the increments of the pressure drop matrix and also determine their signs. In virtue of formulae (1) we calculate the total increments of the effective lengths of one-dimensional images of the blood vessels.

In Appendix it will be explained why the modification of the second Kirchhoff's law by means of the pressure drop matrix unexpectedly deeply increases the accuracy of approach for three-dimensional fluid flow in a system of thin channels by the one-dimensional Reynolds-Poiseuille model. Also, we give proofs of supporting assertions of Sect. 2, 3. Note that considered in 4.2 the Cauchy problem for the homogeneous Stokes system supplemented by the Neumann condition on the part of the boundary it is also of independent interest.

2 Statement of the problem

2.1 Domains with cylindrical outlets to infinity and functional spaces

We introduce the domain Ω with three cylindrical outlets to infinity (see Fig.2). Let Ω be an open unbounded domain with Lipschitz boundary $\partial\Omega$ admitting the representation

$$\Omega = \Omega' \cup \Omega^0 \cup \Omega^+ \cup \Omega^-, \quad \text{where } \Omega^\alpha \cap \Omega^\beta = \emptyset \text{ for } \alpha \neq \beta, \quad \alpha, \beta = 0, \pm. \quad (4)$$

Here $\Omega^\alpha = \{x^\alpha = (y^\alpha, z^\alpha) : y^\alpha \in \omega_\alpha, z^\alpha > L_\alpha\}$ in a certain Cartesian coordinate system $x^\alpha = (y^\alpha, z^\alpha)$ in \mathbb{R}^3 , where y^α are the variables in the cross-section of the outlet Ω^α , z^α is the variable along the axis of Ω^α and ω_α is a bounded domain in \mathbb{R}^2 . The bounded domain Ω' is given by $\Omega' = \{x \in \Omega : z^\alpha < L\}$ for certain L , $L > \max_\alpha L_\alpha$. Henceforth $x = (x_1, x_2, x_3)$ is a global coordinate system in \mathbb{R}^3 related to the whole domain Ω . We define

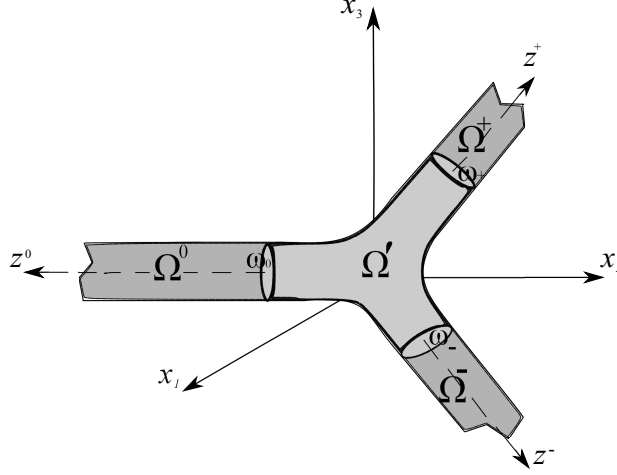


Figure 2: Artery bifurcation (domain Ω)

$L_{2,\beta}(\Omega)$ as the space of measurable functions in Ω with a finite norm

$$\|u\|_{L_{2,\beta}(\Omega)} = \left(\int_{\Omega'} |u(x)|^2 dx + \sum_{\alpha=0,\pm} \int_{\Omega^\alpha} |z^\alpha|^{2\beta} |u(y^\alpha, z^\alpha)|^2 dy^\alpha dz^\alpha \right)^{1/2}.$$

If $\beta = 0$ we will use the usual notation $L_2(\Omega)$ for this space.

By using the Sobolev space $H^1(\Omega)$ together with $L_{2,1}(\Omega)$ we introduce the space of real-valued vector functions in Ω ,

$$\mathcal{H}(\Omega) = \{u = (u_1, u_2, u_3) \in (H^1(\Omega))^3 \mid \operatorname{div} u \in L_{2,1}(\Omega)\} \quad (5)$$

with the norm given by

$$\|u\|_{\mathcal{H}(\Omega)}^2 = \int_{\Omega} (|\nabla u(x)|^2 + |u(x)|^2) dx + \sum_{\alpha=0,\pm} \int_{\Omega^\alpha} |z^\alpha|^2 |\operatorname{div} u(y^\alpha, z^\alpha)|^2 dy^\alpha dz^\alpha. \quad (6)$$

Let also $\mathcal{H}_0(\Omega)$ be the subspace in $\mathcal{H}(\Omega)$ consisting of vector functions equal zero on $\partial\Omega$. The dual space of $\mathcal{H}_0(\Omega)$ is denoted by $(\mathcal{H}_0(\Omega))^*$.

2.2 Formulation of the problem

Consider the Dirichlet problem for the stationary Stokes system with nonzero divergence

$$-\nu\Delta u(x) + \nabla p(x) = F(x), \quad -\operatorname{div} u(x) = G(x), \quad x \in \Omega, \quad (7)$$

$$u(x) = 0, \quad x \in \partial\Omega. \quad (8)$$

Here, $u(x) = (u_1(x), u_2(x), u_3(x))$ is the velocity field and $p(x)$ is the pressure, $\nu > 0$ is the viscosity of fluid, which is assumed to be constant.

In order to define a weak solution of the problem (7), (8), we introduce a bilinear form on $\mathcal{H}(\Omega)$:

$$a(u, w) = \sum_{j=1}^3 \int_{\Omega} \nabla u_j \nabla w_j dx.$$

So if (u, p) is a classical solution of (7), (8), then multiplying the first equation in (7) by $w \in \mathcal{H}_0(\Omega)$ and integrating over Ω , we obtain

$$\nu a(u, w) - \int_{\Omega} p \operatorname{div} w \, dx = \int_{\Omega} F w \, dx \quad \text{for any } w \in \mathcal{H}_0(\Omega). \quad (9)$$

Weak solution of the problem (7), (8) is called a pair $(u, p) \in \mathcal{H}_0(\Omega) \times L_{2,-1}(\Omega)$ satisfying the integral identity (9) for all $w \in \mathcal{H}_0(\Omega)$ and the equation $-\operatorname{div} u = G$ in Ω , where $F \in (\mathcal{H}_0(\Omega))^*$ and $G \in L_{2,1}(\Omega)$ are given.

To prove the main result of this section we need the following

Lemma 2.1. *For arbitrary $g \in L_{2,1}(\Omega)$ subject to*

$$\int_{\Omega} g(x) dx = 0 \quad (10)$$

there exists a vector function $u \in \mathcal{H}_0(\Omega)$ such that $-\operatorname{div} u = g$ in Ω , and

$$\|u\|_{\mathcal{H}(\Omega)} \leq c \|g\|_{L_{2,1}(\Omega)}. \quad (11)$$

Here, c is a constant independent of g .

Lemma's proof is presented in Appendix.

The following theorem on existence and uniqueness of weak solutions to the boundary value problem (7)-(8) is quite standard and we present it here for readers convenience.

Theorem 2.1. Suppose that $F \in (\mathcal{H}_0(\Omega))^*$ and $G \in L_{2,1}(\Omega)$ is such that

$$\int_{\Omega} G(x) dx = 0. \quad (12)$$

Then there exists a weak solution $(u, p) \in \mathcal{H}_0(\Omega) \times L_{2,-1}(\Omega)$ of the problem (7), (8) satisfying the estimate

$$\|u\|_{\mathcal{H}(\Omega)} + \|p\|_{L_{2,-1}(\Omega)} \leq c (\|F\|_{(\mathcal{H}_0(\Omega))^*} + \|G\|_{L_{2,1}(\Omega)}). \quad (13)$$

Here, c is a constant independent of F and G . This solution is defined up to an additive constant in the pressure p .

Proof. Existence. Let $w \in \mathcal{H}_0(\Omega)$ be a solution to the problem

$$-\operatorname{div} w(x) = G(x), \quad x \in \Omega, \quad w(x) = 0, \quad x \in \partial\Omega \quad (14)$$

satisfying estimate (11). Such solution exists due to Lemma 2.1. Then the vector function $V(x) = u(x) - w(x)$ solves the following Stokes problem

$$-\nu \Delta V(x) + \nabla p(x) = \hat{F}, \quad -\operatorname{div} V(x) = 0, \quad x \in \Omega, \quad (15)$$

$$V(x) = 0, \quad x \in \partial\Omega, \quad (16)$$

where $\hat{F}(x) = F(x) + \nu \Delta w(x) \in (\mathcal{H}_0(\Omega))^*$. Introduce the space $\mathcal{H}_0^{\operatorname{div}}(\Omega) = \{W \in H_0^1(\Omega) : \operatorname{div} W = 0 \text{ in } \Omega\}$. Then the vector function $V \in \mathcal{H}_0^{\operatorname{div}}(\Omega)$ is found from the equality

$$\nu a(V, W) = \int_{\Omega} \hat{F} W \, dx \quad \text{for any } W \in \mathcal{H}_0^{\operatorname{div}}(\Omega). \quad (17)$$

By the Riesz theorem such solution exists and satisfies

$$\|V\|_{\mathcal{H}(\Omega)} \leq c \|\hat{F}\|_{(\mathcal{H}_0(\Omega))^*} \leq C (\|F\|_{(\mathcal{H}_0(\Omega))^*} + \|G\|_{L_{2,1}(\Omega)}).$$

To find p we proceed as follows. By Lemma (2.1), for any $g \in L_{2,1}(\Omega)$ subject to (10) there exists a vector function $v_g \in \mathcal{H}_0(\Omega)$ such that $-\operatorname{div} v_g = g$ in Ω , and

$$\|v_g\|_{\mathcal{H}(\Omega)} \leq c \|g\|_{L_{2,1}(\Omega)}.$$

Moreover the correspondence $g \rightarrow v_g$ is linear. We consider the functional

$$G(g) = \int_{\Omega} \hat{F} v_g \, dx - \nu a(V, v_g) \quad (18)$$

on $\mathcal{L}_{2,1}(\Omega) = \{g \in L_{2,1}(\Omega) : \int_{\Omega} g(x) dx = 0\}$. In virtue of

$$|G(g)| \leq c \left(\|\hat{F}\|_{(\mathcal{H}_0(\Omega))^*} + \|V\|_{\mathcal{H}(\Omega)} \right) \|v_g\|_{\mathcal{H}(\Omega)} \leq c \|\hat{F}\|_{(\mathcal{H}_0(\Omega))^*} \|g\|_{L_{2,1}(\Omega)}$$

the linear functional $G(g)$ is continuous on $\mathcal{L}_{2,1}(\Omega)$. Therefore there exist an element p in $L_{2,-1}(\Omega)$ such that

$$G(g) = \int_{\Omega} p g \, dx \quad \text{for all } g \in \mathcal{L}_{2,1}(\Omega)$$

and

$$\|p\|_{L_{2,-1}(\Omega)} \leq c \left(\|F\|_{(\mathcal{H}_0(\Omega))^*} + \|G\|_{L_{2,1}(\Omega)} \right).$$

Clearly, the pair (u, p) is the required weak solution.

Uniqueness. If $F = 0$ and $G = 0$ then from the definition of the weak solution it follows that $a(u, u) = 0$ and hence $u = 0$. This implies that $\int_{\Omega} p \operatorname{div} w \, dx = 0$ for all $w \in \mathcal{H}_0(\Omega)$. Using Lemma 2.1, we conclude that p is constant.

The theorem is proved.

Remark 2.1. Consider a non-homogeneous Dirichlet problem for Stokes system, i.e. equations (7) are supplied with the boundary condition

$$u(x) = H, \quad x \in \partial\Omega, \quad (19)$$

where $H \in \mathcal{H}(\Omega)$ and instead (10) we require

$$\int_{\Omega} G(x) dx + \int_{\partial\Omega} H(x) \cdot n d\Gamma = 0, \quad (20)$$

where n is the unit, outward normal to $\partial\Omega$. Substituting $u(x) = v(x) + H(x)$ into (7), (19) we obtain

$$-\nu \Delta v(x) + \nabla p(x) = f(x), \quad -\operatorname{div} v(x) = g(x), \quad x \in \Omega, \quad (21)$$

$$v(x) = 0, \quad x \in \partial\Omega, \quad (22)$$

where $f(x) = F(x) + \nu \Delta H(x) \in (\mathcal{H}_0(\Omega))^*$ and $g(x) = G(x) + \operatorname{div} H(x) \in L_{2,1}(\Omega)$ verifies (10). Now application of the previous theorem gives the existence of a pair $(v, p) \in \mathcal{H}_0(\Omega) \times L_{2,-1}(\Omega)$ solving problem (7), (19) and satisfying the estimate

$$\|v\|_{\mathcal{H}(\Omega)} + \|p\|_{L_{2,-1}(\Omega)} \leq c \left(\|f\|_{(\mathcal{H}_0(\Omega))^*} + \|g\|_{L_{2,1}(\Omega)} + \|H\|_{\mathcal{H}(\Omega)} \right). \quad (23)$$

Moreover, p is defined up to an additive constant.

2.3 Asymptotics of the variational solution

Let the right-hand sides in (7), (8) satisfy

$$\int_{\Omega'} |F(x)|^2 dx + \sum_{\alpha} \int_{\Omega^{\alpha}} |F(x^{\alpha})|^2 e^{2az^{\alpha}} dx^{\alpha} < \infty \quad (24)$$

and

$$\int_{\Omega'} |G(x)|^2 dx + \sum_{\alpha} \int_{\Omega^{\alpha}} |G(x^{\alpha})|^2 e^{2az^{\alpha}} dx^{\alpha} < \infty, \quad (25)$$

where a is a positive number. Let also G be subject to (12). Then according to Theorem 2.1 the problem (7), (8) has a solution $(u, p) \in \mathcal{H}_0(\Omega) \times L_{2,-1}(\Omega)$. We can conclude that this solution satisfies the following asymptotic representation at infinity

$$(u, p) = \sum_{\alpha=0,\pm} \chi_{\alpha} c_{\alpha}(0, 1) + (\tilde{v}, \tilde{p}), \quad (26)$$

where $\chi_{\alpha} = \chi_{\alpha}(z^{\alpha})$ are smooth functions equal 1 for $z^{\alpha} > L_{\alpha} + 1$ and 0 for $z^{\alpha} < L_{\alpha}$, (\tilde{v}, \tilde{p}) are exponentially decaying terms at $z^{\alpha} \rightarrow \infty$ and c_{α} are real constants. Since this solution is defined up to an additive constant we can (and will) assume $c_0 = 0$. Then the solution is unique.

The remaining part of this section is devoted to finding formulas for evaluation of constants c_+ and c_- . For this purpose we need solutions of homogeneous problem (7), (8), which have a linear growth at infinity, namely we introduce two linear independent solutions (V^{\pm}, P^{\pm}) which have the following asymptotic representations (see [3])

$$(V^{\pm}, P^{\pm}) = -\chi_0(\mathcal{V}^0, \mathcal{P}^0) + \chi_{\pm}(\mathcal{V}^{\pm}, \mathcal{P}^{\pm}) + (v^{\pm}, p^{\pm}), \quad (27)$$

where $(\mathcal{V}^{\alpha}, \mathcal{P}^{\alpha})$ is the Poiseuille flow in the cylinder Ω^{α} , i.e. $\mathcal{V}_{y_i}^{\alpha}(x) = 0$, $i = 1, 2$, $\mathcal{P}^{\alpha}(x) = -C_{\alpha} z^{\alpha}$ and $\mathcal{V}_{z^{\alpha}}^{\alpha} = \mathcal{V}_{z^{\alpha}}^{\alpha}(y^{\alpha})$ solves the following Dirichlet problem in ω_{α}

$$\Delta \mathcal{V}_{z^{\alpha}}^{\alpha} = -C_{\alpha} \quad \text{in } \omega_{\alpha}, \quad \mathcal{V}_{z^{\alpha}}^{\alpha} = 0 \quad \text{on } \partial\omega_{\alpha}.$$

The normalizing constant C_{α} is choosing to satisfy

$$\int_{\omega_{\alpha}} \mathcal{V}_{z^{\alpha}}^{\alpha}(y^{\alpha}) dy^{\alpha} = 1. \quad (28)$$

In the most important case of the circular cylinder, i.e. $\omega_\alpha = \{y^\alpha : |y^\alpha| < r_\alpha\}$,

$$\mathcal{V}_{z^\alpha}^\alpha(x) = \frac{2(r_\alpha^2 - |y^\alpha|^2)}{\pi r_\alpha^4}, \quad \mathcal{P}^\alpha(x) = \frac{-8\nu}{\pi r_\alpha^4} z^\alpha.$$

The remainder term (v^\pm, p^\pm) in (27) satisfies the problem

$$-\nu \Delta v^\pm + \nabla p^\pm = f^\pm, \quad -\operatorname{div} v^\pm = g^\pm \text{ in } \Omega, \quad (29)$$

$$v^\pm = 0 \text{ on } \partial\Omega, \quad (30)$$

where the right-hand sides

$$f^\pm := \nu \Delta(\chi_0 \mathcal{V}^0 - \chi_\pm \mathcal{V}^\pm) - \nabla(\chi_0 \mathcal{P}^0 - \chi_\pm \mathcal{P}^\pm) \quad (31)$$

and

$$g^\pm := \operatorname{div}(\chi_0 \mathcal{V}^0 - \chi_\pm \mathcal{V}^\pm) \quad (32)$$

have compact supports. To verify condition (20) in Theorem (2.1) for g^\pm , we apply the Gauss theorem for the domain $\Omega_R = \{x \in \Omega : z^\alpha < R\}$, where R is a sufficiently large number, and obtain

$$\int_\Omega g^\pm dx = \lim_{R \rightarrow \infty} \int_{\Omega_R} \operatorname{div}(\chi_0 \mathcal{V}^0 - \chi_\pm \mathcal{V}^\pm) dx = \int_{\omega_R^0} \mathcal{V}_{z^0}^0(y^0) d\Sigma - \int_{\omega_R^\pm} \mathcal{V}_{z^\pm}^\pm(y^\pm) d\Sigma = 0.$$

Here we used the normalization condition (28). Therefore, (v^\pm, p^\pm) admits the asymptotic representation (26), where $c_0 = 0$ and (\tilde{v}, \tilde{p}) exponentially tends to zero when $z^\alpha \rightarrow \infty$.

Now we can present formulas for calculation of coefficients in (26)

Theorem 2.2. *Let the functions F and G satisfy (24)-(25) and let the asymptotic formula (26) be valid with $c_0 = 0$. Then*

$$c_\pm = \int_\Omega (FV^\pm + GP^\pm) dx \quad (33)$$

Proof. Let Ω_R be the same domain as before. Multiplying equations (7), (8) by (V^\pm, P^\pm) , integrating over Ω_R and using Green's formula, we obtain

$$\begin{aligned} & \int_{\Omega_R} ((-\nu \Delta v + \nabla p) V^\pm - \operatorname{div} v P^\pm) dx \\ &= \sum_\alpha \int_{\omega_\alpha} \left(-\nu (V^\pm \partial_{z^\alpha} v_{z^\alpha} - v \partial_{z^\alpha} V_{z^\alpha}^\pm) + (p V_{z^\alpha}^\pm - P^\pm v_{z^\alpha}) \right) \Big|_{z^\alpha=R} dy^\alpha. \end{aligned}$$

Taking here limit and using asymptotic formulas for (V^\pm, P^\pm) and (26) we arrive at (33).

Applying formula (33) to the solution (v^\pm, p^\pm) of the problem (7), (8) with the right hand sides given by (31), (32), we obtain the representations

$$(v^\pm, p^\pm) = \chi_\pm Q_{\pm\pm}(0, 1) + \chi_\mp Q_{\pm\mp}(0, 1) + (\tilde{v}_\pm, \tilde{p}_\pm), \quad (34)$$

where the coefficients are evaluated according to

$$Q_{\gamma\tau} = \int_{\Omega} (f^\gamma V^\tau + g^\gamma P^\tau) dx, \quad \gamma, \tau = \pm. \quad (35)$$

From (27) and (34) we get the following representations

$$\begin{aligned} (V^\pm, P^\pm) &= -\chi_0(\mathcal{V}^0, \mathcal{P}^0) + \chi_\pm(\mathcal{V}^\pm, \mathcal{P}^\pm) \\ &+ \chi_\pm Q_{\pm\pm}(0, 1) + \chi_\mp Q_{\pm\mp}(0, 1) + (\tilde{v}_\pm, \tilde{p}_\pm), \end{aligned} \quad (36)$$

with the remainders $(\tilde{v}_\pm, \tilde{p}_\pm)$ exponentially decaying at infinity. Note that a straightforward calculation gives the equality $Q_{\gamma\tau} = Q_{\tau\gamma}$. The coefficients $Q_{\gamma\tau}$ in the expansion (36) of the pressure at infinity in Ω_\pm form the symmetric (2×2) – matrix Q called *the pressure drop matrix*. Another approach to introducing the matrix Q was presented in [3].

3 Asymptotics of the effective lengths

3.1 Regular perturbation of the boundary of Ω

We assume that the boundary $\partial\Omega$ is sufficiently smooth. We introduce coordinates (n, τ) in a neighborhood of the boundary as follows: n is the oriented distance to $\partial\Omega$ ($n > 0$ outside $\bar{\Omega}$) and τ is a local coordinate on $\partial\Omega$. Let $\varphi = \varphi(\tau)$ be a smooth function (positive or negative) with a compact support on $\partial\Omega$. Now define the surface Γ_ε as the perturbation of the surface $\partial\Omega$ as

$$\Gamma_\varepsilon = \{x : n = \varepsilon\varphi(\tau)\}, \quad (37)$$

where $\varepsilon > 0$ is a small parameter. Let Ω_ε is domain with the boundary Γ_ε .

For the perturbed domain Ω_ε we have analog of formula (36)

$$(V_\varepsilon^\pm, P_\varepsilon^\pm) = -\chi_0(\mathcal{V}^0, \mathcal{P}^0) + \chi_\pm(\mathcal{V}^\pm, \mathcal{P}^\pm) + (v_\pm^\varepsilon, p_\pm^\varepsilon), \quad (38)$$

$$(v_\pm^\varepsilon, p_\pm^\varepsilon) = \chi_\pm Q_{\pm\pm}(\varepsilon)(0, 1) + \chi_\mp Q_{\pm\mp}(\varepsilon)(0, 1) + (\tilde{v}_\pm^\varepsilon, \tilde{p}_\pm^\varepsilon). \quad (39)$$

Theorem 3.1. *Let Ω_ε be domain with regular perturbation of the boundary (37). Then formulae (38), (39) have the asymptotic expansion of elements of the matrix Q*

$$Q_{\gamma\tau}(\varepsilon) = Q_{\gamma\tau} + \varepsilon c_\gamma^\tau + O(\varepsilon^2), \quad \gamma, \tau = \pm. \quad (40)$$

Here, $c_\gamma^\tau = \nu \int_{\Gamma_0} \varphi \partial_n V^\gamma \cdot \partial_n V^\tau d\Sigma$.

Proof. Let

$$(w^\varepsilon, q^\varepsilon) = (v_\pm^\varepsilon, p_\pm^\varepsilon) - (v_\pm, p_\pm) = (V_\varepsilon^\pm, P_\varepsilon^\pm) - (V^\pm, P^\pm).$$

If it is needed we extend smoothly functions (v_\pm, p_\pm) outside Ω . Then the pair $(w^\varepsilon, q^\varepsilon)$ satisfies the following problem

$$-\nu \Delta w^\varepsilon + \nabla q^\varepsilon = 0, \quad -\operatorname{div} w^\varepsilon = 0 \text{ in } \Omega_\varepsilon, \quad (41)$$

$$w^\varepsilon = -V^\pm \text{ on } \Gamma_\varepsilon. \quad (42)$$

We take the asymptotic ansatz for a solution (41), (42) as follows:

$$(w^\varepsilon, q^\varepsilon) = \varepsilon(v'_\pm, p'_\pm) + \varepsilon^2(v''_\pm, p''_\pm) + \dots \quad (43)$$

where (v'_\pm, p'_\pm) must satisfy the problem in Ω :

$$-\nu \Delta v'_\pm + \nabla p'_\pm = 0, \quad -\operatorname{div} v'_\pm = 0 \text{ in } \Omega, \quad (44)$$

$$v'_\pm = h^\pm \text{ on } \partial\Omega. \quad (45)$$

Comparing (42) and (45) we see that

$$h^\pm = -\varphi \partial_n V^\pm \text{ on } \Gamma_0. \quad (46)$$

Finally, using that $\partial_n V_n^\pm = 0$ on $\partial\Omega$ (this follows from $-\operatorname{div} V^\pm = 0$) we obtain (40).

Note that the matrix $\{c_\gamma^\tau\}_{\gamma, \tau = \pm}$ in (40) is positive definite for $\varphi > 0$ and is negative definite for $\varphi < 0$. Indeed, if some of the coefficients $c_\gamma^\tau = 0$ then $\partial_n V^\tau = 0$, $\tau = \pm$, on the supp φ and by 4.2 (see Appendix) the homogeneous Stokes problem (7), (8) will have trivial solution $(u, p) = (0, 0, 0; 1)$ only.

Taking into account the formulae (3), (40) we obtain the asymptotic expansion for the total increments of the effective lengths

$$\sum_{\alpha} l_{\alpha}(\varepsilon) = \sum_{\alpha} l_{\alpha} + \varepsilon \Psi + O(\varepsilon^2), \quad (47)$$

where

$$\Psi = \nu \int_{\Gamma_0} \varphi \{ (B_+ + B_- - B_0) \partial_n V^+ \cdot \partial_n V^- - B_+ |\partial_n V^+|^2 - B_- |\partial_n V^-|^2 \} d\Sigma.$$

Let the radii of the blood vessels r_{α} , $\alpha = 0, \pm$, be connected as follows $r_{\pm} = \delta_{\pm} r_0$, $0 < \delta_{\pm} < 1$. We have the following theorem that gives us possibility to estimate by means of the effective lengths the influence of changes in the vessel walls geometry.

Theorem 3.2. *Let δ_{\pm} be real numbers such that $|\delta_+^2 - \delta_-^2| < 1$, then in (47) the matrix of quadratic form Ψ is negative definite for $\varphi > 0$ and is positive definite for $\varphi < 0$.*

Proof. Substituting the asymptotic expansion (40) into (3), using Sylvester's criterion for the matrix of quadratic form Ψ , we immediately prove the assertion of Theorem 3.2.

So we can conclude that if near the bifurcation of an artery the cholesterol plaque (in the case of $\varphi < 0$) is located then the total effective length of the vessels increases. Vascular injury associated with aneurysm (in the case of $\varphi > 0$) corresponds to the decreasing in the total effective length of the vessels, see Appendix.

3.2 Model problem in a half-space

Consider the homogeneous Stokes system in the half-space $\mathbb{R}_+^3 = \{Y = (Y', Y_3) = (Y_1, Y_2, Y_3) : Y_3 > 0\}$:

$$-\nu \Delta U(Y) + \nabla P(Y) = 0, \quad -\operatorname{div} U(Y) = 0, \quad Y \in \mathbb{R}_+^3, \quad (48)$$

$$U(Y', 0) = 0, \quad Y' \in \mathbb{R}^2. \quad (49)$$

We are interested in solutions of (48), (49) having the form $U(x) = r^{\lambda} u(\omega)$, $P(x) = r^{\lambda-1} p(\omega)$, where $r = |Y|$, $\omega = Y/r$ and λ is a complex number. Such solutions exist only when $\lambda = 1, 2, \dots$ or $\lambda = -2, -3, \dots$. Moreover,

the space of such solutions is the same for λ and $-1 - \lambda$, see for example Theorem 5.2.1 in [11].

For $\lambda = 1$ this problem has the following three solutions (V_k, P_k) , $k = 1, 2, 3$, where

$$V_1(Y) = (Y_3, 0, 0), \quad V_2(Y) = (0, Y_3, 0), \quad V_3 = 0, \quad P_1 = P_2 = 0 \quad \text{and} \quad P_3 = 1.$$

The first two vector functions are called the Quette flows and the third one is constant. Using Theorem 5.4.4 [11], we can describe all solutions for $\lambda = -2$. They are given by pairs (U, P) ,

$$U = r^{-2}v(\omega)\omega, \quad P = r^{-3}(2v(\omega) - c),$$

where

$$(\delta + 6)v = 3c \quad \text{in } S_+^2 \quad \text{and} \quad v = 0 \quad \text{on } \partial S_+^2. \quad (50)$$

Here δ is the Laplace-Beltrami operator on S^2 . Solutions to (50) are obtained from solutions to $\Delta v = 3c$ in \mathbb{R}_+^3 with zero Dirichlet boundary conditions and v being second order polynomial. Therefore these solutions are given as $(\mathcal{V}_1, \mathcal{P}_1)$, $(\mathcal{V}_2, \mathcal{P}_2)$ and $(\mathcal{V}_3, \mathcal{P}_3)$, where

$$\mathcal{V}_k(Y) = \frac{\omega_k \omega_3}{\nu r^2}(\omega_1, \omega_2, \omega_3), \quad \mathcal{P}_k(Y) = 2 \frac{\omega_k \omega_3}{r^3}, \quad k = 1, 2,$$

and

$$\mathcal{V}_3(Y) = \frac{\omega_3^2}{\nu r^2}(\omega_1, \omega_2, \omega_3), \quad \mathcal{P}_3(Y) = 2 \frac{\omega_3^2}{r^3} - \frac{2}{3r^3}.$$

These functions verify the following bi-orthogonality conditions

$$\begin{aligned} \langle (\mathcal{V}_k, \mathcal{P}_k), (V_j, P_j) \rangle &= \int_{\mathbb{R}_+^3} \left((-\nu \Delta(\chi \mathcal{V}_k) + \nabla(\chi \mathcal{P}_k)) \cdot V_j - \operatorname{div}(\chi \mathcal{V}_k) P_j \right) dY \\ &= \int_{S_+^2} \left(-\nu(\partial_r \mathcal{V}_k \cdot V_j - \mathcal{V}_k \cdot \partial_r V_j) + \mathcal{P}_k \omega \cdot V_j - \omega \cdot \mathcal{V}_k P_j \right) r^2 dS_\omega \\ &= M_k \delta_k^j, \end{aligned} \quad (51)$$

where χ is a smooth function equals 0 for small $|Y|$ and 1 for large $|Y|$, dS_ω spherical area element and

$$M_k = 5 \int_{S_+^2} \omega_k^2 \omega_3^2 dS_\omega \quad \text{for } k = 1, 2 \quad \text{and} \quad M_3 = \frac{-1}{\nu} \int_{S_+^2} \omega_3^2 dS_\omega.$$

3.3 Domain close to a half-space

Now let us turn to the Stokes system in a domain Ξ which coincides with \mathbb{R}_+^3 outside a ball $B_{2\delta}(0)$ given by $|Y| \leq 2\delta$ and $\Xi = \{Y \in B_\delta(0) : Y_3 > \phi(Y_1, Y_2)\}$, where ϕ is a smooth function equal to 0 for $|Y'| > \delta$:

$$-\nu\Delta u(Y) + \nabla p(Y) = F(Y), \quad -\operatorname{div} u(Y) = G(Y), \quad Y \in \Xi, \quad (52)$$

$$u(Y) = 0, \quad Y \in \partial\Xi, \quad (53)$$

Theorem 3.3. *Let $F \in (H_0^1(\Xi))^*$ and $G \in L^2(\Xi)$. Then the problem (52), (53) has a unique weak solution $(u, p) \in H_0^1(\Xi) \times L^2(\Xi)$. This solution satisfies*

$$\|u\|_{H^1(\Xi)} + \|p\|_{L^2(\Xi)} \leq C \left(\|F\|_{(H_0^1(\Xi))^*} + \|G\|_{L^2(\Xi)} \right).$$

Now, we are interesting in asymptotics of solutions for large $|Y|$.

Theorem 3.4. *Let $F \in (H_0^1(\Xi))^*$ and $G \in L^2(\Xi)$ have compact supports. Then solution $(u, p) \in H_0^1(\Xi) \times L^2(\Xi)$ from the previous theorem satisfies*

$$u(Y) = \sum_{k=1}^3 c_k \mathcal{V}_k + O(|Y|^{-3}), \quad p(Y) = \sum_{k=1}^3 c_k \mathcal{P}_k + O(|Y|^{-4}), \quad \text{for large } |Y|, \quad (54)$$

where $O(\cdot)$ terms can be differentiated.

Proof. Let χ be the smooth cut-off function in \mathbb{R}_+^3 , namely $\chi(Y) = 1$ for $|Y| > R_0$, $\chi(Y) = 0$ for $|Y| < R_0$, where $R_0 = \text{const}$. We transform the problem (52), (53) in Ξ to the following problem in \mathbb{R}_+^3

$$-\nu\Delta(\chi u) + \nabla(\chi p) = \hat{F}, \quad -\operatorname{div}(\chi u) = \hat{G}, \quad Y \in \mathbb{R}_+^3, \quad (55)$$

$$\chi u = 0, \quad Y \in \mathbb{R}^2, \quad (56)$$

where the right-hand sides $\hat{F} = \chi F - \nu((\Delta\chi)u + 2(\nabla\chi, \nabla)u) + p\nabla\chi$ and $\hat{G} = \chi G - (u, \nabla\chi)$ have compact supports.

Let $U = \chi u - v$. Here v is a solution to the problem

$$-\operatorname{div} v = \hat{G}, \quad Y \in \mathbb{R}_+^3, \quad (57)$$

$$v = 0, \quad Y \in \mathbb{R}^2. \quad (58)$$

And we arrive at

$$-\nu\Delta U + \nabla P = f, \quad -\operatorname{div} U = \hat{G}, \quad Y \in \mathbb{R}_+^3, \quad (59)$$

$$U = 0, \quad Y \in \mathbb{R}^2, \quad (60)$$

where $(\mathcal{H}_0(\mathbb{R}_+^3))^* \ni f = \hat{F} + \nu\Delta v$ and $P = \chi p$.

Exploiting the explicit form of the Green's tensor (a tensor field $\mathbf{G}(Y, Z)$ and a vector field $\mathbf{g}(Y, Z)$) for the homogeneous Stokes system in the half-space (see, e.g., [13, Appendix 1]) we find estimates

$$|G_{ij}(Y, Z) - G_{ij}(Y, 0)| \leq C|Y - Z|^{-2}, \quad |g_i(Y, Z) - g_i(Y, 0)| \leq c|Y - Z|^{-3}. \quad (61)$$

Using the estimates (61) for the solution to (59), (60)

$$U_j(Y) = \int_{\mathbb{R}_+^3} G_{ij}(Y, Z) f_i(Z) dZ, \quad P(Y) = \int_{\mathbb{R}_+^3} g_i(Y, Z) f_i(Z) dZ$$

we get formulae (54). The proof is complete.

In order to evaluate the constants c_k in (54) we introduce three solutions $(\hat{V}_k(Y), \hat{P}_k(Y))$, $k = 1, 2, 3$, of the homogeneous problem (52), (53), having the form

$$\hat{V}_k(Y) = V_k(Y) + \tilde{V}_k(Y), \quad \hat{P}_k(Y) = P_k(Y) + \tilde{P}_k(Y) \quad (62)$$

with $(\tilde{V}_k, \tilde{P}_k) \in H^1(\Xi) \times L^2(\Xi)$. Since $V_3 = 0$ and $P_3 = 1$, we have $\tilde{V}_3 = 0$ and $\tilde{P}_3 = 0$. One can verify that

$$-\nu\Delta\tilde{V}_k + \nabla\tilde{P}_k = 0, \quad -\operatorname{div} \tilde{V}_k = 0, \quad Y \in \Xi, \quad (63)$$

and $\tilde{V}_k + V_k = 0$ on $\partial\Xi$. By Theorems 3.3 and 3.4 this problem has solution and it satisfies

$$\tilde{V}_k(Y) = \sum_{j=1}^3 A_j^k \mathcal{V}_j(Y) + O(|Y|^{-3}), \quad \tilde{P}_k(Y) = \sum_{j=1}^3 A_j^k \mathcal{P}_j(Y) + O(|Y|^{-4}) \quad (64)$$

for large $|Y|$. One can see directly that $\hat{V}_3 = 0$ and $\hat{P}_3 = P_3 = 1$.

Theorem 3.5. *Let F and G be the same as in Theorem 3.4 and let c_k be coefficients in (54). Then*

$$M_k c_k = \int_{\partial\Xi} (F\hat{V}_k + G\hat{P}_k) dS. \quad (65)$$

Proof. Let $\Xi_R = \{Y \in \Xi : |Y| < R\}$ and $S_R = \{Y \in \Xi : |Y| = R\}$. Then multiplying the first equation in (52) by \hat{V}_k and the second equation in (52) by \hat{P}_k , summing them up and integrating over Ξ_R , we obtain

$$\int_{\Xi_R} \left((-\nu \Delta u(Y) + \nabla p(Y)) \cdot V_k - \operatorname{div} u(Y) \hat{P}_k \right) dY = \int_{\Xi_R} (F(Y) \cdot \hat{V}_k + G(Y) \hat{P}_k) dY.$$

Using the Green formula in the left-hand side of the last relation, we get

$$\int_{S_R} \left(-\nu (\partial_r u \cdot \hat{V}_k - u \cdot \partial_r \hat{V}_k) + p \omega \cdot \hat{V}_k - \omega \cdot u(Y) \hat{P}_k \right) dS_\omega = \int_{\Xi_R} (F(Y) \cdot \hat{V}_k + G(Y) \hat{P}_k) dY.$$

Replacing vector functions (u, p) and (\hat{V}_k, \hat{P}_k) by their asymptotics, taking the limit as $R \rightarrow \infty$ and using (51), we verify (65).

In the next theorem we prove a positivity property of the coefficients in (64) when the function ϕ in the definition of Ξ has sign.

Theorem 3.6. *Let ϕ be not identically zero. Then the matrix*

$$\{M_k A_j^k\}_{k,j=1}^2,$$

is positive definite for $\phi \leq 0$ and is negative definite for $\phi \geq 0$. Moreover, $A_j^3 = A_3^j = 0$, $j = 1, 2, 3$.

Proof. Since $(\hat{V}_3, \hat{P}_3) = (0, 1)$ we have $A_j^3 = 0$.

Let $\phi \geq 0$ and let Ξ_R be defined as above. Integrating by parts in the right-hand side of

$$0 = \int_{\Xi_R} ((-\nu \Delta \tilde{V}_k + \nabla \tilde{P}_k) \hat{V}_j - \operatorname{div} \tilde{V}_k \hat{P}_j) dY$$

and taking the limit as $R \rightarrow \infty$, we get

$$M_j A_j^k = \int_{\partial G} (-\nu \partial_n \hat{V}_j + n \hat{P}_j) \cdot \tilde{V}_k dS,$$

where n is the unit outward normal to $\partial \Xi$. Using (62) and that $\tilde{V}_k + V_k = 0$ on $\partial \Xi$, we can write the last relation as

$$M_j A_j^k = \int_{\partial \Xi} (-\nu \partial_n \tilde{V}_j + n \tilde{P}_j) \cdot \tilde{V}_k dS - \int_{\partial \Xi} (-\nu \partial_n V_j + n P_j) \cdot V_k dS.$$

Now, applying Green's formula in the domains Ξ and $\mathbb{R}_+^3 \setminus \Xi$, we get

$$M_j A_j^k = -\nu \int_{\Xi} \nabla \tilde{V}_j \cdot \nabla \tilde{V}_k dY - \nu \int_{\mathbb{R}_+^3 \setminus \Xi} \nabla V_j \cdot \nabla V_k dY.$$

If we denote by $\mathcal{Q} = \{\mathcal{Q}_{jk}\}_{k,j=1}^3$ the matrix on the right, then this matrix is symmetric, $\mathcal{Q}_{j3} = 0$ and the matrix $\mathcal{Q} = \{\mathcal{Q}_{jk}\}_{k,j=1}^2$ is negative definite since the vector functions V_j , $j = 1, 2$, are linear independent, and therefore

$$A_j^k = (M_j)^{-1} \mathcal{Q}_{jk}. \quad (66)$$

Now consider the situation when $\phi \leq 0$. Then we represent Ξ as $\mathbb{R}_+^3 \cup \Xi_0$, where $\Xi_0 = \Xi \setminus \mathbb{R}_+^3$. We are looking for the solutions (\hat{V}_k, \hat{P}_k) in the form

$$\hat{V}_k = V'_k + \tilde{V}_k, \quad \hat{P}_k = \tilde{P}_k, \quad k = 1, 2,$$

where $V'_k = V_k$ in \mathbb{R}_+^3 and $V'_k = 0$ in Ξ_0 . Multiplying the equation $-\nu \Delta \tilde{V}_k + \nabla \tilde{P}_k = 0$ by V_j and integrating over \mathbb{R}_+^3 we get

$$M_j A_j^k = \nu \int_{Y_3=0} \tilde{V}_k \cdot \partial_{Y_3} V_j dY' = \nu \int_{Y_3=0} \tilde{V}_k \cdot \partial_{Y_3} (\hat{V}_j - \tilde{V}_j) dY' + \int_{Y_3=0} \tilde{P}_k (\hat{V}_j - \tilde{V}_j)_3 dY'.$$

Hence

$$M_j A_j^k = \int_{Y_3=0} (-\nu \tilde{V}_k \cdot \partial_{Y_3} \tilde{V}_j - \tilde{P}_k (\tilde{V}_j)_3) dY' + \int_{Y_3=0} (\nu \hat{V}_k \cdot \partial_{Y_3} \hat{V}_j + \hat{P}_k (\hat{V}_j)_3) dY'.$$

Using Green's formula in the domains \mathbb{R}_+^3 and Ξ_0 we get

$$M_j A_j^k = \nu \int_{\mathbb{R}_+^3} \nabla \tilde{V}_k \cdot \nabla \tilde{V}_j dY + \nu \int_{\Xi_0} \nabla \hat{V}_k \cdot \nabla \hat{V}_j dY.$$

3.4 Concentrated perturbation of the boundary of Ω

Here we consider a domain Ω_ε which coincides with Ω outside $B_\varepsilon(x_0)$ where x_0 is a fixed point on $\partial\Omega$. Inside $B_\varepsilon(x_0)$ the domain Ω_ε is given by

$$\{y = (y', y_3) \in B_\varepsilon(x_0) : y_3 > \varepsilon \phi(y'/\varepsilon)\},$$

where $y = (y_1, y_2, y_3)$ are Cartesian coordinates with the center at x_0 and ϕ is a smooth function such that $\phi(y') = 0$ for $|y'| > \delta$. Our goal is to find the asymptotics of the matrix Q .

We are looking for solution $(V_\alpha^\varepsilon, P_\alpha^\varepsilon)$ in the form

$$\begin{aligned} V_\alpha^\varepsilon(x) &= (1 - \chi(y/\varepsilon))(V_\alpha(x) + \varepsilon V_\alpha^{(1)}(x)) + \varepsilon^{-1} \zeta(x) W_\alpha(y/\varepsilon) + w_\alpha(x, \varepsilon), \\ P_\alpha^\varepsilon(x) &= (1 - \chi(y/\varepsilon))(P_\alpha(x) + \varepsilon P_\alpha^{(1)}(x)) + \varepsilon^{-1} \zeta(x) R_\alpha(y/\varepsilon) + S_\alpha(x, \varepsilon). \end{aligned}$$

Here $\chi(Y)$ is a smooth cut-off function equals 1 for $|Y| < \delta$ and 0 for $|Y| > 2\delta$, ζ is also a cut-off function equals 1 for $|x| < \delta/2$ and 0 for $|x| > \delta$, where δ is a fixed positive number. Using that

$$V_\alpha(x) = a_\alpha^1 V_1(y) + a_\alpha^2 V_2(y) + O(|x - x_0|^2), \quad P_\alpha(x) = a_\alpha^3 + O(|x - x_0|),$$

we conclude that the vector function (W_α, R_α) must satisfy the system

$$-\nu \Delta W_\alpha(Y) + \nabla R_\alpha(Y) = -\nu \Delta (\chi(Y)(a_\alpha^1 V_1(Y) + a_\alpha^2 V_2(Y))) + a_\alpha^3 \nabla \chi(Y)$$

and

$$-\operatorname{div} W_\alpha(Y) = -\operatorname{div} (\chi(Y)(a_\alpha^1 V_1(Y) + a_\alpha^2 V_2(Y)))$$

in Ξ with the homogeneous Dirichlet boundary condition on $\partial\Xi$. According to Theorems 3.3 and 3.4 this problem has a unique solution in $H_0^1(\Xi) \times L^2(\Xi)$ and this solution satisfies

$$\begin{aligned} W_\alpha(Y) &= \sum_{k=1}^3 C_k^\alpha \mathcal{V}_k(Y) + O(|Y|^{-3}), \\ R_\alpha(Y) &= \sum_{k=1}^3 C_k^\alpha \mathcal{P}_k(Y) + O(|Y|^{-4}) \quad \text{for large } |Y|, \end{aligned} \quad (67)$$

and the coefficients here are evaluated as

$$\begin{aligned} C_k^\alpha &= \int_{\Xi} \left(-\nu \Delta (\chi(Y)(a_\alpha V_1(Y) + b_\alpha V_2(Y))) + c_\alpha \nabla \chi(Y) \right) \hat{V}_k dY \\ &\quad + \nabla \cdot (\chi(Y)(a_\alpha V_1(Y) + b_\alpha V_2(Y))) \hat{P}_k dY. \end{aligned} \quad (68)$$

Now we can write the system for $(V_\alpha^{(1)}, P_\alpha^{(1)})$:

$$-\nu \Delta V_\alpha^{(1)}(x) + \nabla P_\alpha^{(1)} = -\nu \Delta (\zeta(x) \sum_{k=1}^3 C_k^\alpha \mathcal{V}_k(x)) + \nabla (\zeta(x) \sum_{k=1}^3 C_k^\alpha \mathcal{P}_k(x)) \quad (69)$$

and

$$-\operatorname{div} V_\alpha^{(1)}(x) = \operatorname{div}(\zeta(x) \sum_{k=1}^3 C_k^\alpha \mathcal{V}_k(x)) \quad (70)$$

in Ω with zero Dirichlet boundary condition on $\partial\Omega$. Denote the right-hand sides in (69) and (70) by $F_\alpha^{(1)}$ and $G_\alpha^{(1)}$ respectively. In order to apply Theorems 2.1 and 2.2 to problem (69), (70), we must verify that

$$\int_{\Omega} G_\alpha^{(1)} dx = 0.$$

We have

$$\begin{aligned} \int_{\Omega} G_\alpha^{(1)} dx &= \int_{\Omega} \operatorname{div}(\zeta(x) \sum_{k=1}^3 C_k^\alpha \mathcal{V}_k(x)) dx = - \int_{S_+^2} \sum_{k=1}^3 C_k^\alpha \mathcal{V}_k(\omega) \cdot \omega dS_\omega \\ &= -C_3^\alpha \int_{S_+^2} \frac{\omega_3^2}{\nu} dS_\omega. \end{aligned}$$

Furthermore,

$$\begin{aligned} C_3^\alpha &= \int_{\Xi} \nabla \cdot (\chi(Y)(a_\alpha V_1(Y) + b_\alpha V_2(Y))) dY \\ &= \int_{\Xi} \nabla \cdot ((\chi(Y) - 1)(a_\alpha V_1(Y) + b_\alpha V_2(Y))) dY \\ &= -R^3 \int_{S_+^2} \omega \cdot (a_\alpha V_1(\omega) + b_\alpha V_2(\omega)) dS_\omega = 0. \end{aligned}$$

Thus the problem (69), (70) is solvable and its solution satisfies the asymptotics

$$(V_\alpha^{(1)}(x), P_\alpha^{(1)}) = \sum_{\theta=\pm} c_\alpha^{(\theta)} \chi_\theta(0, 1) + (\tilde{V}_\alpha^{(1)}(x), \tilde{P}_\alpha^{(1)}(x)),$$

where the coefficients are evaluated according to

$$c_\alpha^{(\theta)} = \int_{\Omega} (F_\alpha^{(1)} \cdot V^\theta + G_\alpha^{(1)} P^\theta) dx.$$

Taking expressions for $F_\alpha^{(1)}$ and $G_\alpha^{(1)}$ from (69) and (70), replacing Ω by $\Omega_\varepsilon = \{x \in \Omega : |x - x_0| > \varepsilon\}$ and using Green's formula, we obtain

$$c_\alpha^{(\pm)} = \sum_{k=1}^3 C_k^\alpha \langle (\mathcal{V}_k, \mathcal{P}_k), (a_\theta^1 V_1 + a_\theta^2 V_2, a_\theta^3) \rangle = \sum_{k=1}^3 M_k C_k^\alpha a_\theta^k$$

Coefficients C_k^α admit the following interpretation. The vector function (\hat{V}_k, \hat{P}_k) can be constructed also as follows:

$$\hat{V}_k = (1 - \chi)V_k + \check{V}_k, \quad \hat{P}_k = (1 - \chi)P_k + \check{P}_k, \quad k = 1, 2, 3,$$

where $(\check{V}_k, \check{P}_k)$ satisfies the system

$$-\nu\Delta\check{V}_k + \nabla\check{P}_k = \nu\Delta(1 - \chi)V_k - \nabla(1 - \chi)P_k, \quad -\nabla \cdot \check{V}_k = \nabla \cdot (1 - \chi)V_k, \quad \text{in } \Xi,$$

$$\check{V}_k = 0, \quad \text{on } \partial\Xi.$$

Moreover the vector function $(\check{V}_k, \check{P}_k)$ has the same asymptotic representation (64) as $(\tilde{V}_k, \tilde{P}_k)$. Therefore by Theorem 3.5

$$M_k A_j^k = \int_G \left((\nu\Delta(1 - \chi)V_k - \nabla(1 - \chi)P_k) \cdot \hat{V}_j + \nabla \cdot (1 - \chi)V_k \hat{P}_j \right) dY.$$

Comparing this formula and (68) we see that

$$C_k^\alpha = \sum_{m=1}^3 a_\alpha^m M_m A_k^m.$$

Thus

$$c_\alpha^\theta = \sum_{k=1}^3 \sum_{m=1}^3 M_k a_\alpha^m M_m A_k^m a_\theta^k$$

and hence

$$Q_{\alpha\theta}(\varepsilon) = Q_{\alpha\theta} + \varepsilon c_\alpha^\theta + O(\varepsilon^2). \quad (71)$$

Using that $A_k^m = 0$ when m or k is equal to 3, we get that

$$c_\alpha^\theta = \sum_{k=1}^2 \sum_{m=1}^2 M_k a_\alpha^m M_m A_k^m a_\theta^k. \quad (72)$$

Since the matrix $\{A_k^m\}_{k,m=1}^2$ is positive definite we conclude that the matrix $\{c_\alpha^\theta\}_{\alpha,\theta=\pm}$ is also positive definite and

$$\sum_{\alpha,\theta} c_\alpha^\theta \xi_\alpha \xi_\theta = \sum_{k=1}^2 \sum_{m=1}^2 A_k^m \left(\sum_{\theta} M_k a_\theta^k \xi_\theta \right) \left(\sum_{\alpha} M_m a_\alpha^m \xi_\alpha \right), \quad (73)$$

therefore

$$\sum_{\alpha, \theta} c_{\alpha}^{\theta} \xi_{\alpha} \xi_{\theta} \geq c_0 \sum_{k=1}^2 \left(\sum_{\theta} a_{\theta}^k \xi_{\theta} \right)^2. \quad (74)$$

Formulas (71)-(74) describe the asymptotic behavior of the coefficients $Q_{\alpha\theta}$ for the domain Ω_{ε} . Replacing the elements of the pressure drop matrix Q by their asymptotics (71) in (3) we get the asymptotic expansion of the effective lengths of the vessels, i.e. the direct analog of Theorem 3.2 is executed.

4 Appendix

4.1 Pressure drop matrix and modified Kirchhoff transmission conditions

Let's truncate cylindrical outlets in Ω and assume

$$\Omega_h = \Omega' \cup \Omega_h^0 \cup \Omega_h^+ \cup \Omega_h^-,$$

$$\Omega_h^{\alpha} = \{x : |y^{\alpha}| < r_{\alpha}, z^{\alpha} < h^{-1}l_{\alpha}\}, \quad \alpha = 0, \pm, \quad (75)$$

where $h > 0$ is a small dimensionless parameter, $r_{\alpha} > 0$, $l_{\alpha} > 0$ are certain fixed radii and lengths respectively. In domain Ω_h we define the homogeneous ($F = 0, G = 0$) Stokes equations (7), and on its lateral surface $\Sigma_h = \partial\Omega_h \cap \partial\Omega$ we impose the homogeneous ($H = 0$) no-slip conditions (8) (hereinafter refer to these relations, implying that they are restricted to these sets). On the truncated surfaces $\Gamma_h^{\alpha} = \{x : |y^{\alpha}| < r_{\alpha}, z^{\alpha} = h^{-1}l_{\alpha}\}$ assign the following conditions:

$$v^h(x) = -V^0(y^0), \quad x \in \Gamma_h^0, \quad (76)$$

$$v_{y_i}^h(x) = 0, \quad i = 1, 2, \quad -\nu \partial_{z^{\tau}} v_{z^{\tau}}^h(x) + p^h(x) = p^{\infty}, \quad x \in \Gamma_h^{\tau}, \quad \tau = \pm. \quad (77)$$

In other words, at the inlet cross-section of the vessel Ω^0 is assigned the incoming unit flux of fluid, and on the allocated ends of the outlet cross-sections of the vessels Ω^{\pm} peripheral pressure p_{∞} is set. At the same time compression of coordinates by h^{-1} times transforms the problem stated to the usual problem of the blood flow through the artery bifurcation node, which walls, as already explained, it is assumed to be rigid (cf. [1]). In the new coordinates the vessels become smaller radii hr_{α} and fixed lengths l_{α} . We emphasize that the problem (7), (8), (75), (76), (77) is still included in

the symmetric Green's formula in Ω_h . Its interpretation in the framework of the weighted spaces technique with detached asymptotics is given in [3].

As an approximate solution of the problem stated in Ω_h we take the sums

$$\hat{v}^h = a_+^h V^+ + a_-^h V^-, \quad \hat{p}^h = a_+^h P^+ + a_-^h P^- + a_0^h, \quad (78)$$

where (V^\pm, P^\pm) are introduced special solutions (27) and the last term refers to the constant pressure. Using the asymptotic representations (see Sect. 2.3), satisfy the boundary conditions (76), (77) up to exponentially small terms for $h \rightarrow +0$ and we obtain the following relations

$$1 = a_+^h + a_-^h, \quad (79)$$

$$p^\infty = a_0^h - h^{-1} L_\tau a_\tau^h + \sum_{\alpha=\pm} Q_{\tau\alpha} a_\alpha^h, \quad \tau = \pm, \quad (80)$$

where $L_\alpha = \frac{8\nu}{\pi r_\alpha^4} l_\alpha$, $\alpha = 0, \pm$, and henceforth $L = \text{diag}\{L_+, L_-\}$.

Let $\mathbf{e} = (1, 1)$ and $\mathbf{a}^h = (a_+^h, a_-^h)$ be columns. In virtue of (80) we deduce

$$(p^\infty - a_0^h) \mathbf{e} = (Q - h^{-1} L) \mathbf{a}^h,$$

hence

$$\mathbf{a}^h = (h^{-1} L - Q)^{-1} \mathbf{e} (a_0^h - p^\infty),$$

and thus equality (79) rewritten in the form $\mathbf{e} \cdot \mathbf{a}^h = 1$ leads to the relations

$$1 = T_h(a_0^h - p^\infty),$$

$$\begin{aligned} T_h &= \mathbf{e} \cdot (h^{-1} L - Q)^{-1} \mathbf{e} = h^{-1} \mathbf{e} \cdot L^{-1} (I - hQL^{-1})^{-1} \mathbf{e} \\ &= h^{-1} \mathbf{e} \cdot (L^{-1} + hL^{-1}QL^{-1} + O(h^2)) \mathbf{e}. \end{aligned}$$

We finally find that

$$a_0^h = p^\infty + T_h^{-1},$$

$$T_h = h(t_0 + ht_1 + O(h)), \quad t_0 = \mathbf{e} \cdot L^{-1} \mathbf{e} > 0, \quad t_1 = \mathbf{e} \cdot L^{-1} Q L^{-1} \mathbf{e}, \quad (81)$$

so we get

$$a_0^h = p^\infty + ht_0^{-1} (1 - ht_0^{-1} t_1 + O(h^2)). \quad (82)$$

Thus, the pressure at the "input" Γ_h^0 up to the smaller terms is equal to

$$h^{-1} L_0 + p^\infty + ht_0^{-1} - h^2 t_0^{-2} t_1 + O(h^3). \quad (83)$$

The first term of (83) is the pressure drop, which provides the unit flux delivery to the artery bifurcation, the second term is also positive, it is necessary to supply the fluxes to the points $z^\pm = h^{-1}l_\pm$, and the third term, the sign of which depends on the pressure drop matrix Q , corresponds to just the shape of the node.

If in the vicinity of the node is formed plaque ($\varphi < 0$ in (37)) then according to the formula (40), the magnitude of (81) decreases, i.e. the pressure (83) increases. This fact is consistent with an obvious observation: constriction of the channel requires the growth of pressure in the input. At a constant pressure, decreases the flux supplied into the vessels Ω_h^\pm . Many reasons of hypertensive pressure doctors associated with the clogging of blood vessels.

At first glance it seems that an aneurysm ($\varphi > 0$ in (37)) facilitates the passage of blood through a bifurcation node, i.e. the correction term $O(h^2)$ in (83) is reduced due to the minus sign in front of it. This impression is erroneous, i.e. statement of the problem adopted in the article does not consider elasticity of the vessel walls. As it is known from medical reference books and explained in [12], using the one-dimensional model of the types of aneurysms, vascular permeability may be reduced due to the growth of the hematoma in the aneurysm cavity. At the same time, increase of the pressure at the input (increased heart rate while running and stress) provokes the wall rupture and leads to a variety of sad consequences, such as death due to a rapid expiration of a false aneurysm in a large femoral artery.

4.2 The Cauchy problem

Here we assume that the boundaries of the domains ω_α are analytic. Let also Γ' is the part of $\partial\Omega$ which is $\partial\Omega \cap \overline{\Omega}'$.

Here we prove that the homogeneous Stokes problem (7), (8) with the additional Neumann boundary condition

$$\partial_n u(x) = 0, \quad x \in \Gamma', \quad (84)$$

has a trivial bounded solution (u, p) , i.e. identically equals $(0, 0; 1)$.

We assume that $\partial\Omega \setminus \Gamma_0$ is analytic. Here Γ_0 is a compact subset of Γ . Since $\partial_n u(x)$ is analytic on the surface $\partial\Omega \setminus \Gamma'$ as a solution to (7), (8) and due to (84), we conclude that $\partial_n u = 0$ on the whole $\partial\Omega$. This implies $\nabla u = 0$ and hence $\text{rot } u = 0$ on $\partial\Omega$. Application of the operator rot to the homogeneous

Stokes system (7), (8) gives the Dirichlet boundary value problem for $\text{rot } u$, i.e.

$$\Delta(\text{rot } u(x)) = 0, \quad x \in \Omega, \quad (85)$$

$$\text{rot } u(x) = 0, \quad x \in \partial\Omega. \quad (86)$$

Since we look for the bounded solution to (85), (86) (by using a local estimate this implies that the gradient is also bounded at infinity), one can conclude that $\text{rot } u \equiv 0$, and hence we get $u = \nabla\Phi$, where Φ is a scalar potential. Thus we arrive at the following system of equations

$$\nabla(-\nu\Delta\Phi(x) + p(x)) = 0, \quad -\Delta\Phi(x) = 0, \quad x \in \Omega, \quad (87)$$

$$\nabla\Phi(x) = 0, \quad x \in \partial\Omega. \quad (88)$$

In virtue of (87), (88) we obtain $(u, p) = (0, 0, 0; 1)$.

4.3 Proof of Lemma (2.1)

First we consider an auxiliary problem, namely we look for a solution of the boundary value problem

$$-\text{div } u_\alpha(x^\alpha) = \eta_\alpha(y^\alpha)G_\alpha(z^\alpha), \quad x^\alpha \in \Omega^\alpha, \quad (89)$$

$$u_\alpha(x^\alpha) = 0, \quad x^\alpha \in \partial\Omega^\alpha, \quad (90)$$

where

$$\eta_\alpha(y^\alpha) \in C_0^\infty(\omega^\alpha), \quad \int_{\omega_\alpha} \eta_\alpha(y^\alpha) dy^\alpha = 1 \quad \text{and} \quad G_\alpha(z^\alpha) = \int_{\omega_\alpha} g(y^\alpha, z^\alpha) dy^\alpha.$$

One can verify directly that the vector function

$$u_\alpha(x^\alpha) = (0, 0, \eta_\alpha(y^\alpha)w_\alpha(z^\alpha)), \quad w_\alpha(z^\alpha) = - \int_{z^\alpha}^\infty G_\alpha(t) dt,$$

solves the problem (89), (90). Moreover, by Hardy's inequality (see Theorem 5.2 [[14]])

$$\|w_\alpha\|_{L_2(L_\alpha, \infty)} \leq c \int_{L_\alpha}^\infty |z^\alpha|^2 |G_\alpha|^2 dz^\alpha \leq C \|g\|_{L_{2,1}(L_\alpha, \infty)}. \quad (91)$$

Using (91), we obtain

$$\|u_\alpha\|_{\mathcal{H}(\Omega^\alpha)} \leq c_\alpha \|g\|_{L_{2,1}(\Omega^\alpha)}. \quad (92)$$

We are looking for a solution to $-\operatorname{div} u = g$ in the form

$$u = \sum_{\alpha=0,\pm} \chi_\alpha(z^\alpha) u_\alpha(y^\alpha, z^\alpha) + U(x) \quad (93)$$

for the equation $-\operatorname{div} u = g$ in Ω . Here the cut-off functions χ_α are defined by

$$\chi_\alpha(z^\alpha) = 1 \text{ for } z^\alpha > 2L_\alpha, \quad \chi_\alpha(z^\alpha) = 0 \text{ for } z^\alpha < L_\alpha.$$

The function U in (93) satisfies the equation

$$-\operatorname{div} U = g + \operatorname{div} \left(\sum_{\alpha=0,\pm} \chi_\alpha(z^\alpha) u_\alpha(y^\alpha, z^\alpha) \right) \equiv \hat{g} \in L_{2,1}(\Omega)$$

in Ω . It is easy to see that

$$\int_{\omega^\alpha} \hat{g} dy^\alpha = 0 \quad \text{for } z^\alpha > L_\alpha. \quad (94)$$

Let us introduce a local covering of Ω . Let $\Omega_j^\alpha = \{(y^\alpha, z^\alpha) : y^\alpha \in \omega_\alpha, L_\alpha + j - 1 < z^\alpha < L_\alpha + j + 3/2\}$, $j = 1, \dots$. Then

$$\Omega = \Omega' + \sum_{\alpha} \sum_{j=1}^{\infty} \Omega_j^\alpha.$$

Let us consider the partition of unity corresponding to this covering:

$$1 = \phi'(x) + \sum_{\alpha} \sum_{j=1}^{\infty} \phi_j^\alpha(z^\alpha), \quad \text{where } \phi_j^\alpha \in C_0^\infty(L_\alpha + j - 1, L_\alpha + j + 3/2),$$

and ϕ' is a smooth function supported in $\overline{\Omega'}$. We can write

$$\hat{g} = \hat{g}' + \sum_{\alpha} \sum_{j=1}^{\infty} \hat{g}_j^\alpha, \quad \text{where } \hat{g}_j^\alpha = \phi_j^\alpha \hat{g}, \quad \hat{g}' = \phi'(x) \hat{g}.$$

By (94) and (10) we have

$$\int_{\Omega_j^\alpha} \hat{g}_j^\alpha dx^\alpha = 0, \quad \text{and,} \quad \int_{\Omega'} \hat{g}' dx = 0.$$

So we obtain the problem in each bounded domain Ω_j^α :

$$-\operatorname{div} U_j^\alpha = \hat{g}_j^\alpha \text{ in } \Omega_j^\alpha, \quad (95)$$

$$U_j^\alpha = 0 \text{ on } \partial\Omega_j^\alpha, \quad (96)$$

and similar problem for U' in Ω' . These problems have solutions $U_j^\alpha \in \mathcal{H}_0(\Omega_j^\alpha)$ (see [5]) which satisfy

$$\|U_j^\alpha\|_{\mathcal{H}(\Omega_j^\alpha)} \leq C \|\hat{g}_j^\alpha\|_{L_{2,1}(\Omega_j^\alpha)}.$$

Thus the vector function

$$U = U' + \sum_{\alpha} \sum_{j=1}^{\infty} U_j^\alpha \in \mathcal{H}_0(\Omega)$$

solves problem (95), (96) and satisfies

$$\|U\|_{\mathcal{H}(\Omega)} \leq C_1 \left(\|U'\|_{\mathcal{H}(\Omega')} + \sum_{\alpha} \sum_{j=1}^{\infty} \|U_j^\alpha\|_{\mathcal{H}(\Omega_j^\alpha)} \right) \leq C_2 \|g\|_{L_{2,1}(\Omega)}.$$

The proof is complete.

Acknowledgements. V. K. acknowledges the support of the Swedish Research Council (VR) grant EO418401. S. N. was supported by the Russian Foundation for Basic Research, project no. 18-01-00325, and by Linköping University (Sweden). G. Z. was supported by Linköping University, and by RFBR grant 16-31-60112.

References

- [1] *Kozlov, V. A., Nazarov, S. A.*, Transmission conditions in a one-dimensional model of bifurcating arteries with elastic walls, *J Math Sci*, 224:1, 2017, 94–118.
- [2] *Kozlov, V. A., Nazarov, S. A.*, One-dimensional model of flow in a junction of thin channels, including arterial trees, *Sbornik: Mathematics*, 208:8, 2017, 1138–1186.
- [3] *Nazarov, S. A., Pileckas, K.*, Asymptotic conditions at infinity for the Stokes and Navier-Stokes problems in domains with cylindrical outlets to infinity, *Quaderny di matematica*, 4, 1999, 141–243.
- [4] *Berntsson, F., Karlsson, M., Kozlov, V. A., Nazarov, S. A.*, A modification to the Kirchhoff conditions at a bifurcation and loss coefficients, *LiU electronic press, MAI, LiTH-MAT-R-2018/05-SE*, 2018, 1–10.
- [5] *Solonnikov, V. A.*, On the solvability of boundary and initial-boundary value problems for the Navier-Stokes system in domains with noncompact boundaries, *Pacific J. Math.*, Vol. 93, Number 2, 1981, 443–458.
- [6] *Ladyzhenskaya, O. A., Solonnikov, V. A.*, Determination of the solutions of boundary value problems for stationary Stokes and Navier-Stokes equations having an unbounded Dirichlet integral, *J. Sov. Math.*, Vol. 21, 1983, 728–761.
- [7] *Amick, C. J.*, Steady solutions of the Navier-Stokes equations in unbounded channels and pipes, *Ann. Scuola Norm. Sup. Pisa*, 4, 1977, 473–513.
- [8] *Bogovskii, M. E.*, On solution of certain problems of vector analysis associated with operators div and grad , *Trudy sem. S. L. Soboleva*, Novosibirsk, Vol. 1, 1980, 5–40.

- [9] *Amick, C. J.*, Properties of steady Navier-Stokes solutions for certain unbounded channels and pipes, *Nonlinear analysis, Theory, Meth., Appl.*, 2, 1978, 689–720.
- [10] *Maz'ya, V. G., Nazarov, S. A., Plamenevskij, B. A.*, Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. 2. Basel: Birkhäuser Verlag, 2000.
- [11] *Kozlov, V. A., Maz'ya, V. G., Rossmann J.*, Spectral problems associated with corner singularities of solutions to elliptic equations. Vol. 85 AMS, 2001.
- [12] *Kozlov, V. A., Nazarov, S. A.*, A simple one-dimensional model of a false aneurysm in the femoral artery, *J Math Sci*, 214:3, (2016), 287-301
- [13] *Mazja, V.G., Plamenevskii, B.A., Stupyalis, L.T.*, The three-dimensional problem of steady-state motion of a fluid with a free surface, *Trans. Am. Math. Soc.*, 123. 1984, 171–268.
- [14] *Kufner, A.*, Weighted Sobolev Spaces, Teubner Texte zur Mathematik, 31, Leipzig, 1980.