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Computational aspects of Hamburger's theorem

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Abstract. Riemann's zeta-function (defined by a certain Dirichlet series) satisfies an identity known as the functional equation. H. Hamburger established that the function is identified by the equation inside a wide class of functions defined by Dirichlet series.

Riemann's zeta-function is a member of a large family of functions with similar properties, in particular, satisfying certain functional equations. Hamburger's theorem can be extended to some (but not to all) of these equations.

The paper address the following question: how could we discover the Dirichlet series satisfying given functional equation? Two "rules of thumb" for performing such discoveries via numerical computations are demonstrated for functional equations satisfied by Dirichlet eta-function, Ramanujan tau L-function, and Davenport–Heilbronn function.

A conjectured discrete version of Hamburger's theorem is stated.

Key words: Hamburger's theorem, functional equation, Riemann's zeta function, Ramanujan tau L-function, Davenport–Heilbronn function

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1 Number-theoretical backgrounds

This introductory section presents some well-known definitions and results required for understanding the rest of the paper.

1.1 Riemann's zeta-function

One of the most important open problems in Number Theory is the celebrated *Riemann Hypothesis*. It is a prediction about positions of the zeroes of *Riemann's zeta-function*. This function can be defined via *Dirichlet series*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.1}$$

This series converges for $\mathrm{Re}(s) > 1$ but the function can be extended to the whole complex plane with the exception of the point $s = 1$ (at this point the zeta-function has its only pole).

B. Riemann [17] conjectured that all non-real zeros of the zeta-function lie on the *critical line* $\mathrm{Re}(s) = 1/2$.

While the function is named after Riemann, it was studied (for real values of the argument) already by L. Euler. He also worked with closely related entire function

$$\eta(s) = (1 - 2 \times 2^{-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \tag{1.2}$$

named *alternating zeta-function* or *Dirichlet eta-function*. The alternating series in (1.2) has the advantage over the series (1.1) of being convergent in the wider region $\mathrm{Re}(s) > 0$. Respectively, at this half-plane the zeta-function can be calculated as

$$\zeta(s) = \frac{\eta(s)}{1 - 2 \times 2^{-s}} = \frac{\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}}{1 - 2 \times 2^{-s}}. \tag{1.3}$$

1.2 Euler product

Euler also gave another, rather different from (1.1) and (1.3), definition of the zeta-function:

$$\zeta(s) \quad = \quad \prod_{p \ \text{prime}} \frac{1}{1 - p^{-s}}. \tag{1.4}$$

Similar to the series (1.1), the product in (1.4) also converges for $\mathrm{Re}(s) > 1$ only. The right hand side of (1.4) is nowadays known as *Euler product*.

In order to see why (1.4) is true one can at first observe that

$$\prod_{p \text{ prime}} \frac{1}{1^{-s} - p^{-s}} \quad = \quad \prod_{p \text{ prime}} \left(1 + p^{-s} + p^{-2s} + \dots\right), \tag{1.5}$$

and then apply the *Fundamental Theorem of Arithmetic*. This theorem states that every natural number has a unique factorization into product of powers of primes. This is equivalent to the fact that expanding the right hand side of (1.5) one gets exactly the right hand side of (1.1)!

The equivalence of two definitions, (1.1) and (1.4), explains why the zeta-function is a very important tool in the study of prime numbers.

1.3 The functional equation

Euler began his study of the zeta-function by determining its values at positive even integers. At first, he computationally discovered an approximate equality

$$\zeta(2) \approx \frac{\pi^2}{6} \tag{1.6}$$

by calculating (without computer![1]) many decimal digits of the left- and right-hand sides in (1.6). Later, he proved that the equality is in fact exact, and, more generally, that

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2(2m)!}, \qquad m = 1, 2, \dots; \tag{1.7}$$

here $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{12}$, $B_3 = 0$, ... are the *Bernoulli numbers*.

Euler also indicated values of the zeta-function at negative integers:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \qquad n = 1, 2, \dots \tag{1.8}$$

In particular,

$$\zeta(-2m) = 0, \tag{1.9}$$

[1]In this respect it is interesting to note that in [21] A. Turing used the word "computer" having in mind "a man performing computations"; in this sense a computer (namely, Euler) was involved in the discovery of (1.6).

and today even negative integers are called the *trivial zeros* of the zeta-function.

Comparison of (1.7) and (1.8) allows one to eliminate Bernoulli numbers and get the equality

$$2(2m-1)!\zeta(2m) = (-1)^m(2\pi)^{2m}\zeta(1-2m), \qquad m = 1, 2, \dots \tag{1.10}$$

Euler stated that, more generally, for every real s

$$g(s)\zeta(s) = g(1-s)\zeta(1-s) \tag{1.11}$$

where[2]

$$g(s) \;=\; \pi^{-\frac{s}{2}}(s-1)\Gamma(\tfrac{s}{2}+1). \tag{1.12}$$

The identity (1.11) is known today as the *functional equation*; its validity for all complex s was proved by Riemann [17].

1.4 Hamburger's theorem

H. Hamburger [10] established that the functional equation (1.11) identifies the zeta-function inside a wide class of functions defined by Dirichlet series. In particular, the zeta-function is the only function $D(s)$ such that

- $D(s)$ can be defined for $\mathrm{Re}(s) > 1$ by convergent Dirichlet series of the form
$$D(s) = 1 + \sum_{n=2}^{\infty} a_n n^{-s}; \tag{1.13}$$

- $(s-1)D(s)$ is an entire function of finite order;

- $D(s)$ satisfies the functional equation
$$g(s)D(s) = g(1-s)D(1-s) \tag{1.14}$$
where $g(s)$ is defined by (1.12).

[2]This form of writing this identity is due to Riemann [17]; Euler [9, Sect. 10] published an equivalent formula in terms of function $\eta(s)$.

1.5 Futher generalizations

Riemann's zeta-function is (historically first) member of a large family of functions with similar properties. A. Selberg [18] axiomatically described what is now known as *Selberg class* S. Each function from class S can be defined by a Dirichlet series as well as by (a counterpart of) Euler product, satisfies certain functional equation, and has some other feature akin to the zeta-function.

Also all functions from the class S are expected to satisfy corresponding analogs of the Riemann Hypothesis, and for this the existence of Euler products and functional equations is believed to be indispensable.

Original Hamburger's results were extended to other functional equations and improved by weakening certain restrictions on the function; for a recent survey of such *converse theorem* see [16]. However, in general case the linear space of Dirichlet series satisfying certain functional equation has dimension greater than 1.

2 Our objective

We will look at converse theorems of Hamburger type from computational point of view. In this paper we confine ourselves to consideration of functional equations of the simplest form

$$h(s)D(s) = h(c-s)D(c-s). \tag{2.1}$$

Here c and $h(s)$ are given number and function respectively, and $D(s)$ is an unknown Dirichlet series with real coefficients,

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}. \tag{2.2}$$

Suppose that we expect that (2.1) has a unique solution under certain extra restrictions on $D(s)$ (but we may not know these restrictions). The main question is: *how could we discover this series?*

We can distinguish two subquestions:

- *how could we calculate (approximate) values of the initial coefficients,* $a_1,\ a_2,\ \ldots;$

- *how could we calculate (approximate) value of function $D(s)$ and its derivatives for a given s (which need not lie in the half-plane of the convergence of the series)?*

In a more general situation we can expect only that the linear space of functions satisfying (2.1) has a finite dimension. Then we can ask:

- *could we select in a natural way a single "canonical" solution of* (2.1)*?*
- *how could we discover a basis for the linear space of solution of* (2.1)*?*

The author found (by computer experiments) several rather unexpected ways to answer such questions. These procedures work at least for a number of already studied functional equations; hopefully, this technique can be used for "solving" other functional equations as well.

Below the efficiency of some of such "rules of thumb" is demonstrated on a number of examples.

3 Calculation of the eta-function

Within this section we presuppose that

$$c = 1 \qquad \text{and} \qquad h(s) = \frac{g(s)}{1 - 2 \times 2^{-s}} = \frac{\pi^{-\frac{s}{2}}(s-1)\Gamma(\frac{s}{2}+1)}{1 - 2 \times 2^{-s}}. \tag{3.1}$$

With this choice, the equation (2.1) is satisfied by the function $\eta(s)$ (according to (1.2) and (1.11)) but this fact is used here only as a motivation to consider this particular functional equation.

3.1 First general construction

In order to discover coefficients of $D(s)$ (of the form (2.2) satisfying (2.1)) we will try to approximate this *infinite* series by *finite* series

$$D_N(s) = \sum_{n=1}^{N} a_{N,n} n^{-s} \tag{3.2}$$

with sufficiently large N. The latter series should imitate the expected solution of equation (2.1) in the following sense: we formally replace $D(s)$ by $D_N(s)$ in (2.1) and require that the equality

$$h(s)D_N(s) = h(1-s)D_N(1-s). \tag{3.3}$$

should hold, but for certain values of s only.

Functional equation (2.1) can determine $D(s)$ up to a multiplicative constant only; thus we need a kind of normalization condition, and we impose that

$$a_1 = 1. \tag{3.4}$$

Respectively, we put

$$a_{N,1} = 1, \tag{3.5}$$

and define the remaining $N-1$ coefficients, $a_{N,2}, \dots, a_{N,N}$, by solving the linear system consisting of $N-1$ replicas of equation (3.3) for s from a certain set $\mathfrak{S}_N$ containing $N-1$ elements.

3.2 Our specialization

Such a set $\mathfrak{S}_N$ can be chosen in many ways. Within this section we opt for

$$\mathfrak{S}_N = \{3/2, 5/2, \dots, N-1/2\}. \tag{3.6}$$

The reason for such a choice is as follows.

The gamma-function (entering in (3.3)) satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s). \tag{3.7}$$

According to Bohr–Mollerup theorem [7], this equation (together with some other mild restrictions) uniquely determines the gamma-function.

Equality (3.7) can be easily generalized: for a natural number m

$$\Gamma(z+m) = \left(\prod_{k=z}^{z+m-1} k\right)\Gamma(z). \tag{3.8}$$

The difference of the arguments of the gamma-factors in the left- and right-hand sides in (3.3) is equal to $s-1/2$, which is a positive integer whenever $s \in \mathfrak{S}_N$. Respectively, applying (3.8) we can make both arguments equal and cancel the gamma-factors. Thus for $s \in \mathfrak{S}_N$ equation (3.3) reduces to

$$h_1(s)D_N(s) = h_2(1-s)D_N(1-s) \tag{3.9}$$

where

$$h_1(s) \;=\; (2s-2)!!\left(1-2^{-s}\right), \tag{3.10}$$

$$h_2(s) \;=\; (-1)^{(2s-3)(2s-1)/8}\pi^{\frac{1}{2}-s}\left(1-2^{-s}\right). \tag{3.11}$$

3.3 Explicit formulas

We can write down an explicit expression for $D_N(s)$. Consider $N \times N$ matrix

$$M_N(s) = \left(\mu_{m,n}(s)\right)\Big|_{m=1}^{N}\Big|_{n=1}^{N} \tag{3.12}$$

where

$$\mu_{m,n}(s) = \begin{cases} n^{-s}, & \text{if } m = 1, \\ h_1(m - 1/2)n^{1/2-m} - h_2(3/2 - m)n^{m-3/2}, & \text{otherwise .} \end{cases} \tag{3.13}$$

Let L_N be the $(N-1) \times (N-1)$ matrix resulting from $M_N(s)$ by deleting the first row and the first column. Then

$$D_N(s) = \frac{\det(M_N(s))}{\det(L_N)}. \tag{3.14}$$

3.4 Numerical data

At first sight the idea of using numbers from the set (3.6) as values of s in (3.3) looks crazy – the infinite series in (1.2) does not converge at points $1 - s$. Astoundingly, this trick works!

Table 1 shows the coefficients of $D_N(s)$ for $N = 50$. Examining the initial coefficients one can surmise that alternating $a_n = (-1)^{n+1}$ should give a solution of (2.1), and we know that this is indeed so.

More important is the observation that $D_N(s)$ gives good approximations to $\eta(s)$ for a large range of values of s — see Table 2. Respectively, (approximate) values of the zeta-function and its derivatives can be calculated as

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\zeta(s) \approx \frac{\mathrm{d}^k}{\mathrm{d}s^k}\frac{D_N(s)}{1 - 2 \times 2^{-s}}. \tag{3.15}$$

Table 1: Coefficients of $D_{50}(s)$ for (2.1)

n	$b_{50,n}$	n	$b_{50,n}$
1	1.0000000000000000...	2	-0.9999999999999517...
3	0.9999999999989788...	4	-0.9999999999849114...
5	0.9999999998274060...	6	-0.9999999984068907...
7	0.9999999877970622...	8	-0.9999999207764316...
9	0.9999995568036840...	10	-0.9999978351017259...
11	0.9999906661069766...	12	-0.9999641614198762...
13	0.9998765303771978...	14	-0.9996158801266156...
15	0.9989149039834088...	16	-0.9972031802946021...
17	0.9933942170370522...	18	-0.9856470926223216...
19	0.9712077550029767...	20	-0.9464927728130374...
21	0.9075744066371679...	22	-0.8511115240177615...
23	0.7755520378458773...	24	-0.6822017143861291...
25	0.5756614095764715...	26	-0.4632926781019783...
27	0.3537541018518465...	28	-0.2550737531447883...
29	0.1729474483683150...	30	-0.1098431344988656...
31	0.0651146675641168...	32	-0.0359028510830116...
33	0.0183498194062661...	34	-0.0086628349972250...
35	0.0037635967925978...	36	-0.0014987102715588...
37	0.0005445947606385...	38	-0.0001796757789624...
39	0.0000535126895000...	40	-0.0000142901860447...
41	0.0000033941685251...	42	-0.0000007100592081...
43	0.0000001292559287...	44	-0.0000000201600405...
45	0.0000000026399504...	46	-0.0000000002822698...
47	0.0000000000236649...	48	-0.0000000000014588...
49	0.0000000000000588...	50	-0.0000000000000011...

Table 2: Approximation of $\eta(s)$ by $D_{50}(s)$ for (3.1)

s	$\left\lvert\frac{D_{50}(s)}{\eta(s)} - 1\right\rvert$	s	$\left\lvert\frac{D_{50}(s)}{\eta(s)} - 1\right\rvert$
-35	$1.05021...\cdot 10^{-8}$	$10i$	$4.59065...\cdot 10^{-13}$
$-33 + 1i$	$1.97364...\cdot 10^{-9}$	$16i$	$7.65086...\cdot 10^{-11}$
$-31 + 3i$	$2.06803...\cdot 10^{-9}$	$22i$	$9.21145...\cdot 10^{-9}$
$-29 + 5i$	$2.22572...\cdot 10^{-9}$	0.5	$1.26385...\cdot 10^{-15}$
$-27 + 7i$	$2.61390...\cdot 10^{-9}$	$0.5 + 6i$	$9.76838...\cdot 10^{-15}$
$-25 + 8i$	$1.06488...\cdot 10^{-9}$	$0.5 + 8i$	$8.51062...\cdot 10^{-14}$
$-23 + 10i$	$1.62068...\cdot 10^{-9}$	$0.5 + 11i$	$7.51379...\cdot 10^{-13}$
$-21 + 12i$	$2.69873...\cdot 10^{-9}$	$0.5 + 13i$	$5.86538...\cdot 10^{-12}$
$-19 + 13i$	$1.78099...\cdot 10^{-9}$	$0.5 + 15i$	$4.16476...\cdot 10^{-11}$
$-17 + 14i$	$1.37291...\cdot 10^{-9}$	$0.5 + 18i$	$7.69482...\cdot 10^{-10}$
$-15 + 15i$	$1.22591...\cdot 10^{-9}$	$0.5 + 20i$	$2.09350...\cdot 10^{-9}$
$-13 + 16i$	$1.25674...\cdot 10^{-9}$	1	$1.24841...\cdot 10^{-15}$
$-11 + 17i$	$1.46323...\cdot 10^{-9}$	$1 + 6i$	$9.42471...\cdot 10^{-15}$
$-9 + 18i$	$1.90802...\cdot 10^{-9}$	$1 + 12i$	$1.28347...\cdot 10^{-12}$
$-5 + 19i$	$1.86971...\cdot 10^{-9}$	$1 + 20i$	$1.20238...\cdot 10^{-9}$
0	$1.24841...\cdot 10^{-15}$	$3 + 26i$	$1.07592...\cdot 10^{-9}$
$5i$	$4.81114...\cdot 10^{-15}$	$5 + 38i$	$1.02395...\cdot 10^{-9}$

3.5 Conjectures

Numerical data (presented in this paper and other calculations performed by the author) allow one to state a number of conjectures.

Conjecture A. *For every n*

$$\lim_{N\to\infty} a_{N,n} = (-1)^{n+1}. \tag{3.16}$$

Conjecture B. *For every s*

$$\eta(s) = \lim_{N\to\infty} \frac{\det(M_N(s))}{\det(L_N)}. \tag{3.17}$$

It was known before that smooth truncations (similar to that in Table 1) of series (1.2) can produce good approximations. For example, P. Borwein [8] described a class of such smooth truncations giving exponentially close approximations to $\eta(s)$.

Borwein's truncations are defined via selection of certain polynomials with special properties, and he indicated two particular choices of such polynomials. However, these polynomials are not directly connected with the zeta-function, so the resulting smooth truncations are not inherent to it.

A kind of smooth truncation intrinsic to the zeta-function appeared in [13] (for further development see [12, 6, 15]). The coefficients of arising Dirichlet series encode a lot of information about prime numbers. However, from computational point of view this method is very complicated because it requires precalculations of the zeta-zeros with high accuracy.

In our case, entries to matrices $M_N(s)$ and L_N arise in a natural way from the functional equations (1.11) and (3.7) but definitions of these entries (given by (3.10), (3.11) and (3.13)) use only "simple" functions like the exponentiation and the double factorial.

It is interesting to study other properties of matrices $M_N(s)$, in particular, their eigenvalues and singular values (they "feel" zeta-zeros).

Conjectures A and B say that the coefficients of the Dirichlet series for $\eta(s)$ and values of this function can be calculated from the meager information contained in (3.5) and (3.9)–(3.11) for $s \in \mathfrak{S}_N$. Does it indicate that Bohr–Mollerup and Hamburger's theorems could be combined and produce the following discrete version of the latter theorem?

Conjecture C. *Riemann's zeta-function is the only function $D(s)$ such that*

- *$D(s)$ can be defined for $\mathrm{Re}(s) > 1$ by a convergent Dirichlet series of the form*
$$D(s) = 1 + \sum_{n=2}^{\infty} a_n n^{-s}; \tag{3.18}$$

- *$(s-1)D(s)$ is an entire function of finite order;*

- *for $m = 1, 2, \ldots$ function $D(s)$ satisfies the numerical equalities*
$$g(m+1/2)D(m+1/2) = g(1/2-m)D(1/2-m) \tag{3.19}$$
where $g(s)$ is defined by (1.12).

3.6 Other options

The selection of the set (3.6) is not rigid, it can be replaced by many other sets. For example, for values of s we could use integers greater than 1. In this case equation (3.3) reduces to counterparts of equalities (1.9) and (1.7) found already to Euler. Namely, for an odd $s = 2m+1$ equation (3.3) simplifies to
$$D_N(-2m) = 0, \tag{3.20}$$
and to
$$\begin{multline} 2(2m-1)!(1 - 2 \times 2^{2m-1})D_N(2m) = \\ (-1)^m (2\pi)^{2m}(1 - 2 \times 2^{-2m})D_N(1-2m) \end{multline} \tag{3.21}$$
for an even $s = 2m$.

Integers can be uses for values of s both instead of half-integers or together with them; in the latter case the accuracy of approximation of $\eta(s)$ by $D_N(s)$ is considerably higher.

Naturally, one can extend Conjectures A, B, and C for other choices of the set $\mathfrak{S}_N$.

Table 3: Initial coefficients of $D_N(s)$ for (4.1)

n	N	$a_{N,n}$
2	30	0.2841393450505322423648802...
	60	0.2840790438403573189026424...
	90	0.2840790438404122960282913...
3	30	-0.2844747272382600399086622...
	60	-0.2840790438400370082649792...
	90	-0.2840790438404122960282888...
4	30	-0.9977157059817277186689871...
	60	-1.0000000000024692403274721...
	90	-1.0000000000000000000000198...
5	30	-0.0142683988631866552023641...
	60	0.0000000000201869092463668...
	90	0.0000000000000000000001553...
6	30	1.0920687449236902877982957...
	60	0.9999999998079004381711738...
	90	0.9999999999999999999991173...
7	30	-0.2827413866159128279052885...
	60	0.2840790456867030872580591...
	90	0.2840790438404122960080168...
8	30	2.8784730710492549088446329...
	60	-0.2840790592273102172511881...
	90	-0.2840790438404122947139839...
9	30	-16.5679500529887487367574618...
	60	-0.9999999192097203175804888...
	90	-1.0000000000000000487473701...
10	30	66.7426105379517569482375592...
	60	0.0000003828326636970813932...
	90	0.0000000000000014726845966...
11	30	-246.7604018799068158985862084...
	60	0.9999815553656583309079810...
	90	0.9999999999999613898393090...
12	30	794.8300805378296122198943507...
	60	0.2843997083405965241718633...
	90	0.2840790438413085216718576...

4 Davenport–Heilbronn function

Within this section we presuppose that

$$c = 1 \qquad \text{and} \qquad h(s) = \left(\frac{5}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right). \tag{4.1}$$

4.1 Second general construction

Again, in order to construct a finite approximation (3.2) we will use (2.1) not in its full generality but only a few of its consequences. Namely, (2.1) is equivalent to saying that the function

$$F(z) = h(c/2 + z)D(c/2 + z) \tag{4.2}$$

is even, which can be expressed by the infinite system of numerical equalities

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}z^k} F(z)\right|_{z=0} = 0, \qquad k = 1, 3, \ldots, 2m+1, \ldots \tag{4.3}$$

In terms of $D(s)$ this corresponds to

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big(h(s)D(s)\Big)\right|_{s=c/2} = 0, \qquad k = 1, 3, \ldots, 2m+1, \ldots \tag{4.4}$$

We again impose normalizing condition (3.5) and define coefficients $a_{N,2}, \ldots, a_{N,N}$ from (3.2) by solving the system consisting of $N-1$ analogs of (4.4):

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big(h(s)D_N(s)\Big)\right|_{s=c/2} = 0, \qquad k = 1, 3, \ldots, M \tag{4.5}$$

where $M = 2N - 3$.

4.2 Numerical data

Table 3 shows corresponding values of $a_{N,2}, \ldots, a_{N,12}$ for $N = 30,\ 60,\ 90$. The numerical data suggest the following surmises for $N \to \infty$:

- coefficients $a_{N,2}$, $a_{N,7}$, and $a_{N,12}$ approach certain limiting value $\alpha = 0.2840790404\ldots$;

- coefficients $a_{N,3}$ and $a_{N,8}$ approach $-\alpha$;
- coefficients $a_{N,4}$ and $a_{N,9}$ approach -1;
- coefficients $a_{N,5}$ and $a_{N,10}$ approach 0;
- coefficients $a_{N,6}$ and $a_{N,11}$ approach 1.

Table 4: Comparison of even derivatives of $h(s)f(s)$ and $h(s)D_{30}(s)$ for (4.1)

m	$\frac{\frac{\mathrm{d}^m}{(\mathrm{d}s)^m}(h(s)D_{30}(s))\big\vert_{s=1/2}}{\frac{\mathrm{d}^m}{(\mathrm{d}s)^m}(h(s)f(s))\big\vert_{s=1/2}} - 1$	m	$\frac{\frac{\mathrm{d}^m}{(\mathrm{d}s)^m}(h(s)D_{30}(s))\big\vert_{s=1/2}}{\frac{\mathrm{d}^m}{(\mathrm{d}s)^m}(h(s)f(s))\big\vert_{s=1/2}} - 1$
0	$4.10785\ldots\cdot 10^{-6}$	26	$3.38965\ldots\cdot 10^{-12}$
2	$8.41657\ldots\cdot 10^{-7}$	28	$1.43867\ldots\cdot 10^{-12}$
4	$2.13426\ldots\cdot 10^{-7}$	30	$6.20969\ldots\cdot 10^{-13}$
6	$6.15266\ldots\cdot 10^{-8}$	32	$2.64828\ldots\cdot 10^{-13}$
8	$1.93507\ldots\cdot 10^{-8}$	34	$1.34226\ldots\cdot 10^{-13}$
10	$6.48438\ldots\cdot 10^{-9}$	36	$-1.05994\ldots\cdot 10^{-14}$
12	$2.28126\ldots\cdot 10^{-9}$	38	$3.41245\ldots\cdot 10^{-13}$
14	$8.34335\ldots\cdot 10^{-10}$	40	$-1.99099\ldots\cdot 10^{-12}$
16	$3.15036\ldots\cdot 10^{-10}$	42	$1.61377\ldots\cdot 10^{-11}$
18	$1.22191\ldots\cdot 10^{-10}$	44	$-1.70056\ldots\cdot 10^{-10}$
20	$4.84975\ldots\cdot 10^{-11}$	46	$2.41375\ldots\cdot 10^{-9}$
22	$1.96387\ldots\cdot 10^{-11}$	48	$-4.79564\ldots\cdot 10^{-8}$
24	$8.09412\ldots\cdot 10^{-12}$	50	$1.40744\ldots\cdot 10^{-6}$

The above surmises can be generalized by guessing that for all n coefficients $a_{N,n}$ approach certain limiting quantity a_n which depends only on the value of n modulo 5. Respectively, we can expect that the Dirichlet series

$$\sum_{m=0}^{\infty}(5m+1)^{-s}+\alpha(5m+2)^{-s}-\alpha(5m+3)^{-s}-(5m+4)^{-s} \tag{4.6}$$

is a solution of (2.1) for (4.1).

As for the nature of α, both The Inverse Symbolic Calculator [2] and WolframAlpha [3] suggest that α is a root of the equation

$$z^4+2z^3-6z^2-2z+1=0, \tag{4.7}$$

Table 5: Coefficients of $D_{30}(s)$ for (4.1)

n	$a_{30,n}$	n	$a_{30,n}$
1	1.00000000...	16	19241.29315524...
2	0.28413934...	17	-29407.07560910...
3	-0.28447472...	18	38570.85113607...
4	-0.99771570...	19	-43253.85469735...
5	-0.01426839...	20	41265.28452795...
6	1.09206874...	21	-33287.54237140...
7	-0.28274138...	22	22535.10552513...
8	2.87847307...	23	-12681.87163010...
9	-16.56795005...	24	5858.25583683...
10	66.74261053...	25	-2182.96010798...
11	-246.76040187...	26	639.95094404...
12	794.83008053...	27	-142.11869475...
13	-2199.87496770...	28	22.47824068...
14	5254.16299598...	29	-2.25669057...
15	-10831.19871227...	30	0.10811767...

that is

$$\alpha = \frac{-1-\sqrt{5}+\sqrt{10+2\sqrt{5}}}{2}. \tag{4.8}$$

With this value of α function (4.6) is the well-known *Davenport–Heilbronn function* $f(s)$. It indeed satisfies functional equation (2.1) with c and $h(s)$ defined by (4.1) (see [20, 10.25]) and is the only solution of this equation (see [1, 5.1] or [5, Sect. 8]).

When calculating coefficients $a_{30,n}$ we imposed restrictions of two kinds:

- normalization $a_{30,1} = 1$;
- vanishing of *odd* derivatives of the product $h(s)D_{30}(s)$ at $s = 1/2$.

Surprisingly, the values of *even* derivatives of $h(s)D_{30}(s)$ give very good approximations to the values of corresponding derivatives of the product $h(s)f(s)$ at $s = 1/2$ – see Table 4. This fact is more peculiar than the good approximations of $\eta(s)$ shown in Table 2, and the reason why it is so startling is as follows. Table 5 presents all coefficients of $D_{30}(s)$; we see that, except for a few initial, these coefficients differ very much from the coefficients in (4.6).

5 Ramanujan tau Lfunction

Within this section we presuppose that

$$c = 12 \qquad \text{and} \qquad h(s) = (2\pi)^{-s}\,\Gamma(s). \tag{5.1}$$

We have two way for "solving" a functional equation – via replicas of the equation itself for particular values of s (as in Section 3), and via vanishing of the odd derivatives at one point (as in Section 4). In this paper we will use the latter way (the former one was used in [14]; of course, one can combine equations of both types, (3.3) and (4.5), in one system).

5.1 Numerical data

To begin with we define coefficients $a_{N,n}, \ldots, a_{N,N}$ of $D_N(s)$ by (3.5) and (4.5) with $M = 2N - 3$.

Table 6 shows corresponding values of $a_{N,2}, \ldots, a_{N,7}$ for $N = 50, \ldots, 250$. It does not looks like that the coefficients approach some limiting values. More likely, they behave as partial sums of an asymptotic series — at first approaching "correct" value, but then retreating it.

The values of $a_{N,2}$, esspecially $a_{100,2}$, are very close to an integer, so we can make a guess that

$$a_2 = -24. \tag{5.2}$$

Similar but less confident guesses could be made about the values of $a_{N,3}$, $a_{N,4}$, $a_{N,5}$, and $a_{N,6}$. But already for $a_{N,7}$ the data from the table are not sufficient in order to make choice between -16744 and -16745.

At the moment we make only commitment (5.2), that is, from now on we assume not only (3.5) but

$$a_{N,2} = -24 \tag{5.3}$$

as well; respectively, we reduce the number of other equations 1, that is, we proceed with the system (4.5) with $M = 2N - 5$.

Table 7 shows values of $a_{N,3}, \ldots, a_{N,8}$ recalculated under the two assumptions, (3.5) and (5.3). We get greater confidence that

$$a_3 = 252 \tag{5.4}$$

Table 6: Values of $a_{N,n}$ from solutions of system (4.5) for (5.1) with $M = 2N - 3$ under assumption (3.5)

n	N	$a_{N,n}$
2	50	-24.000000000118497...
	100	-23.999999999999942...
	150	-23.999999999999770...
	200	-23.999999998866933...
	250	-24.000199961334035...
3	50	252.000000057374527...
	100	251.999999999961931...
	150	251.999999999836542...
	200	251.999999165844212...
	250	252.149430632741081...
4	50	-1472.000012515395811...
	100	-1471.999999986251471...
	150	-1471.999999931279260...
	200	-1471.999626780797076...
	250	-1540.912343773167466...
5	50	4830.001582240256756...
	100	4829.999996590998579...
	150	4829.999978579614166...
	200	4829.871536668917347...
	250	29755.868246403074758...
6	50	-6048.129472974338049...
	100	-6047.999374391124392...
	150	-6047.994675315311927...
	200	-6011.297392336898792...
	250	-7657439.816197617182839...
7	50	-16736.650298606985052...
	100	-16744.088289724678448...
	150	-16745.089444954449710...
	200	-25731.482054790443951...
	250	2061626557.103562626814415...

Table 7: Values of $a_{N,n}$ from solutions of system (4.5) for (5.1) with $M = 2N-5$ under assumptions (3.5) and (5.3)

n	N	$a_{N,n}$
3	50	252.000000001276967...
	100	251.999999999999973...
	150	252.000000000000027...
	200	251.999999999845893...
	250	252.000030423879946...
4	50	-1472.000000679159479...
	100	-1471.999999999973598...
	150	-1472.000000000034434...
	200	-1471.999999792994376...
	250	-1472.042583958137485...
5	50	4830.000145096068699...
	100	4829.999999987556299...
	150	4830.000000022057612...
	200	4829.999851805405458...
	250	4862.471493291169755...
6	50	-6048.016996351377522...
	100	-6047.999996384861074...
	150	-6048.000009448467037...
	200	-6047.926302953560375...
	250	-23666.925084998437692...
7	50	-16742.744014527296735...
	100	-16744.000722228039628...
	150	-16743.997009796162146...
	200	-16772.110633412127340...
	250	7502100.474170648682726...
8	50	84416.317314370117715...
	100	84480.105533105112501...
	150	84479.263918168986183...
	200	93122.012655507139459...
	250	-2651916509.556645449374102...

Table 8: Values of $a_{N,n}$ from solutions of system (4.5) for (5.1) with $M = 2N - 7$ under assumptions (3.5), (5.3), and (5.5)

n	N	$a_{N,n}$
4	50	-1472.000000014705761...
	100	-1472.000000000000225...
	150	-1472.000000000000278...
	200	-1471.999999999999752...
	250	-1472.000010741048018...
5	50	4830.000006623631354...
	100	4830.000000000255668...
	150	4830.000000000433781...
	200	4829.999999999870874...
	250	4830.020863090813102...
6	50	-6048.001194719567767...
	100	-6048.000000130026647...
	150	-6048.000000326036551...
	200	-6048.000000262414081...
	250	-6068.908783173152161...
7	50	-16743.881211654650012...
	100	-16743.999960465762010...
	150	-16743.999844150211253...
	200	-16743.999559817499552...
	250	-2503.262375742032465...
8	50	84472.490032773918378...
	100	84479.991888819148223...
	150	84479.947056235365133...
	200	84479.656617018282086...
	250	-7254372.111906719899883...
9	50	-113314.578115188801735...
	100	-113641.796404697915159...
	150	-113629.420756134702361...
	200	-113465.380053378057042...
	250	3025890243.971514185540493...

Table 9: Values of $a_{N,n}$ from solutions of system (4.5) for (5.1) with $M = 2N - 9$ under assumptions (3.5), (5.3), (5.5), and (5.7)

n	N	$a_{N,n}$
5	50	4830.000000079105807...
	100	4830.000000000004008...
	150	4830.000000000001438...
	200	4830.000000000316194...
	250	4830.000000161564014...
6	50	-6048.000027589401480...
	100	-6048.000000004386464...
	150	-6048.000000002441836...
	200	-6048.000000665629448...
	250	-6048.000349947935199...
7	50	-16743.996092066542882...
	100	-16743.999997890350852...
	150	-16743.999998065045899...
	200	-16743.999318063485484...
	250	-16743.633330978509881...
8	50	84479.691889532967686...
	100	84479.999400486350227...
	150	84479.999048286456341...
	200	84479.549703982396722...
	250	84236.978331976094194...
9	50	-113627.478816657760268...
	100	-113642.885788007170829...
	150	-113642.673364788701395...
	200	-113428.463299544266846...
	250	-2200.273147271369263...
10	50	-116460.643498323631911...
	100	-115935.685178325567478...
	150	-116003.489243160027771...
	200	-194313.225722385114160...
	250	-35879210.651607157671389...

and from now on we assume also that

$$a_{N,3} = 252. \tag{5.5}$$

Further recalculation (see Table 8) performed under the three assumptions, (3.5), (5.3) and (5.5), suggests that

$$a_4 = -1472 \tag{5.6}$$

and from now on we assume that

$$a_{N,4} = -1472. \tag{5.7}$$

The next recalculation with this additional assumption (see Table 9) allows us to guess that

$$a_5 = 4830 \qquad \text{and} \qquad a_6 = -6048. \tag{5.8}$$

The On-Line Encyclopedia of Integer Sequences [19] recognizes (3.4), (5.2), (5.4), (5.6), and (5.8) as the beginning of Sequence A000594 of *tau numbers of Ramanujan*, usually denoted as $\tau(n)$. They can be defined in many ways, in particular, via the formal expansion

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n. \tag{5.9}$$

Values $\tau_7 = -17644$ and $\tau_8 = 84480$ are in a sufficiently good agreement with Table 9.

The Dirichlet generating function for the tau numbers,

$$L_\tau(s) = \sum_{n=1}^{\infty}\tau_n n^{-s}, \tag{5.10}$$

is called *Ramanujan tau L-function*. It indeed satisfies the functional equation (2.1) for parameters (5.1) as it was shown by J. R. Wilton [22].

6 An equation with many solutions

Within this section we presuppose that

$$c = 1 \qquad \text{and} \qquad h(s) = 5^{s/2}\pi^{-s/2}\Gamma(s/2). \tag{6.1}$$

For these parameters the functional equation (2.1) is satisfied by Dirichlet L-function

$$\begin{aligned} L(\xi_5^{\langle 3\rangle}, s) &= 1^{-s} - 2^{-s} - 3^{-s} + 4^{-s} + \\ &\quad 6^{-s} - 7^{-s} - 8^{-s} + 9^{-s} + \dots \end{aligned} \tag{6.2}$$

and by the product

$$\begin{aligned} &F(s) = (1 + \sqrt{5} \times 5^{-s})\zeta(s) = \\ &\quad 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + (1 + \sqrt{5})5^{-s} + \\ &\quad 6^{-s} + 7^{-s} + 8^{-s} + 9^{-s} + (1 + \sqrt{5})10^{-s} + \dots \end{aligned} \tag{6.3}$$

(this example of a pair of functions solving the same functional equation was considered by E. P. Balanzario and J. Sánchez-Ortiz in [4, 5]). Thus there are infinitely many Dirichlet series (for example, any linear combination of (6.2) and (6.3)) satisfying (2.1) for c and $h(s)$ from (6.1), so it is not evident what will be the behavior of the coefficients of our finite Dirichlet series $D_N(s)$ in this case.

6.1 Numerical data I

We begin by defining coefficients $a_{N,n}, \dots, a_{N,N}$ of $D_N(s)$ via (3.5) and (3.3) for $N - 1$ integral and half-integral values of s, namely, for

$$s \in \mathfrak{S}_N = \left\{ \frac{3}{2},\ 2,\ \frac{5}{2},\ \dots,\ \frac{N+1}{2} \right\}. \tag{6.4}$$

Table 10 shows corresponding values of $a_{150,1}, \dots, a_{150,25}$. They clearly proffer series (6.2) as a solution of our functional equation.

It would be interesting to find the "reason" why numbers $a_{N,n}$ "vote" so strongly in favour of (6.2). One possible explanation is as follows: this series defines an entire function while (6.3) has a pole.

Another elucidation can be due to the following fact proved (in a greater generality) by J. Kaczorowski, G. Molteni, and A. Alberto in [11]: *among all functions satisfying the functional equation* (2.1) *for c and $h(s)$ from* (6.1), *which are defined by Dirichlet series and fulfill some other natural conditions, only one (up to a multiplicative constant) function has an Euler product, namely, Dirichlet L-function* (6.2). Thus we can say that, in a sense, our method of solving the functional equation "is aware of" the existence of the Euler product.

Table 10: Initial coefficients of $D_{150}(s)$ for (6.1)

n	$a_{150,n}$
1	1.0000000000000000000000000000000000000000...
2	-1.0000000000000000000000000000000000503818...
3	-0.9999999999999999999999999999999993511570...
4	0.9999999999999999999999999999999920664778...
5	0.0000000000000000000000000000001297531999...
6	0.9999999999999999999999999999971313135966...
7	-0.9999999999999999999999999999219413414513...
8	-1.0000000000000000000000000023824492196236...
9	1.0000000000000000000000000762228891639139...
10	-0.0000000000000000000000024432067893709509...
11	1.0000000000000000000000761946711258008472...
12	-1.0000000000000000000022691776540366859765...
13	-0.9999999999999999999362164265029050271163...
14	0.9999999999999999983198462791825519218419...
15	0.0000000000000000413050105627139882799632...
16	0.9999999999999990543012952193696130700355...
17	-0.9999999999999798506875667681724768850477...
18	-1.0000000000003995494978841229526902410874...
19	1.0000000000073787608204712203349645563369...
20	-0.0000000001270467782293793017009278338761...
21	1.0000000020421174897272628354810859712222...
22	-1.0000000306878756845227901699845834134406...
23	-0.9999995681947151741927104494293965174994...
24	0.9999943019805495162540206709295242723444...

Table 11: Initial coefficients of $\tilde{D}_{150}(s)$

n	$\tilde{a}_{150,n}$
1	1
2	$2.11051...\cdot 10^{32}$
3	$2.11051...\cdot 10^{32}$
4	$-4.22102...\cdot 10^{32}$
5	$3.41488...\cdot 10^{32}$
6	$-4.22102...\cdot 10^{32}$
7	$2.11051...\cdot 10^{32}$
8	$2.11051...\cdot 10^{32}$
9	$-9.73518...\cdot 10^{6}$
10	$-3.41488...\cdot 10^{32}$
11	$-1.01328...\cdot 10^{10}$
12	$2.11051...\cdot 10^{32}$

n	$\tilde{a}_{150,n}$
13	$2.11051...\cdot 10^{32}$
14	$-4.22102...\cdot 10^{32}$
15	$3.41488...\cdot 10^{32}$
16	$-4.22102...\cdot 10^{32}$
17	$2.11051...\cdot 10^{32}$
18	$2.11051...\cdot 10^{32}$
19	$-1.14278...\cdot 10^{21}$
20	$-3.41488...\cdot 10^{32}$
21	$-3.27188...\cdot 10^{23}$
22	$2.11051...\cdot 10^{32}$
23	$2.11051...\cdot 10^{32}$
24	$-4.22101...\cdot 10^{32}$

6.2 Numerical data II

In order to discover another solution, linear independent from (6.2), we need to work with a different functional equation.

Similar to what was done in Section 1.1, let us consider function

$$\tilde{h}(s) = \frac{h(s)}{1 - 2 \times 2^{-s}} = \frac{5^{s/2}\pi^{-s/2}\Gamma(s/2)}{1 - 2 \times 2^{-s}} \tag{6.5}$$

and functional equation

$$\tilde{h}(s)\tilde{D}(s) = \tilde{h}(1-s)\tilde{D}(1-s) \tag{6.6}$$

where

$$\tilde{D}(s) = \sum_{n=1}^{\infty} \tilde{a}_n n^{-s}. \tag{6.7}$$

Clearly, solutions of (2.1) and (6.6) are related in the following way:

$$D(s) = \frac{\tilde{D}(s)}{1 - 2 \times 2^{-s}}. \tag{6.8}$$

Again we introduce finite Dirichlet series

$$\tilde{D}_N(s) = \sum_{n=1}^{N} \tilde{a}_{N,n} n^{-s} \tag{6.9}$$

Table 12: Initial coefficients of the renormalized solution of (6.10)

n	$\tilde{a}_{150,n}/\tilde{a}_{150,2}$
1	0.0000000000000000000000000000000047381833...
2	1.0000000000000000000000000000000000000000...
3	1.0000000000000000000000000000000090824864...
4	-2.0000000000000000000000000000000097325743...
5	1.6180339887498948482045868343655872871425...
6	-1.9999999999999999999999999999983642221007...
7	0.9999999999999999999999999999544680005399...
8	1.0000000000000000000000000014144310868154...
9	-0.0000000000000000000000000461271066102859...
10	-1.6180339887498948482045853259151114036191...
11	-0.0000000000000000000000480114725108261323...
12	1.0000000000000000000014592559737179796915...
13	0.9999999999999999999581485098189617462655...
14	-1.9999999999999999988755724360180755618786...
15	1.6180339887498948200221432474848726640823...
16	-1.9999999999999993424569952410769364467755...
17	0.9999999999999857298476669180527403065751...
18	1.0000000000002881000325750133038334736396...
19	-0.0000000000054147331057448374478772689388...
20	-1.6180339886550517602553639788386115014737...
21	-0.0000000015502803369016449669398574304662...
22	1.0000000236828778008923877292406545597831...
23	0.9999996613464755355483477982666804654266...
24	-1.9999954599136616169250566952909379210637...

and imitate (6.6) by

$$\tilde{h}(s)\tilde{D}_N(s) = \tilde{h}(1-s)\tilde{D}_N(1-s). \tag{6.10}$$

Table 11 shows values of $\tilde{a}_{N,1}, \ldots, \tilde{a}_{N,24}$ obtained by solving the system consisting of equations (6.10) for $s \in \mathfrak{S}_N$ and normalization condition

$$\tilde{a}_{N,1} = 1 \tag{6.11}$$

for $N = 150$. Extremely large values of all coefficients, different from the default (6.11), suggest that this normalization was not felicitous. So we perform renormalization via dividing all the coefficients by $\tilde{a}_{N,2}$. Resulting ratios (presented in Table 12) also give a solution to (6.10) for $s \in \mathfrak{S}_N$.

Examination of the values in Table 12) produces the following surmises about the coefficients of a solution of (6.6):

- $\tilde{a}_1 = \tilde{a}_9 = \tilde{a}_{11} = \tilde{a}_{19} = \tilde{a}_{21} = 0$;
- $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_7 = \tilde{a}_8 = \tilde{a}_{12} = \tilde{a}_{13} = \tilde{a}_{17} = \tilde{a}_{18} = \tilde{a}_{22} = \tilde{a}_{23} = 1$;
- $\tilde{a}_4 = \tilde{a}_6 = \tilde{a}_{14} = \tilde{a}_{16} = \tilde{a}_{24} = -2$;
- $\tilde{a}_5 = -\tilde{a}_{10} = \tilde{a}_{15} = -\tilde{a}_{20} = \phi$ where $\phi = 1.618033988...$

Both The Inverse Symbolic Calculator [2] and WolframAlpha [3] recognize 1.618033988 as the familiar golden ratio, $\phi = (1+\sqrt{5})/2$.

Now performing formal division in (6.8) we get the following values for the 24 initial coefficients of $D(s)$:

- $a_1 = a_4 = a_6 = a_9 = a_{11} = a_{14} = a_{16} = a_{19} = a_{21} = a_{24} = 0$;
- $a_2 = a_3 = a_7 = a_8 = a_{12} = a_{13} = a_{17} = a_{18} = a_{22} = a_{23} = 1$;
- $a_5 = a_{10} = a_{15} = a_{20} = \phi$.

It is quite natural to make a general guess that for all k

- $a_{5k+1} = a_{5k+4} = 0$;
- $a_{5k+2} = a_{5k+3} = 1$;
- $a_{5k} = \phi$.

But this is equivalent to saying that

$$D(s) = F(s)/2 - L(\xi_5^{\langle 3 \rangle}, s)/2, \tag{6.12}$$

thus we have discovered a second solution of the functional equation (2.1) for parameters (6.1). According to [5], all solutions of this equation with periodic coefficients are linear combination of the two functions (6.2) and (6.3).

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