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Large contractible subgraphs of a 3-connected graph

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Abstract

Let $m \geq 5$ be a positive integer and G be a 3-connected graph on at least $2m+1$ vertices. We prove, that G has a contractible set W, such that $m \leq |W| \leq 2m-4$. (Recall, that a set $W \subset V(G)$ of a 3-connected graph G is contractible, if the graph $G(W)$ is connected and the graph $G-W$ is 2-connected.) A particular case for $m=4$ is that any 3-connected graph on at least 11 vertices has a contractible set of 5 or 6 vertices.

Keywords: connectivity, 3-connected graph, contractible subgraph.

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Basic definitions

Before introducing results of our paper let us recall main definitions that we need. We consider undirected graphs without loops and multiple edges and use standard notation.

For a graph G we denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. We use notation $v(G)$ for the number of vertices of G. For disjoint sets $X, Y \subset V(G)$ we denote by $E_G(X, Y)$ the set of all edges of the graph G, joining X and Y. A notation $xy \in E_G(X, Y)$ means that $x \in X$ and $y \in Y$.

We denote the *degree* of a vertex x in the graph G by $d_G(x)$.

Let $\mathrm{N}_G(w)$ denote the *neighbourhood* of a vertex $w \in V(G)$ (i.e. the set of all vertices of the graph G, adjacent to w).

For a set of vertices $U \subset V(G)$ we denote by $G(U)$ the *induced subgraph* of the graph G on the set U.

We say that a vertex $u \in V(G)$ is *adjacent* to a set $W \subset V(G)$, if $u \notin W$ and W contains a vertex, adjacent to u. We say, that two disjoint sets $U, W \subset V(G)$ are *adjacent*, if there exist adjacent vertices $u \in U$ and $w \in W$.

In this paper, every *path* or *cycle* is simple, i.e. passes any vertex at most once. An *xy-path* is a path between vertices x and y. If P is a path containing vertices x, y, then xPy denote the part of P between x and y.

Definition 1. 1) Let $R \subset V(G)$. We denote by $G - R$ the graph obtained from G upon deleting all vertices of the set R and all edges incident to vertices of R. The set R is a *cutset*, if the graph $G - R$ is disconnected.

2) A graph G is *k-connected*, if $|V(G)| > k$ and G has no cutset of size less than k.

Definition 2. 1) A vertex set $W \subset V(G)$ is *connected*, if the graph $G(w)$ is connected.

2) Let G be a 3-connected graph. A set $W \subset V(G)$ is *contractible*, if W is connected and the graph $G - W$ is 2-connected.

1 Introduction and main results

Consider a 2-connected graph G on n vertices, let n_1, n_2 be positive integers with $n_1 + n_2 = n$. It is rather clear, that there exists a connected vertex set decomposition of the vertex set of G into two disjoint connected sets V_1 and V_2, such that $|V_1| = n_1$, $|V_2| = n_2$.

In 1994, McCuaig and Ota [4] have formulated the following conjecture for 3-connected graphs. This conjecture was mentioned in Mader's survey on connectivity [3].

Conjecture. *Let $m \in \mathrm{N}$. Then there exists an integer n such that every 3-connected graph G on at least n vertices has a contractible set of m vertices.*

For $m = 1$ this statement is clear, for $m = 2$ it is rather easy and well-known (it was proved by Tutte). The case $m = 3$ was proved by authors of this conjecture [4], the case $m = 4$ was proved by M. Kriesell [5]. For any $m \geq 5$ Conjecture is open now. It is only known [6], that in case $m = 5$ Conjecture is true for cubic graphs and graphs of average degree close to 3.

We suggest a new result on existence of large contractible sets in 3-connected graphs.

Theorem 1. *Let $m \geq 5$ be a positive integer and G be a 3-connected graph on at least $2m + 1$ vertices. Then G has a contractible set W, such that $m \leq |W| \leq 2m - 4$.*

A particular case of this theorem for $m = 5$ is the following.

Corollary 1. *A 3-connected graph on $n \geq 11$ vertices has a contractible set of 5 or 6 vertices.*

In what follows we formulate several facts on the structure of 2-connected graphs and after that, with the help of them, we prove Theorem 1.

2 Necessary tools

We start with well known definitions of blocks and cutpoints.

2.1 Blocks and cutpoints of a connected graph

For connected graphs, we have a classic instrument to study graph's structure — blocks and cutpoints. Let's recall the definitions.

Definition 3. Let G be a connected graph.

A vertex $a \in V(G)$ is a *cutpoint* of G, if the graph $G - a$ is disconnected.

A *block* of the graph G is a maximal up to inclusion subgraph, having no cutpoints.

The *interior* $\mathrm{Int}(B)$ of a block B is the set of all its vertices which are not cutpoints of G.

The structure of mutual disposition of blocks and cutpoints of a connected graph can be described by the *tree of blocks and cutpoints* (see [7]). Recall, that the tree of blocks and cutpoints of a graph G is a bipartite graph. Vertices of the first partition are all cutpoints $a_1, \dots, a_n$ of the graph G, vertices of the second partition are all blocks $B_1, \dots, B_m$ of the graph G. Vertices a_i and B_j are adjacent if and only if $a_i \in V(B_j)$. It is easy to prove, that this graph is a tree, all leaves of which correspond to blocks (which are called *pendant blocks*).

2.2 The decomposition of a graph by a set of cutsets

We need to describe the structure of decomposition of a 2-connected graph by its 2-vertex cutsets. We define the *block tree* of a 2-connected graph as in [12]. In general, this structure is similar to Tutte's one [1]. Let's start with the *decomposition of a graph by a set of cutsets*, defined in [10].

In our paper, *connected components* of a graph are vertex sets of maximal up to inclusion connected subgraphs.

Definition 4. Let $R \subset V(G)$ be a cutset.

1) Let $X, Y \subset V(G)$, $X \not\subset R$, $Y \not\subset R$. We say that R *separates* the set X from Y, if no two vertices $v_x \in X$ and $v_y \in Y$ belong to the same connected component of the graph $G - R$.

2) We say that R *splits* a set $X \subset V(G)$, if the set $X \setminus R$ is not contained in one connected component of the graph $G - R$.

In this section, $k \geq 2$ and G is a k-connected graph. Denote by $\mathfrak{R}_k(G)$ the set of all k-vertex cutsets of G.

Definition 5. Let $\mathfrak{S} \subset \mathfrak{R}_k(G)$.

1) A set $A \subset V(G)$ is a *part of decomposition* of G by $\mathfrak{S}$, if A is a maximal up to inclusion set, such that no cutset of $\mathfrak{S}$ splits A.

The set of all parts of decomposition of the graph G by $\mathfrak{S}$ we denote by $\mathrm{Part}(G; \mathfrak{S})$.

2) A vertex of a part $A \in \mathrm{Part}(G)$ is *inner*, if it does not belong to a cutset of $\mathfrak{S}$. The set of all inner vertices of the part A is called the *interior* of A and denoted by $\mathrm{Int}(A)$.

The *boundary* of A is the set $\mathrm{Bound}(A) = A \setminus \mathrm{Int}(A)$.

It is clear that if two parts of $\mathrm{Part}(G; \mathfrak{S})$ have nonempty intersection, then their intersection is a subset of a certain cutset of $\mathfrak{S}$.

It is easy to prove [11], that $\mathrm{Bound}(A)$ consists of all vertices of the part A, which are adjacent to $V(G) \setminus A$. If $\mathrm{Int}(A) \neq \varnothing$, then $\mathrm{Bound}(A)$ separates $\mathrm{Int}(A)$ from $V(G) \setminus A$.

Definition 6. Two cutsets $S, T \in \mathfrak{R}_k(G)$ are *independent*, if S does not split T and T does not split S. Otherwise, these sets are *dependent.*

It is proved [2, 8] that only two variants are possible for cutsets $S, T \in \mathfrak{R}_k(G)$ of a k-connected graph G: either S and T are independent, or each of them splits the other. The proof of this fact is easy.

2.3 The block tree of a 2-connected graph

In this section, the graph G is 2-connected.

Definition 7. 1) A cutset $S \in \mathfrak{R}_2(G)$ is *single*, if S is independent with all other cutsets of $\mathfrak{R}_2(G)$. Denote by $\mathfrak{O}(G)$ the set of all single cutsets of the graph G.

2) We will write $\mathrm{Part}(G)$ instead of $\mathrm{Part}(G; \mathfrak{O}(G))$. Parts of this decomposition will be called simply *parts of the graph G.*

Definition 8. The *block tree* $\mathrm{BT}(G)$ of a 2-connected graph G is a bipartite graph. The first partition of this graph is $\mathfrak{O}(G)$, the second partition is $\mathrm{Part}(G)$. Vertices $S \in \mathfrak{O}(G)$ and $A \in \mathrm{Part}(G)$ are adjacent if and only if $S \subset A$.

In what follows we list several properties of $\mathrm{BT}(G)$. Most of them are similar to properties of the classic tree of blocks and cutpoints of a connected graph.

Lemma 1. [13, Lemma 1] *For a 2-connected graph G the following statements hold.*

1) $\mathrm{BT}(G)$ *is a tree. Every leaf of* $\mathrm{BT}(G)$ *corresponds to a part of* $\mathrm{Part}(\mathfrak{G})$.

2) *Let $B, B' \in \mathrm{Part}(G)$. Then a cutset $S \in \mathfrak{O}(G)$ separates B from B' in G if and only if S separates B from B' in* $\mathrm{BT}(G)$.

Definition 9. A part $A \in \mathrm{Part}(G)$ is *pendant*, if it corresponds to a leaf of $\mathrm{BT}(G)$.

Remark 1. If $A \in \mathrm{Part}(G)$ is a pendant part, then $\mathrm{Bound}(A)$ is a single cutset of the graph G.

Definition 10. 1) For a 2-connected graph G, we denote by G' the graph obtained from G upon adding all edges of type ab where $\{a, b\} \in \mathfrak{O}(G)$.

2) A part $A \in \mathrm{Part}(G)$ is called a *cycle*, if the graph $G'(A)$ is a cycle. A is called a *block*, if $G'(A)$ is a 3-connected graph. If A is a cycle, then $|A|$ is the *length* of A.

Lemma 2. [13, Lemma 2] *For a 2-connected graph G the following statements hold.*

1) *Every part of* $\mathrm{Part}(G)$ *is either a cycle or a block.*

2) *If $A \in \mathrm{Part}(G)$ is a cycle, then all vertices of* $\mathrm{Int}(A)$ *have degree 2 in the graph G.*

3) *Let $A \in \mathrm{Part}(G)$ be a cycle of length at least 4. Then any pair of its non-neighbouring vertices form a non-single cutset of the graph G. All non-single cutsets of G are of such type.*

Lemma 3. [12, Theorem 2] *Let G be a 2-connected graph without single cutsets. Then either G is 3-connected, or G is a cycle.*

3 Proof of Theorem 1

In what follows, the graph G will be 3-connected.

Definition 11. A contractible set $W \subset V(G)$ of a 3-connected graph G is *maximal*, if there exists no vertex $x \in V(G) \setminus W$, such that the set $W \cup \{x\}$ is contractible.

Remark 2. Let $W \subset V(G)$ be a maximal contractible set and $x \in V(G) \setminus W$ be a vertex adjacent to W. Then the graph $G - W - x$ is not 2-connected.

Lemma 4. *Let G be a 3-connected graph, and $W \subset V(G)$ be a maximal contractible set, such that the graph $H = G - W$ is not a cycle. Then the following statements hold.*

1) *The set W is adjacent to all inner vertices of all parts-cycles of H.*

2) *There are at least two pendant parts in* $\mathrm{Part}(H)$*, all these parts are cycles of length at least 4. The boundary of every pendant part is a single cutset of H.*

3) *Let $A \in \mathrm{Part}(H)$ be a pendant part. Then $H - \mathrm{Int}(A)$ is 2-connected.*

Proof. 1) Let x be an inner vertex of a part-cycle of $H = G - W$. Then $d_H(x) = 2$. Since G is 3-connected, the vertex x must be adjacent in G to the set W.

2) Since W is maximal, the graph H is not 3-connected. Since H is not a cycle, by Lemma 4 this graph has single cutsets. Hence, the tree $\mathrm{BT}(H)$ has at least two leaves, which correspond to pendant parts of $\mathrm{Part}(H)$. The boundary of a pendant part is a single cutset of the graph H.

Assume that a pendant part $A \in \mathrm{Part}(H)$ is a block and $\mathrm{Bound}(A) = S$. If W is not adjacent to $\mathrm{Int}(A)$, then a 2-vertex cutset S separates $\mathrm{Int}(A)$ in a 3-connected graph G. Since this is impossible, there exists a vertex $x \in \mathrm{Int}(A)$

adjacent to W in G. By item 3 of Lemma 2, no cutset of $\mathfrak{R}_2(H)$ contains an inner vertex of a part-block. Hence, $H - x$ is a 2-connected graph. This contradicts maximality of W.

3) Let $\mathrm{Bound}(A) = \{x, x'\}$. By item 2, vertices of the set $\mathrm{Int}(A)$ form a simple xx'-path in H. Assume, that the graph $H' = H - \mathrm{Int}(A)$ is not 2-connected. Since H' becomes 2-connected after adding an xx'-path, there is a cutvertex in H', which separates x from x'.

On the other side, $\mathrm{Bound}(A) = \{x, x'\}$ is a single cutset in H. Therefore, no cutset of $\mathfrak{R}_2(H)$ separates x from x'. By Menger's theorem there exist three independent xx'-paths in H. At most one of these paths intersects the xx'-path formed by vertices of $\mathrm{Int}(A)$. Thus there are two disjoint xx'-paths in H', i.e. no cutvertex separates x from x' in $H - \mathrm{Int}(A)$. The contradiction obtained shows that H' is a 2-connected graph. □

Theorem 1 is a consequence of the following lemma.

Lemma 5. *Let a 3-connected graph G on n vertices have a contractible set of $m \geq 4$ vertices, and $n \geq 2m + 3$. Then G has a contractible set of m' vertices, where $m + 1 \leq m' \leq 2m - 2$.*

The proof of Lemma 5 is rather complicated. We start with several claims, considering cases of the proof. In all these claims let G satisfy the condition of Lemma 5, i.e. let G be a 3-connected graph with $v(G) \geq 2m+3$. We assume that G has a contractible set of $m \geq 4$ vertices. Each such set is maximal, otherwise, Theorem is proved. We try to find in the graph G a *suitable* vertex set W', i.e. a contractible set of size $m + 1 \leq |W'| \leq 2m - 2$.

Claim 1. *Let $W \subset V(G)$ be a maximal contractible set of m vertices. Assume, that the graph $G - W$ is not a cycle and has a pendant part D with $|\mathrm{Int}(D)| \leq m - 2$. Then the assertion of Lemma 5 holds.*

Proof. Consider the set $W' = W \cup \mathrm{Int}(D)$. By item 1 of Lemma 4 the graph $G(W')$ is connected. By item 3 of Lemma 4 the graph $G - W' = (G - W) - \mathrm{Int}(D)$ is 2-connected. Since $m = |W| < |W'| \leq 2m - 2$, the set W' is suitable. □

Claim 2. *Let $M \subset V(G)$ be a maximal contractible set of at most m vertices with $|\mathrm{N}_G(M)| = p \leq m + 2$. Then the graph $G - M$ is not a cycle and has pendant parts $D_1, \ldots, D_k$, such that*

$$\sum_{i=1}^{k} |\mathrm{Int}(D_i)| \leq p.$$

Proof. Let $G' = G - M$. If G' is a cycle, then all vertices of this cycle are adjacent to M in G. Therefore, $V(G) \subset M \cup \mathrm{N}_G(M)$, whence $v(G) \le |M| + |\mathrm{N}_G(M)| \le 2m + 2$. This contradicts the condition of Theorem.

Thus, G' is not a cycle. Then the graph G and the set M satisfy the condition of Lemma 4, therefore, the graph G' has at least two pendant parts $D_1, \dots, D_k$, which interiors are disjoint. By item 1 of Lemma 4 we have $\cup_{i=1}^{k} \mathrm{Int}(D_i) \subset \mathrm{N}_G(M)$, whence our Claim follows. $\square$

Claim 3. *Let $M, W \subset V(G)$ be two maximal contractible sets, such that $|M| = m$, $|W| \le m$ and $|\mathrm{N}_G(M) \setminus W| \le 2$. Then the assertion of Lemma 5 holds.*

Proof. The contractible set M satisfies the condition of Claim 2. Let $D_1, \dots, D_k$ be pendant parts of the graph $G - M$ and $D = \cup_{i=1}^{k} \mathrm{Int}(D_i)$. If $W \not\subset D$, then

$$|D| \le |W| - 1 + 2 = m + 1,$$

whence by $k \ge 2$ the graph G' has a pendant part, which interior contains at most $\frac{m+1}{2} < m - 1$ vertices. In this case, by Claim 1 the assertion of Lemma 5 holds.

Now let $W \subset D$. By Lemma 4, $\mathrm{Int}(D_1), \dots, \mathrm{Int}(D_k)$ are connected components of the graph $G(D)$. Since the graph $G(W)$ is connected, we have $W \subset \mathrm{Int}(D_i)$ for a certain i. Hence, the union of all other interiors consists of at most 2 vertices. Therefore, G' has a pendant part, which interior has at most $2 \le m - 2$ vertices and by Claim 1 the assertion of Lemma 5 holds. $\square$

Claim 4. *Let $W \subset V(G)$ be a maximal contractible set of at most m vertices and the graph $H = G - W$ be a cycle. Then the assertion of Lemma 5 holds.*

Proof. Let H be a cycle $h_1 h_2 \dots h_k$, where $k \ge m + 3$ (numeration is cyclic modulo k). Since G is a 3-connected graph, $d_G(h_i) \ge 3$. Hence, all vertices of H are adjacent in G to the set W. Recall, that the graph $F = G(W)$ is connected

Consider vertices h_i and h_{i+m+1}, let they be adjacent to vertices $x, y \in W$, respectively. Let $L = \{h_{i+1}, h_{i+2} \dots, h_{i+m}\}$ and P be a xy-path in the graph F. Then, in the graph $G' = G - L$, all vertices of the path P and the set $V(H) \setminus L$ lie on a cycle (see figure 1a). Hence, these vertices lie in the same block B of the graph G'.

Let U be the set of all vertices of G' which don't belong to B. Then $U \subset W \setminus \{x, y\}$. If $U \ne \varnothing$, then every connected component of $G(U)$ is separated in the graph G' from B by a cutpoint and, therefore, is adjacent

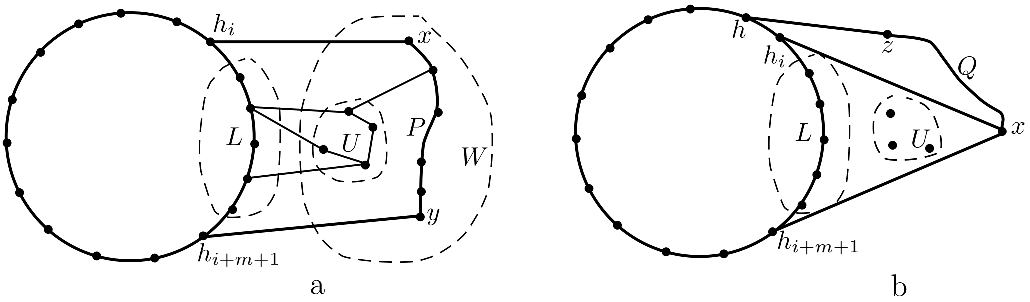

Рис. 1: H is a cycle

to L (since the graph G is 3-connected). Hence, for a set $W' = L \cup U$ the graph $G(W')$ is connected. The graph $G - W' = B$ is 2-connected. Moreover,

$$m \leq |L| \leq |W'| = |L| + |U| \leq |L| + |W \setminus \{x, y\}| \leq 2m - 1. \tag{1}$$

Let $|W'| = m$. Then $W' = L$ is a contractible set in G. If the assertion of Lemma 5 does not hold, L is a maximal contractible set. Note, that $\mathrm{N}_G(L) \subset \{h_i, h_{i+m+1}\} \cup W$. Then the assertion of Lemma 5 holds by Claim 3, applied to the set L.

In what follows let $|W'| \geq m + 1$. If the last inequality in (1) is strict, then the set W' is suitable. The case where this inequality turns to equality remains. Then $|U| = m-1 = |W|-1$, whence, $x = y$ and $U = W \setminus \{x\}$. Thus, either Lemma 5 is proved, or vertices x and y cannot be chosen distinct, i.e., each of the vertices h_i and h_{i+m+1} is adjacent in W to exactly one vertex — to the vertex x (see figure 1b). Assume, that a vertex h of the set $V(H) \setminus L$ is adjacent to a vertex $z \in W$ different from x. Consider a zx-path Q in the connected graph F. Since $h, x \in V(B)$, all vertices of the path Q belong to B. In particular, $z \in B$, hence, $U \subset W \setminus \{x, z\}$ and $|U| \leq m - 2$. Then the set W' is suitable.

The only case remaining is where all vertices h_{i+m+1}, $h_{i+m+2} \dots$, h_i (their number is at least 3) are adjacent in W only to x. Then we consider vertices h_{i-1} and h_{i+m} instead of h_i and h_{i+m+1} and by the same reasoning as above obtain, that h_{i+m} is adjacent in W only to x. Similarly, we assure that all vertices of the cycle H are adjacent in W only to x. However, this is impossible for a 3-connected graph G. Claim is proved. $\square$

Claim 5. *Let $W \subset V(G)$ be a maximal contractible set. Assume, that $|W| \leq m$ and the graph $H = G - W$ is not a cycle. Let $A \in \mathrm{Part}(H)$ be a pendant part, such that $|\mathrm{Int}(A)| \geq m$. Then the assertion of Lemma 5 holds.*

Proof. Recall, that the graph $F = G(W)$ is connected. By item 2 of Lemma 4, the pendant part A is a cycle and $\mathrm{Bound}(A) = \{s, t\}$ is a single cutset of H. Let vertices of $\mathrm{Int}(A)$ follow $a_1, \ldots, a_k$ in cyclic order from s to t.

Set $L = \{a_1, \ldots, a_m\}$, $G' = G - L$. If $k = m$, set $t' = t$. If $k > m$, set $t' = a_{m+1}$. Consider two cases.

1. *The graph G' is 2-connected.*
Then L is a contractible set of m vertices. The set L is maximal, otherwise Lemma 5 is proved. The neighbourhood $\mathrm{N}_G(L)$ can include the set W and two other vertices: s and t'. Then the assertion of Lemma 5 is proved by Claim 3, applied to the set L.

2. *The graph G' is not 2-connected.*
By item 3 of Lemma 4 the graph $H - \mathrm{Int}(A)$ is 2-connected, hence, all vertices of this graph lie in the same block B of the graph G'. Let $N = V(H) \setminus L$. The graph H has a pendant part $A' \neq A$, all vertices of the set $\mathrm{Int}(A') \subset N$ are adjacent to W. Thus, N contains two vertices adjacent to W. Let $d, d' \in N$ be vertices adjacent to $x, x' \in W$, respectively (see figure 2a). If it is possible, choose d and d' such that $x \neq x'$. There is an xx'-path P in the connected graph F. Since ends of the path $P' = dxPx'd'$ are two distinct vertices of the block B, we have $x, x' \in V(B)$.

Let $k > m$ and none of the vertices d, d' coincide with a_{m+1}. Then there exists a vertex $y \in W$ adjacent to $a_{m+1} \in \mathrm{Int}(A)$. There is an xy-path Q in the connected graph F (see figure 2a). One may assume that $d \neq t$. (If $d = t$, then $d' \neq t$. In this case, we exchange pairs d, x and d', x'). Then the ends of the path $Q' = dxQya_{m+1} \ldots a_k t$ are distinct vertices of B, hence, all inner vertices of Q' lie in B.

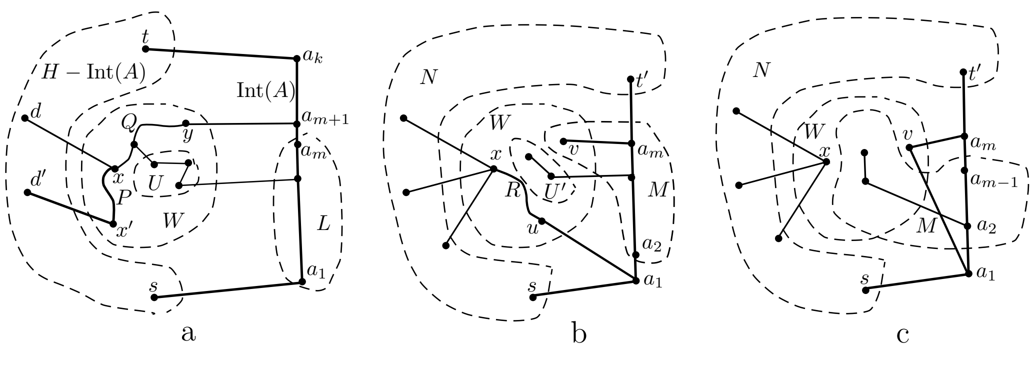

Рис. 2: $|\mathrm{Int}(A)| \geq m$

Thus, all vertices of the graph G', which do not lie in B, belong to a nonempty set $U \subset W \setminus \{x, x'\}$. Every connected component of $G(U)$ is separated in the graph G' from B by a cutpoint and, therefore, is adjacent to L (since the graph G is 3-connected). Hence, for a set $W' = L \cup U$ the graph $G(W')$ is connected. The graph $G - W' = B$ is 2-connected. Let's estimate the number of vertices in U and W':

$$|U| \le |W \setminus \{x, x'\}| \le m - 1 \quad \text{and} \quad m = |L| < |L| + |U| = |W'| \le 2m - 1.$$

Thus, the set W' is suitable, except for the case $|U| = m - 1$, which can occur only if $x = x'$. By the construction, then no vertex of the set $W \setminus \{x\}$ is adjacent to N.

Consider two cases.

2.1. *There exist two distinct vertices $u, v \in W$, such that $ua_1, va_m \in E(G)$.*
If one of the vertices u, v coincides with x, let it be u. If none of the vertices u, v coincides with x, we can assume, that there is an ux-path R in F, which does not contain v. (Otherwise, v separates u from x in F. Hence, u does not separate v from x in F and there is a vx-path in F, which does not contain u. In this case, we exchange pairs u, a_1 and v, a_m and obtain a similar case.) Let $M = \{a_2, \dots, a_m, v\}$ (see figure 2b). Clearly, the graph $G(M)$ is connected.

Let $G_1 = G - M$. Consider two cases.

2.1.1. *The graph G_1 is 2-connected.*
Then M is a contractible set of m vertices. We may assume that M is maximal (otherwise, Theorem is proved). Recall, that v is not adjacent to the set N. Therefore, $\mathrm{N}_G(M) \setminus W$ can contain two vertices: a_1 and t'. Then Lemma 5 follows from Claim 3.

2.1.2. *The graph G_1 is not 2-connected.*
In the graph G_1, all vertices of the set $N \cup \{x\}$ lie in the same block, say B'. Since there is a path sa_1uRx in G_1, which ends are distinct vertices of B' (see figure 2b), the vertex u lie in B'. Let U' be the set of all vertices of G_1 which do not belong to B' and $W'' = M \cup U'$. Then $U' \subset W \setminus \{x, u, v\}$ and $m + 1 \le |W''| \le 2m - 2$. Any connected component of the graph $F(U')$ is separated in G_1 from B' by a cutpoint, therefore, it is adjacent to M. Hence, the graph $G(W'')$ is connected and the graph $G - W'' = B'$ is 2-connected. Thus, the set W'' is suitable.

2.2. *There exists a vertex $v \in W$, such that a_1 and a_m are not adjacent to $W \setminus \{x, v\}$ (the vertex v can coincide with x).*
By Lemma 4, then $a_1v, a_mv \in E(G)$. Set $M = \{a_2, \dots, a_{m-1}\} \cup (W \setminus \{x, v\})$ (see figure 2c). Let's prove, that the graph $G_1 = G - M$ is 2-connected. Recall, that $V(G_1) = N \cup \{x, v, a_1, a_m\}$. By proved above, all vertices of the set $N \cup \{x\}$ lie in the same block of G_1, say B'. In the graph G_1, there is

a path sa_1va_mt', which ends are distinct vertices of the block B'. Therefore, $a_1, v, a_m \in B'$, i.e. $G_1 = B'$ is a 2-connected graph.

Consider a connected component U of the graph $F - \{x, v\}$. Since G is 3-connected, U is adjacent in G to $N \cup L$. In the case we consider, U can be adjacent neither to N nor to $\{a_1, a_m\}$. Hence, U is adjacent to $\{a_2, \dots, a_{m-1}\}$. Therefore, the graph $G(M)$ is connected.

If $v = x$, then $|M| = 2m - 3$ and this set is suitable. Assume, that $v \neq x$. Then $|M| = 2m - 4$. If $m > 4$, this set is suitable. Consider the last case $m = 4$. Then M is a maximal contractible set. Since $(W \setminus \{x, v\})$ is not adjacent to N, we have $\mathrm{N}_G(M) \subset W \cup \{a_1, a_m\}$ and Lemma 5 follows from Claim 3. $\square$

Proof of Lemma 5. Let $W \subset V(G)$ be a contractible set of m vertices in a 3-connected graph G and $H = G - W$. We can assume that W is maximal, otherwise Theorem is proved. Recall, that the graph $F = G(W)$ is connected.

If H is a cycle, apply Claim 4. In what follows, H is not a cycle and by Lemma 4 has at least two pendant part-cycles. The interior of any pendant part of H consists of $m-1$ vertices, otherwise one can apply Claims 1 and 5. Let $A, A' \in \mathrm{Part}(H)$ be two pendant parts,

$$\mathrm{Bound}(A) = \{s, t\}, \quad L = \mathrm{Int}(A) = \{a_1, \dots, a_{m-1}\},$$
$$\mathrm{Bound}(A') = \{s', t'\}, \quad L' = \mathrm{Int}(A') = \{a'_1, \dots, a'_{m-1}\},$$

where vertices of L are enumerated in cyclic order from s to t and vertices of L' are enumerated in cyclic order from s' to t'. Set the notation $N = V(H) \setminus (L \cup L')$.

Consider several cases.

1. *For any vertex $w \in W$ and any part $B \in \mathrm{Part}(H)$, there is at most one edge from w to $\mathrm{Int}(B)$.*
For each vertex $a \in L \cup L'$ we choose one edge from a to W. The chosen edges are called *good*. Then exactly one vertex $z \in W$ is not an end of a good edge incident to L and exactly one vertex $z' \in W$ is not an end of a good edge incident to L'.

1.1. *There exist two adjacent vertices $x, y \in W$, different from z'.*
Consider the set $W' = L \cup (W \setminus \{x, y\})$ (see figure 3a). Then $|W'| = 2m - 3$. The graph $G - W'$ is 2-connected, since it can be obtained from a 2-connected (by item 3 of Lemma 4) graph $H - L$ upon adding adjacent vertices x, y, which are adjacent to different vertices of the set $L' \subset V(H - L)$. If the graph $G(W')$ is connected, the set W' is suitable.

Let the graph $G(W')$ be disconnected. Then the only vertex of the set W not adjacent to L — the vertex z — is separated in the graph F by the

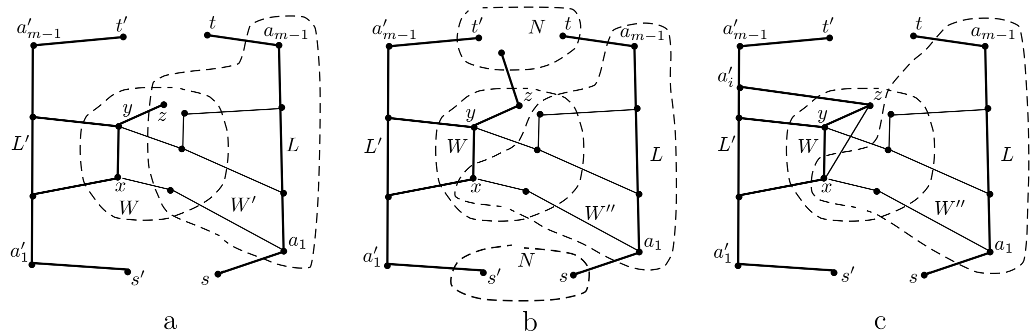

Рис. 3: Case 1.1

set $\{y, x\}$ from all other vertices. The vertex z is adjacent in the connected graph F to at least one of the vertices x and y — say, to y. Since G is a 3-connected graph, $d_G(z) \geq 3$. Thus, z is adjacent to $L' \cup N$. If z is adjacent to N (see figure 3b), the graph $G(N \cup L' \cup \{z, y\})$ is 2-connected. In the remaining case, z is not adjacent to N. Then z is adjacent to exactly one vertex of the set $N \cup L'$ — a certain vertex $a'_i \in L'$, whence $zy, zx \in E(G)$. One of the vertices x and y (say, y) is adjacent to a vertex of the set L', different from a'_i (see figure 3c). Then the graph $G(N \cup L' \cup \{z, y\})$ is again 2-connected. In both cases, the set $W'' = L \cup (W \setminus \{z, y\})$ is suitable: the graph $G - W'' = G(N \cup L' \cup \{z, y\})$ is 2-connected, the graph $G(W'')$ is connected (all vertices of the set $W \setminus \{z, y\}$ are adjacent to L) and $|W''| = 2m - 3$.

1.2. *All edges of the graph F are incident to the vertex z'.*
Similarly, we may assume that all edges of F are incident to z. Thus, $z = z'$ and F is a star with the center z (see figure 4a). In this case, consider a vertex $y \in W$, adjacent to a'_2 and the set $M = L \cup \{y\}$. Let's prove, that the graph $G_1 = G - M$ is 2-connected. By Lemma 4, vertices of the set $N \cup L' = V(H - L)$ lie in the same block B of G_1. Recall, that $F - y$ is a star, all its leaves are incident to good edges, and other ends of these edges are distinct vertices of the set $L' \subset V(B)$. Therefore, we have $W \setminus \{y\} \subset V(B)$. Hence, $G_1 = B$ is a 2-connected graph.

Note, that $|M| = m$, the graph $G(M)$ is connected and

$$\mathrm{N}_G(M) \subset (W \setminus \{y\}) \cup \{a'_2, s, t\}.$$

Thus, M is a maximal contractible set. By Claim 2, the graph $G - M$ is not a cycle and has pendant parts $D_1, \ldots, D_k$, where $k \geq 2$ and $\sum_{i=1}^{k} |\mathrm{Int}(D_i)| \leq m + 2$. The interior of each of these pendant parts must contain exactly $m - 1$ vertices, since all other cases are analysed above. This is possible only if $m = 4$ and $k = 2$. Then the graph $G(\mathrm{N}_G(M))$ must have two connected components $\mathrm{Int}(D_1)$ and $\mathrm{Int}(D_2)$, such that $|\mathrm{Int}(D_1)| = |\mathrm{Int}(D_2)| = m-1$. However, the graph $G(W \setminus \{y\}) = F - y$ is connected and has exactly $m-1$ vertices, and the vertex a'_2 can be adjacent only to a'_1, a'_3 and vertices of W. Hence, a'_2 is not adjacent to $\{s,t\}$. Therefore, $G(\mathrm{N}_G(M))$ has a connected component of at most 2 vertices, that contradicts proved above. The case is analysed.

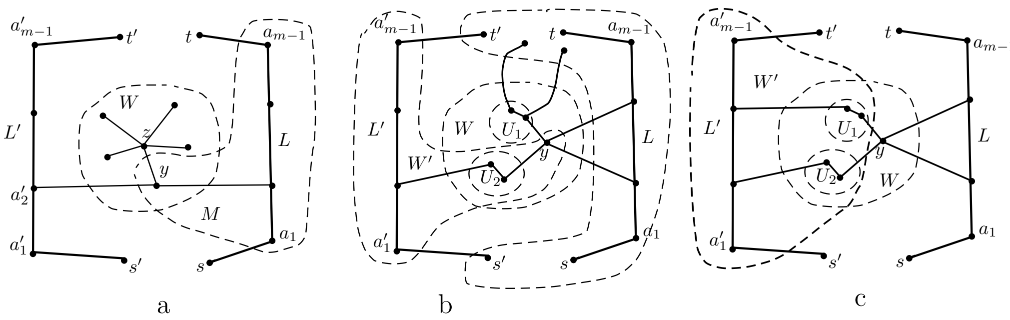

Рис. 4: Cases 1.2 and 2.1

2. *A vertex $y \in W$ is adjacent to two vertices of the set* $\mathrm{Int}(A)$.
Recall, that the graph $H - L' = G(N \cup L)$ is 2-connected by Lemma 4. Consider two cases.

2.1. *The graph $F - y$ is disconnected.*
Let $U_1, \ldots, U_p$ be all connected components of $F - y$.

Assume, that the component U_1 is not adjacent to L' (see figure 4b). Since the graph $G - y$ is 2-connected, for every block B' of the graph $G(U_1)$ by Menger's Theorem in $G - y$ there exist two disjoint paths from B' to $N \cup L$. Moreover, none of these two paths contains vertices of L' and other connected components of $F - y$. Therefore, the graph $G' = G(N \cup L \cup U_1)$ is 2-connected. Then set $W' = L' \cup (W \setminus U_1)$. The graph $G - W' = G'$ is 2-connected, the graph $G(W')$ is connected (all components $U_2, \ldots, U_k$ are adjacent to y, the set $W \setminus U_1$ is adjacent to L') and $m+1 \le |W'| \le 2m-2$. Hence, the set W' is suitable.

The only case remaining is where all components $U_1, \ldots, U_k$ are adjacent to L' (see figure 4c). Then set $W' = L' \cup (W \setminus \{y\})$. The graph $G - W' = G(N \cup L \cup \{y\})$ is 2-connected, the graph $G(W')$ is connected and $|W'| = 2m-2$. Hence, the set W' is suitable.

2.2. *The graph $F - y$ is connected.*
Assume, that the sets $W \setminus \{y\}$ and L' are adjacent (see figure 5a). Then set $W' = L' \cup (W \setminus \{y\})$. The graph $G - W' = G(N \cup L \cup \{y\})$ is 2-connected, the graph $G(W')$ is connected and $|W'| = 2m-2$. Hence, the set W' is suitable.

In what follows the sets L' and $W \setminus \{y\}$ are not adjacent. Then every vertex of the set L' is adjacent to the only vertex of the set W — to the vertex y. Let us exchange the sets L and L' and with the help of similar reasoning assure, that all vertices of the set L are adjacent to the only vertex of the set W, and this vertex is also y (see figure 5b). The same statement holds for the interior of any other pendant part of $\mathrm{Part}(H)$, if such a part exists.

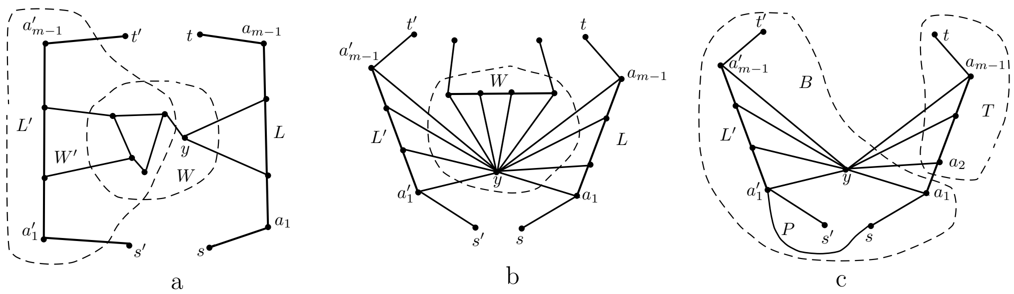

Рис. 5: Case 2.2

Since $G - y$ is a 2-connected graph and the set $L \cup L'$ is not adjacent to $W \setminus \{y\}$, the graph $G(N \cup (W \setminus \{y\}))$ is also 2-connected. Thus for a set $M = L \cup \{y\}$ the graph $G - M$ is 2-connected. Since $|M| = m$ and the graph $G(M)$ is connected, M is a maximal contractible set. Let us perform the same reasoning for the graph $G - M$. We either prove the theorem, or assure, that all pendant parts of the graph $G - M$ have $m - 1$ inner vertices, and all these vertices are adjacent to y. The interior of one of these pendant parts is $W \setminus \{y\}$, other parts are different from A parts of the graph $H = G - W$. Thus, (see figure 5b)

the graph $F = G(W)$ contains a path of $m - 1$ vertices, all vertices of which are adjacent to a certain vertex $y \in W$. $(*)$

Now consider the set $T = \{a_2, \dots, a_{m-1}, t\}$ of $m-1$ vertices (see figure 5c). The graph $G(T)$ is connected. Prove, that the graph $G - T$ is 2-connected. Indeed, this graph is obtained from a 2-connected graph $G - t$ upon deleting vertices of the set $T' = T \setminus \{t\}$, adjacent in the graph $G - t$ only to y and a_1. In the 2-connected graph $H' = H - L$, there are two disjoint a'_1s-paths and at most one of them contains t. Thus, in the graph $H' - t$ there is an a'_1s-path P, which forms together with the path $sa_1ya'_1$ a cycle Z. Thus, in the graph $G - T$ there is a cycle Z, which contains a_1 and y. Therefore, vertices a_1 and y belong to a certain block B of the graph $G - T$. If $G - T$ is not 2-connected, it has a cutpoint x, separating B from another block B'. Since vertices of the set T' are adjacent in $G - T = G - t - T'$ only to vertices of the block B, the vertex x also separates B from B' in the 2-connected graph $G - t$. This is impossible. The contradiction obtained shows us that the graph $G - T$ is 2-connected.

Thus, the set T of $m - 1$ vertices is contractible. At the end of the proof consider two cases.

2.2.1. *The set T is not maximal.*
Then there exists a vertex u adjacent to T, such that the graph $G - T - u$ is 2-connected. Note, that $u \neq y$, since $d_{G-T-y}(a_1) = 1$. But any other vertex u is adjacent to none of the vertices $a_3, \dots, a_{m-1}$. Since $a_2 t \notin E(G)$, the graph $G(T \cup \{u\})$ has no vertex adjacent to all other vertices. Therefore condition $(*)$ fails for $G(T \cup \{u\})$. Thus, if we consider $T \cup \{u\}$ instead of W and repeat above reasoning, we will prove our Theorem.

2.2.2. *The set T is maximal.*
The graph $H_0 = G - T$ is 2-connected. If H_0 is a cycle, apply Claim 4. Let H_0 be not a cycle. Consider a pendant part $D \in \mathrm{Part}(H)$. If $|\mathrm{Int}(D)| \le m - 1$, consider a set $W' = T \cup \mathrm{Int}(D)$. By Lemma 4 the graph $G - W' = H_0 - \mathrm{Int}(D)$ is 2-connected and the graph $G(W')$ is connected. Since $2 \le |\mathrm{Int}(D)| \le m - 1$, we have $m + 1 \le |W'| \le 2m - 2$, i.e. the set W' is suitable. If $|\mathrm{Int}(D)| \ge m$, our Theorem is proved by Claim 5. $\square$

Proof of Theorem 1. Consider the maximal $s \le m$, such that the graph G has a contractible set U of s vertices. If $s = m$ we are done. Assume that $s \le m - 1$. By Lemma 5, there exists another contractible set U', such that $s + 1 \le |U'| \le 2s - 2 \le 2m - 4$. By the maximality of s, we have $|U'| > m$. Thus, the set U' is suitable for Theorem 1. $\square$

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