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BOUNDS FOR L_p-DISCREPANCIES OF POINT DISTRIBUTIONS IN COMPACT METRIC SPACES

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We consider finite point subsets (distributions) in compact connected metric measure spaces. The spaces under study are specialized by conditions on the volume of metric balls as a function of radii. The conditions are not hard and hold, particularly, for all compact Riemannian manifolds. Under these conditions we prove nontrivial upper bounds for the L_p-discrepancies of point distributions for any $p > 0$ and $p = \infty$ (Theorem 1.1 and Corollary 1.1). The order of these bounds is sharp, at least, for compact Riemannian symmetric manifolds of rank one and $2 \leqslant p < \infty$.

Key words and phrases: Metric measure spaces, random distribution.

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1. Introduction

In the present paper we continue the study of point distributions in compact metric measure spaces. In the previous papers [11, 12] the L_2-discrepancies of point distributions were investigated. In [11] the upper bounds for the L_2-discrepancies were given for general compact rectifiable spaces while the lower bounds were established in [12] for compact Riemannian symmetric manifolds of rank one.

In the present paper we consider the L_p-discrepancies of point distributions in compact connected metric measure spaces satisfying simple conditions on the volume of metric balls. Particularly, these conditions hold for all compact Riemannian manifolds. For such spaces and for $0 < p \leqslant \infty$, we prove nontrivial upper bounds for the L_p-discrepancies. For $2 \leqslant p < \infty$ and for compact Riemannian symmetric manifolds of rank one, the order of these bounds turns out to be sharp as it follows from the lower bounds for the L_2-discrepancies given in [12].

Let $\mathcal{M}$ be a compact metric measure space with a fixed metric θ and a finite non-negative Borel measure μ, normalized, for convenience, by

$$\mu(\mathcal{M}) = 1, \quad \operatorname{diam} \mathcal{M} = 1, \tag{1.1}$$

where $\operatorname{diam} \mathcal{E} = \sup\{\theta(y_1, y_2), y_1, y_2 \in \mathcal{E}\}$ denotes the diameter of a set $\mathcal{E} \subseteq \mathcal{M}$.

Since $\mathcal{M}$ is connected and satisfies (1.1), the set of values of θ coincides with the interval $\mathcal{I} = [0, 1]$. We write $B(y, r) = \{x : \theta(x, y) < r\}$ for the ball in $\mathcal{M}$ of radius $r \in \mathcal{I}$ centered at $y \in \mathcal{M}$ and of volume $v(y, r) = \mu(B(y, r))$. We can conveniently write $B(y, r) = \emptyset$ and $v(y, r) = 0$ if $r \leqslant 0$ and $B(y, r) = \mathcal{M}$ and $v(y, r) = 1$ if $r \geqslant 1$.

We specialize the spaces $\mathcal{M}$ by the following two conditions.

Condition A. The volume $v(y, r)$ satisfies the bounds

$$c_1^{-1} r^d \leqslant v(y, r) \leqslant c_1 r^d, \quad y \in \mathcal{M},\ r \in \mathcal{I}, \tag{1.2}$$

with positive constants d and c_1 independent of $y \in \mathcal{M}$ and $r \in \mathcal{I}$

The spaces satisfying the Condition A are known as Ahlfors regular spaces, see, for example, [7].

In the following, we write consecutively $c_1, c_2, c_3, \dots$ for positive constants depending only on $\mathcal{M}$.

Condition B. The volume as a function of r is Lipschitz continuous:

$$|v(y, r_1) - v(y, r_2)| \leqslant c_2 |r_1 - r_2|, \quad y \in \mathcal{M},\ r_1, r_2 \in \mathcal{I}. \tag{1.3}$$

It is not difficult to give many examples of compact spaces satisfying both Conditions A and B. Particularly, a Riemannian manifold can be thought of as a mertric measure space with respect to the Riemannian distance and measure, and the following is true.

Proposition 1.1. *Any compact d-dimensional Riemannian manifold satisfies the Conditions A and B.*

The Condition A is well-known for compact Riemannian manifolds, see, for example, [8, 10], while the Condition B is a little more specific, it can be derived from the Bishop–Gromov volume comparison theorem. For completeness, we shall give a short proof of Proposition 1.1 in Appendix in Section 4.

The *local discrepancy* of an N-point subset $\mathcal{D}_N \subset \mathcal{M}$ (distribution) in a metric ball $B(y,r)$ is defined by

$$L[B(y,r),\mathcal{D}_N] = \#(B(y,r)\cap\mathcal{D}_N) - Nv(y,r)) \\ = \sum_{x\in\mathcal{D}_N} L(y,r,x), \tag{1.4}$$

where

$$L(y,r,x) = \chi(B(y,r),x) - v(y,r), \tag{1.5}$$

and $\chi(\mathcal{E},x)$ denotes the characteristic function of a subset $\mathcal{E}\subset\mathcal{M}$.

The L_p-*discrepancy* is defined by

$$\mathcal{L}_p[\xi,\mathcal{D}_N] = \left(\iint_{\mathcal{M}\times\mathcal{I}} L[y,r,\mathcal{D}_N]^p \,\mathrm{d}\mu(y)\,\mathrm{d}\xi(r)\right)^{1/p}, \quad 0<p<\infty, \tag{1.6}$$

where ξ is a finite (non-negative) measure on $\mathcal{I}$ normalized by $\xi(\mathcal{I})=1$. For $p=\infty$, we put

$$\mathcal{L}_\infty[\mathcal{D}_N] = \sup_{y,r} L[y,r,\mathcal{D}_N], \tag{1.7}$$

where the supremum is taken over all balls $B(y,r)\subset\mathcal{M}$.

We introduce also the following extremal discrepancies

$$\lambda_p[\xi,N] = \inf_{\mathcal{D}_N} \mathcal{L}_p[\xi,\mathcal{D}_N], \quad \lambda_\infty[N] = \inf_{\mathcal{D}_N}\mathcal{L}_\infty[\mathcal{D}_N], \tag{1.8}$$

where the infimum is taken over all N-point subsets $\mathcal{D}_N\subset\mathcal{M}$.

Now we are in position to state our main results.

Theorem 1.1. *Let $\mathcal{M}$ be a compact connected metric measure space satisfying the Conditions A and B. Then for all N we have the bound*

$$\lambda_p[\xi,N] \leqslant c_3(p+1)^{\frac{1}{2}} N^{\frac{1}{2}-\frac{1}{2d}}, \quad 0<p<\infty, \tag{1.9}$$

where ξ is an arbitrary normalized measure on $\mathcal{I}$.

Particularly, the bound (1.9) *holds for any compact Riemannian manifold of dimension d.*

The proof of Theorem 1.1 is given in Section 3. In its proof, special random N-point distributions will be used. Such random distributions are constructed in terms of partitions of the space $\mathcal{M}$ into N subsets of equal measure and small diameters. The local discrepancies of such distributions can be written as sums of random independent variables, and the Marcinkiewicz–Zigmund inequality can be applied to obtain the bound (1.9).

In (1.9), the dependence on the exponent p is described explicitly. This allows us to obtain upper bounds for the extremal L_∞-discrepancy. For this purpose, we use the following *a priory estimate*, which is also of interest by itself.

Proposition 1.2. *Let the assumptions of Theorem* 1.1 *hold. Then for an arbitrary N-point subset $\mathcal{D}_N\subset\mathcal{M}$, we have*

$$\mathcal{L}_\infty[\mathcal{D}_N] \leqslant 2m^{2/p}\mathcal{L}_p[\xi_0,\mathcal{D}_N] + c_4\,Nm^{-1/d}, \tag{1.10}$$

where ξ_0 is the standard Lebesgue measure on $\mathcal{I}$, while $p>1$ and integer $m\geqslant c_5$ are arbitrary parameters. Particularly, we have

$$\lambda_\infty[N] \leqslant 2m^{2/p}\lambda_p[\xi_0,N] + c_4\,Nm^{-1/d}. \tag{1.11}$$

The proof of Proposition 1.2 is given in Section 2.

Comparing Theorem 1.1 and Proposition 1.2, we arrive at the following.

Corollary 1.1. *Let the assumptions of Theorem* 1.1 *hold. Then for all N we have*

$$\lambda_\infty[N] \leqslant c_6\, N^{\frac{1}{2}-\frac{1}{2d}}\,(\log N)^{1/2}. \tag{1.12}$$

Proof. Putting $m = N^d$ in (1.11) and using (1.9), we obtain

$$\lambda_\infty[N] \leqslant 2N^{\frac{2d}{p}}\lambda_p[\xi_0, N] + c_4 \leqslant 2c_3 N^{\frac{2d}{p}}(p+1)^{1/2} N^{\frac{1}{2}-\frac{1}{2d}} + c_4.$$

Now, we choose $p = 2d\log N$ (with the log in base 2, say) to obtain

$$\lambda_\infty[N] \leqslant 2c_3 N^{\frac{1}{2}-\frac{1}{2d}}(2d\log N + 1)^{1/2} + c_4 \leqslant c_6 N^{\frac{1}{2}-\frac{1}{2d}}\,(\log N)^{1/2},$$

that completes the proof. □

Under such general assumptions one cannot expect that the bounds (1.9) are best possible. The corresponding counterexample can be found in [5, 11, 12]. In this counterexample the space is the d-dimensional Euclidean sphere S^d, the measure ξ is atomic and concentrated at the point $r = 1/2$ and all discrepancies $\lambda_p[N]$ are bounded by a constant independent of N and p.

One can conjecture that if the measure ξ is absolutely continuous on $\mathcal{I}$, then the order of the bounds (1.9) is the best possible. In the paper [12] this conjecture was proved for $2 \leqslant p < \infty$ and all compact Riemannian symmetric manifolds of rank one (two-point homogeneous spaces). Recall that these manifolds are the spheres S^d, the real, complex and quaternionic projective spaces $\mathbb{F}P^n, \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and the octonionic projective plane $\mathbb{O}P^2$, see, for example, [10].

Point distributions on the spheres S^d have been studied by many authors, see the surveys [2, 6] and references therein. Mention should be made of the following results intimately related with the context of the present paper. For the L_2-discrepancy of point distributions on the spheres S^d, the bound (1.9) with the special measure $d\xi(r) = \frac{\pi}{2}\sin(\pi r)dr, r \in \mathcal{I}$ has been established by Alexander [1] and Stolarsky [13], see also [4, pp.237–239]. Beck [3] proved that this bound is sharp, see also [4, Corollary 24C]. For the L_∞-discrepancy of point distributions on the spheres S^d, Beck proved the bound (1.12), see [4, Theorem 24D]; the proof is based on the large-deviation inequalities for sums of random variables.

The present paper is organized as follows. In Section 2 we describe the necessary facts on partitions of metric measure spaces and prove Proposition 1.2. In Section 3 we describe the construction of random point distributions and prove Theorem 1.1. Finally, in Section 4 we prove Proposition 1.1.

2. Partitions of metric spaces. Proof of Proposition 1.2

The following general result is due to Gigante and Leopardi [9, Theorem 2].

Lemma 2.1. *Let $\mathcal{M}$ be a compact connected metric measure space satisfying the Condition A. Then for all sufficiently large $m > c_7$ there exists a partition $P_m = \{\mathcal{P}_j\}_1^m$ of $\mathcal{M}$ into m subsets $\mathcal{P}_j$ with the following properties*

$$\mathcal{M} = \bigcup_{1\leqslant j\leqslant m} \mathcal{P}_j, \quad \mathcal{P}_j \cap \mathcal{P}_i = 0, \quad j \neq i, \quad \mu(\mathcal{P}_j) = m^{-1}, \quad 1 \leqslant j \leqslant m \tag{2.1}$$

and

$$c_8^{-1}\, m^{-1/d} \leqslant \operatorname{diam}\mathcal{P}_j \leqslant c_8\, m^{-1/d}, \quad 1 \leqslant j \leqslant m \tag{2.2}$$

Partitions with such properties occur in many fields of geometry and analysis. For special spaces, such as the spheres S^d, they have long been in use. In the general case, the proof of Lemma 2.1 given in [9] relies on a nontrivial construction of the so-called 'dyadic cubes' in Alhfors regular spaces [8].

We wish to give some simple corollaries of Lemma 2.1 needed for the proofs of Theorem 1.1 and Proposition 1.1. We write $\Sigma(y,r)=\{x:\theta(x,y)=r\}$ for the sphere in $\mathcal{M}$ of radius $r\in\mathcal{I}$ centered at $y\in\mathcal{M}$. For a partition $P_m=\{\mathcal{P}_j\}_1^m$ of $\mathcal{M}$, we put

$$\left.\begin{aligned} J_m=J_m(y,r)=\{j:\Sigma(y,r)\cup\mathcal{P}_j\neq\emptyset\},\\ K_m=K_m(y,r)=\#\{J_m(y,r)\}.\end{aligned}\right\} \tag{2.3}$$

Thus, K_m is the number of subsets $\mathcal{P}_j\in P_m$ entirely covering the sphere $\Sigma(y,r)$.

Lemma 2.2. *Let $\mathcal{M}$ be a compact connected metric measure space satisfying the Conditions A and B and let $P_m=\{\mathcal{P}_j\}_1^m$ be the partition of $\mathcal{M}$ from Lemma* 2.1. *Then, we have*

$$K_m(y,r)\leqslant c_9\, m^{1-\frac{1}{d}}. \tag{2.4}$$

Proof. Put $\tilde\Sigma(y,r)=\bigcup_{j\in J_m}\mathcal{P}_j$. In view of (2.1), $\mu(\tilde\Sigma(y,r))=m^{-1}K_m$. From the other hand, in view of (2.2), the union $\tilde\Sigma(y,r)$ is a subset in the spherical shell $B(y,r+c_8\,m^{-1/d})\setminus B(y,r-c_8\,m^{-1/d})$. By the Condition B, we obtain

$$K_m\leqslant m\left(v(y,r+c_8\,m^{-1/d})-v(y,r-c_8\,m^{-1/d})\right)\leqslant 4c_2c_8\,m^{1-\frac{1}{d}},$$

that completes the proof. □

Introduce the following kernels

$$\delta_m^{\mathcal{M}}(y,z)=m\sum_{1\leqslant j\leqslant m}\chi(\mathcal{P}_j,y)\chi(\mathcal{P}_j,z)\quad y,z\in\mathcal{M}, \tag{2.5}$$

where $P_m=\{\mathcal{P}_j\}_1^m$ is an equal measure partition of $\mathcal{M}$, see (2.1),

$$\delta_m^{\mathcal{I}}(r,u)=m\sum_{1\leqslant i\leqslant m}\chi(\mathcal{Q}_i,y)\chi(\mathcal{Q}_i,z)\quad r,u\in\mathcal{I}, \tag{2.6}$$

where $Q_m=\{\mathcal{Q}_i\}_1^m$ is the partition of $\mathcal{I}\setminus\{0\}$ into the segments $\mathcal{Q}=(\frac{i-1}{m},\frac{i}{m}]$ of equal length m^{-1}. We put

$$\delta_m(y,z;r,u)=\delta_m^{\mathcal{M}}(y,z)\ \delta_m^{\mathcal{I}}(r,u). \tag{2.7}$$

The kernel (2.7) is non-negative and one can easily check the following relations

$$\iint_{\mathcal{M}\times\mathcal{I}}\delta_m(y,z;r,u)\,\mathrm{d}\mu(z)\,\mathrm{d}u=1, \tag{2.8}$$

$$\left(\iint_{\mathcal{M}\times\mathcal{I}}\delta_m(y,z;r,u)^q\,\mathrm{d}\mu(z)\,\mathrm{d}u\right)^{1/q}=m^{2/p}, \tag{2.9}$$

where $1<q<\infty,\ 1<p<\infty$ and $\frac{1}{q}+\frac{1}{p}=1$.

For the characteristic function and the volume of a ball $B(y,r)$, we consider the following approximations (piece-wise on the partition $\mathcal{P}_m\times\mathcal{I}_m$)

$$\chi_m(B(y,r),x)=\iint_{\mathcal{M}\times\mathcal{I}}\delta_m(y,z;r,u)\chi(B(z,u),x)\,\mathrm{d}\mu(z)\,\mathrm{d}u, \tag{2.10}$$

$$v_m(y,r) = \iint_{\mathcal{M}\times\mathcal{I}} \delta_m(y,z;r,u)v(z,u)\,\mathrm{d}\mu(z)\,\mathrm{d}u. \tag{2.11}$$

Lemma 2.3. *Let the assumptions of Lemma 2.2 hold. Then, we have*

$$\chi_m(B(y,r-\varepsilon_m),x) \leqslant \chi(B(y,r),x) \leqslant \chi_m(B(y,r+\varepsilon_m),x), \tag{2.12}$$

$$v_m(y,r-\varepsilon_m) \leqslant v(y,r) \leqslant v_m(y,r+\varepsilon_m), \tag{2.13}$$

where $\varepsilon_m = 2c_8\, m^{-1/d}$.

Proof. By the triangle inequality, the ball $B(z,u)$ contains the ball $B(y,r^-)$ with $r^- = u - \theta(y,z)$ and is contained in the ball $B(y,r^+)$ with $r^+ = u + \theta(y,z)$.

From the definitions (2.5) and (2.6), we conclude that the kernel (2.7) does not vanish, if and only if both centers y and z belong to the same subset $\mathcal{P}_j \in P_m$ and both radii r and u belong to the same subset $\mathcal{Q}_i \in Q_m$. In such a situation, from (2.2) and the definition of the partition P_m, we obtain

$$r^- \geqslant r - c_8\, m^{-1/d} - m^{-1/d} \geqslant r - \varepsilon_m,$$

$$r^+ \leqslant r + c_8\, m^{-1/d} + m^{-1/d} \leqslant r + \varepsilon_m.$$

Therefore, the ball $B(z,u)$ contains the ball $B(y,r-\varepsilon_m)$ and is contained in the ball $B(y,r+\varepsilon_m)$. For the characteristic functions, this means

$$\chi(B(y,r-\varepsilon_m),x) \leqslant \chi(B(z,u),x) \leqslant \chi(B(y,r+\varepsilon_m),x).$$

Substituting these inequalities into (2.10) and using (2.8), we obtain

$$\chi(B(y,r-\varepsilon_m),x) \leqslant \chi_m(B(y,r),x) \leqslant \chi(B(y,r+\varepsilon_m),x).$$

Replacing in these inequalities r with $r-\varepsilon_m$ and next with $r+\varepsilon_m$, we obtain (2.12). Integrating (2.12) with respect to $x \in \mathcal{M}$, we obtain (2.13). $\square$

Proof of Proposition 1.2. Substituting (2.12) and (2.13) into (1.4), we obtain

$$\begin{aligned} L_m[B(y,r-\varepsilon_m),\mathcal{D}_N] - N\alpha_m^-(y,r) &\leqslant L[B(e,r),\mathcal{D}_N] \\ &\leqslant L_m[B(y,r+\varepsilon_m),\mathcal{D}_N] - N\alpha_m^+(y,r), \end{aligned} \tag{2.14}$$

where

$$\left.\begin{aligned} \alpha_m^-(y,r) &= v(y,r) - v(y,r-\varepsilon_m) \geqslant 0, \\ \alpha_m^+(y,r) &= v(y,r+\varepsilon_m) - v(y,r) \geqslant 0 \end{aligned}\right\} \tag{2.15}$$

and

$$\begin{aligned} L[B(y,r),\mathcal{D}_N] &= \sum_{x\in\mathcal{D}_N} \chi_m(B(y,r),x) - Nv_m(y,r) \\ &= \iint_{\mathcal{M}\times\mathcal{I}} \delta_m(y,z;r,u)L[(B(z,u),\mathcal{D}_N].\,\mathrm{d}\mu(z)\,\mathrm{d}u, \end{aligned} \tag{2.16}$$

From (2.14), we obtain the bound

$$\begin{aligned} |L[B(y,r),\mathcal{D}_N]| \leqslant{} & |L[B(y,r-\varepsilon_m),\mathcal{D}_N]| + |L[B(y,r+\varepsilon_m),\mathcal{D}_N]| \\ & + N\alpha_m^-(y,r) + N\alpha_m^+(y,r). \end{aligned} \tag{2.17}$$

The quantities (2.15) can be easily estimated by the Condition B

$$\alpha_m^-(y,r) \leqslant 2c_2\, c_8\, m^{-1/d}, \quad \alpha_m^+(y,r) \leqslant 2c_2\, c_8\, m^{-1/d}. \tag{2.18}$$

Applying Hölder's inequality to the integral (2.16) and using (2.9), we obtain

$$|L[B(y,r),\mathcal{D}_N]| \leqslant m^{2/p}\; \mathcal{L}_p[\xi_0,\mathcal{D}_N], \tag{2.19}$$

where ξ_0 is the standard Lebesgue measure on the interval $\mathcal{I}$.

Notice that the right hand sides in (2.18) and (2.19) are independent of y and r. Substituting (2.18) and (2.19) into (2.17) and using the definition of L_∞-discrepancy (1.7), we obtain

$$\mathcal{L}_\infty[\mathcal{D}_N] \leqslant 2m^{2/p}\ \mathcal{L}_p[\xi_0, \mathcal{D}_N] + 4c_2\, c_8\, N m^{-1/d}. \tag{2.20}$$

This proves the bound (1.10) with $c_4 = 4c_2c_8$ and $c_5 = c_7$. □

3. Random point distributions. Proof of Theorem 1.1

Random N-point distributions can be constructed as follows. Suppose that a partition $\mathcal{P}_N = \{\mathcal{P}_j\}_1^N$ of the space $\mathcal{M}$ into N parts $\mathcal{P}_j \subset \mathcal{M}$ of equal measure N^{-1} is given. Introduce the probability space

$$\Omega_N = \prod_{1\leqslant j\leqslant N} \mathcal{P}_j = \{X_N = (x_1, \dots, x_N) : x_j \in \mathcal{P}_i, 1 \leqslant i \leqslant N\}, \tag{3.1}$$

with a probability measure $\omega_N = \prod_{1\leqslant j\leqslant N} \tilde{\mu}_j$, where $\tilde{\mu}_j = N\mu|_{\mathcal{P}_j}$, and $\mu|_{\mathcal{P}_j}$ denotes the restriction of the measure μ to a subset $\mathcal{P}_j \subset \mathcal{M}$. We write $\mathbb{E}F[\,\cdot\,]$ for the expectation of a random variable $F[X_N]$, $X_N \in \Omega_N$:

$$\begin{aligned}\mathbb{E}F[\,\cdot\,] &= \int_{\Omega_N} F[X_N]\,\mathrm{d}\omega_N \\ &= N^N \int\dots\int_{\mathcal{P}_1\times\dots\times\mathcal{P}_N} F(x_1, \dots, x_N)\,\mathrm{d}\mu(x_1)\dots\mathrm{d}\mu(x_N).\end{aligned} \tag{3.2}$$

Particularly, if $F[X_N] = f(x_j)$, where j is a fixed index and $f(x)$, $x \in \mathcal{M}$, is a summable function, then

$$\mathbb{E}F[\,.\,] = N\int_{\mathcal{P}_j} f(x)\,\mathrm{d}\mu(x). \tag{3.3}$$

Elements $X_N = (x_1, \dots, x_N) \in \Omega_N$ can be thought of as random N-point distributions in the space $\mathcal{M}$, and their local discrepancies $\mathcal{L}_p[\xi, X_N]$ as random variables on the probability space Ω_N. We shall prove the following

Lemma 3.1. *Let $\mathcal{M}$ be a compact connected metric measure space satisfying the Conditions A and B, and let the probability space Ω_N in* (3.1) *be constructed by the partition $P_N = \{\mathcal{P}_j\}_1^N$ of $\mathcal{M}$ from Lemma 2.1 with $m = N$. Then, we have*

$$(\,\mathbb{E}\,|\,\mathcal{L}_p[\xi, .]\,|^p\,)^{1/p} \leqslant c_{10}\,(p+1)^{1/2}\,N^{\frac{1}{2}-\frac{1}{2d}}, \quad 0 < p < \infty, \tag{3.4}$$

where ξ is an arbitrary normalized measure on $\mathcal{I}$.

Theorem 1.1 is a direct corollary of Lemma 3.1.

Proof of Theorem 1.1. It follows from (3.4) that for each $0 < p < \infty$ there exists an N-point subset $X_N^{(p)} \in \Omega_N$ such that

$$\mathcal{L}_p[\xi, X_N^{(p)}] \leqslant c_{10}\,(p+1)^{1/2}\,N^{\frac{1}{2}-\frac{1}{2d}},$$

and the bound (1.9) follows for $N > c_7$ with $c_3 = c_{10}$, while for $N \leqslant c_7$, we have $\lambda_p[\xi, N] \leqslant 2c_7$. This proves Theorem 1.1. □

For the proof of Lemma 3.1, we need the Marcinkiewicz–Zigmund inequality, which can be stated as follows.

Lemma 3.2. *Let ζ_j, $j \in J, \#\{J\} < \infty$, be a finite collection of real-valued independent random variables on a probability space Ω with expectations $\mathbb{E}\,\zeta_j = 0, j \in J$. Then, we have*

$$\mathbb{E}\,|\sum_{j\in J} \zeta_j\,|^p \leqslant 2^p\,(p+1)^{p/2}\,\mathbb{E}\,(\sum_{j\in J} \zeta_j^2\,)^{p/2}, \quad 1 \leqslant p < \infty. \tag{3.5}$$

The proof of Lemma 3.2 can be found in [7, Section 10.3,Theorem 2].

Proof of Lemma 3.1. Introduce the notation

$$\left.\begin{aligned} J^0_m = J^0_m(y,r) = \{j : \mathcal{P}_j \subset B(y,r)\}, \\ K^0_m = K^0_m(y,r) = \#\{J^0_m(y,r)\}. \end{aligned}\right\} \tag{3.6}$$

In the notation (2.3) and (3.6) the characteristic function and volume of a ball can be written as

$$\chi(B(y,r),x) = \sum_{j\in J^0_N} \chi(\mathcal{P}_j,x) + \sum_{j\in J_N} \chi(B(y,r)\cap\mathcal{P}_j,x),$$
$$v(y,r) = N^{-1}K^0_N + \sum_{j\in J_N} \mu(B(y,r)\cap\mathcal{P}_j).$$

With the help of these formulas we can calculate the local discrepancy (1.4) for the random point distribution $X_N = (x_1,\dots,x_N) \in \Omega_N$:

$$\begin{aligned} &L[B(y,r),X_N] = \#(B(y,r)\cap X_N) - Nv(y,r)) \\ &= K^0_N + \sum_{j\in J_N} \chi(B(y,r)\cap\mathcal{P}_j,x_j) - K^0_N - N\sum_{j\in J_N} \mu(B(y,r)\cap\mathcal{P}_j) \\ &= \sum_{j\in J_N} \chi(B(y,r)\cap\mathcal{P}_j,x_j) - N\sum_{j\in J_N} \mu(B(y,r)\cap\mathcal{P}_j), \end{aligned}$$

and we can write

$$L[B(y,r),X_N] = \sum_{j\in J_N} \zeta[X_N]. \tag{3.7}$$

where

$$\zeta_j[X_N] = \zeta_j[y,r,X_N] = \chi(B(y,r)\cap\mathcal{P}_j,x_j) - N\mu(B(y,r)\cap\mathcal{P}_j). \tag{3.8}$$

are random variables on the probability space Ω_N.

The random variables (3.8) are independent, $|\zeta_j[X_N]| < 1$ and, in view of (3.3), their expectations $\mathbb{E}\zeta_j[\,.\,] = 0$, $j \in J_N$. Hence, the Marcinkiewicz–Zigmund inequality (3.5) can be applied to the sum (3.7), and taking the bound (2.4) into account, we obtain

$$\begin{aligned} \mathbb{E}\,|\sum_{j\in J_N} \zeta_j[\,.\,]\,|^p &\leqslant 2^p(p+1)^{1/2}K_N^{p/2} \\ &\leqslant 2^p\,(p+1)^{1/2}\;c_9^{p/2}\;N^{(\frac{1}{2}-\frac{1}{2d})p}, \quad 1 \leqslant p < \infty. \end{aligned} \tag{3.9}$$

Notice that the right hand side in (3.9) is independent of y and r. Integrating the inequality (3.9) with respect to the measure $\mu\times\xi$ on $\mathcal{M}\times\mathcal{I}$, we obtain

$$\begin{aligned} &\iint_{\mathcal{M}\times\mathcal{I}} \mathbb{E}\,|\sum_{j\in J_N} \zeta_j[y,r,\,.\,]\,|^p\,\mathrm{d}\mu(y)\,\mathrm{d}u \\ &= \mathbb{E}(\mathcal{L}_p[\xi,.])^p \;\leqslant\; 2^p\,(p+1)^{1/2}\;c_9^{p/2}\;N^{(\frac{1}{2}-\frac{1}{2d})p}, \quad 1 \leqslant p < \infty. \end{aligned} \tag{3.10}$$

This proves the bound (3.4) for $1 \leqslant p < \infty$. Since the left hand side in (3.4) is a non-decreasing function of p, the bound (3.4) holds for all $0 \leqslant p < \infty$. $\square$

4. Appendix: Proof of Proposition 1.1

In this Section we consider a compact d-dimensional Riemanniam manifold $\mathcal{M}$ with the standard Riemannian geodesic distance θ and measure μ defined by the corresponding metric tensor on $\mathcal{M}$, see, [10]. Notice that for such θ and μ the normalization (1.1) fails but this is of no importance for the present discussion, because the choose of normalization has effect only on the constants in the bounds (1.2) and (1.3). We keep the same notation $v(y, r)$ for the volume of a ball with respect to the metric θ and the measure μ.

The bounds (1.2) for a compact Riemanniam manifold are well-known, see, for example, [8, 10]. Recall that local consideration of any (compact or non-compact) Riemannian manifold shows that at each point $y \in \mathcal{M}$ and for small r, one has the asymptotic $v(y, r) = \kappa_d r^d + O(r^{d-1})$, where κ_d is the volume of unit ball in $\mathbb{R}^d$, see [10, Chapter 5]. This implies the bounds (1.2) for small radii r. Since $\mathcal{M}$ is compact, the bounds (1.2) can be easily extended to all $0 < r \leqslant \operatorname{diam} \mathcal{M}$.

In order to prove the bound (1.3), we compare $v(y, r)$ with the volume $v_k(r)$ of a geodesic ball in the d-dimensional simply connected hyperbolic space of constant negative sectional curvature $-k^2$. The volume $v_k(r)$ is independent of the position of its center and is given explicitly by

$$v_k(r) = \sigma_d \int_0^r \left(\frac{\sinh ku}{k} \right)^{d-1} \mathrm{d}u, \quad 0 \leqslant r < \infty,$$

where σ_d is the $(d-1)$-dimensional area of the unit sphere in $\mathbb{R}^d$.

Lemma 4.1. *For any compact Riemannian manifold $\mathcal{M}$, there exists a constant $k_{\mathcal{M}} \geqslant 0$ depending only on $\mathcal{M}$, such that for all $k > k_{\mathcal{M}}$ the ratio $\frac{v(y,r)}{v_k(r)}$ as a function of r is non-increasing and tends to 1 as $r \to 0$.*

Lemma 4.1 is a very special case of the Bishop–Gromov volume comparison theorem, see [10, Chapter 9, Lemma 36]. The constant $k_{\mathcal{M}}$ is the smallest $k_0 \geqslant 0$ such that the matrix $R(y) + k_0^2(d-1)I_d$ is not-negative defined for all $y \in \mathcal{M}$, here $R(y)$ is the Ricci tensor at $y \in \mathcal{M}$ and I_d is the identity $d \times d$ matrix.

By Lemma 4.1, for $0 < r_1 \leqslant r_2 \leqslant \operatorname{diam} \mathcal{M}$, we have

$$\frac{v(y, r_2)}{v_k(r_2)} \leqslant \frac{v(y, r_1)}{v_k(r_1)} \leqslant 1.$$

Therefore,

$$v(y, r_2) - v_k(r_1) \leqslant \frac{v(y, r_1)}{v_k(r_1)} (v_k(r_2) - v_k(r_1)) \leqslant v_k(r_2) - v_k(r_1),$$

and the bound (1.3) follows, since $v_k(r)$ is smooth and increasing.

The proof of Proposition 1.1 is competed.

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