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STOLARSKY'S INVARIANCE PRINCIPLE FOR PROJECTIVE SPACES

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We show that Stolarsky's invariance principle, known for point distributions on the Euclidean spheres, can be extended to the real, complex, and quaternionic projective spaces and the octonionic projective plane.

Key words and phrases: Projective spaces, geometry of distances, discrepancies

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1. Introuction and main results

Let $S^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ be the standard d-dimensional unit sphere in $\mathbb{R}^{d+1}$ with the geodesic (great circle) metric θ and the Lebesgue measure μ normalized by $\mu(S^d) = 1$. We write $C(y,t) = \{x \in S^d : \langle x, y\rangle > t\}$ for the spherical cap of height $t \in [-1,1]$ centered at $y \in S^d$, here $\langle .,.\rangle$ is the inner product in $\mathbb{R}^{d+1}$.

For an N-point subset $\mathcal{D}_N \subset S^d$, the spherical cap quadratic discrepancy is defined by

$$\lambda[\mathcal{D}_N] = \int_{-1}^{1}\int_{S^d} \left(\#\{C(y,t)\cap\mathcal{D}_N\} - N\mu(C(y,t)) \right)^2 \mathrm{d}\mu(y)\,\mathrm{d}t, \tag{1.1}$$

Introduce the following sum of pairwise Euclidean distances $\|.\|$ between points of $\mathcal{D}_N$

$$\tau[\mathcal{D}_N] = \frac{1}{2}\sum_{x_1,x_2\in\mathcal{D}_N} \|x_1 - x_2\| = \sum_{x_1,x_2\in\mathcal{D}_N} \sin\frac{1}{2}\theta(x_1,x_2), \tag{1.2}$$

and write $\langle\tau\rangle$ for the average value of the Euclidean distance on S^d,

$$\langle\tau\rangle = \frac{1}{2}\iint_{S^d\times S^d} \|y_1 - y_2\|\, d\mu(y_1)\, d\mu(y_2). \tag{1.3}$$

The study of the quantities (1.1) and (1.2) falls within the subjects of the discrepancy theory and geometry of distances, see [1,6,21] and references therein. It turns out that the quantities (1.1) and (1.2) are not independent and are intimately related by the following remarkable identity

$$\gamma(S^d)\lambda[\mathcal{D}_N] + \tau[\mathcal{D}_N] = \langle\tau\rangle N^2, \tag{1.4}$$

for an arbitrary N-point subset $\mathcal{D}_N \subset S^d$. Here $\gamma(S^d)$ is a positive constant independent of $\mathcal{D}_N$,

$$\gamma(S^d) = \frac{d\sqrt{\pi}\,\Gamma(d/2)}{2\,\Gamma((d+1)/2)} \sim \sqrt{\pi d/2}\,. \tag{1.5}$$

The identity (1.4) was established by Stolarsky in [24], and known in the literature as Stolarsky's invariance principle. Its original proof has been essentially simplified in [9, 11], particularly, the explicit formula (1.5) has been given in [9, 11]. In our notation $\gamma(S^d) = (2C_d)^{-1}$, where C_d is the constant in [9, Theorem 2.2] and [11, Eq. (6)]. In the present paper we consider the relations of this type in a more general setting. Let $\mathcal{M}$ be a compact metric measure space with a fixed metric θ and a finite Borel measure μ, normalized, for convenience, by

$$\operatorname{diam}(\mathcal{M},\theta) = \pi, \quad \mu(\mathcal{M}) = 1, \tag{1.6}$$

where $\operatorname{diam}(\mathcal{E},\rho) = \sup\{\rho(x_1,x_2) : x_1,x_2 \in \mathcal{E}\}$ denotes the diameter of a subset $\mathcal{E} \subseteq \mathcal{M}$ with respect to a metric ρ.

We write $B(y,r) = \{x : \theta(x,y) < r\}$ for the ball of radius $r \in \mathcal{R}$ centered at $y \in \mathcal{M}$ and of volume $v(y,r) = \mu(B(y,r))$, here $\mathcal{R} = \{r = \theta(x_1,x_2) : x_1,x_2 \in \mathcal{M}\}$ is the set of all possible radii. If the space $\mathcal{M}$ is connected, we have $\mathcal{R} = [0,\pi]$.

We consider distance-invariant metric spaces. Recall that a metric space $\mathcal{M}$ is called distance-invariant, if the volume of any ball $v(r) = v(y,r)$ is independent of $y \in \mathcal{M}$, see [20, p. 504]. The typical examples of distance-invariant spaces are homogeneous spaces $\mathcal{M} = G/K$, where G is a compact group, $K \subset G$ is a closed subgroup, and θ and μ are G-invariant metric and measure on $\mathcal{M}$.

For an N-point subset $\mathcal{D}_N \subset \mathcal{M}$, the ball quadratic discrepancy is defined by

$$\lambda[\xi, \mathcal{D}_N] = \int_{\mathcal{R}} \int_{\mathcal{M}} \left(\#\{B(y,r) \cap \mathcal{D}_N\} - N v(r) \right)^2 \mathrm{d}\mu(y)\, \mathrm{d}\xi(r), \tag{1.7}$$

where ξ is a measure on the set of radii $\mathcal{R}$.

Notice that for spheres S^d spherical caps and balls are related by $C(y,t) = B(y,r)$, $t = \cos r$, and for the the spherical cap and ball discrepancies (1.1) and (1.7) one has $\lambda[\mathcal{D}_N] = \lambda[\xi^\natural, \mathcal{D}_N]$, where $\mathrm{d}\xi^\natural(r) = \sin r\, \mathrm{d}r$, $r \in \mathcal{R} = [0,\pi]$.

The ball quadratic discrepancy (1.7) can be written in the form

$$\lambda[\xi, \mathcal{D}_N] = \sum_{x_1, x_2 \in \mathcal{D}_N} \lambda(\xi, x_1, x_2), \tag{1.8}$$

with the kernel

$$\lambda(\xi, x_1, x_2) = \int_{\mathcal{R}} \int_{\mathcal{M}} \Lambda(B(y,r), x_1)\, \Lambda(B(y,r), x_2)\, \mathrm{d}\mu(y) \mathrm{d}\xi(r), \tag{1.9}$$

where

$$\Lambda(B(y,r), x) = \chi(B(y,r), x) - v(r), \tag{1.10}$$

and $\chi(\mathcal{E}, .)$ denotes the characteristic function of a subset $\mathcal{E} \subseteq \mathcal{M}$.

The symmetry of the metric θ implies the following relation

$$\chi(B(y,r), x) = \chi(B(x,r), y) = \chi^+(r - \theta(x,y)), \tag{1.11}$$

where $\chi^+(.)$ is the characteristic function of the half-axis $(0,\infty)$. Substituting (1.10) into (1.9) and using (1.11), we obtain

$$\lambda(\xi, x_1, x_2) = \int_{\mathcal{R}} \Big(\mu(B(x_1,r) \cap B(x_2,r)) - v(r)^2 \Big)\, \mathrm{d}\xi(r) \tag{1.12}$$

For an arbitrary metric ρ on $\mathcal{M}$ we introduce the sum of pairwise distances

$$\rho[\mathcal{D}_N] = \sum_{x_1, x_2 \in \mathcal{D}_N} \rho(x_1, x_2). \tag{1.13}$$

We write

$$\langle \rho \rangle = \int\limits_{\mathcal{M} \times \mathcal{M}} \rho(y_1, y_2)\, \mathrm{d}\mu(y_1)\, \mathrm{d}\mu(y_2). \tag{1.14}$$

for the average value of a metric ρ.

Introduce the following symmetric difference metrics on the space $\mathcal{M}$

$$\begin{aligned} \theta^\Delta(\xi, y_1, y_2) &= \frac{1}{2} \int_{\mathcal{R}} \mu(B(y_1,r) \Delta B(y_2,r))\, \mathrm{d}\xi(r) \\ &= \frac{1}{2} \int_{\mathcal{R}} \int_{\mathcal{M}} \chi(B(y_1,r) \Delta B(y_2,r), y) \mathrm{d}\mu(y) \mathrm{d}\xi(r), \end{aligned} \tag{1.15}$$

where $B(y_1,r) \Delta B(y_2,r) = B(y_1,r) \cup B(y_2,r) \setminus B(y_1,r) \cap B(y_2,r)$ is the symmetric difference of the balls $B(y_1,r)$ and $B(y_2,r)$. We have

$$\chi(B(y_1,r) \Delta B(y_2,r), y) = |\chi(B(y_1,r), y) - \chi(B(y_2,r), y)|, \tag{1.16}$$

Therefore,

$$\theta^\Delta(\xi, y_1, y_2) = \frac{1}{2} \int_{\mathcal{R}} \int_{\mathcal{M}} |\chi(B(y_1,r), y) - \chi(B(y_2,r), y)| \mathrm{d}\mu(y) \mathrm{d}\xi(r) \tag{1.17}$$

From the other hand, we have

$$\begin{aligned}&\chi(B(y_1,r)\Delta B(y_2,r))\\&=\chi(B(y_1,r),y)+\chi(B(y_2,r),y)-2\chi(B(y_1,r),y)\chi(B(y_2,r),y)\end{aligned}\tag{1.18}$$

Substituting (1.18) into (1.15) and using (1.11), we obtain

$$\theta^{\Delta}(\xi,y_1,y_2)=\int_{\mathcal{R}}\Big(v(r)-\mu(B(y_1,r)\cap B(y_2,r))\Big)\mathrm{d}\xi(r),\tag{1.19}$$

and

$$\langle\theta^{\Delta}(\xi)\rangle=\int_{\mathcal{R}}\Big(v(r)-v(r)^2\Big)\,\mathrm{d}\xi(r).\tag{1.20}$$

Comparing the relations (1.12), (1.19), and (1.20), we arrive at the following Proposition 1.1. Other versions and applications of this result have been discussed in [22].

Proposition 1.1. *Let a compact metric measure space $\mathcal{M}$ with a metric θ and a measure μ be distance-invariant. Then we have*

$$\lambda(\xi,y_1,y_2)+\theta^{\Delta}(\xi,y_1,y_2)=\langle\theta^{\Delta}(\xi)\rangle.\tag{1.21}$$

Particularly, we have the following L_1-invariance principle

$$\lambda(\xi,\mathcal{D}_N)+\theta^{\Delta}(\xi,\mathcal{D}_N)=\langle\theta^{\Delta}(\xi)\rangle N^2\tag{1.22}$$

for an arbitrary N-point subset $\mathcal{D}_N\subset\mathcal{M}$.

The identities (1.21) *and* (1.22) *hold with any measure ξ om the set of radii $\mathcal{R}$ such that the integrals* (1.12), (1.19) *and* (1.20) *converge (for example, with any finite measure ξ).*

Racall that a metric space $\mathcal{M}$ with a metric ρ is called isometrically L_q-embeddable (for $q=1$ or 2), if there exists a map $\varphi:\mathcal{M}\ni x\to\varphi(x)\in L_q$, such that $\rho(x_1,x_2)=\|\varphi(x_1)-\varphi(x_2)\|_{L_q}$ for all $x_1,x_2\in\mathcal{M}$. Notice that the L_2-embeddability is stronger and implies the L_1-embeddability, see [15, Sec. 6.3].

Since the space $\mathcal{M}$ is isometrically L_1-embeddable with respect to the symmetric difference metrics $\theta^{\Delta}(\xi)$, see (1.17), the identity (1.22) is called the L_1-invariance principle, while Stolarsky's invariance principle should be called the L_2-invariance principle, because it involves the Euclidean metric.

In the present paper we shall prove the L_2-invariance principles for compact Riemannian symmetric manifolds of rank one. All such manifolds are homogeneous spaces $\mathcal{M}=G/K$, with compact Lie groups G and $K\subset G$. The complete list of these manifolds is the following, see, for example, [27, Sec. 8.12]:

(i) The d-dimensional Euclidean spheres $S^d=SO(d+1)/SO(d)\times\{1\}$, $d\geqslant 2$, and $S^1=O(2)/O(1)\times\{1\}$.

(ii) The real projective spaces $\mathbb{R}P^n=O(n+1)/O(n)\times O(1)$.

(iii) The complex projective spaces $\mathbb{C}P^n=U(n+1)/U(n)\times U(1)$.

(iv) The quaternionic projective spaces $\mathbb{H}P^n=Sp(n+1)/Sp(n)\times Sp(1)$,

(v) The octonionic projective plane $\mathbb{O}P^2=F_4/\operatorname{Spin}(9)$.

Here we use the standard notation from the theory of Lie groups; particularly, F_4 is one of the exceptional Lie groups in Cartan's classification.

The indicated projective spaces $\mathbb{F}P^n$ as compact Riemannian manifolds have dimensions d,

$$d=\dim_{\mathbb{R}}\mathbb{F}P^n=nd_0,\quad d_0=\dim_{\mathbb{R}}\mathbb{F},\tag{1.23}$$

where $d_0 = 1, 2, 4, 8$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, correspondingly.

For spheres S^d we put $d_0 = d$ by definition. Projective spaces of dimension d_0 $(n = 1)$ are isomorphic to the spheres S^{d_0}: $\mathbb{R}P^1 \approx S^1, \mathbb{C}P^1 \approx S^2, \mathbb{H}P^1 \approx S^4, \mathbb{O}P^1 \approx S^8$. We can conveniently agree that $d > d_0$ $(n \geqslant 2)$ for projective spaces, while the equality $d = d_0$ holds only for spheres. Under this convention, the dimensions $d = nd_0$ and d_0 define uniquely (up to isomorphism) the corresponding two-point homogeneous space which we denote by $Q = Q(d, d_0)$.

We consider $Q(d, d_0)$ as a metric measure space with the metric θ and measure μ proportional to the invariant Riemannian distance and measure on $Q(d, d_0)$. The coefficients of proportionality are defined to satisfy (1.6). In what follows we always assume that $n = 2$ if $\mathbb{F} = \mathbb{O}$, since projective spaces $\mathbb{O}P^n$ do not exist for $n > 2$.

The spaces $Q(d, d_0)$ have a very rich geometrical structure and can be also characterized as compact connected two-point homogeneous spaces. This means that for any two pairs of points x_1, x_2 and y_1, y_2 in $Q(d, d_0)$ with $\theta(x_1, x_2) = \theta(y_1, y_2)$ there exists an isometry $g \in G$, such that $y_1 = gx_1$, $y_2 = gx_2$. In more detail the geometry of spaces $\mathbb{F}P^n$ will be outlined in Section 2.

Any space $Q(d, d_0)$ is distance-invariant and the volume of balls is given by

$$v_r = B(d/2, d_0/2)^{-1} \int_0^r (\sin \frac{1}{2} u)^{d-1} (\cos \frac{1}{2} u)^{d_0 - 1} \, \mathrm{d}u, \quad r \in [0, \pi], \tag{1.24}$$

where $B(.,.)$ is the beta function, see (4.10) Equivalent forms of the relation (1.24) can be found in the literature, see [16, pp. 177–178], [19, pp. 165–168], [20, pp. 508–510].

The chordal metric on the spaces $Q(d, d_0)$ can be defined by

$$\tau(x_1, x_2) = \sin \frac{1}{2} \theta(x_1, x_2), \quad x_1, x_2 \in Q(d, d_0). \tag{1.25}$$

Notice that the expression (1.25) defines a metric because the function $\varphi(\theta) = \sin \theta/2$, $0 \leqslant \theta \leqslant \pi$, is concave, increasing, and $\varphi(0) = 0$, that implies the triangle inequality. For the sphere S^d we have $\cos \theta(x_1, x_2) = (x_1, x_2), x_1, x_2 \in S^d$ and $\tau(x_1, x_2) = \sin \frac{1}{2} \theta(x_1, x_2) = \frac{1}{2} \| x_1 - x_2 \|$.

Each projective space $\mathbb{F}P^n$ can be canonically imbedded into the unit sphere

$$\Pi : Q(d, d_0) \ni x \to \Pi(x) \in S^{m-1} \subset \mathbb{R}^m, \ m = \frac{1}{2}(n+1)(d+2), \tag{1.26}$$

such that

$$\tau(x_1, x_2) = \frac{1}{\sqrt{2}} \| \Pi(x_1) - \Pi(x_2) \|, \quad x_1, x_2 \in \mathbb{F}P^n, \tag{1.27}$$

where $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^m$. Hence, the metric $\tau(x_1, x_2)$ is proportional to the Euclidean length of a segment joining the corresponding points $\Pi(x_1)$ and $\Pi(x_2)$ on the unit sphere and normalized by $\mathrm{diam}(Q(d, d_0), \tau) = 1$. The imbedding (1.26) will be described explicitly in Section 2.

The chordal metric τ on the complex projective space $\mathbb{C}P^n$ is known as the Fubini–Study metric. In connection with special point configurations in two-point homogeneous spaces the chordal metric on projective spaces has been discussed in the papers [12, 13], see also the paper [14], where the chordal metric has been defined for Grassmannian manifolds.

Now we are in position to state our main results.

Theorem 1.1. *For any space $Q = Q(d, d_0)$ the chordal metric* (1.25) *and the symmetric difference metric* (1.15) *are related by*

$$\tau(x_1, x_2) = \gamma(Q)\ \theta^{\Delta}(\eta^{\natural}, x_1, x_2),\ \ x_1, x_2 \in Q, \tag{1.28}$$

where $\eta^{\natural}(r) = \sin r$, $r \in [0, \pi]$, and

$$\gamma(Q) = \frac{\langle \tau \rangle}{\langle \theta^{\Delta}(\eta^{\natural}) \rangle} = \frac{\operatorname{diam}(Q, \tau)}{\operatorname{diam}(Q,\, \theta^{\Delta}(\eta^{\natural}))}. \tag{1.29}$$

The proof of Theorem 1.1 is given in Section 2. It is clear that the equalities (1.29) follow immediately from (1.28). It suffices to calculate the average values (1.14) of both metrics in (1.28) to obtain the first equality in (1.29). Similarly, writing (1.28) for any pair of antipodal points x_1, x_2, $\theta(x_1, x_2) = \pi$, we obtain the second equality in (1.29). Recall that points x_1, x_2 are antipodal for a metric ρ if $\rho(x_1, x_2) = \operatorname{diam}(Q, \rho)$. If points x_1, x_2 are antipodal for the metric θ, then, in view of (1.25) and (1.28), they are also antipodal for the metrics τ and $\theta^{\Delta}(\eta^{\natural})$.

Comparing Theorem 1.1 and Proposition 1.1, we arrive at the following.

Corollary 1.1. *For any space $Q = Q(d, d_0)$ we have the L_2-invariance principle*

$$\gamma(Q)\lambda[\eta^{\natural}, \mathcal{D}_N] + \tau[\mathcal{D}_N] = \langle \tau \rangle N^2 \tag{1.30}$$

for an arbitrary N-point subset $\mathcal{D}_N \subset Q$.

Corollary 1.1 can be thought of as an extension of Stolarsky's invariance principle to projective spaces.

The average value of the chordal metric can be easily calculated in terms of the beta function, see (4.10). Using (1.24), we obtain

$$\begin{aligned}\langle \tau \rangle &= B(d/2, d_0/2)^{-1} \int_0^{\pi} \sin \frac{1}{2}u\ (\sin \frac{1}{2}u)^{d-1}\, (\cos \frac{1}{2}u)^{d_0-1}\, \mathrm{d}u \\ &= B((d+1)/2, d_0/2)\, B(d/2, d_0/2)^{-1}. \end{aligned}\tag{1.31}$$

The explicit calculation of the constant $\gamma(\mathbb{F}P^n)$ is more specific, especially for the real projective space $\mathbb{R}P^n$. Relying on the spherical function expansions for the symmetric difference metrics, see [23, Theorem 8.1(ii)], we shall prove the following general formula.

Theorem 1.2. *For any space $Q(d, d_0)$, we have*

$$\begin{aligned}\gamma(Q(d, d_0)) &= \frac{\sqrt{\pi}}{4}\,(d + d_0)\,\frac{\Gamma(d_0/2)}{\Gamma((d_0+1)/2)} \\ &= \frac{d + d_0}{2d_0}\,\gamma(S^{d_0})\,,\end{aligned}\tag{1.32}$$

where $\gamma(S^{d_0})$ is defined by (1.5).

Theorem 1.2 is proved in section 4. For the sphere $S^d = Q(d, d)$, the relation (1.32) coincides with the formula (1.5). For projective spaces, from (1.32) we obtain the following.

Corollary 1.2. *For projective spaces $\mathbb{F}P^n = Q(nd_0, d_0)$, $d_0 = \dim_{\mathbb{R}} \mathbb{F}$, we have*

$$\gamma(\mathbb{F}P^n) = \frac{n+1}{2}\,\gamma(S^{d_0}). \tag{1.33}$$

Explicitly, we have

$$\left.\begin{aligned} \gamma(\mathbb{R}P^n) &= \frac{n+1}{2}\,\gamma(S^1) = \frac{\pi}{4}\,(n+1), \\ \gamma(\mathbb{C}P^n) &= \frac{n+1}{2}\,\gamma(S^2) = n+1, \\ \gamma(\mathbb{H}P^n) &= \frac{n+1}{2}\,\gamma(S^4) = \frac{4}{3}\,(n+1), \\ \gamma(\mathbb{O}P^2) &= \ \frac{3}{2}\,\gamma(S^8) = \frac{192}{35}. \end{aligned}\right\} \tag{1.34}$$

Comparing the formulas (1.5) and (1.33), (1.34), we observe that for spheres and for projective spaces the behavior of the constant $\gamma(Q(d,d_0))$ differs essentially in large dimensions.

The paper is organized as follows. In Section 2 we define the chordal metrics on the projective spaces $\mathbb{F}P^n$, $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $n\geqslant 2$, and the octonionic projective plane $\mathbb{O}P^2$ in terms of special models for these spaces. For the reader convenience, we describe such models in close detail and give the necessary references.

In Section 3 we prove Theorem 1.1 relaying on the results of Section 2 and a special representation for symmetric difference metrics (Lemma 3.1). For completeness, in Section 3 we also give a simple proof of Stolarsky's invariance principle for the spheres S^d.

In Section 4 we calculate the constants $\gamma(Q(d,d_0))$ and prove Theorem 1.2. As a by-product of our calculations, we obtain explicit formulas for some integrals with Jacobi polynomials. Perhaps, such formulas are known but the author could not find them in the literature. We briefly discuss these formulas at the end of Section 4.

2. Models of projective spaces and chordal metrics

Recall the general facts on the algebras $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$ over the field of real numbers. We have the natural inclusions $\mathbb{R}\subset\mathbb{C}\subset\mathbb{H}\subset\mathbb{O}$. where the octonions $\mathbb{O}$ are a nonassociative and noncommutative algebra of dimension 8 with a basis $1,e_1,e_2,e_3,$ e_4,e_5,e_6,e_7 (their multiplication table can be found in [4, p. 150] and [7, p. 90]), the quaternions $\mathbb{H}$ are an associative but noncommutative subalgebra of dimension 4 spanned by $1,e_1,e_2,e_3$, finally, $\mathbb{C}$ and $\mathbb{R}$ are associative and commutative subalgebras of dimensions 2 and 1 spanned by $1,e_1$ and 1, correspondingly. From the multiplication table one can easily see that for any two indexes $1\leqslant i,j\leqslant 7, i\neq j$, there exists an index $1\leqslant k\leqslant 7$, such that

$$e_ie_j=-e_je_i=e_k,\quad i\neq j,\quad e_i^2=-1. \tag{2.1}$$

Let $a=\alpha_0+\sum_{i=1}^7\alpha_ie_i\in\mathbb{O}$, $\alpha_i\in\mathbb{R}$, $0\leqslant i\leqslant 7$, be a typical octonion. We write $\operatorname{Re}a=\alpha_0$ for the real part, $\bar a=\alpha_0-\sum_{i=1}^7\alpha_ie_i$ for the conjugation, $|a|=\big(\alpha_0^2+\sum_{i-1}^7\alpha_i^2\big)^{1/2}$ fot the norm. Using (2.1), one can easily check that

$$\operatorname{Re}ab=\operatorname{Re}ba,\quad \overline{ab}=\overline{ba},\quad |a|^2=a\bar a=\bar a a,\quad |ab|=|a|\,|b|. \tag{2.2}$$

The last equality in (2.2) implies that $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$ are division algebras. Notice also that by a theorem of Artin a subalgebra in $\mathbb{O}$ generated by any two octonions is associative and isomorphic to one of the algebras $\mathbb{H}$, $\mathbb{C}$, or $\mathbb{R}$, see [4].

The usual model of projective spaces over the associative algebras $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ is the following, see, for example, [4,7,17,27]. Let $\mathbb{F}^{n+1}$ be a linear space of vectors

$\mathbf{a} = (a_0, \ldots, a_n)$, $a_i \in \mathbb{F}$, $1 \leqslant i \leqslant n$ with the right multiplication by scalars $a \in \mathbb{F}$, the Hermitian inner product

$$(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{n} \bar{a}_i b_i, \quad \mathbf{a}, \mathbf{b} \in \mathbb{F}^{n+1}, \tag{2.3}$$

and the norm $|\mathbf{a}|$,

$$|\mathbf{a}|^2 = (\mathbf{a}, \mathbf{a}) = \sum_{i=0}^{n} |a_i|^2. \tag{2.4}$$

In view of associativity of the algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, a projective space $\mathbb{F}P^n$ can be defined as a set of one-dimensional (over $\mathbb{F}$) subspaces in $\mathbb{F}^{n+1}$:

$$\mathbb{F}P^n = \{p(\mathbf{a}) = \mathbf{a}\mathbb{F} : \mathbf{a} \in \mathbb{F}^{n+1},\ |\mathbf{a}| = 1\}. \tag{2.5}$$

The metric θ on $\mathbb{F}P^n$ is defined by

$$\cos \frac{1}{2}\theta(\mathbf{a}, \mathbf{b}) = |(\mathbf{a}, \mathbf{b})|, \quad \mathbf{a}, \mathbf{b} \in \mathbb{F}^{n+1}, \quad |\mathbf{a}| = |\mathbf{b}| = 1,\ 0 \leqslant \theta(\mathbf{a}, \mathbf{b}) \leqslant \pi, \tag{2.6}$$

i.e. $\frac{1}{2}\theta(\mathbf{a}, \mathbf{b})$ is the angle between the subspaces $p(\mathbf{a})$ and $p(\mathbf{b})$. The transitive group of isometries $U(n+1, \mathbb{F})$ for the metric θ consists of nondegenerate linear transformations of the space $\mathbb{F}^{n+1}$, preserving the inner product (2.3), and the stabilizer of a point is isomorphic to the subgroup $U(n, \mathbb{F}) \times U(1, \mathbb{F})$. Hence,

$$\mathbb{F}P^n = U(n+1, \mathbb{F})/U(n, \mathbb{F}) \times U(1, \mathbb{F}). \tag{2.7}$$

The groups $U(n+1, \mathbb{F})$ can be easily determined (they were indicated in section 2).

There is another model where a projective space $\mathbb{F}P^n$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, is identified with the set of orthogonal projectors onto the one-dimensional subspaces $p(\mathbf{a}) \in \mathbb{F}^{n+1}$. This model admits a generalization to the octonionic projective plane $\mathbb{O}P^2$ and in its terms the chordal metric can be naturally defined for all projective spaces.

Let $\mathcal{H}(\mathbb{F}^{n+1})$ denote the set of all Hermitian $(n+1) \times (n+1)$ matrices with the entries in $\mathbb{F}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$,

$$\mathcal{H}(\mathbb{F}^{n+1}) = \{A = ((a_{ij})) : a_{ij} = \overline{a}_{ji},\ a_{ij} \in \mathbb{F},\ 0 \leqslant i, j \leqslant n\} \tag{2.8}$$

where $n = 2$ if $\mathbb{F} = \mathbb{O}$. It is clear that $\mathcal{H}(\mathbb{F}^{n+1})$ is a linear space over $\mathbb{R}$ of dimension

$$m = \dim_{\mathbb{R}} \mathcal{H}(\mathbb{F}^{n+1}) = \frac{1}{2}(n+1)(d+2), \quad d = nd_0. \tag{2.9}$$

The linear space $\mathcal{H}(\mathbb{F}^{n+1})$ is equipped with the symmetric real-valued inner product

$$\langle A, B\rangle = \frac{1}{2}\operatorname{Tr}(AB + BA) = \operatorname{Re}\operatorname{Tr} AB = \operatorname{Re} \sum_{i,j=0}^{n} a_{ij}\overline{b_{ij}} \tag{2.10}$$

and the norm

$$\|A\| = (\operatorname{Tr} A^2)^{1/2} = \left(\sum_{i,j=0}^{n} |a_{ij}|^2\right)^{1/2}, \tag{2.11}$$

here $\operatorname{Tr} A = \sum_{i=0}^{n} a_{ii}$ denotes the trace of a matrix A. For the distance $\|A - B\|$ between matrices $A, B \in \mathcal{H}(\mathbb{F}^{n+1})$, we have

$$\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2\langle A, B\rangle. \tag{2.12}$$

Thus, $\mathcal{H}(\mathbb{F}^{n+1})$ can be thought of as the m-dimensional Euclidean space.

If $\mathbb{F} \neq \mathbb{O}$, the orthogonal projector $\Pi_{\mathbf{a}} \in \mathcal{H}(\mathbb{F}^{n+1})$ onto a one-dimensional subspace $p(\mathbf{a}) = \mathbf{a}\mathbb{F}$, $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{F}^{n+1}$, $|\mathbf{a}| = 1$, can be given by $\Pi_{\mathbf{a}} = \mathbf{a}(\mathbf{a}, \cdot)$ or in the matrix form $\Pi_{\mathbf{a}} = [a_i \bar{a}_j]$, $0 \leqslant i, j \leqslant n$. Therefore, the projective space (2.5) can be written as follows

$$\mathbb{F}P^n = \{\Pi \in \mathcal{H}(\mathbb{F}^{n+1}) : \Pi^2 = \Pi,\ \operatorname{Tr}\Pi = 1\}. \tag{2.13}$$

The group of isometries $U(n+1, \mathbb{F})$ acts on such projectors by the formula $g(\Pi) = g\Pi g^{-1}$, $g \in U(n+1, \mathbb{F})$.

For the octonionic projective plane $\mathbb{O}P^2$ the similar model is also known. A detailed discussion of this model can be found in [4, 7, 17] including an explanation why octonionic projective spaces $\mathbb{O}P^n$ do not exist if $n > 2$. In this model one puts by definition

$$\mathbb{O}P^2 = \{\Pi \in \mathcal{H}(\mathbb{O}^3) : \Pi^2 = \Pi,\ \operatorname{Tr}\Pi = 1\}. \tag{2.14}$$

Thus, the formulas (2.13) and (2.14) are quite similar. One can check that each matrix in (2.14) can be written as $\Pi_{\mathbf{a}} \in \mathbb{O}P^2$ for a vector $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{O}^3$, where $\Pi_{\mathbf{a}} = [a_i \bar{a}_j]$, $0 \leqslant i, j \leqslant 2$, $|\mathbf{a}|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 = 1$, and additionally $(a_0 a_1) a_2 = a_0 (a_1 a_2)$, see [17, Lemma 14.90]. The additional condition means that the subalgebra in $\mathbb{O}$ generated by the coordinates a_0, a_1, a_2 is associative. Using this fact, one can easily show that $\mathbb{O}P^2$ is a 16-dimensional compact connected Riemannian manifold, see [4, 7, 17].

The group of nondegenerate linear transformations g of the space $\mathcal{H}(\mathbb{O}^3)$ preserving the squares $g(A^2) = g(A)^2$, $A \in \mathcal{H}(\mathbb{O}^3)$, is isomorphic to the 52-dimensional exceptional Lie group F_4. This group also preserves the trace, inner product (2.10) and norm (2.11) of matrices $A \in \mathcal{H}(\mathbb{O}^3)$. The group F_4 is transitive on $\mathbb{O}P^2$, and the stabilizer of a point is isomorphic to the spinor group Spin(9), see [17, Lemma 14.96 and Theorem 14.99]. Hence, $\mathbb{O}P^2 = F_4 / \operatorname{Spin}(9)$ is a homogeneous space, and one can prove that $\mathbb{O}P^2$ is a two-point homogeneous space.

For our discussion we need to describe the structure of geodesics in projective spaces. Such a description can be easily done in terms of models (2.13) and (2.14). It is known, see [7, 18, 27], that all geodesics on a two-point homogeneous space $Q(d, d_0)$ are closed and homeomorphic to the unit circle. The group of isometries is transitive on the set of geodesics and the the stabilizer of a point is transitive on the set of geodesics passing through this point. Therefore, all geodesics have the same length 2π (under the normalization (1.6)).

The inclusions $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$ induce the following inclusions of the corresponding projective spaces

$$\mathbb{F}_1 P^{n_1} \subseteq \mathbb{F}P^n, \quad \mathbb{F}_1 \subseteq \mathbb{F}, \quad n_1 \leqslant n, \tag{2.15}$$

moreover, the subspace $\mathbb{F}_1 P^{n_1}$ is a geodesic submanifold in $\mathbb{F}P^n$, see [7, Sec. 3.24]. Particularly, the real projective line $\mathbb{R}P^1$, homeomorphic to the unit circle S^1, is embedded as a geodesic into all projective spaces $\mathbb{F}P^n$,

$$S^1 \approx \mathbb{R}P^1 \subset \mathbb{F}P^n, \tag{2.16}$$

see [7, Proposition 3.32]. In (2.16) $n = 2$ if $\mathbb{F} = \mathbb{O}$. These facts can also be immediately derived from a general description of geodesic submanifolds in Riemannian symmetric spaces, see [18, Chap. VII, Corollary 10.5].

Using the models (2.13) and (2.14), we can write the real projective line $\mathbb{R}P^1$ as the following set of 2×2 matrices:

$$\mathbb{R}P^1 = \{\zeta(u), u \in \mathbb{R}/\pi\mathbb{Z}\}, \tag{2.17}$$

where

$$\zeta(u) = \begin{pmatrix} \cos^2 u & \sin u \cos u \\ \sin u \cos u & \sin^2 u \end{pmatrix} = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos u & \sin u \\ \sin u & \cos u \end{pmatrix}.$$

For each $u \in \mathbb{R}$ the matrix $\zeta(u)$ is an orthogonal projector onto the one-dimensional subspace $x\mathbb{R}$, $x = (\cos u, \sin u) \in S^1$. The embedding $\mathbb{R}P^1$ into $\mathbb{F}P^n$ can be written as the following set of $(n+1) \times (n+1)$ matrices:

$$Z = \{Z(u), u \in \mathbb{R}/\pi\mathbb{Z}\} \subset \mathbb{F}P^n, \tag{2.18}$$

where

$$Z(u) = \begin{pmatrix} \zeta(u) & 0_{n-1,2} \\ 0_{2,n-1} & 0_{n-1,n-1} \end{pmatrix},$$

where $0_{k,l}$ denotes the zero matrix of size $k \times l$. The set of matrices (2.18) is a geodesic in $\mathbb{F}P^n$. All other geodesics are of the form $g(Z)$, where $g \in G$ is an isometry of the space $\mathbb{F}P^n$. The parameter u in (2.18) and the geodesic distance θ on the space $\mathbb{F}P^n$ are related by

$$\theta(Z(u), Z(0)) = 2|u|, \quad -\pi/2 < u \leqslant \pi/2, \tag{2.19}$$

and for all $u \in \mathbb{R}$ this formula can be extended by periodicity. Particularly, we have

$$\theta(Z(u/2), Z(-u/2)) = \begin{cases} 2\min\{u, \pi - u\} & \text{if} \quad 0 \leqslant u \leqslant \pi, \\ 2u & \text{if} \quad 0 \leqslant u \leqslant \pi/2. \end{cases}$$

Therefore,

$$\theta(Z(v), Z(-v)) = 4v, \quad 0 \leqslant v \leqslant \pi/4. \tag{2.20}$$

The relation (2.20) will be needed in the next section.

Now, we define the chordal distance on projective spaces. The formulas (2.13), (2.14) and (2.11) imply

$$\|\Pi\|^2 = \operatorname{Tr} \Pi^2 = \operatorname{Tr} \Pi = 1. \tag{2.21}$$

for any $\Pi \in \mathbb{F}P^n$. Therefore, the projective spaces $\mathbb{F}P^n$, defined by (2.13) and (2.14), are submanifolds in the unit sphere

$$\mathbb{F}P^n \subset S^{m-1} = \{A \in \mathcal{H}(\mathbb{F}^{n+1}) : \|A\| = 1\} \subset \mathcal{H}(\mathbb{F}^{n+1}) \approx \mathbb{R}^m. \tag{2.22}$$

It fact, this is an embedding of $\mathbb{F}P^n$ into the $(m-2)$-dimensional sphere, the intersection of the sphere S^{m-1} with the hyperplane in $\mathcal{H}(\mathbb{F}^{n+1})$ defined by $\operatorname{Tr} A = 1$, see (2.21).

The *chordal distance* $\tau(\Pi_1, \Pi_2)$ between $\Pi_1, \Pi_2 \in \mathbb{F}P^n$ is defined as the Euclidean distance (2.12):

$$\tau(\Pi_1, \Pi_2) = \frac{1}{\sqrt{2}} \|\Pi_1 - \Pi_2\| = (1 - \langle \Pi_1, \Pi_2 \rangle)^{1/2}. \tag{2.23}$$

The coefficient $1/\sqrt{2}$ is chosen to satisfy $\operatorname{diam}(\mathbb{F}P^n, \tau) = 1$.

It is clear from (2.23) that $\tau(g(\Pi_1), g(\Pi_2)) = \tau(\Pi_1, \Pi_2)$ for all isometries $g \in G$ of the space $\mathbb{F}P^n$. Since $\mathbb{F}P^n$ is a two-point homogeneous space, for any $\Pi_1, \Pi_2 \in \mathbb{F}P^n$ with $\theta(\Pi_1, \Pi_2) = 2u$, $0 \leqslant u \leqslant \frac{1}{2}\pi$, there exists $g \in G$, such that $g(\Pi_1) = Z(u)$, $g(\Pi_2) = Z(0)$. From (2.23), (2.18) and (2.17), we obtain $\tau(Z(u), Z(0)) = \sin u = \sin \frac{1}{2}\theta(\Pi(u), \Pi(0))$. Therefore,

$$\tau(\Pi_1, \Pi_2) = \sin \frac{1}{2} \theta(\Pi_1, \Pi_2), \tag{2.24}$$

as it was defined before in (1.25).

Notice also that antipodal points $\Pi_+, \Pi_- \in \mathbb{F}P^n$, i.e. $\theta(\Pi_+, \Pi_-) = \pi$ and $\tau(\Pi_+, \Pi_-) = 1$, can be characterized by the orthogonality condition $\langle \Pi_+, \Pi_- \rangle = 0$, see (2.23), (2.24).

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is relying on the following special representation of the symmetric difference metric (1.15), given earlier in see [22, Lemma 2.1]. Here this representation is given in a form adapted to the chordal metric (1.25).

Lemma 3.1. *We have*

$$\theta^{\Delta}(\xi, y_1, y_2) = \frac{1}{2} \int_{\mathcal{M}} |\sigma(\theta(y_1, y)) - \sigma(\theta(y_2, y))| \, \mathrm{d}\mu(y) \tag{3.1}$$

with the non-increasing function

$$\sigma(r) = \xi([r, \pi]) = \int_r^{\pi} \mathrm{d}\xi(u). \tag{3.2}$$

Particularly, if $\mathcal{M}$ is a two-point homogeneous space $Q = Q(d, d_0)$ and the measure $\mathrm{d}\xi^{\natural}(r) = \sin r \, \mathrm{d}r$, then

$$\theta^{\Delta}(\xi^{\natural}, y_1, y_2) = \int_Q |\tau(y_1, y)^2 - \tau(y_2, y)^2| \, \mathrm{d}\mu(y), \tag{3.3}$$

where $\tau(\cdot, \cdot)$ is the chordal metric (1.25) *on $Q(d, d_0)$.*

Proof. For brevity, we write $\theta(y_1, y) = \theta_1$ and $\theta(y_2, y) = \theta_2$. Using (1.15), (1.11) and (1.18), we obtain

$$\begin{aligned}
&\theta^{\Delta}(\xi, y_1, y_2) \\
&= \frac{1}{2} \int_{\mathcal{M}} \left(\int_0^{\pi} (\chi(r - \theta_1) + \chi(r - \theta_2) - 2\chi(r - \theta_1)\chi(r - \theta_2)) \mathrm{d}\xi(r) \, dr \right) \mathrm{d}\mu(y) \\
&= \frac{1}{2} \int_{\mathcal{M}} (\sigma(\theta_1) + \sigma(\theta_2) - 2\sigma(\max\{\theta_1, \theta_2\})) \, \mathrm{d}\mu(y).
\end{aligned} \tag{3.4}$$

Since σ is a non-increasing function, we have

$$2\sigma(\max\{\theta_1, \theta_2\}) = 2\min\{\sigma(\theta_1), \sigma(\theta_2)\} = \sigma(\theta_1) + \sigma(\theta_2) - |\sigma(\theta_1) - \sigma(\theta_2)|. \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain (3.1).

If $\mathrm{d}\,\xi^{\natural}(r) = \sin r \, \mathrm{d}r$, then $\sigma^{\natural}(r) = 2 - 2(\sin r/2)^2$. Substituting this expression into (3.1) and using the definition (2.24), we obtain (3.3). □

For completeness, we give in the beginning a very short proof of Theorem 1.1 in the case of spheres.

Proof of Theorem 1.1 for spheres. For the sphere S^d the chordal metric τ is defined by (1.25). We have

$$\begin{aligned}
\tau(y_1, y)^2 - \tau(y_2, y)^2 &= \frac{1}{4}(\|y_1 - y\|^2 - \|y_2 - y\|^2) \\
&= \frac{1}{2}(y_2 - y_1, y) = \tau(y_1, y_2)(x, y), \; y_1, y_2 \in S^d,
\end{aligned} \tag{3.6}$$

where $x = \|y_2 - y_1\|^{-1}(y_2 - y_1) \in S^d$. Substituting (3.6) into (3.3), we obtain

$$\theta^{\Delta}(\xi^{\natural}, y_1, y_2) = \tau(y_1, y_2) \int_{S^d} |(x, y)| \, \mathrm{d}\mu(y). \tag{3.7}$$

It is clear that the integral in (3.7) is independent of $x \in S^d$. This proves the equality (1.28) for S^d with the constant

$$\gamma(S^d) = \left(\int_{S^d} |(x,y)| \, d\mu(y) \right)^{-1} . \tag{3.8}$$

This completes the proof. $\square$

Notice that the integral (3.8) over S^d can be easily calculated to obtain (1.5), see [9, 11].

Proof of Theorem 2.1 for projective spaces. We write Π_1, Π_2, Π for points in the models of projective spaces (2.13) and (2.14). With this notation, the relation (3.3) takes the form

$$\theta^{\Delta}(\xi^{\natural}, \Pi_1, \Pi_2) = \int_{\mathbb{F}P^n} |\tau(\Pi_1, \Pi)^2 - \tau(\Pi_2, \Pi)^2| \, \mathrm{d}\mu(\Pi). \tag{3.9}$$

Since $\mathbb{F}P^n$ is a two-point homogeneous space, for $\Pi_1, \Pi_2 \in \mathbb{F}P^n$ with $\theta(\Pi_1, \Pi_2) = 4v$, $0 \leqslant v \leqslant \pi/4$, there exists an isometry $g \in G$, such that $g(\Pi_1) = Z(v)$, $g(\Pi_2) = Z(-v)$, see (2.20). Therefore,

$$\int_{\mathbb{F}P^n} |\tau(\Pi_1, \Pi)^2 - \tau(\Pi_2, \Pi)^2| \, \mathrm{d}\mu(\Pi) = \int_{\mathbb{F}P^n} |\tau(Z(v), \Pi)^2 - \tau(Z(-v), \Pi)^2| \, \mathrm{d}\mu(\Pi). \tag{3.10}$$

From the definition (2.23), we obtain

$$\begin{aligned} \tau(Z(v), \Pi)^2 - \tau(Z(-v), \Pi)^2 &= \frac{1}{2}(\|Z(v) - \Pi\|^2 - \|Z(-v) - \Pi\|^2) \\ &= \langle Z(v) - Z(-v), \Pi \rangle. \end{aligned} \tag{3.11}$$

The formulas (2.17) and (2.18) imply

$$Z(v) - Z(-v) = \begin{pmatrix} \zeta(v) - \zeta(-v) & 0_{n-1,2} \\ 0_{2,n-1} & 0_{n-1,n-1} \end{pmatrix}$$

and

$$\zeta(v) - \zeta(-v) = \begin{pmatrix} 0 & \sin 2u \\ \sin 2u & 0 \end{pmatrix} = \sin 2u (\zeta_+ - \zeta_-),$$

where

$$\zeta_+ = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \zeta_- = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Therefore,

$$Z(v) - Z(-v) = \sin 2v (Z_+ - Z_-), \tag{3.12}$$

where

$$Z_{\pm} = \begin{pmatrix} \zeta_{\pm} & 0_{n-1,2} \\ 0_{2,n-1} & 0_{n-1,n-1} \end{pmatrix}.$$

We have $Z_{\pm}^* = Z_{\pm}$, $Z_{\pm}^2 = Z_{\pm}$, $\operatorname{Tr} Z_{\pm} = 1$, i.e. $Z_{\pm} \in \mathbb{F}P^n$, and $\langle Z_+, Z_- \rangle = 0$, i.e. Z_+ and Z_- are antipodal points. Using (2.24), we can write

$$\tau(\Pi_1, \Pi_2) = \tau(Z(v), Z(-v)) = \sin 2v,$$

and the equality (3.12) takes the form

$$Z(v) - Z(-v) = \tau(\Pi_1, \Pi_2)(Z_+ - Z_-). \tag{3.13}$$

Substituting (3.13) into (3.11), we find that

$$\tau(Z(v), \Pi)^2 - \tau(Z(-v), \Pi)^2 = \tau(\Pi_1, \Pi_2) \langle Z_+ - Z_-, \Pi \rangle. \tag{3.14}$$

Substituting (3.14) into (3.10) and using (3.9), we obtain

$$\theta^{\Delta}(\xi^{\natural}, \Pi_1, \Pi_2) = \tau(\Pi_1, \Pi_2)\theta^{\Delta}(\xi^{\natural}, Z_+, Z_-), \tag{3.15}$$

where

$$\theta^{\Delta}(\xi^{\natural}, Z_+, Z_-) = \int_{\mathbb{F}P^n} |\langle Z_+ - Z_-, \Pi\rangle| \,\mathrm{d}\mu(\Pi). \tag{3.16}$$

The integral (3.16) is independent of Π_1, Π_2, This proves the equality (1.28) for $\mathbb{F}P^n$ with the constant

$$\gamma(\mathbb{F}P^n) = \left(\int_{\mathbb{F}P^n} |\langle Z_+ - Z_-, \Pi\rangle| \,\mathrm{d}\mu(\Pi)\right)^{-1}. \tag{3.17}$$

Notice that in this formula any pair of antipodal points in $\mathbb{F}P^n$ can be taken instead of Z_+, Z_-. The proof of Theorem 1.1 is complete. □

It is not quite clear how the integral (3.17) over $\mathbb{F}P^n$ could be calculated immediately. In the next section we shall use another way to calculate the constant $\gamma(\mathbb{F}P^n)$.

4. Proof of Theorem 1.2

The zonal spherical functions ϕ_l for the spaces $Q = Q(d, d_0)$ are eigenfunctions of the radial part of the Laplace–Beltrami operator on Q and can be found explicitly, see [16, p. 178], [19, Chap. V, Theorem 4.5], [20, pp. 514–512, 543–544], [28, Theorem. 11.4.21]. We have

$$\phi_l(Q, x_1, x_2) = \frac{P_l^{(\alpha,\beta)}(\cos\theta(x_1, x_2))}{P_l^{(\alpha,\beta)}(1)}, \quad l \geqslant 0,\ x_1, x_2 \in Q, \tag{4.1}$$

where $P_l^{(\alpha,\beta)}(t)$, $t \in [-1, 1]$, denotes the Jacobi polynomial of degree l with parameters

$$\alpha = d/2 - 1, \quad \beta = d_0/2 - 1, \quad \alpha,\, \beta \geqslant -1/2. \tag{4.2}$$

They can be given by Rodrigues' formula

$$P_l^{(\alpha,\beta)}(t) = \frac{(-1)^l}{2^l l!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^l}{dt^l}\left\{(1-t)^{l+\alpha}(1+t)^{l+\beta}\right\}. \tag{4.3}$$

We also have the bound

$$|P_l^{(\alpha,\beta)}(t)| \leqslant P_l^{(\alpha,\beta)}(1) = \binom{\alpha+l}{l} = \frac{(\alpha+1)\dots(\alpha+l)}{l!} \simeq l^{\alpha}, \quad t \in [-1, 1]. \tag{4.4}$$

Jacobi polynomials form a complete orthogonal system in the L_2 on the segment $[-1, 1]$ with the weight $(1-t)^{\alpha}(1+t)^{\beta}$. We have the orthogonality relations

$$\int_{-1}^{1} P_l^{(\alpha,\beta)}(t)P_{l'}^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta}\,\mathrm{d}t = \delta_{ll'}\, 2^{\alpha+\beta+1}\, M_l^{-1}, \tag{4.5}$$

where $\delta_{ll'}$ is Kronecker's symbol and

$$M_l = (2l + \alpha + \beta + 1)\frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)} \simeq l. \tag{4.6}$$

Using the orthogonality relations (4.5), we formally obtain for a function $f(t)$, $t \in [-1, 1]$, the following expansion

$$f(t) = \sum_{l \geqslant 0} 2^{-\alpha-\beta-1}\, M_l\, c_l\, P_l^{(\alpha,\beta)}(t), \tag{4.7}$$

where

$$c_l = \int_{-1}^{1} f(t)\,(1-t)^{\alpha}(1+t)^{\beta}\, P_l^{(\alpha,\beta)}(t)\,\mathrm{d}t \tag{4.8}$$

If the function $f(t)$ is differentiable for $t \in (-1,1)$ and $f^{(l)}(t)\,(1-t)^{\alpha+l}(1+t)^{\beta+l}$ vanish at $t=\pm 1$ for all l, then substituting Rodrigues' formula (4.3) into (4.8) and integrating l times by part, we obtain

$$c_l = \frac{1}{2^l l!}\int_{-1}^{1} f^{(l)}(t)\,(1-t)^{\alpha+l}(1+t)^{\beta+l}\,\mathrm{d}t. \tag{4.9}$$

A detailed consideration of Jacobi polynomials can be found in [2,3,25].

In what follows, we always assume that the parameters α,β and the dimensions d, d_0 are related by (4.2). We shall also use the formulas for the beta function

$$\begin{aligned} B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} &= \int_0^{\pi} (\sin\frac{1}{2}u)^{2a-1}(\cos\frac{1}{2}u)^{2b-1}\,\mathrm{d}u \\ &= 2^{1-a-b}\int_{-1}^{1}(1-t)^{a-1}\,(1+t)^{b-1}\,\mathrm{d}t\,, \end{aligned} \tag{4.10}$$

and the standard notation

$$(a)_0 = 1,\; (a)_l = a(a+1)\dots(a+l-1) = \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)}. \tag{4.11}$$

Lemma 4.1. *For any space $Q = Q(d,d_0)$, the symmetric difference metric $\theta^{\Delta}(\xi,y_1,y_2)$, see* (1.15)*, and the chordal metric $\tau(y_1,y_2)$, see* (1.25)*, have the following zonal spherical function expansions*

$$\theta^{\Delta}(\xi,y_1,y_2) = B(d/2,d_0/2)^{-1}\sum_{l\geqslant 1} l^{-2} M_l A_l(\xi)\,[1-\phi_l(Q,x_1,x_2)]\,, \tag{4.12}$$

where

$$A_l(\xi) = \int_0^{\pi}(\sin\frac{1}{2}r)^{2d}(\cos\frac{1}{2}r)^{2d_0}\left\{P_{l-1}^{(\alpha+1,\beta+1)}(\cos r)\right\}^2\,\mathrm{d}\xi(r), \tag{4.13}$$

and

$$\tau(y_1,y_2) = \frac{1}{2}\sum_{l\geqslant 1} M_l\, C_l\,\left[1-\phi_l(Q,x_1,x_2)\right], \tag{4.14}$$

where

$$C_l = B(\alpha+3/2,\beta+l+1)\,\Gamma(l+1)^{-1}\,(1/2)_{l-1}\,P_l^{(\alpha,\beta)}(1)\,. \tag{4.15}$$

The series (4.12) *and* (4.14) *converge absolutely and uniformly.*

Proof. The expansion (4.12) has been established in [23, Theorem 9.1(ii)]. The proof is based on the observation that the term $\mu(B(y_1,r)\cap B(y_2,r))$ in the formula (1.19) can be thought of as a convolution of the characteristic functions of the balls on the homogeneous space $Q(d,d_0)$. Since the spaces $Q(d,d_0)$ have rank one, such a convolution can be calculated explicitly. We refer to [23, Sections 8 and 9] for details.

The proof of the expansion (4.14) is much easily. Applying the formulas (4.7) – (4.9) to the function $f(t)=(1-t)^{1/2}$ and using (4.10), we obtain the expansion

$$\begin{aligned} (1-t)^{1/2} &= 2^{1/2}\,\Gamma(\alpha+3/2)\times \\ &\sum_{l\geqslant 0}\frac{(2l+\alpha+\beta+1)\,\Gamma(l+\alpha+\beta+1)\,(-1/2)_l}{\Gamma(l+\alpha+1)\,\Gamma(l+\alpha+\beta+3/2)}\;P_l^{(\alpha,\beta)}(t). \end{aligned} \tag{4.16}$$

This expansion can be found in [3, Sec.10.20, Eq.(3)]. However, it should be noted that the co-factor $(2l+\alpha+\beta+1)$ in (4.16) is misprinted in [3, Sec.10.20, Eq.(3)] as $\Gamma(2l+\alpha+\beta+1)$.

Applying Stirling's approximation to the gamma functions in (4.16), and taking into account the bound (4.4), we observe that the coefficients in (4.16) are of the order $O(l^{-2})$. Therefore, the series (4.16) converges absolutely and uniformly.

Since $(-1/2)_0 = 1$ and $(-1/2)_l = -1/2\,(1/2)_{l-1}$ for $l \geqslant 1$, the series (4.16) can be written as follows

$$\left(\frac{1-t}{2}\right)^{1/2} = M_0\, C_0 - \sum_{l\geqslant 1} M_l\, C_l\, \frac{P_l^{(\alpha,\beta)}(t)}{P_l^{(\alpha,\beta)}(1)}\,, \tag{4.17}$$

where $C_0 = B(\alpha+3/2, \beta+1)$ and $C_l, l \geqslant 1$ are given in (4.15). Putting $t = 0$, we find

$$M_0\, C_0 = \sum_{l\geqslant 1} M_l\, C_l\,. \tag{4.18}$$

Combining (4.17) and (4.18), we obtain

$$\left(\frac{1-t}{2}\right)^{1/2} = \sum_{l\geqslant 1} M_l\, C_l\, \left[1 - \frac{P_l^{(\alpha,\beta)}(t)}{P_l^{(\alpha,\beta)}(1)}\right]\,. \tag{4.19}$$

For $t = \cos(x_1, x_2)$, the equality (4.19) coincides with (4.14). $\square$

Proof of Theorem 1.2. Substituting (4.12) and (4.14) into (1.28) and equating coefficients at ϕ_l, we obtain the following series of equations

$$\gamma(Q) A_l(\xi^\natural) = \frac{(1/2)_{l-1} l^2}{2\Gamma(l+1)} \binom{\alpha+l}{l} B(d/2, d_0/2)\, B((d+1)/2, l+d_0/2)\,, \tag{4.20}$$

where

$$A_l(\xi^\natural) = 2\int_0^\pi (\sin\frac{1}{2}r)^{2d+1} (\cos\frac{1}{2}r)^{2d_0+1} \left\{P_{l-1}^{(\alpha+1,\beta+1)}(\cos r)\right\}^2\, \mathrm{d}r\,. \tag{4.21}$$

Each of these equations can be used to determine the constant $\gamma(Q)$. In the simplest case of $l = 1$, we have

$$\gamma(Q) A_1(\xi^\natural) = \frac{d}{4}\, B(d/2, d_0/2)\, B((d+1)/2, 1+d_0/2) \tag{4.22}$$

and

$$A_1(\xi^\natural) = 2\int_0^\pi (\sin\frac{1}{2}r)^{2d+1} (\cos\frac{1}{2}r)^{2d_0+1}\, \mathrm{d}r = 2B(d+1, d_0+1), \tag{4.23}$$

see (4.10). Therefore,

$$\gamma(Q) = \frac{d\, B(d/2, d_0/2)\, B((d+1)/2, 1+d_0/2)}{8B(d+1, d_0+1)} \tag{4.24}$$

In the terms of gamma functions, we have

$$\gamma(Q) = \frac{d\Gamma(d/2)\,\Gamma(d_0/2)^2\,\Gamma((d+1)/2)\,\Gamma(d+d_0+2)}{16\,\Gamma(d)\,\Gamma(d_0)\,\Gamma((d+d_0)/2)\,\Gamma((d+d_0+3)/2)}\,, \tag{4.25}$$

where the relation $\Gamma(z+1) = z\Gamma(z)$ has been used. Applying the duplication formula $\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z)\Gamma(z+1/2)$ to the terms $\Gamma(d)\,, \Gamma(d_0)$ and $\Gamma(d+d_0+2)$, we obtain

$$\gamma(Q) = \frac{\sqrt{\pi}}{4}\,(d+d_0)\,\frac{\Gamma(d_0/2)}{\Gamma((d_0+1)/2)}\,. \tag{4.26}$$

This completes the proof. $\square$

In conclusion, it is worth noting that the equalities (4.20) with the constant (4.26) define explicit formulas for the integrals (4.21). We have

$$\int_{-1}^{1} \left(P_{l-1}^{(d/2,d_0/2)}(t)\right)^2 \left(\frac{1-t}{2}\right)^d \left(\frac{1+t}{2}\right)^{d_0} \mathrm{d}t$$

$$= 4l^2 (1/2)_{l-1} \binom{d/2+l-1}{l} \frac{B(d/2+1/2, d_0/2+l)\, B(2d+1, 2d_0+1)}{dB(d/2+1/2, d_0/2+1)}\,, \tag{4.27}$$

where $l = 1, 2, \dots,$ and d, d_0 are the following.

First of all, $d_0 = d$ and d is a natural number. In this case $P_l^{(d/2,d/2)}(t)$ are proportional to Gegenbauer polynomials. Next, $d_0 = 1, 2, 4$ and $d = nd_0$, where n is a natural number. Finally, the equality (4.27) holds in the exceptional case of $d_0 = 8, d = 16$.

Notice that in the first two cases both sides of (4.27) are holomorphic functions of polynomial growth for complex d, $\operatorname{Re} d > 0$, and the well-known interpolation arguments, see [2, Section 2.8], show that the equalities (4.27) hold also for all complex d, $\operatorname{Re} d > 0$.

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