

ПРЕПРИНТЫ ПОМИ РАН

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**ON THE MODEL PROBLEM ARISING
IN THE STUDY OF MOTION OF VISCOUS COMPRESSIBLE
AND INCOMPRESSIBLE FLUIDS WITH A FREE INTERFACE**

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Abstract

The paper is concerned with a model problem arising in the study of the evolution of two viscous capillary fluids of different types: compressible and incompressible contained in a bounded vessel and separated by a free interface. The estimates of solution in the Sobolev–Slobodetskii spaces of functions are obtained that can be useful for the proof of stability of the rest state.

Keywords: compressible and incompressible fluids, free boundary, Sobolev–Slobodetskii spaces.

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1 Introduction

Free boundary problem for compressible and incompressible viscous fluids was considered for the first time by I.V.Denisova [1]. In this paper, local in time solvability of the problem was established under some additional restrictions on the coefficients of viscosity of both fluids. These results were obtained as a consequence of a detailed treatment of the model problem in two half-spaces, which was connected with some technical difficulties in the analysis of the explicit formula of solution of this problem. It was the motivation of imposing the above-mentioned additional requirements on the viscosity coefficients. When the surface tension is absent, these requirements were removed by direct calculations in the papers of I.Denisova [2] and of Prof. Y.Shibata and his colleagues [3,4], who has used uniqueness of the solution of the model problem (i.e., the Lopatinski condition); see also the review article [5].

The aim of the present paper is to analyze the model problem for the capillary fluids. We concentrate on the problem of stability of the rest state and obtain necessary estimates in the infinite time interval for the solution of the model problem with a small compact support. This assumption was used also in [5]. Our main attention is given to the problem that arises in estimating the solution near the interface as the most complicated one.

In Section 1 the statement of the nonlinear problem is given and the linearization procedure is described. Section 2 is devoted to the construction and estimate of the solution of the model problem.

We pass to the statement of the free boundary problem. Assume that two fluids, compressible and incompressible, are contained in a bounded vessel $\Omega = \Omega_t^+ \cup \Gamma_t \cup \Omega_t^- \subset \mathbb{R}^3$ with the boundary Σ and separated by a free interface Γ_t , $t \geq 0$. It is assumed that the domains Ω_t^+ and Ω_t^- are filled with the compressible and incompressible fluids, respectively, Ω_t^- being located inside Ω , and $\partial\Omega_t^- = \Gamma_t$ is bounded away from Σ . The problem reduces to the determination of Γ_t for $t > 0$ together with the velocity vector field $\mathbf{v}^\pm(x, t)$, $x \in \Omega_t^\pm$, of both fluids, the pressure $p^-(x, t)$ of the incompressible fluid and the density $\rho^+(x, t)$ of the compressible one satisfying the system of the Navier-Stokes equation, the initial and boundary conditions on Σ and Γ_t :

$$\left\{ \begin{array}{l} \rho^-(\mathcal{D}_t \mathbf{v}^- + (\mathbf{v}^- \cdot \nabla) \mathbf{v}^-) - \nabla \cdot \mathbb{T}^-(\mathbf{v}^-) + \nabla p^- = \rho^- \mathbf{f}, \\ \nabla \cdot \mathbf{v}^- = 0 \text{ in } \Omega_t^-, \\ \rho^+(\mathcal{D}_t \mathbf{v}^+ + (\mathbf{v}^+ \cdot \nabla) \mathbf{v}^+) - \nabla \cdot \mathbb{T}^+(\mathbf{v}^+) + \nabla p^+(\rho^+) = \rho^+ \mathbf{f}, \\ \mathcal{D}_t \rho^+ + \nabla \cdot (\rho^+ \mathbf{v}^+) = 0 \text{ in } \Omega_t^+, \\ \mathbf{v}^\pm|_{t=0} = \mathbf{v}_0^\pm \text{ in } \Omega_0^\pm, \quad \rho^+(x, 0) = \rho_0^+(x) \text{ in } \Omega_0^+, \\ \mathbf{v}^+|_\Sigma = 0, \quad [\mathbf{v}] = 0 \quad V_n = \mathbf{v} \cdot \mathbf{n}, \\ (-p^+(\rho^+) + p^-) \mathbf{n} + [\mathbb{T}^\pm(\mathbf{v}) \mathbf{n}] = \sigma H \mathbf{n} \text{ on } \Gamma_t. \end{array} \right. \quad (1.1)$$

In these relations, $\mathbb{T} = \mathbb{T}^\pm$ is the viscous part of the stress tensor, i.e.,

$$\mathbb{T}^-(\mathbf{v}^-) = \mu^- \mathbb{S}(\mathbf{v}^-), \quad \mathbb{T}^+(\mathbf{v}^+) = \mu^+ \mathbb{S}(\mathbf{v}^+) + \mu_1^+ \mathbb{I} \nabla \cdot \mathbf{v}^+,$$

$\mu^\pm > 0$, $\mu_1^+ > -2\mu^+/3$ are constant viscosity coefficients, $\mathbb{S}(\mathbf{w}) = (\nabla \mathbf{w}) + (\nabla \mathbf{w})^T$ is the doubled rate-of-strain tensor, the superscript T means transposition, \mathbb{I} is the identity matrix, $p^+(\rho^+(x, t))$ is the pressure function of the density ρ^+ of the compressible fluid that is positive and increasing, ρ^- is a constant density of the incompressible fluid, \mathbf{f} is the vector field of the mass forces given in Ω , V_n is the velocity of the evolution of the free surface Γ_t in the direction of the exterior normal \mathbf{n} to Γ_t . At the initial instant $t = 0$ the surface Γ_0 is given.

By $[u]|_{\Gamma_t}$ we mean the jump of the function $u(x)$ given in $\Omega_t^+ \cup \Omega_t^-$ across Γ_t :

$$[u] = u^+ - u^-|_{\Gamma_t}.$$

We write (1.1) as a non-linear problem in a domain with fixed interface Γ_0 by passing to the Lagrangian coordinates $y \in \Omega_0^+ \cup \Gamma_0 \cup \Omega_0^-$. They are connected with the Eulerian coordinates $x \in \Omega_t^+ \cup \Gamma_t \cup \Omega_t^-$ by the equation

$$x = y + \int_0^t \mathbf{u}(y, \tau) d\tau \equiv X_{\mathbf{u}}(y, t), \quad (1.2)$$

where $\mathbf{u}(y, \tau)$ is the velocity vector field written as a function of the Lagrangian coordinates. One of the advantages of the transformation (1.2) is that the space derivatives of ρ^+ disappear from the continuity equation after the change of variables, which is not the case for the Hanzawa transformation. In this case it is necessary to work in the spaces of somewhat more regular functions and impose additional compatibility conditions on the data of the problem.

For small $\int_0^t \mathbf{u} d\tau$, the mapping (1.2) establishes one-to-one correspondence between $\Omega_0^+ \cup \Gamma_0 \cup \Omega_0^-$ and $\Omega_t^+ \cup \Gamma_t \cup \Omega_t^-$. In the coordinates (y, t) , problem (1.1) takes the form

$$\begin{cases} \rho^- \mathcal{D}_t \mathbf{u}^- - \nabla_{\mathbf{u}} \cdot \mathbb{T}_{\mathbf{u}}^-(\mathbf{u}^-) + \nabla_{\mathbf{u}} q^- = \rho^- \widehat{\mathbf{f}}, & \nabla_{\mathbf{u}} \cdot \mathbf{u}^- = 0 \text{ in } \Omega_0^-, \\ \widehat{\rho}^+ \mathcal{D}_t \mathbf{u}^+ - \nabla_{\mathbf{u}} \cdot \mathbb{T}_{\mathbf{u}}^+(\mathbf{u}^+) + \nabla_{\mathbf{u}} p^+(\widehat{\rho}^+) = \widehat{\rho}^+ \widehat{\mathbf{f}}, \\ \mathcal{D}_t \widehat{\rho}^+ + \widehat{\rho}^+ \nabla_{\mathbf{u}} \cdot \mathbf{u}^+ = 0 \text{ in } \Omega_0^+, \\ \mathbf{u}^\pm|_{t=0} = \mathbf{u}_0^\pm \equiv \mathbf{v}_0^\pm \text{ in } \Omega_0^\pm, & \widehat{\rho}^+|_{t=0} = \rho_0^+ \text{ in } \Omega_0^+, \\ \mathbf{u}^-|_{\Sigma} = 0, & [\mathbf{u}] = 0, \\ (-p^+(\widehat{\rho}^+) + q^-) \mathbf{n} + [\mathbb{T}_{\mathbf{u}}^\pm(\mathbf{u}) \mathbf{n}] = \sigma H \mathbf{n} \text{ on } \Gamma_0, \end{cases} \quad (1.3)$$

where $\widehat{\mathbf{f}}(y, t) = \mathbf{f}(X_{\mathbf{u}}, t)$, $q^- = p^-(X_{\mathbf{u}}, t)$, $\widehat{\rho}^+ = \rho^+(X_{\mathbf{u}}, t)$, $\nabla_{\mathbf{u}} = \mathbb{L}^{-T} \nabla_y = (\mathbb{L}^{-1})^T \nabla$ is the transformed gradient ∇_x , \mathbb{L} is the Jacobi matrix of the transformation (1.2), $\widehat{\mathbb{L}} = L\mathbb{L}$, $L = \det \mathbb{L}$,

$$\mathbb{T}_{\mathbf{u}}^-(\mathbf{u}^-) = \mu^- \mathbb{S}_{\mathbf{u}^-}(\mathbf{u}^-), \quad \mathbb{T}_{\mathbf{u}}^+(\mathbf{u}^+) = \mu^+ \mathbb{S}_{\mathbf{u}^+}(\mathbf{u}^+) + \mu_1^+ \mathbb{I} \nabla_{\mathbf{u}} \cdot \mathbf{u}^+,$$

$\mathbb{S}_{\mathbf{u}}(\mathbf{u}) = \nabla_{\mathbf{u}} \otimes \mathbf{u} + (\nabla_{\mathbf{u}} \otimes \mathbf{u})^T$ is the transformed doubled rate-of-strain tensor, $H = H(X_{\mathbf{u}}, t)$. The kinematic condition $V_n = \mathbf{u} \cdot \mathbf{n}$ is fulfilled automatically. The normal $\mathbf{n}(X_{\mathbf{u}}, t)$ to Γ_t is connected with the normal \mathbf{n}_0 to Γ_0 by the formula

$$\mathbf{n} = \frac{\widehat{\mathbb{L}}^T \mathbf{n}_0(y)}{|\widehat{\mathbb{L}}^T \mathbf{n}_0(y)|}. \quad (1.4)$$

Since one of the fluids is incompressible, the volumes $|\Omega_t^\pm|$ of Ω_t^\pm are independent of t , as well as the mean value $\rho_m^+ = M^+ / |\Omega_t^+|$ of ρ^+ , where M^+ is the total mass of the compressible fluid. We set

$$|\Omega_t^-| = \frac{4\pi R_0^3}{3}, \quad \widehat{\rho}^+ = \rho_m^+ + \theta^+, \quad q^- = p(\rho_m^+) + \theta^- - \frac{2\sigma}{R_0};$$

hence $\int_{\Omega_t^+} (\rho^+ - \rho_m^+) dx = \int_{\Omega_0^+} L\theta^+(y, t) dy = 0$ and the last boundary condition in (1.3) can be written in the form

$$\left(- (p^+(\rho_m^+ + \theta^+) - p^+(\rho_m^-)) + \theta^- \right) \mathbf{n} + [\mathbb{T}_{\mathbf{u}}(\mathbf{u}) \mathbf{n}] = \sigma \left(H + \frac{2}{R_0} \right) \mathbf{n}.$$

If $\mathbf{n} \cdot \mathbf{n}_0 > 0$, then this condition is equivalent to

$$\begin{aligned} & [\mu^\pm \Pi_0 \Pi \mathbb{S}_u(\mathbf{u}) \mathbf{n}]|_{\Gamma_0} = 0, \\ & - (p^+(\rho_m^+ + \theta^+) - p^+(\rho_m^+)) + \theta^- + [\mathbf{n} \cdot \mathbb{T}_u(\mathbf{u}) \mathbf{n}]|_{\Gamma_0} = \sigma(H + \frac{2\sigma}{R_0})|_{\Gamma_0}, \end{aligned} \quad (1.5)$$

where $\Pi \mathbf{g} = \mathbf{g} - \mathbf{n}(\mathbf{g} \cdot \mathbf{n})$ and $\Pi_0 \mathbf{g} = \mathbf{g} - \mathbf{n}_0(\mathbf{g} \cdot \mathbf{n}_0)$ are projections of \mathbf{g} onto the tangential planes to Γ_t and Γ_0 , respectively.

We also notice that $\int_{\Omega_t^+} \nabla \cdot \mathbf{v}^+(x, t) dx = \int_{\Omega} \nabla \cdot \mathbf{v}(x, t) dx = 0$. Since

$$H \mathbf{n} = \Delta(t) \mathbf{X}_u, \quad (1.6)$$

where $\Delta(t)$ is the Laplace-Beltrami operator on Γ_t , problem (1.3) can be written as follows:

$$\left\{ \begin{aligned} & \rho^- \mathcal{D}_t \mathbf{u}^- - \nabla \cdot T^-(\mathbf{u}^-) + \nabla \theta^- = l_1^-(\mathbf{u}^-, \theta^-), \quad \nabla \cdot \mathbf{u}^- = l_2^-(\mathbf{u}^-) \quad \text{in } \Omega_0^-, \quad t > 0, \\ & \rho_m^+ \mathcal{D}_t \mathbf{u}^+ - \nabla \cdot T^+(\mathbf{u}^+) + p_1 \nabla \theta^+ = l_1^+(\mathbf{u}^+, \theta^+), \\ & \mathcal{D}_t \theta^+ + \rho_m^+ (\nabla \cdot \mathbf{u}^+ - \frac{1}{|\Omega_0|} \int_{\Omega_0} \nabla \cdot \mathbf{u}^+(z, t) dz) = l_2^+(\mathbf{u}^+, \theta^+) \quad \text{in } \Omega_0^+, \quad t > 0, \\ & \mathbf{u}^\pm|_{t=0} = \mathbf{u}_0^\pm \equiv \mathbf{v}_0^\pm \quad \text{in } \Omega_0^\pm, \quad \theta^\pm|_{t=0} = \theta_0^\pm \equiv \rho_0^+ - \rho_m^+ \quad \text{in } \Omega_0^+, \\ & [\mathbf{u}]|_{\Gamma_0} = 0, \quad [\mu^\pm \Pi_0 \mathbb{S}(\mathbf{u}) \mathbf{n}_0]|_{\Gamma_0} = l_3(\mathbf{u}), \\ & - p_1 \theta^+ + \theta^- + [\mathbf{n}_0 \cdot \mathbb{T}(\mathbf{u}) \mathbf{n}_0]|_{\Gamma_0} - \sigma \mathbf{n}_0 \cdot \int_0^t \Delta(0) \mathbf{u}(y, \tau) d\tau|_{\Gamma_0} \\ & = l_4(\mathbf{u}) + \int_0^t (l_5(\mathbf{u}) + l_6(\mathbf{u})) d\tau + \sigma(H_0 + \frac{2}{R_0}), \quad \mathbf{u}|_{\Sigma} = 0, \end{aligned} \right. \quad (1.7)$$

where $p_1 = p'(\rho_m^+) > 0$,

$$\begin{aligned} l_1^-(\mathbf{u}, \theta) &= \nabla_{\mathbf{u}} \cdot T_{\mathbf{u}}^-(\mathbf{u}^-) - \nabla \cdot T^-(\mathbf{u}^-) + (\nabla - \nabla_{\mathbf{u}}) \theta^- + \rho^- \widehat{\mathbf{f}}, \\ l_1^+(\mathbf{u}, \theta) &= \nabla_{\mathbf{u}} \cdot T_{\mathbf{u}}^+(\mathbf{u}^+) - \nabla \cdot T^+(\mathbf{u}^+) \\ &+ p_1 (\nabla - \nabla_{\mathbf{u}}) \theta^+ - \nabla_{\mathbf{u}} (p(\rho_m^+ + \theta^+) - p_1 \theta^+) - \theta^+ \mathcal{D}_t \mathbf{u}^+ + (\rho_m^+ + \theta^+) \widehat{\mathbf{f}}, \\ l_2^-(\mathbf{u}) &= (\nabla - \nabla_{\mathbf{u}}) \cdot \mathbf{u}^- = \nabla \cdot \mathbf{L}(\mathbf{u}^-), \quad \mathbf{L}(\mathbf{u}^-) = (\mathbb{I} - \widehat{\mathbb{L}}) \mathbf{u}^-, \\ l_2^+(\mathbf{u}, \theta) &= \rho_m (\nabla - \nabla_{\mathbf{u}}) \cdot \mathbf{u}^+ - \theta^+ \nabla_{\mathbf{u}} \cdot \mathbf{u}^+ - \frac{\rho_m^+}{|\Omega_0^+|} \int_{\Omega_0^+} (\nabla \cdot \mathbf{u}^+ - L \nabla_{\mathbf{u}} \cdot \mathbf{u}^+) dz, \\ l_3(\mathbf{u}) &= [\mu^\pm \Pi_0 (\Pi_0 \mathbb{S}(\mathbf{u}) \mathbf{n}_0 - \Pi \mathbb{S}_u(\mathbf{u}) \mathbf{n})]|_{\Gamma_0}, \\ l_4(\mathbf{u}) &= [\mathbf{n}_0 \cdot \mathbb{T}(\mathbf{u}, q) \mathbf{n}_0 - \mathbf{n} \cdot \mathbb{T}_u(\mathbf{u}, q) \mathbf{n}] - (p^+(\rho_m^+ + \theta^+) - p^+(\rho_m^+) - p_1 \theta^+)|_{\Gamma_0}, \\ l_5(\mathbf{u}) &= \sigma \mathcal{D}_t (\mathbf{n} \Delta(t)) \cdot \int_0^t \mathbf{u}(y, \tau) d\tau + \sigma (\mathbf{n} \cdot \Delta(t) - \mathbf{n}_0 \cdot \Delta(0)) \mathbf{u}(y, \tau), \\ l_6(\mathbf{u}) &= \sigma (\dot{\mathbf{n}} \Delta(t) + \mathbf{n} \dot{\Delta}(t)) \cdot \mathbf{y}|_{\Gamma_0}, \quad \dot{\mathbf{n}} = \mathcal{D}_t \mathbf{n}, \quad \dot{\Delta}(t) = \mathcal{D}_t \Delta(t). \end{aligned} \quad (1.8)$$

By replacing the expressions (1.8) in (1.7) with some given functions, we arrive at the linear problem

$$\begin{cases}
\rho_m^+ \mathcal{D}_t \mathbf{v}^+ - \mu^+ \nabla^2 \mathbf{v}^+ - (\mu^+ + \mu_1^+) \nabla(\nabla \cdot \mathbf{v}^+) + p_1 \nabla \theta^+ = \mathbf{f}^+, \\
\mathcal{D}_t \theta^+ + \rho_m^+ (\nabla \cdot \mathbf{v}^+ - \frac{1}{|\Omega_0^+|} \int_{\Omega_0^+} \nabla \cdot \mathbf{v}^+ dz) = h^+ \text{ in } \Omega_0^+, \\
\rho^- \mathcal{D}_t \mathbf{v}^- - \mu^- \nabla^2 \mathbf{v}^- + \nabla \theta^- = \mathbf{f}^-, \quad \nabla \cdot \mathbf{v}^- = h^- \text{ in } \Omega_0^-, \\
\mathbf{v}|_{t=0} = \mathbf{v}_0 \text{ in } \Omega_0^+ \cup \Omega_0^-, \quad \theta^+|_{t=0} = \theta_0^+ \text{ in } \Omega_0^+, \\
\mathbf{v}^+|_{\Sigma} = 0, \quad [\mathbf{v}]|_{\Gamma_0} = 0, \quad [\mu^\pm \Pi_0 \mathbb{S}(\mathbf{v}) \mathbf{n}_0]|_{\Gamma_0} = \mathbf{b}, \\
-p_1 \theta^+ + \theta^- + [\mathbf{n}_0 \cdot \mathbb{T}(\mathbf{v}) \mathbf{n}_0] - \sigma \mathbf{n}_0 \cdot \int_0^t \Delta(0) \mathbf{v}(y, \tau) d\tau|_{\Gamma_0} = b.
\end{cases} \quad (1.9)$$

We recall the definition of the Sobolev–Slobodetskii spaces. The isotropic space $W_2^r(\Omega)$, $\Omega \subset \mathbb{R}^n$, is the space with the norm

$$\|u\|_{W_2^r(\Omega)}^2 = \sum_{0 \leq |j| \leq r} \|\mathcal{D}^j u\|_{L_2(\Omega)}^2 \equiv \sum_{0 \leq |j| \leq r} \int_{\Omega} |\mathcal{D}^j u(x)|^2 dx,$$

if $r = [r]$, i.e. r is an integral number, and

$$\|u\|_{W_2^r(\Omega)}^2 = \|u\|_{W_2^{[r]}(\Omega)}^2 + \sum_{|j|=[r]} \int_{\Omega} \int_{\Omega} |\mathcal{D}^j u - \mathcal{D}^j u(y)|^2 \frac{dx dy}{|x-y|^{n+2\rho}},$$

if $r = [r] + \rho$, $\rho \in (0, 1)$. As usual, $\mathcal{D}^j u$ denotes a (generalized) partial derivative $\frac{\partial^{j_1} u}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}$ where $j = (j_1, j_2, \dots, j_n)$ and $|j| = j_1 + \dots + j_n$. The anisotropic space $W_2^{r,r/2}(Q_T)$, $Q_T = \Omega \times (0, T)$, can be defined as

$$L_2((0, T), W_2^r(\Omega)) \cap W_2^{r/2}((0, T), L_2(\Omega))$$

and supplied with the norm

$$\|u\|_{W_2^{r,r/2}(Q_T)}^2 = \int_0^T \|u(\cdot, t)\|_{W_2^r(\Omega)}^2 dt + \int_{\Omega} \|u(x, \cdot)\|_{W_2^{r/2}(0, T)}^2 dx. \quad (1.10)$$

There exist many other equivalent norms in $W_2^{r,r/2}(Q_T)$; some of them will be used below. Sobolev spaces of functions given on smooth surfaces, in particular, on Γ_0 and on $\Gamma_0 \times (0, T)$, are introduced in a standard way, i.e., with the help of local maps and partition of unity. We find it convenient to introduce the spaces

$$W_2^{r,0}(Q_T) = L_2((0, T); W_2^r(\Omega)), \quad W_2^{0,r/2}(Q_T) = W_2^{r/2}((0, T); L_2(\Omega));$$

the squares of norms in these spaces coincide, respectively, with the first and the second term in (1.10). We also introduce the notation

$$\begin{aligned}
\|u\|_{Q_T}^{(r+l, l/2)} &= \|u\|_{W_2^{r+l,0}(Q_T)} + \|u\|_{W_2^{l/2}(0, T); W_2^r(\Omega)}, \\
\|u\|_{W_2^r(\cup \Omega^\pm)} &= \|u\|_{W_2^r(\Omega^+)} + \|u\|_{W_2^r(\Omega^-)},
\end{aligned} \quad (1.11)$$

if $u(x)$ can be discontinuous on $\bar{\Omega}^+ \cap \bar{\Omega}^-$. The norm $\|u\|_{W_2^{l, l/2}(\cup Q_T^\pm)}$ is defined in a similar way.

It is well known that the norms $\|u\|_{W_2^\lambda(\mathbb{R}^n)}$ and $(\int_{\mathbb{R}^n} (1 + |\xi|^2)^\lambda |\tilde{u}(\xi)|^2 d\xi)^{1/2}$ are equivalent, in view of the Parseval identity (by $\tilde{u}(\xi)$ we mean the Fourier transform of $u(x)$). In what follows we deal with the functions $u(x, t)$ from the Sobolev spaces with the exponential weight $e^{\beta t}$, $\beta > 0$, and the following lemma will be useful.

Proposition 1. *Let $e^{\beta t}u \in W_2^\lambda(\mathbb{R})$ with $\lambda \in (0, 1)$, and $u(t) = 0$ for $t < 0$. Then*

$$\begin{aligned} c_1 \int_{s_1-i\infty}^{s_1+i\infty} (|s|^{2\lambda} + 1) |\tilde{u}(s)|^2 ds_2 &\leq \int_0^\infty e^{2\beta t} dt \int_0^1 |u(t-\tau) - u(t)|^2 \frac{d\tau}{\tau^{1+2\lambda}} \\ &+ \int_0^\infty e^{2\beta t} |u(t)|^2 dt \leq c_2 \int_{s_1-i\infty}^{s_1+i\infty} (|s|^{2\lambda} + 1) |\tilde{u}(s)|^2 ds_2, \end{aligned} \quad (1.12)$$

$$\begin{aligned} c_3 \int_{s_1-i\infty}^{s_1+i\infty} (|s|^{2\lambda} + 1) |\tilde{u}(s)|^2 ds_2 &\leq \int_0^\infty dt \int_0^1 |e^{\beta(t-\tau)}u(t-\tau) - e^{\beta t}u(t)|^2 \frac{d\tau}{\tau^{1+2\lambda}} \\ &+ \int_0^\infty e^{2\beta t} |u(t)|^2 dt \leq c_4 \int_{s_1-i\infty}^{s_1+i\infty} (|s|^{2\lambda} + 1) |\tilde{u}(s)|^2 ds_2, \end{aligned} \quad (1.13)$$

where

$$\tilde{u}(s) = \int_0^\infty e^{-st} u(t) dt$$

is the Laplace transform of $u(t)$, $s = s_1 + is_2$, $s_1 = -\beta$.

Proof. Clearly,

$$u(t) - u(t-\tau) = \frac{1}{2\pi} \int_{-\beta-i\infty}^{-\beta+i\infty} \tilde{u}(s)(1 - e^{-s\tau})e^{st} ds_2.$$

By the Parseval identity

$$2\pi \int_0^\infty |u(t)|^2 e^{2\beta t} dt = \int_{-\beta-i\infty}^{-\beta+i\infty} |\tilde{u}(s)|^2 ds_2,$$

we have

$$2\pi \int_0^\infty e^{2\beta t} dt \int_0^1 |u(t) - u(t-\tau)|^2 \frac{d\tau}{\tau^{1+2\lambda}} = \int_{-\beta-i\infty}^{-\beta+i\infty} |\tilde{u}(s)|^2 ds_2 \int_0^1 |e^{-s\tau} - 1|^2 \frac{d\tau}{\tau^{1+2\lambda}}. \quad (1.14)$$

Since

$$|e^{-s\tau} - 1|^2 = (e^{-s_1\tau} - 1)^2 + 4e^{-s_1\tau} \sin^2 \frac{\tau s_2}{2},$$

there holds

$$\begin{aligned} \int_0^1 |e^{-s\tau} - 1|^2 \frac{d\tau}{\tau^{1+2\lambda}} &\leq c(|s_1|^{2\lambda} + \int_0^\infty \sin^2 \frac{\tau s_2}{2} \frac{d\tau}{\tau^{1+2\lambda}}) \leq c|s|^{2\lambda}, \\ \int_0^1 |e^{-s\tau} - 1|^2 \frac{d\tau}{\tau^{1+2\lambda}} &\geq c(|s_1|^2 + \int_0^\infty \sin^2 \frac{\tau s_2}{2} \frac{d\tau}{\tau^{1+2\lambda}} - \int_1^\infty \sin^2 \frac{\tau s_2}{2} \frac{d\tau}{\tau^{1+2\lambda}}) \geq c(|s|^{2\lambda} - 1). \end{aligned}$$

These inequalities and (1.14) imply (1.12). Since $e^{-\beta}e^{\beta t} \leq e^{\beta(t-\tau)} \leq e^{\beta t}$ for $\tau \in (0, 1)$, estimates (1.13) can be easily deduced from (1.12) and the relation

$$e^{\beta(t-\tau)}u(t-\tau) - e^{\beta t}u(t) = e^{\beta(t-\tau)}(u(t-\tau) - u(t)) + u(t)(e^{\beta(t-\tau)} - e^{\beta t}).$$

The proposition is proved.

Proposition 1 extends in an obvious way to the spaces $W_2^l(\mathbb{R})$ with $l = [l] + \lambda$, $[l] > 0$ and to anisotropic spaces.

In what follows we set

$$\|u\|_{W_2^\lambda(\mathbb{R})}^2 = \int_0^\infty dt \int_0^\infty |u(t) - u(t-\tau)|^2 \frac{d\tau}{\tau^{1+2\lambda}} + \int_0^\infty |u(t)|^2 dt \quad (1.15)$$

and take the expression

$$\left(\int_0^\infty dt \int_0^\infty |e^{\beta(t-\tau)}u(t-\tau) - e^{\beta t}u(t)|^2 \frac{d\tau}{\tau^{1+2\lambda}} + \int_0^\infty e^{2\beta t}|u(t)|^2 dt \right)^{1/2} \equiv \|e^{\beta t}u\|_{W_2^\lambda(\mathbb{R})}$$

as the norm in the corresponding weighted space. Since

$$\begin{aligned} \int_0^\infty dt \int_0^\infty |e^{\beta(t-\tau)}u(t-\tau) - e^{\beta t}u(t)|^2 \frac{d\tau}{\tau^{1+2\lambda}} &\leq \int_0^\infty dt \int_0^1 |e^{\beta(t-\tau)}u(t-\tau) - e^{\beta t}u(t)|^2 \frac{d\tau}{\tau^{1+2\lambda}} \\ &+ c \int_0^\infty e^{2\beta t}|u(t)|^2 dt \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty dt \int_0^\infty |e^{\beta(t-\tau)}u(t-\tau) - e^{\beta t}u(t)|^2 \frac{d\tau}{\tau^{1+2\lambda}} &\geq \int_0^\infty dt \int_0^1 |e^{\beta(t-\tau)}u(t-\tau) - e^{\beta t}u(t)|^2 \frac{d\tau}{\tau^{1+2\lambda}} \\ &- c \int_0^\infty e^{2\beta t}|u(t)|^2 dt, \end{aligned}$$

the norm $\|e^{\beta t}u\|_{W_2^\lambda(\mathbb{R})}$ is equivalent to

$$\left(\int_{s_1-i\infty}^{s_1+i\infty} (|s|^{2\lambda} + 1)|\tilde{u}(s)|^2 ds_2 \right)^{1/2}.$$

In the case $s_1 = \gamma > 0$ this equivalence is established in [6].

2 Linear model problems

The study of the linear problem (1.9) is based on the analysis of the model problem in the domain $\mathbb{R}_+^3 \cup \mathbb{R}_-^3$, in which the interface is fixed and coincides with the plane $y_3 = 0$. Assuming that the

compressible fluid is located in \mathbb{R}_+^3 , we write this problem in the form

$$\left\{ \begin{array}{l} \mathcal{D}_t \mathbf{v}^- - \nu^- \nabla^2 \mathbf{v}^- + \frac{1}{\rho^-} \nabla \theta^- = \mathbf{f}^-, \quad \nabla \cdot \mathbf{v}^- = h^- \quad \text{in } R_T^- = \mathbb{R}_-^3 \times (0, T), \\ \mathcal{D}_t \mathbf{v}^+ - \nu^+ \nabla^2 \mathbf{v}^+ - (\nu^+ + \nu_1^+) \nabla (\nabla \cdot \mathbf{v}^+) + \frac{p_1}{\rho_m^+} \nabla \theta^+ = \mathbf{f}^+, \\ \mathcal{D}_t \theta^+ + \rho_m^+ \nabla \cdot \mathbf{v}^+ = h^+ \quad \text{in } R_T^+ = \mathbb{R}_+^3 \times (0, T), \\ \mathbf{v}|_{t=0} = 0 \quad \text{in } \mathbb{R}_+^3 \cup \mathbb{R}_-^3, \quad \theta^+|_{t=0} = 0 \quad \text{in } \mathbb{R}_+^3, \quad \mathbf{v} \xrightarrow{|y_3| \rightarrow \infty} 0, \quad \theta \xrightarrow{|y_3| \rightarrow \infty} 0, \\ [\mathbf{v}]|_{y_3=0} = 0, \quad \left[\mu^\pm \left(\frac{\partial v_\alpha}{\partial y_3} + \frac{\partial v_3}{\partial y_\alpha} \right) \right] \Big|_{y_3=0} = b_\alpha(y', t), \quad \alpha = 1, 2; \\ -p_1 \theta^+ + \theta^- + \mu_1^+ \nabla \cdot \mathbf{v}^+ + \left[2\mu^\pm \frac{\partial v_3}{\partial y_3} \right] + \sigma \Delta' \int_0^t v_3 \, d\tau \Big|_{y_3=0} \\ = b_3 + \int_0^t B(y', \tau) \, d\tau, \quad t \leq T \end{array} \right. \quad (2.1)$$

Here, $T \in (0, \infty]$, $\mathbb{R}_\pm^3 = \{\pm x_3 > 0\}$, $\nu^- = \mu^-/\rho^-$, $\nu^+ = \mu^+/\rho_m^+$, $\nu_1^+ = \mu_1^+/\rho_m^+$, $\rho^-, \rho_m^+ = \text{const} > 0$, $\Delta' = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$, $y' = (y_1, y_2)$. Our goal is to obtain estimates of solutions of (2.1) that are compactly supported in the Sobolev spaces with exponential weight $e^{\beta t}$, $\beta > 0$. To this end, we extend the solutions periodically with respect to $y' = (y_1, y_2)$ into \mathbb{R}^2 and expand into the Fourier series

$$u(y') = \frac{1}{(2d_0)^2} \sum_{\mathbf{k}' \in \mathbb{Z}^2} \tilde{u}(\xi') e^{i\xi' \cdot y'}, \quad \xi' = \frac{\pi}{d_0} \mathbf{k}', \quad \mathbf{k}' = (k_1, k_2), \quad (2.2)$$

(cf. [5,7]), where

$$\tilde{u}(\xi') = \int_{\mathfrak{Q}'} e^{-i\xi' \cdot y'} u(y') \, dy', \quad y' = (y_1, y_2),$$

are the Fourier coefficients and $\mathfrak{Q}' = \{|y_\alpha| \leq d_0, \alpha = 1, 2, \}$ is a periodic cell. Problem (2.1) will be treated separately in the spaces of function satisfying the condition

$$\int_{\mathfrak{Q}'} u \, dy' = 0 \quad (2.3)$$

and of functions of the type $(2d_0)^{-1} \tilde{u}(0)$ (i.e., independent of y_1 and y_2). We introduce the notation $I^\pm = \{\pm y_3 \in (0, d_1)\}$, $\mathfrak{Q}^\pm = \{y' \in \mathfrak{Q}', y_3 \in I^\pm\}$, $Q^\pm = \{y' \in \mathfrak{Q}', \pm y_3 \geq 0\}$, $\mathfrak{Q}'_T = \mathfrak{Q}' \times (0, T)$, $Q^\pm_T = Q^\pm \times (0, T) \forall T \leq \infty$ and prove the following propositions.

Theorem 1. *Assume that $e^{\beta t} \mathbf{f} \in W_2^{l, l/2}(\cup Q_T^\pm)$, $e^{\beta t} h^+ \in W_2^{l+1, 0}(Q_T^+) \cap W_2^{l/2}((0, T); W_2^1(Q^+))$, $e^{\beta t} \nabla h^- \in W_2^{l+1, l/2+1/2}(Q_T^-)$, $h^- = \nabla \cdot \mathbf{H} + H_0$, $e^{\beta t} \mathbf{H}$, $e^{\beta t} H_0 \in W_2^{0, 1+l/2}(Q_T^-)$, $e^{\beta t} b_\alpha \in W_2^{l+1/2, l/2+1/4}(\mathfrak{Q}'_T)$, $\alpha = 1, 2$, $e^{\beta t} b_3 \in W_2^{l+1/2, 0}(\mathfrak{Q}'_T) \cap W_2^{l/2}((0, T); W_2^{1/2}(\mathfrak{Q}'))$, $e^{\beta t} B \in W_2^{l-1/2, l/2-1/4}(\mathfrak{Q}'_T)$, where β and d_0 are certain positive numbers, d_0 is small and $l \in (1/2, 1)$. Assume also that these functions satisfy the compatibility conditions*

$$h^-|_{t=0} = 0, \quad \mathbf{H}|_{t=0} = 0, \quad H_0|_{t=0} = 0, \quad b_\alpha|_{t=0} = 0, \quad \alpha = 1, 2,$$

and the orthogonality condition (2.3). Then problem (2.1) has a unique solution also satisfying (2.3) and such that $e^{\beta t} \mathbf{v} \in W_2^{2+l, 1+l/2}(\cup Q_T^\pm)$, $e^{\beta t} \nabla \theta^- \in W_2^{l, l/2}(Q_T^-)$, $e^{\beta t} \theta^- \in W_2^{l/2}((0, T); W_2^{1/2}(Q'))$,

$e^{\beta t}\theta^+, e^{\beta t}\mathcal{D}_t\theta^+ \in W_2^{l+1,0}(Q_T^+) \cap W_2^{l/2}((0,T); W_2^1(Q^+))$. It is subject to the inequality

$$\begin{aligned} & \|e^{\beta t}\mathbf{v}\|_{W_2^{2+l,1+l/2}(\cup Q_T^\pm)} + \|e^{\beta t}\nabla\theta^-\|_{W_2^{l,1/2}(Q_T^-)} + |e^{\beta t}\theta^-|_{Q_T^-}^{(l+1/2,l/2)} + |e^{\beta t}\theta^+|_{Q_T^+}^{(1+l,l/2)} \\ & + |e^{\beta t}\mathcal{D}_t\theta^+|_{Q_T^+}^{(1+l,l/2)} \leq c(\|e^{\beta t}\mathbf{f}\|_{W_2^{l,1/2}(\cup Q_T^\pm)} + \|e^{\beta t}h^-\|_{W_2^{l+1,0}(Q_T^-)} + \|e^{\beta t}\mathbf{H}\|_{W_2^{0,1+l/2}(Q_T^-)} \\ & + \|e^{\beta t}H_0\|_{W_2^{0,1+l/2}(Q_T^-)} + |e^{\beta t}h^+|_{Q_T^+}^{(1+l,l/2)} + \sum_{\alpha=1,2} \|e^{\beta t}b_\alpha\|_{W_2^{l+1/2,l/2+1/4}(Q_T^-)} \\ & + |e^{\beta t}b_3|_{Q_T^-}^{(l+1/2,l/2)} + \|e^{\beta t}B\|_{W_2^{l-1/2,l/2-1/4}(Q_T^-)}) \end{aligned} \quad (2.4)$$

with the constant independent of T .

We also consider one-dimensional problems

$$\begin{cases} \mathcal{D}_t v_\alpha^\pm - \nu^\pm \mathcal{D}_{y_3}^2 v_\alpha = f_\alpha^\pm & \text{in } I_T^\pm, \\ v_\alpha|_{t=0} = 0, \quad v_\alpha|_{y_3=\pm d_1} = 0, \\ [v_\alpha]|_{y_3=0} = 0, \quad [\mu \frac{dv_\alpha}{dy_3}]|_{y_3=0} = b_\alpha(t), \quad \alpha = 1, 2, \end{cases} \quad (2.5)$$

$$\begin{cases} \mathcal{D}_t v_3^+ - (2\nu^+ + \nu_1^+) \mathcal{D}_{y_3}^2 v_3 + \frac{p_1^+}{\rho_m^+} \mathcal{D}_{y_3} \theta^+ = f_3^+, \quad \mathcal{D}_t \theta^+ + \rho_m^+ \mathcal{D}_{y_3} v_3 = h^+ & \text{in } I_T^+, \\ \mathcal{D}_t v_3^- - \nu^- \mathcal{D}_{y_3}^2 v_3^- + \frac{1}{\rho^-} \mathcal{D}_{y_3} \theta^- = f_3^-, \quad \mathcal{D}_{y_3} v_3 = h^- & \text{in } I_T^-, \\ [v_3]|_{y_3=0} = 0, \quad v_3|_{y_3=\pm d_1} = 0, \\ -p_1 \theta^+ + \theta^- + (2\mu^+ + \mu_1^+) \mathcal{D}_{y_3} v_3^+ - 2\mu^- \mathcal{D}_{y_3} v_3^- \Big|_{y_3=0} = b_3(t), \end{cases} \quad (2.6)$$

Theorem 2. 1. If $e^{\beta t} f_\alpha^\pm \in W_2^{l,l/2}(\cup I_T^\pm)$, $e^{\beta t} b_\alpha \in W_2^{l/2+1/4}((0,T))$, and d_0, d_1 are sufficiently small, then problem (2.5) has a unique solution such that $e^{\beta t} v_\alpha \in W_2^{2+l,1+l/2}(\cup I_T^\pm)$, where $\alpha = 1, 2$, $I_T^\pm = I^\pm \times (0, T)$, and the solution satisfies the inequality

$$\sum_{\alpha=1}^2 \|e^{\beta t} v_\alpha\|_{W_2^{2+l,1+l/2}(\cup I_T^\pm)} \leq c \sum_{\alpha=1}^2 (\|e^{\beta t} f_\alpha^\pm\|_{W_2^{l,l/2}(\cup I_T^\pm)} + \|e^{\beta t} b_\alpha\|_{W_2^{l/2+1/4}(0,T)}). \quad (2.7)$$

2. If $e^{\beta t} f_3^\pm \in W_2^{l,l/2}(\cup I_T^\pm)$, $e^{\beta t} h^+ \in W_2^{l+1,0}(I_T^+) \cap W_2^{l/2}((0,T), W_2^1(I^+))$, $e^{\beta t} \mathcal{D}_{y_3} h^- \in W_2^{l,l/2}(I_T^-)$, $h^- = \mathcal{D}_{y_3} \mathfrak{H}_3 + \mathfrak{H}_0$, $e^{\beta t} \mathfrak{H}_3, e^{\beta t} \mathfrak{H}_0 \in W_2^{0,1+l/2}(I_T^+)$, $e^{\beta t} b_3 \in W_2^{l/2+1/4}((0,T))$, then problem (2.6) has a unique solution such that $e^{\beta t} v_3 \in W_2^{2+l,1+l/2}(\cup I_T^\pm)$, $e^{\beta t} \mathcal{D}_{y_3} \theta^+ \in W_2^{l,l/2}(I_T^+)$, $e^{\beta t} \mathcal{D}_t \theta^+ \in W_2^{l+1,0}(I_T^+) \cap W_2^{l/2}((0,T); W_2^1(I^+))$, $e^{\beta t} \mathcal{D}_{y_3} \theta^- \in W_2^{l,l/2}(I_T^-)$, $e^{\beta t} \theta^-|_{y_3=0} \in W_2^{l/2+1/4}(0,T)$, and the solution satisfies the inequality

$$\begin{aligned} & \|e^{\beta t} v_\alpha\|_{W_2^{2+l,1+l/2}(\cup I_T^\pm)} + \|e^{\beta t} \mathcal{D}_{y_3} \theta^+\|_{W_2^{l,l/2}(I_T^+)} + |e^{\beta t} \mathcal{D}_t \theta^+|_{I_T^+}^{(1+l,l/2)} \\ & + \|e^{\beta t} \mathcal{D}_{y_3} \theta^-\|_{W_2^{l,l/2}(I_T^-)} + \|e^{\beta t} \theta^-\|_{y_3=0} \|W_2^{l/2+1/4}(0,T)\| \leq c(\|e^{\beta t} f_3\|_{W_2^{l,l/2}(\cup I^\pm)} \\ & + \|e^{\beta t} h^-\|_{W_2^{l+1,0}(I_T^-)} + \|e^{\beta t} \mathfrak{H}_3\|_{W_2^{0,1+l/2}(I_T^-)} + \|e^{\beta t} \mathfrak{H}_0\|_{W_2^{0,1+l/2}(I_T^-)} \\ & + |e^{\beta t} h^+|_{I_T^+}^{(1+l,l/2)} + \|b_3\|_{W_2^{l/2+1/4}(0,T)}). \end{aligned} \quad (2.8)$$

The constants in (2.7) and (2.8) are independent of T .

Theorem 3. *The solution $(\mathbf{v}^\pm, \theta^\pm)$ of problem (2.1) supported in Ω_T^\pm and such that $e^{\beta t} \mathbf{v} \in W_2^{2+l, 1+l/2}(\cup \Omega_T^\pm)$, $e^{\beta t} \nabla \theta^- \in W_2^{l, l/2}(\Omega_T^-)$, $e^{\beta t} \theta^- \in W_2^{l/2}((0, T); W_2^{1/2}(\Omega'))$, $e^{\beta t} \nabla \theta^+ \in W_2^{l, l/2}(\Omega_T^+)$, $e^{\beta t} \mathcal{D}_t \theta^+ \in W_2^{l+1, 0}(\Omega_T^+) \cap W_2^{l/2}((0, T); W_2^1(\Omega^+))$, satisfies the inequality*

$$\begin{aligned}
& \|e^{\beta t} \mathbf{v}\|_{W_2^{2+l, 1+l/2}(\cup \Omega_T^\pm)} + \|e^{\beta t} \nabla \theta^-\|_{W_2^{l, l/2}(\Omega_T^-)} + |e^{\beta t} \theta^-|_{\Omega_T^-}^{(l+1/2, l/2)} + \|e^{\beta t} \nabla \theta^+\|_{W_2^{l, l/2}(\Omega_T^+)} \\
& + |e^{\beta t} \mathcal{D}_t \theta^+|_{\Omega_T^+}^{(1+l, l/2)} \leq c(\|e^{\beta t} \mathbf{f}\|_{W_2^{l, l/2}(\cup \Omega_T^\pm)} + \|e^{\beta t} h^-\|_{W_2^{l+1, 0}(\Omega_T^-)} + \|e^{\beta t} \mathbf{H}\|_{W_2^{0, 1+l/2}(\Omega_T^-)} \\
& + \|e^{\beta t} H_0\|_{W_2^{0, 1+l/2}(\Omega_T^-)} + |e^{\beta t} h^+|_{\Omega_T^+}^{(1+l, l/2)} + \sum_{\alpha=1, 2} \|e^{\beta t} b_\alpha\|_{W_2^{l+1/2, l/2+1/4}(\Omega_T^-)} \\
& + |e^{\beta t} b_3|_{\Omega_T^+}^{(l+1/2, l/2)} + \|e^{\beta t} B\|_{W_2^{l-1/2, l/2-1/4}(\Omega_T^-)})
\end{aligned} \tag{2.9}$$

with the constant independent of T , provided that $\int_{\Omega'} B \, dy' = 0$.

Proof. We extend $\mathbf{v}^\pm, \theta^\pm$ periodically with respect to (y_1, y_2) into \mathbb{R}^2 and expand in the series (2.2). It is easily seen that (2.1) is decomposed into two problems: the first one for the projection

$$\mathbf{v}' = \mathbf{v} - \frac{1}{(2d_0)^2} \int_{\Omega'} \mathbf{v} \, dy', \quad \theta' = \theta - \frac{1}{(2d_0)^2} \int_{\Omega'} \theta \, dy'$$

of (\mathbf{v}, θ) onto the space of functions satisfying (2.3) and another one, of the type (2.5), (2.6), for

$$\mathbf{v}'' = \frac{1}{(2d_0)^2} \int_{\Omega'} \mathbf{v} \, dy_3, \quad \theta'' = \frac{1}{(2d_0)^2} \int_{\Omega'} \theta \, dy_3$$

(it is obtained by integration of (2.1) over Ω' , taking account of $\int_{\Omega'} B \, dy' = 0$). Clearly, $\mathfrak{H}_3 = \int_{\Omega'} H_3 \, dy'$, $\mathfrak{H}_0 = \int_{\Omega'} H_0 \, dy'$. Applying Theorems 1 and 2, we obtain the desired estimate.

Remark. The norm $\|e^{\beta t} \nabla \theta^+\|_{W_2^{l, l/2}(\Omega_T^+)}$ in (2.9) can be replaced by a stronger norm $|e^{\beta t} \mathcal{D}_t \theta^+|_{\Omega_T^+}^{(1+l, l/2)}$, if the condition $\int_{\Omega^+} \theta^+ \, dy' = 0$ is satisfied. This can be achieved by the construction of a special decomposition of θ^+ in (1.9) (as in [9]).

The proof of theorems 1 and 2 is given in the rest of the section.

2.1. Homogeneous Lamé-Stokes problem.

We proceed with the proof of Theorem 1 for $T = \infty$. We start with analysis of the homogeneous problem

$$\left\{ \begin{array}{l} \mathcal{D}_t \mathbf{v}^- - \nu^- \nabla^2 \mathbf{v}^- + \frac{1}{\rho_0} \nabla \theta^- = 0, \quad \nabla \cdot \mathbf{v}^- = 0 \quad \text{in } R_\infty^- = \mathbb{R}_-^3 \times (0, \infty) \\ \mathcal{D}_t \mathbf{v}^+ - \nu^+ \nabla^2 \mathbf{v}^+ - (\nu^+ + \nu_1^+) \nabla (\nabla \cdot \mathbf{v}^+) = 0 \quad \text{in } R_\infty^+ = \mathbb{R}_+^3 \times (0, \infty), \\ \mathbf{v}|_{t=0} = 0 \quad \text{in } \mathbb{R}_+^3 \cup \mathbb{R}_-^3, \quad \mathbf{v} \xrightarrow{|y_3| \rightarrow \infty} 0, \quad \theta^- \xrightarrow{y_3 \rightarrow -\infty} 0, \\ [\mathbf{v}]|_{y_3=0} = 0, \quad \left[\mu^\pm \left(\frac{\partial v_\alpha}{\partial y_3} + \frac{\partial v_3}{\partial y_\alpha} \right) \right] \Big|_{y_3=0} = b_\alpha(y', t), \quad \alpha = 1, 2; \\ \theta^- + \mu_1^+ \nabla \cdot \mathbf{v}^+ + \left[2\mu^\pm \frac{\partial v_3}{\partial y_3} \right] + \sigma \Delta' \int_0^t v_3 \, d\tau \Big|_{y_3=0} = b_3 + \sigma \int_0^t B \, d\tau \equiv b'_3, \end{array} \right. \tag{2.10}$$

assuming that \mathbf{b} and B are $2d_0$ -periodic functions of y_1 and y_2 decaying exponentially as $t \rightarrow \infty$. We expand the data (\mathbf{b}, B) in the Fourier series (2.2) and take the Fourier-Laplace transform in (2.2), which converts this problem into a system of ordinary differential equations and some jump conditions for the functions $\tilde{\mathbf{v}}, \tilde{\theta}^-$, namely,

$$\left\{ \begin{array}{l} (s + \nu^+ |\xi'|^2) \tilde{\mathbf{v}}^+ - \nu^+ \frac{d^2 \tilde{\mathbf{v}}^+}{dy_3^2} - (\nu^+ + \nu_1^+) \tilde{\nabla} (\tilde{\nabla} \cdot \tilde{\mathbf{v}}^+) = 0 \text{ for } y_3 > 0, \\ (s + \nu^- |\xi'|^2) \tilde{\mathbf{v}}^- - \nu^- \frac{d^2 \tilde{\mathbf{v}}^-}{dy_3^2} + \frac{1}{\rho^-} \tilde{\nabla} \tilde{\theta}^- = 0, \quad \tilde{\nabla} \cdot \tilde{\mathbf{v}}^- = 0 \text{ for } y_3 < 0, \\ [\tilde{\mathbf{v}}]_{y_3=0} = 0, \quad [\mu^\pm (\frac{d\tilde{v}_\alpha}{dy_3} + i\xi_\alpha \tilde{v}_3)]_{y_3=0} = \tilde{b}_\alpha, \quad \alpha = 1, 2, \\ \tilde{\theta}^- + [2\mu^\pm \frac{d\tilde{v}_3}{dy_3}] + \mu_1^\pm \tilde{\nabla} \cdot \tilde{\mathbf{v}} - \sigma \frac{|\xi'|^2}{s} \tilde{v}_3 \Big|_{y_3=0} = \tilde{b}_3 + \frac{\sigma}{s} \tilde{B} \equiv \tilde{b}'_3, \\ \tilde{\mathbf{v}} \rightarrow 0, \quad \tilde{\theta}^- \rightarrow 0 \text{ as } |y_3| \rightarrow \infty, \end{array} \right. \quad (2.11)$$

where $\tilde{\nabla} = (i\xi_1, i\xi_2, \frac{d}{dy_3})$.

As above, this system is studied separately for $\xi' = (\frac{\pi}{d_0} k_1, \frac{\pi}{d_0} k_2)$ with $|\mathbf{k}'| = |k_1| + |k_2| > 0$ and $\mathbf{k}' = 0$, but for the time being it is assumed that $\xi' \in \mathbb{R}^2$. The general form of the solution of (2.11) is

$$\begin{aligned} \tilde{\mathbf{v}}^+(\xi', s, y_3) &= C_1^+ \begin{pmatrix} r^+ \\ 0 \\ i\xi_1 \end{pmatrix} e^{-r^+ y_3} + C_2^+ \begin{pmatrix} 0 \\ r^+ \\ i\xi_2 \end{pmatrix} e^{-r^+ y_3} + C_3^+ \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1^+ \end{pmatrix} e^{-r_1^+ y_3} \\ &= \begin{pmatrix} C_1 r^+ + i\xi_1 C_3^+ \\ C_2 r^+ + i\xi_2 C_3^+ \\ C^+ - r_1^+ C_3 \end{pmatrix} e^{-r^+ y_3} + C_3^+ \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1^+ \end{pmatrix} (e^{-r_1^+ y_3} - e^{-r^+ y_3}) \text{ for } y_3 > 0, \\ \tilde{\mathbf{v}}^-(\xi', s, y_3) &= C_1^- \begin{pmatrix} -r^- \\ 0 \\ i\xi_1 \end{pmatrix} e^{r^- y_3} + C_2^- \begin{pmatrix} 0 \\ -r^- \\ i\xi_2 \end{pmatrix} e^{r^- y_3} + C_3^- \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ |\xi'| \end{pmatrix} e^{|\xi'| y_3} \\ &= \begin{pmatrix} -C_1 r^- + i\xi_1 C_3^- \\ -C_2 r^- + i\xi_2 C_3^- \\ C^- + |\xi'| C_3^- \end{pmatrix} e^{r^- y_3} + C_3^- \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ |\xi'| \end{pmatrix} (e^{|\xi'| y_3} - e^{r^- y_3}), \\ \tilde{\theta}^- &= -C_3^- \rho_0^- s e^{|\xi'| y_3} \text{ for } y_3 < 0, \end{aligned}$$

where $r^\pm(s, \xi') = \sqrt{s/\nu^\pm + |\xi'|^2}$, $r_1^+ = \sqrt{s/(2\nu^+ + \nu_1^+) + |\xi'|^2}$, $|\arg \sqrt{z}| \leq \pi/2$ for arbitrary complex-valued z , $C^\pm = \sum_{\gamma=1}^2 i\xi_\gamma C_\gamma$.

The coefficients C_i^\pm are found from the jump conditions in (2.11). Since

$$\begin{aligned} \frac{d\tilde{v}_\alpha^+}{dy_3} \Big|_{y_3=0} &= -r^{+2} C_\alpha^+ - i\xi_\alpha r_1^+ C_3^+, \quad \frac{d\tilde{v}_\alpha^-}{dy_3} \Big|_{y_3=0} = -r^{-2} C_\alpha^- + i\xi_\alpha |\xi'| C_3^-, \\ \frac{d\tilde{v}_3^+}{dy_3} \Big|_{y_3=0} &= -r^+ C^+ + r_1^{+2} C_3^+, \quad \frac{d\tilde{v}_3^-}{dy_3} \Big|_{y_3=0} = r^- C^- + |\xi'|^2 C_3^-, \quad \tilde{v}^+|_{y_3=0} = \tilde{v}^-|_{y_3=0}, \end{aligned}$$

these conditions reduce to

$$\begin{aligned}
& \mu^+((-r^{+2}C_\alpha^+ - i\xi_\alpha r_1^+ C_3^+) + i\xi_\alpha(\mathbb{C}^+ - r_1^+ C_3^+)) \\
& - \mu^-((-r^{-2}C_\alpha^- + i\xi_\alpha |\xi'| C_3^-) + i\xi_\alpha(\mathbb{C}^- + |\xi'| C_3^-)) = \tilde{b}_\alpha, \quad \alpha = 1, 2, \\
& (2\mu^+ + \mu_1^+)(-r^+ \mathbb{C}^+ + r_1^{+2} C_3^+) + \mu_1^+ \sum_{\gamma=1,2} i\xi_\gamma (r^+ C_\gamma^+ + i\xi_\gamma C_3^+) \\
& - 2\mu^-(r^- \mathbb{C}^- + |\xi'|^2 C_3^-) - C_3^- \rho_0^- s - \frac{\sigma |\xi'|^2}{s} (\mathbb{C}^+ - r_1^+ C_3^+) = \tilde{b}'_3, \\
& C_\alpha^+ r^+ + i\xi_\gamma C_3^+ = -C_\alpha^+ r^- + i\xi_\alpha C_3^-, \quad \alpha = 1, 2, \quad \mathbb{C}^+ - r_1^+ C_3^+ = \mathbb{C}^- + |\xi'| C_3^-.
\end{aligned} \tag{2.12}$$

From (2.12) it follows that

$$\begin{aligned}
& \mu^+(-r^{+2} \mathbb{C}^+ + |\xi'|^2 r_1^+ C_3^+ - |\xi'|^2 (\mathbb{C}^+ - r_1^+ C_3^+)) \\
& - \mu^-(-r^{-2} \mathbb{C}^- - |\xi'|^3 C_3^- - |\xi'|^2 (\mathbb{C}^- + |\xi'| C_3^-)) = \tilde{\mathbb{B}} = \sum_{\gamma=1,2} i\xi_\gamma \tilde{b}_\gamma, \\
& \mathbb{C}^+ r^+ - |\xi'|^2 C_3^+ = -\mathbb{C}^- r^- - |\xi'|^2 C_3^-.
\end{aligned} \tag{2.13}$$

We transform (2.12), (2.13) into an algebraic system with respect to C_3^\pm and $\mathbb{C}^+ - r_1^+ C_3^+$. We replace \mathbb{C}^+ with $(\mathbb{C}^+ - r_1^+ C_3^+) + r_1^+ C_3^+$, \mathbb{C}^- with $(\mathbb{C}^- + |\xi'| C_3^-) - |\xi'| C_3^-$ and make use of the formulas

$$r^{\pm 2} - |\xi'|^2 = \frac{s}{\nu^\pm} = \frac{\rho^\pm s}{\mu^\pm} \quad \text{and} \quad r_1^+ r^+ - |\xi'|^2 = \frac{r_1^+ + \varkappa^+ r^+}{r_1^+ + r^+} \frac{s}{\nu^+} \equiv \frac{s}{\nu^+} R^+, \tag{2.14}$$

where $\varkappa^+ = \nu^+ / (2\nu^+ + \nu_1^+)$. This leads to

$$\begin{aligned}
& A_1(\mathbb{C}^+ - r_1^+ C_3^+) - r_1^+ \rho^+ s C_3^+ - |\xi'| \rho^- s C_3^- = \tilde{\mathbb{B}}, \\
& (A_2 - \frac{\sigma |\xi'|^2}{s})(\mathbb{C}^+ - r_1^+ C_3^+) + \rho^+ s(1 - 2R^+) C_3^+ - \rho^- s(1 - \frac{2|\xi'|}{r^- + |\xi'|}) C_3^- = \tilde{b}'_3, \\
& A_3(\mathbb{C}^+ - r_1^+ C_3^+) + \rho^+ s \frac{R^+}{\mu^+} C_3^+ - \rho^- s \frac{|\xi'| C_3^-}{\mu^-(r^- + |\xi'|)} = 0,
\end{aligned} \tag{2.15}$$

where

$$A_1 = -\mu^+(r^{+2} + |\xi'|^2) + \mu^-(r^{-2} + |\xi'|^2), \quad A_2 = -2\mu^+ r^+ - 2\mu^- r^-, \quad A_3 = r^+ + r^-.$$

In order to write (2.15) in a uniform way, we set $r_1^- = |\xi'|$ and $R^- = |\xi'| / (r^- + |\xi'|)$; then (2.15) takes the form

$$\begin{aligned}
& A_1(\mathbb{C}^+ - r_1^+ C_3^+) - r_1^+ \rho^+ s C_3^+ - r_1^- \rho^- s C_3^- = \tilde{\mathbb{B}}, \\
& (A_2 - \frac{\sigma |\xi'|^2}{s})(\mathbb{C}^+ - r_1^+ C_3^+) + \rho^+ s(1 - 2R^+) C_3^+ - \rho^- s(1 - 2R^-) C_3^- = \tilde{b}'_3, \\
& A_3(\mathbb{C}^+ - r_1^+ C_3^+) + \rho^+ s \frac{R^+}{\mu^+} C_3^+ - \rho^- s \frac{R^-}{\mu^-} C_3^- = 0,
\end{aligned} \tag{2.16}$$

We see that $(\rho^+ s C_3^+, \rho^- s C_3^-)$ satisfy the equation

$$\mathcal{M}(\rho^+ s C_3^+, \rho^- s C_3^-)^T = (\tilde{\mathbb{B}}, \tilde{b}'_3)^T,$$

where

$$\mathcal{M} = \begin{pmatrix} -\frac{A_1 R^+}{A_3 \mu^+} - r_1^+ & \frac{A_1 R^-}{A_3 \mu^-} - r_1^- \\ -\frac{A'_2 R^+}{A_3 \mu^+} + 1 - 2R^+ & \frac{A'_2 R^-}{A_3 \mu^-} - (1 - 2R^-) \end{pmatrix}, \quad A'_2 = A_2 - \frac{\sigma|\xi'|^2}{s}.$$

Hence

$$\begin{aligned} C_3^+ &= \frac{1}{\rho^+ s P} \left\{ (A'_2 \frac{R^-}{\mu^-} - A_3(1 - 2R^-)) \tilde{\mathbb{B}} - (A_1 \frac{R^-}{\mu^-} - A_3 r_1^-) \tilde{b}'_3 \right\}, \\ C_3^- &= \frac{1}{\rho^- s P} \left\{ (A'_2 \frac{R^+}{\mu^+} - A_3(1 - 2R^+)) \tilde{\mathbb{B}} - (A_1 \frac{R^+}{\mu^+} + A_3 r_1^+) \tilde{b}'_3 \right\}, \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} P &= A_3 \det \mathcal{M} = (-\mu^+(r^{+2} + |\xi'|^2) + \mu^-(r^{-2} + |\xi'|^2)) \left(-\frac{R^-}{\mu^-} (1 - 2R^+) + \frac{R^+}{\mu^+} (1 - 2R^-) \right) \\ &+ 2(\mu^+ r^+ + \mu^- r^-) \left(\frac{r_1^+ R^-}{\mu^-} + \frac{r_1^- R^+}{\mu^+} \right) + (r^+ + r^-) (r_1^+ (1 - 2R^-) + r_1^- (1 - 2R^+)) \\ &+ \frac{\sigma|\xi'|^3}{s} \left(\frac{r_1^+}{\mu^-(r^- + |\xi'|)} + \frac{R^+}{\mu^+} \right). \end{aligned} \quad (2.18)$$

Following [2], we write the solution of (2.11) in the form convenient for the forthcoming estimates:

$$\begin{aligned} \tilde{\mathbf{v}} &= \mathbf{W} e_0^\pm(y_3) + \mathbf{V}^\pm e_1^\pm(y_3), \quad \text{if } \pm y_3 > 0, \\ \tilde{\theta}^- &= -C_3^- \rho^- s e^{|\xi'| y_3}, \quad \text{if } y_3 < 0, \end{aligned} \quad (2.19)$$

where C_3^\pm are given by (2.17),

$$e_0^\pm = e^{\mp r^\pm y_3}, \quad e_1^- = \frac{e^{r^- y_3} - e^{|\xi'| y_3}}{r^- - |\xi'|}, \quad e_1^+ = \frac{e^{-r^+ y_3} - e^{-r_1^+ y_3}}{r^+ - r_1^+}, \quad (2.20)$$

$$\mathbf{W} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad \mathbf{V}^- = -C_3^-(r^- - |\xi'|) \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ |\xi'| \end{pmatrix}, \quad \mathbf{V}^+ = -C_3^+(r^+ - r_1^+) \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1^+ \end{pmatrix}.$$

In view of (2.12) and (2.16),

$$\begin{aligned} \omega_\alpha &= -\frac{\tilde{b}_\alpha}{\mu^+ r^+ + \mu^- r^-} + \frac{i\xi_\alpha (\mu^+ - \mu^-) \omega_3}{\mu^+ r^+ + \mu^- r^-} + \frac{i\xi_\alpha (\mu^+ C_3^+(r^+ - r_1^+) + \mu^- C_3^-(r^- - r_1^-))}{\mu^+ r^+ + \mu^- r^-}, \\ \omega_3 &= -\frac{1}{A_3} (\rho^+ s C_3^+ \frac{R^+}{\mu^+} - \rho^- s C_3^- \frac{R^-}{\mu^-}). \end{aligned} \quad (2.21)$$

Now, we simplify the function

$$\begin{aligned} M &= (-\mu^+(r^{+2} + |\xi'|^2) + \mu^-(r^{-2} + |\xi'|^2)) \left(-\frac{R^-}{\mu^-} (1 - 2R^+) + \frac{R^+}{\mu^+} (1 - 2R^-) \right) \\ &+ 2(\mu^+ r^+ + \mu^- r^-) \left(\frac{r_1^+ R^-}{\mu^-} + \frac{r_1^- R^+}{\mu^+} \right) + (r^+ + r^-) (r_1^+ (1 - 2R^-) + r_1^- (1 - 2R^+)) \end{aligned} \quad (2.22)$$

that is the principal part of P . We write it as the sum

$$M = \mu^{+2} E^+ + \mu^{-2} E^- + \mu^+ \mu^- E_0. \quad (2.23)$$

It is easily seen that

$$\begin{aligned}
E^+ &= \frac{1}{\mu^+\mu^-}(r^{+2} + |\xi'|^2)R^-(1 - 2R^+) + 2r_1^+r^+R^- \\
&= \frac{1}{\mu^+\mu^-}\left(\frac{s}{\nu^+}R^-(1 - 2R^+) + 2\frac{s}{\nu^+}R^-R^+ + 2|\xi'|^2R^-(1 - 2R^+) + 2|\xi'|^2R^-\right) \\
&= \frac{1}{\mu^+\mu^-}\left(\frac{sR^-}{\nu^+} + 4|\xi|^2R^-(1 - R^+)\right), \\
E^- &= \frac{1}{\mu^+\mu^-}\left(\frac{sR^+}{\nu^-} + 4|\xi'|^2R^+(1 - R^-)\right),
\end{aligned}$$

$$\begin{aligned}
E_0 &= \frac{1}{\mu^+\mu^-}(-(r^{+2} + |\xi'|^2)R^+(1 - 2R^-) - (r^{-2} + |\xi'|^2)R^-(1 - 2R^+)) \\
&\quad + 2(r^+r_1^-R^+ + r^-r_1^+R^-) + (r^+ + r^-)(r_1^+(1 - 2R^-) + r_1^-(1 - 2R^+)) \\
&= \frac{1}{\mu^+\mu^-}\left(-\frac{s}{\nu^+}R^+(1 - 2R^-) - \frac{s}{\nu^-}R^-(1 - 2R^+) + 2\left(\frac{s}{\nu^+}R^+R_+^- + \frac{s}{\nu^-}R^-R_-^+\right) \right. \\
&\quad \left. + \left(\frac{s}{\nu^+}R^+ + \frac{s}{\nu^-}R_-^+\right)(1 - 2R^-) + \left(\frac{s}{\nu^+}R_+^- + \frac{s}{\nu^-}R^- \right)(1 - 2R^+) \right. \\
&\quad \left. - 2|\xi'|^2(R^+(1 - 2R^-) + R^-(1 - 2R^+)) + 2|\xi|^2(R^+ + R^-) + 2|\xi'|^2((1 - 2R^-) + (1 - 2R^+))\right),
\end{aligned}$$

where

$$R_-^+ = \frac{r_1^+ + \varkappa_-^+r^-}{r_1^+ + r^-}, \quad R_+^- = \frac{|\xi'|}{|\xi'| + r^+}, \quad \varkappa_-^+ = \frac{\nu^-}{2\nu^+ + \nu_1^+}. \quad (2.24)$$

Easy calculation shows that

$$E_0 = \frac{1}{\mu^+\mu^-}\left(\frac{s}{\nu^-}R_-^+ + \frac{s}{\nu^+}R_+^- + 4|\xi'|^2(1 - R^-(1 - R^+) - R^+(1 - R^-))\right),$$

hence

$$\begin{aligned}
M &= \frac{1}{\mu^+\mu^-}\left(\mu^{+2}\frac{sR^-}{\nu^+} + \mu^{-2}\frac{sR^+}{\nu^-} + \mu^+\mu^-\left(\frac{sR_-^+}{\nu^-} + \frac{sR_+^-}{\nu^+}\right) \right. \\
&\quad \left. + 4|\xi'|^2(\mu^{+2}R^-(1 - R^+) + \mu^{-2}R^+(1 - R^-) + \mu^+\mu^-(1 - R^-(1 - R^+) - R^+(1 - R^-)))\right), \quad (2.25)
\end{aligned}$$

$$P = M + \frac{\sigma|\xi'|^3}{s}g, \quad g = \frac{r_1^+}{\mu^-(r^- + |\xi'|)} + \frac{R^+}{\mu^+}.$$

Remark. Formula (2.25) for M coincides, up to a factor, with the expression given (without extended proof) in the paper [5] by I.V.Denisova; it has the following form in our notation:

$$\begin{aligned}
M &= \rho^+\mu^+\frac{sR^-}{\nu^+} + \rho^-\mu^-\frac{sR^+}{\nu^-} + \rho^+\mu^-\left(\frac{sR_-^+}{\nu^-} + \frac{sR_+^-}{\nu^+}\right) \\
&\quad + 4\rho^+\mu^+|\xi'|^2 - 4(\mu^+ - \mu^-)\frac{|\xi'|^2}{\nu^+}(\mu^-(1 - R^-)R^+ - \mu^+(1 - R^+)R^-).
\end{aligned}$$

We pass to the estimates of M and P . The estimate of M (2.26) obtained below is proved in the papers [2] and [3] for $\text{Res} > 0$; we need it also for negative (small) Res .

Proposition 2. *If $\text{Res} = s_1 \geq 0$ or $0 > \text{Res} = s_1 > -\delta|\xi'|^2$ with small $\delta > 0$ and $s_1 < 0$, then*

$$c_1(|s| + |\xi'|^2) \leq |M| \leq c_2(|s| + |\xi'|^2). \quad (2.26)$$

Proof. Let $\text{Im}s \equiv s_2 \geq 0$, for definiteness (the case $\text{Im}s \equiv s_2 \leq 0$ is treated in a similar way). We notice that for small δ

$$\begin{aligned} c_3(|s| + |\xi'|^2)^{1/2} &\leq \text{Re}r^\pm \leq |r^\pm| \leq c_4(|s| + |\xi'|^2)^{1/2}, \\ c_5 &\leq \text{Re}R^\pm \leq |R^\pm| \leq c_6, \end{aligned} \quad (2.27)$$

and similar inequalities hold for R_-^+ , R_-^- . This can be proved by elementary calculations. In addition, R^\pm and R_\pm^\pm possess the following properties:

- (i) $\arg R^-, \arg R_+^- \in (-\pi/4, 0]$;
- (ii) there exists $\omega \in (0, \pi/4)$ such that $\arg R^+, \arg R_-^+ \in (-\omega, 0)$, $\arg(1 - R^\pm) \in (0, \omega)$;
- (iii) $\arg R_\pm^+ \in (-\omega, 0)$ both for $\varkappa_\pm^+ = \nu^-/(2\nu^+ + \nu_1^+) > 1$ and for $\varkappa_\pm^+ < 1$.

The statements (i) are obvious; (ii) follow from inequality $\nu^+ < 2\nu^+ + \nu_1^+$, that implies $0 < \varkappa^+ < 1$; moreover, the relations

$$\lim \arg R^+, \quad \lim \arg R_-^+, \quad \lim \arg(1 - R^\pm) = 0 \quad \text{as } s_2 \rightarrow \infty$$

should be taken into account. They do not allow to $R^+, R_-^+, (1 - R^\pm)$ to reach the limiting value of $\arg R = -\pi/4$ as $s_2 \rightarrow \infty$. The simplest way to verify (ii) and (iii) is to represent R^\pm and R_\pm^\pm as vectors on the complex plane s .

The value of ω is different for R^+, R_-^+ and $1 - R^\pm$; we shall choose as ω the maximal of these numbers.

Assume that $\text{Res} \geq 0$, $\mu^+ > \mu^-$ and write M in the form $M = sM_s + |\xi'|^2 M_\xi$, where

$$\begin{aligned} M_s &= \frac{\mu^+ R^-}{\mu^- \nu^+} + \frac{\mu^- R^+}{\mu^+ \nu^-} + \frac{R_+^+}{\nu^-} + \frac{R_+^-}{\nu^+}, \\ M_\xi &= \frac{4}{\mu^+ \mu^-} ((\mu^+ - \mu^-) \mu^+ R^- (1 - R^+) + \mu^{-2} R^+ (1 - R^-) + \mu^+ \mu^- (1 - R^+ + R^+ R^-)). \end{aligned}$$

In view of (i) and (ii), we have: $\arg sM_s \in (-\pi/4, \pi/2)$, $\arg |\xi'|^2 M_\xi \in (-\pi/4 - \omega, \omega)$, which implies

$$|\arg sM_s - |\xi'|^2 M_\xi| \leq (3\pi/4 + \omega) < \pi$$

and

$$|M| \geq c(|s||M_s| + |\xi'|^2 |M_\xi|) \geq c(|s| + |\xi'|^2), \quad (2.28)$$

because $|M_s| \geq \text{Re}M_s > c > 0$, $|M_\xi| \geq \text{Re}M_\xi > c > 0$. The case $\mu^+ < \mu^-$ is treated in a similar way. If $\text{Res} = s_1 < 0$ and s_1, δ are small, then, in view of homogeneity and boundedness of the functions R , we have

$$\begin{aligned} |M(s, \xi') - M(is_2, \xi')| &\leq |s_1| |M_s(s, \xi')| + |s_2| |M_s(s, \xi') - M_s(is_2, \xi')| + |\xi'|^2 |M_\xi(s, \xi') - M_\xi(is_2, \xi')| \\ &\leq c\delta|\xi|^2 + |s_2||s_1| \int_0^1 |\mathcal{D}_a M_s(is_2 + as_1, \xi')| da + |\xi'|^2 |s_1| \int_0^1 |\mathcal{D}_a M_\xi(is_2 + as_1, |\xi'|^2)| da \\ &\leq \delta|\xi'|^2 + c \frac{|s_2| + |\xi'|^2}{\sqrt{|s_2| + |\xi'|^2}} |s_1| \leq \delta|\xi'|^2 + c\sqrt{|s_2| + |\xi'|^2} \sqrt{\delta|s_1|} |\xi'| \leq \alpha(|s| + |\xi'|^2) \end{aligned}$$

with $\alpha \ll 1$. Together with (2.28) (for $s = is_2$) this proves (2.23) also for small $s_1 = \text{Res} < 0$. The inverse inequality is obvious.

Proposition 3. *If the assumptions of Proposition 2 are satisfied and, in addition, $|\xi'| > A_0 \gg 1$, then*

$$c_7(|s|(|s| + |\xi'|^2) + \sigma|\xi'|^3) \leq |s||P| \leq c_8(|s|(|s| + |\xi'|^2) + \sigma|\xi'|^3). \quad (2.29)$$

Proof. Assume that $|s| > \sigma A_1 |\xi'|$ with A_1 so large that $|s||M| \geq c_1 |s| |\xi'|^2 \geq 2\sigma |\xi'|^3 |g|$. Then

$$|sP| \geq |sM| - \sigma |\xi'|^3 |g| \geq c |s| (|s| + |\xi'|^2) \leq c_9 (|s| (|s| + |\xi'|^2) + \sigma |\xi'|^3). \quad (2.30)$$

Before treating the case $|s| \leq \sigma A_1 |\xi'|$, we consider the function $P_1 = |\xi'|^2 M_\xi + \sigma |\xi'|^3 g/s$. Since $|\arg g| \leq \omega < \pi/4$, we have (assuming again that $\text{Im}s \geq 0$):

$$\arg(\sigma |\xi'|^3 g/s) \in (-\pi/2 - \omega, \omega), \quad \arg |\xi'|^2 M_\xi \in (-\pi/4 - \omega, \omega),$$

which implies $|\arg(\sigma |\xi'|^3 g/s) - \arg |\xi'|^2 M_\xi| < \pi/2 + 2\omega < \pi$, $|P_1| \geq c(|\xi'|^2 |M_\xi| + \sigma |\xi'|^3 |g/s|)$ and $|s||P_1| \geq c(|s| |\xi'|^2 + \sigma |\xi'|^3)$ for arbitrary ξ' and s satisfying the assumptions of Proposition 2.

Now we consider the case $|s| \leq \sigma A_1 |\xi'|$ assuming $|\xi'|$ to be so large that $|s| \leq \alpha |\xi'|^{3/2}$ with a small α . We have

$$|sP| \geq |s||P_1| - |s|^2 |M_s| \geq c(|s| |\xi'|^2 + \sigma |\xi'|^3) - \alpha^2 |\xi'|^3 \geq c(|s| (|s| + |\xi'|^2) + \sigma |\xi'|^3),$$

which completes the proof of the proposition.

Remark. Inequality (2.29) holds, if the assumption $|\xi'| \geq A_0$ is replaced with $\text{Res} \geq \gamma \gg 1$. Indeed, we have (2.30) for $|s| > \sigma A_1 |\xi'|$ and, in the case $|s| \leq \sigma A_1 |\xi'|$, there holds

$$|sP| \geq |s||P_1| - |s|^2 |M_s| \geq c(|s| |\xi'|^2 + \sigma |\xi'|^3) - c_1 |\xi'|^2 \geq c(|s| (|s| + |\xi'|^2) + \sigma |\xi'|^3),$$

if γ is large (see also [5], Lemma 10).

2.2 Proof of Theorem 1.

We proceed with analysis of a model problem (2.1), where $\mathbf{f}^\pm = 0$, $h^\pm = 0$ and \mathbf{b} and B are periodic and satisfy the condition (2.3):

$$\left\{ \begin{array}{l} \mathcal{D}_t \mathbf{v}^- - \nu^- \nabla^2 \mathbf{v}^- + \frac{1}{\rho^-} \nabla \theta^- = 0, \quad \nabla \cdot \mathbf{v}^- = 0, \quad \text{for } y_3 < 0, \\ \mathcal{D}_t \mathbf{v}^+ - \nu^+ \nabla^2 \mathbf{v}^+ - (\nu^+ + \nu_1^+) \nabla (\nabla \cdot \mathbf{v}^+) + \frac{p_1}{\rho_m^+} \nabla \theta^+ = 0, \\ \mathcal{D}_t \theta^+ + \rho_m^+ \nabla \cdot \mathbf{v}^+ = 0, \quad \text{for } y_3 > 0, \\ \mathbf{v}|_{t=0} = 0, \quad \theta^+|_{t=0} = 0, \quad \mathbf{v} \xrightarrow{|y_3| \rightarrow \infty} 0, \quad \theta^\pm \xrightarrow{|y_3| \rightarrow \infty} 0, \\ [\mathbf{v}]|_{y_3=0} = 0, \quad \left[\mu^\pm \left(\frac{\partial v_\alpha}{\partial y_3} + \frac{\partial v_3}{\partial y_\alpha} \right) \right] \Big|_{y_3=0} = b_\alpha(y', t), \quad \alpha = 1, 2; \\ -p_1 \theta^+ + \theta^- + \mu_1^+ \nabla \cdot \mathbf{v}^+ + \left[2\mu^\pm \frac{\partial v_3}{\partial y_3} \right] \Big|_{y_3=0} + \sigma \Delta' \int_0^t v_3|_{y_3=0} d\tau = b_3 + \sigma \int_0^t B d\tau. \end{array} \right. \quad (2.31)$$

The Fourier-Laplace transform in (y_1, y_2, t) converts (2.31) into

$$\left\{ \begin{array}{l} (s + \nu^- |\xi'|^2) \tilde{v}_\alpha^- - \nu^- \frac{d^2}{dy_3^2} \tilde{v}_\alpha^- + \frac{1}{\rho^-} i \xi_\alpha \tilde{\theta}^- = 0, \quad \alpha = 1, 2, \\ (s + \nu^- |\xi'|^2) \tilde{v}_3^- - \nu^- \frac{d^2}{dy_3^2} \tilde{v}_3^- + \frac{1}{\rho^-} \frac{d\tilde{\theta}^-}{dy_3} = 0, \quad \tilde{\nabla} \cdot \tilde{v}^- = 0 \text{ for } y_3 < 0, \\ (s + \nu^+ |\xi'|^2) \tilde{v}_\alpha^+ - \nu^+ \frac{d^2}{dy_3^2} \tilde{v}_\alpha^+ - (\nu^+ + \nu_1^+) i \xi_\alpha \tilde{\nabla} \cdot \tilde{v}^+ + \frac{p_1}{\rho_m^+} i \xi_\alpha \tilde{\theta}^+ = 0, \\ \alpha = 1, 2, \\ (s + \nu^+ |\xi'|^2) \tilde{v}_3^+ - \nu^+ \frac{d^2}{dy_3^2} \tilde{v}_3^+ - (\nu_1 + \nu_1^+) \frac{d}{dy_3} \tilde{\nabla} \cdot \tilde{v}^+ + \frac{p_1}{\rho_m^+} \frac{d}{dy_3} \tilde{\theta}^+ = 0, \\ s \tilde{\theta}^+ + \rho_m^+ \tilde{\nabla} \cdot \tilde{v}^+ = 0 \text{ for } y_3 > 0, \\ [\tilde{v}]|_{y_3=0} = 0, \quad \left[\mu^\pm \left(\frac{d\tilde{v}_\alpha}{dy_3} + i \xi_\alpha \tilde{v}_3 \right) \right] \Big|_{y_3=0} = \tilde{b}_\alpha, \quad \alpha = 1, 2, \\ -p_1 \tilde{\theta}^+ + \tilde{\theta}^- + \mu_1^+ \tilde{\nabla} \cdot \tilde{v}^+ + \left[2\mu^\pm \frac{d\tilde{v}_3}{dy_3} \right] \Big|_{y_3=0} - \sigma \frac{|\xi'|^2}{s} \tilde{v}_3 \Big|_{y_3=0} = \tilde{b}_3 + \frac{\sigma}{s} \tilde{B}. \end{array} \right. \quad (2.32)$$

By eliminating the function $\tilde{\theta}^+$ we reduce (2.32) to the system (2.11) with the parameter ν_1^+ replaced by the function $\nu_1^+(s) = \nu_1^+ + p_1/s$. With this replacement, the expressions $1/(2\nu^+ + \nu_1^+)$ and r_1^+ go over into $s/(as + p_1)$ and $r_{11}^+ = \sqrt{(s^2/(as + p_1) + \xi^2)}$, respectively, where $a = 2\nu^+ + \nu_1^+ > 0$, R^- , R_+^- remain invariant and R^+ , R_-^+ are transformed into

$$\begin{aligned} R_1^+ &= \frac{r_{11}^+ + \varkappa^+(s)r^+}{r_{11}^+ + r^+} \quad \text{with} \quad \varkappa^+(s) = \frac{\nu^+ s}{as + p_1} = \frac{\nu^+}{a} \frac{s}{s + p_1/a}, \\ R_{1-}^+ &= \frac{r_{11}^+ + \varkappa_-^+(s)r^-}{r_{11}^+ + r^-} \quad \text{with} \quad \varkappa_-^+(s) = \frac{\nu^- s}{as + p_1} = \frac{\nu^-}{a} \frac{s}{s + p_1/a}. \end{aligned} \quad (2.33)$$

The solution of the problem obtained is given by (2.19), (2.20), (2.21) with ν_1^+ substituted by $\nu_1^+(s)$, which converts M and P into

$$\mathbb{M}(s, \xi') = s\mathbb{M}_s(s, \xi') + |\xi'|^2 \mathbb{M}_\xi(s, \xi'), \quad \mathbb{P} = s\mathbb{M} + \sigma |\xi'|^3 q(s, \xi'), \quad (2.34)$$

where $\xi' = (\frac{\pi}{d_0} k_1, \frac{\pi}{d_0} k_2)$, $|\mathbf{k}'| > 0$,

$$\begin{aligned} \mathbb{M}_s &= \frac{\mu^+}{\mu^-} \frac{R^-}{\nu^+} + \frac{\mu^-}{\mu^+} \frac{R_1^+}{\nu^-} + \frac{R_{1-}^+}{\nu^-} + \frac{R_+^-}{\nu^+}, \\ \mathbb{M}_\xi &= \frac{4}{\mu^+ \mu^-} ((\mu^{+2} R^- (1 - R_1^+) + \mu^{-2} R_1^+ (1 - R^-) + \mu^+ \mu^- (1 - R^- (1 - R_1^+) - R_1^+ (1 - R^-))), \\ q(s, \xi') &= \frac{r_{11}^+}{\mu^- (r^- + |\xi'|)} + \frac{R_1^+}{\mu^+}. \end{aligned} \quad (2.35)$$

Proposition 4. *If the assumptions of Proposition 3 are satisfied (maybe with greater A_0), then \mathbb{M} and \mathbb{P} satisfy inequalities (2.26) and (2.29). The condition $|\xi'| > A_0$ can be replaced with $\text{Res} \geq \gamma \gg 1$.*

Proof. We start with auxiliary estimates of the function $r_{11}^+(s, \xi')$ the square of which is given by

$$r_{11}^{+2} = \frac{s^2}{as + p_1} + |\xi'|^2 = \frac{s_1 a |s|^2 + p_1 (s_1^2 - s_2^2)}{|as + p_1|^2} + i s_2 \frac{a |s|^2 + 2p_1 s_1}{|as + p_1|^2} + |\xi'|^2,$$

where $s_1 = \text{Res}$, $s_2 = \text{Im}s \geq 0$. The expressions $\frac{a|s|^2 + 2p_1 s_1}{|as + p_1|^2}$ and $\frac{p_1(s_1^2 - s_2^2)}{|as + p_1|^2}$ are bounded from above and from below uniformly with respect to s . Hence for large $|\xi'|$ and for $s_1 \geq 0$ or for small negative s_1 (or in the case $s_1 \geq \gamma \gg 1$) inequality $\text{Re} r_{11}^{+2} > 0$ holds. The expression $\frac{a|s|^2 + 2p_1 s_1}{|as + p_1|^2}$ is positive and bounded for $s_1 \geq 0$, but it takes small negative values for small negative s_1 such that $2p_1 s_1 < -a|s|^2$. Hence it can be assumed that $\arg r_{11}^+ = \arg r_{11}^{+2}/2 > -\omega_0$, $\omega_0 < \pi/4$. Thus,

$$\begin{aligned} c_1(|s| + |\xi'|^2) &\leq |r_{11}^{+2}| \leq c_2(|s| + |\xi'|^2), \\ c_3(|s| + |\xi'|^2)^{1/2} &\leq \text{Re} r_{11}^+ \leq |r_{11}^+| \leq c_4(|s| + |\xi'|^2)^{1/2}. \end{aligned}$$

Moreover, the differences $r_{11}^+ - r_1^+$ and $\varkappa^+(s) - \varkappa^+$ satisfy

$$\begin{aligned} |r_{11}^+ - r_1^+| &\leq \frac{1}{|r_{11}^+ + r_1^+|} \left| \frac{s^2}{as + p_1} - \frac{s}{a} \right| \leq \frac{c|s|}{|as + p_1| |r_{11}^+ + r_1^+|} \leq c \frac{(|s| + |\xi'|^2)^{1/2}}{|as + p_1|}, \\ |\varkappa^+(s) - \varkappa^+| &= \left| \frac{\nu^+ s}{as + p_1} - \frac{\nu^+}{a} \right| \leq \frac{c}{|as + p_1|}, \quad |\varkappa_-^+(s) - \frac{\nu^-}{a}| \leq \frac{c}{|as + p_1|}, \end{aligned} \quad (2.36)$$

which implies

$$|R_1^+ - R^+| + |R_{1-}^+ - R_-^+| \leq \frac{c}{|as + p_1|}.$$

As a consequence, we obtain after elementary calculations:

$$|\mathbb{M} - M| \leq \frac{c(|s| + |\xi'|^2)}{|as + p_1|}, \quad |q - g| \leq \frac{c}{|as + p_1|},$$

which proves (2.26) and (2.29) for \mathbb{M} and \mathbb{P} in the case of large s : $|s| \geq H$.

Now, we treat the case $|s| \leq H$. We assume that $|\xi'|$ is so large that $|s| \leq \alpha |\xi'|^2$ with small α . We have

$$r^\pm(s, \xi') = |\xi'| r^\pm(s/|\xi'|^2, 1), \quad r_{11}^+(s, \xi') = |\xi'| \left(\frac{s}{\nu^+ |\xi'|^2} \varkappa^+(s) + 1 \right)^{1/2} = |\xi'| r^+ \left(\frac{s \varkappa^+(s)}{|\xi'|^2}, 1 \right),$$

which implies

$$\begin{aligned} R_1^+(s, \xi') &= \frac{r^+(s \varkappa^+(s)/|\xi'|^2, 1) + \varkappa^+(s) r^+(s/|\xi'|^2, 1)}{r^+(s \varkappa^+(s)/|\xi'|^2, 1) + r^+(s/|\xi'|^2, 1)}, \\ R_{1-}^+(s, \xi') &= \frac{r^+(s \varkappa^+(s)/|\xi'|^2, 1) + \varkappa_-^+(s) r^-(s/|\xi'|^2, 1)}{r^+(s \varkappa^+(s)/|\xi'|^2, 1) + r^-(s/|\xi'|^2, 1)}, \\ R^-(s, \xi') &= \frac{1}{r^-(s/|\xi'|^2, 1) + 1}, \quad R_-^-(s, \xi') = \frac{1}{r^+(s/|\xi'|^2, 1) + 1}. \end{aligned}$$

where $\varkappa_-^+(s) = \frac{\nu^-}{a} \frac{s}{s + p_1/a}$. Clearly, these functions are uniformly bounded for $|s| < H \leq \alpha |\xi'|^2$. It follows that

$$|\mathbb{M}(s, \xi')| \geq |\xi'|^2 |\mathbb{M}_\xi| - |s| |\mathbb{M}_s| \geq c |\xi'|^2 \geq c(|s| + |\xi'|^2).$$

To estimate \mathbb{P} , we compute $q(s, \xi')$ replacing $s/|\xi'|^2$ with zero, which yields

$$|q|_{s/|\xi'|^2=0} = \left| \frac{1}{2\mu^-} + \frac{1 + \varkappa^+(s)}{2\mu^+} \right| \geq c.$$

It is easily seen that

$$|r_{11}^+(s\varkappa^+(s)/|\xi|^2, 1) - 1| + |r^-(s/|\xi|^2, 1) - 1| \leq c\alpha,$$

which implies

$$|R_1^+(s, \xi') - \frac{1 + \varkappa^+(s)}{2}| \leq c\alpha, \quad |q(s, \xi') - q|_{s/|\xi'|^2=0}| \leq c\alpha.$$

Other terms in the expression for $s\mathbb{P}$ are estimated as follows:

$$|s||\xi'|^2 \mathbb{M}_\xi \leq c\sqrt{H\alpha}|\xi'|^2|\xi'|^2|\mathbb{M}_\xi| \leq c\sqrt{\alpha}|\xi'|^3, \quad |s|^2|\mathbb{M}_s| \leq c\alpha^{3/2}|\xi'|^3.$$

Hence

$$|s\mathbb{P}(s, \xi')| \geq \sigma|\xi'|^3|q| - \alpha'|\xi'|^3 \geq c|\xi'|^3 \geq c(|s|(|s| + |\xi'|^2) + \sigma|\xi'|^3),$$

where $\alpha' \leq c\sqrt{\alpha} \ll 1$. This completes the proof of Proposition 4.

Proposition 5. *The solution of Problem (2.11) with parameters ν_1^+ and μ_1^+ replaced by $\nu_1^+(s) = \nu_1^+ + p_1/s$ and $\rho_m^+\nu_1^+(s)$, respectively, satisfies the inequality*

$$\begin{aligned} & \sum_{j=0}^2 \sum_{\pm} \|\mathcal{D}_{y_3}^j \tilde{v}^\pm\|_{L_2(\mathbb{R}^\pm)}^2 \mathfrak{r}^{2(2+l-j)} + \sum_{\pm} \|\mathbf{v}^\pm\|_{W_2^{2+l}(\mathbb{R}^\pm)}^2 + \|\tilde{\nabla}\tilde{\theta}^-\|_{L_2(\mathbb{R}^-)}^2 \mathfrak{r}^{2l} \\ & + |\tilde{\theta}^-|_{y_3=0}^2 |\xi'|^{1/2} \mathfrak{r}^l \leq c(|\tilde{\mathbf{b}}|^2 \mathfrak{r}^{2l+1} + |\tilde{b}_3|^2 \mathfrak{r}^{2l} |\xi'| + |\tilde{B}|^2 \mathfrak{r}^{2l-1}), \end{aligned} \quad (2.37)$$

where $\tilde{\nabla} = (i\xi_1, i\xi_2, \mathcal{D}_{y_3})$.

Proof. As shown above, the solution $(\tilde{v}, \tilde{\theta}^-)$ of the problem we are treating is given by formulas (2.19), where

$$\begin{aligned} A_1 &= -\mu^+(r^{+2} + |\xi'|^2)^2 + \mu^-(r^{-2} + |\xi'|^2)^2, \quad A_2 = -2\mu^+r_{11}^+ - 2\mu^-r^-, \quad A_3 = r^+ + r^-, \\ \omega_\alpha &= -\frac{\tilde{b}_\alpha}{\mu^+r^+ + \mu^-r^-} + \frac{i\xi_\alpha(\mu^+ - \mu^-)\omega_3}{\mu^+r^+ + \mu^-r^-} + \frac{i\xi_\alpha(\mu^+C_3^+(r^+ - r_{11}^+) + \mu^-C_3^-(r^- - |\xi'|))}{\mu^+r^+ + \mu^-r^-}, \\ \omega_3 &= -\frac{1}{A_3}(\rho^+sC_3^+\frac{R^+}{\mu^+} - \rho^-sC_3^-\frac{R^-}{\mu^-}), \\ C_3^+s &= \frac{1}{\rho^+\mathbb{P}}\{(A_2'\frac{R^-}{\mu^-} - A_3(1 - 2R^-))\tilde{\mathbb{B}} - (A_1\frac{R^-}{\mu^-} - A_3|\xi'|)\tilde{b}'_3\}, \\ C_3^-s &= \frac{1}{\rho^-\mathbb{P}}\{(A_2'\frac{R_1^+}{\mu^+} - A_3(1 - 2R_1^+))\tilde{\mathbb{B}} - (A_1\frac{R_1^+}{\mu^-} + A_3r_{11}^+)\tilde{b}'_3\}, \\ \mathbb{P} &= \mathbb{M} + \sigma|\xi'|^3q/s, \quad q = \frac{r_{11}^+}{\mu^-(r^- + |\xi'|)} + \frac{R_1^+}{\mu^+}, \quad A_2' = A_2 - \frac{\sigma|\xi'|^2}{s}, \end{aligned} \quad (2.38)$$

and $\mathbb{P}, \mathbb{M}, q$ are given in (2.34), (2.35).

Let $\mathfrak{r} = (|s| + |\xi'|^2)^{1/2}$. If the parameter d_0 is sufficiently small, then $|\xi'|$ is large and \mathbb{M}, \mathbb{P} satisfy (2.26), (2.29). By (2.38),

$$\begin{aligned} |C_3^+ s| &\leq c \left((r + \frac{\sigma|\xi|^2}{|s|}) \frac{|\xi'|}{|\mathbb{P}|} |\tilde{\mathbf{b}}'| + \frac{|\xi|\mathfrak{r}}{|\mathbb{P}|} |\tilde{b}_3| + \frac{|\xi'\mathfrak{r}}{|s||\mathbb{P}|} |\tilde{B}| \right) \leq c \left(|\tilde{\mathbf{b}}'| + \frac{|\xi'|}{\mathfrak{r}} |\tilde{b}_3| + \frac{\mathfrak{r}^2}{|s||\mathbb{P}|} |\tilde{B}| \right), \\ |C_3^- s| &\leq c \left(|\tilde{\mathbf{b}}'| + |\tilde{b}_3| + \frac{\mathfrak{r}^2}{|s||\mathbb{P}|} |\tilde{B}| \right) \\ |\tilde{\omega}| &\leq \frac{c}{\mathfrak{r}} \left(|\tilde{\mathbf{b}}'| + \frac{|\xi'|}{\mathfrak{r}} |\tilde{b}_3| + \frac{|\xi'\mathfrak{r}}{|s||\mathbb{P}|} |\tilde{B}| \right), \quad |\mathbf{V}^+| \leq c|C_3^+ s|, \quad |\mathbf{V}^-| \leq c \frac{|\xi'|}{\mathfrak{r}} |C_3^- s|, \end{aligned}$$

where $\tilde{\mathbf{b}}' = (\tilde{b}_1, \tilde{b}_2)$. Making use of

$$\begin{aligned} \int_0^\infty |e_0^+(y_3)|^2 dy_3 + \int_{-\infty}^0 |e_0^-(y_3)|^2 dy_3 &\leq c\mathfrak{r}^{-1}, \\ \int_0^\infty |e_1^+(y_3)|^2 dy_3 &\leq c\mathfrak{r}^{-3}, \quad \int_{-\infty}^0 |e_1^-(y_3)|^2 dy_3 \leq c\mathfrak{r}^{-2}|\xi'|^{-1}, \end{aligned}$$

we obtain

$$\begin{aligned} \left(\int_0^\infty |\tilde{\mathbf{v}}^+|^2 dy_3 \right)^{1/2} \mathfrak{r}^{2+l} &\leq c(|\omega|\mathfrak{r}^{3/2+l} + |\mathbf{V}^+|\mathfrak{r}^{l+1/2}) \\ &\leq c(\mathfrak{r}^{l+1/2}|\tilde{\mathbf{b}}'| + \mathfrak{r}^l|\xi'|^{1/2}|\tilde{b}_3| + \frac{|\xi'\mathfrak{r}^2}{|s\mathbb{P}|} |\tilde{B}|\mathfrak{r}^{l-1/2}) \leq c(\mathfrak{r}^{l+1/2}|\tilde{\mathbf{b}}'| + \mathfrak{r}^l|\xi'|^{1/2}|\tilde{b}_3| + |\tilde{B}|\mathfrak{r}^{l-1/2}) \\ \left(\int_{-\infty}^0 |\tilde{\mathbf{v}}^-|^2 dy_3 \right)^{1/2} \mathfrak{r}^{2+l} &\leq c(|\omega|\mathfrak{r}^{3/2+l} + |\mathbf{V}^-|\mathfrak{r}^{l+1}|\xi|^{-1/2}) \\ &\leq c(\mathfrak{r}^{l+1/2}|\tilde{\mathbf{b}}'| + \mathfrak{r}^l|\xi'|^{1/2}|\tilde{b}_3| + \frac{|\xi'|^{1/2}\mathfrak{r}^{5/2}}{|s\mathbb{P}|} |\tilde{B}|\mathfrak{r}^{l-1/2}) \\ &\leq c(\mathfrak{r}^{l+1/2}|\tilde{\mathbf{b}}'| + \mathfrak{r}^l|\xi'|^{1/2}|\tilde{b}_3| + |\tilde{B}|\mathfrak{r}^{l-1/2}), \end{aligned} \tag{2.39}$$

because

$$\begin{aligned} |\xi'\mathfrak{r}^2| &\leq |s||\xi'|^2 + |\xi'|^3 \leq c|s||\mathbb{P}|, \\ |\xi'|^{1/2}\mathfrak{r}^{5/2} &\leq c|s|^{5/4}|\xi'|^{1/2} \leq c(|s|^2)^{5/8}(|\xi'|^3)^{3/8} \leq c|s||\mathbb{P}|. \end{aligned} \tag{2.40}$$

Moreover, by using the inequalities

$$\begin{aligned}
& \int_0^\infty \left| \frac{d^j e_0^+(y_3)}{dy_3^j} \right|^2 dy_3 \leq c|r^+|^{2j-1}, \quad \int_0^\infty \left| \frac{d^j e_1^+(y_3)}{dy_3^j} \right|^2 dy_3 \leq c|r^+|^{2j-3}, \\
& \int_0^\infty \int_0^\infty \left| \frac{d^j e_0^+(y_3+z)}{dy_3^j} - \frac{d^j e_0^+(y_3)}{dy_3^j} \right|^2 \frac{dy_3 dz}{z^{1+2l}} \leq c|r^+|^{2(j+l)-1}, \\
& \int_0^\infty \int_0^\infty \left| \frac{d^j e_1^+(y_3+z)}{dy_3^j} - \frac{d^j e_1^+(y_3)}{dy_3^j} \right|^2 \frac{dy_3 dz}{z^{1+2l}} \leq c|r^+|^{2(j+l)-3}, \quad j = 1, 2, \\
& \int_{-\infty}^0 \left| \frac{d^j e_0^-(y_3)}{dy_3^j} \right|^2 dy_3 \leq c|r^-|^{2j-1}, \quad \int_{-\infty}^0 \left| \frac{d^j e_1^-(y_3)}{dy_3^j} \right|^2 dy_3 \leq c \frac{|r^-|^{2j-1} + |\xi'|^{2j-1}}{|r^-|^2}, \\
& \int_{-\infty}^0 \int_0^\infty \left| \frac{d^j e_0^-(y_3-z)}{dy_3^j} - \frac{d^j e_0^-(y_3)}{dy_3^j} \right|^2 \frac{dy_3 dz}{z^{1+2l}} \leq c|r^-|^{2(j+l)-1}, \\
& \int_{-\infty}^0 \int_0^\infty \left| \frac{d^j e_1^-(y_3-z)}{dy_3^j} - \frac{d^j e_1^-(y_3)}{dy_3^j} \right|^2 \frac{dy_3 dz}{z^{1+2l}} \leq c \frac{|r^-|^{2(j+l)-1} + |\xi'|^{2(j+l)-1}}{|r^-|^2}, \quad j = 1, 2,
\end{aligned}$$

we estimate in a similar way the sum

$$\sum_{\pm} \left(\|\mathcal{D}_{y_3}^j \tilde{\mathbf{v}}^\pm\|_{L_2(\mathbb{R}^\pm)}^2 \mathfrak{r}^{2(2+l-j)} + \|\tilde{\mathbf{v}}^\pm\|_{W_2^{2+l}(\mathbb{R}^\pm)}^2 \right)$$

and the norms of $\tilde{\theta}^- = -C_3^- \rho^- s e^{|\xi'|y_3}$.

As for the function $\tilde{\theta}^+$ that was eliminated from (2.32), it can be estimated by using the third, fourth and fifth lines in (2.32). Taking also (2.3) into account we prove that

$$\begin{aligned}
& \|\tilde{\theta}^+\|_{L_2(\mathbb{R}^+)}^2 \mathfrak{r}^{2l}(1+|s|^2) + \|\tilde{\nabla} \theta^+\|_{L_2(\mathbb{R}^+)}^2 \mathfrak{r}^{2l}(1+|s|^2) \\
& \leq c(|\tilde{\mathbf{b}}|^2 \mathfrak{r}^{2l+1} + |\tilde{\mathbf{b}}_3|^2 \mathfrak{r}^{2l} |\xi'| + |\tilde{B}|^2 \mathfrak{r}^{2l-1}).
\end{aligned} \tag{2.41}$$

Together with (2.37), this inequality implies estimate (2.4) in the case $\mathbf{f}^\pm, h^\pm = 0, T = \infty$, in view of Proposition 1 and the Parseval identity

$$\int_{s_1-i\infty}^{s_1+i\infty} \sum_{\mathbf{k}' \in \mathbb{Z}^2} |\tilde{u}(\xi', s)|^2 ds_2 = 2\pi(2d_0)^2 \int_0^\infty \int_{\Omega'} e^{2\beta t} |u(y', t)|^2 dy' dt, \quad \beta = -s_1.$$

Proposition 5 is proved.

2.2. On non-homogeneous problem (2.1) with $T = \infty$.

We reduce non-homogeneous problem (2.1) to a similar problem with $\mathbf{f} = 0$, $h = 0$ by constructing auxiliary functions $(\mathbf{u}^\pm, \sigma^\pm)$ such that

$$\begin{cases} \mathcal{D}_t \mathbf{u}^- - \nu^- \nabla^2 \mathbf{u}^- + \frac{1}{\rho^-} \nabla \sigma^- = \mathbf{f}^-, & \nabla \cdot \mathbf{u}^- = h^- & \text{in } Q_\infty^-, \\ \mathbf{u}^-|_{t=0} = 0, & \mathbf{u}^-, \sigma^- \xrightarrow{y_3 \rightarrow -\infty} 0, & \int_{\Omega'} \mathbf{u}^- dy' = 0, \end{cases} \quad (2.42)$$

$$\begin{cases} \mathcal{D}_t \mathbf{u}^+ - \nu^+ \nabla^2 \mathbf{u}^+ - (\nu^+ + \nu_1^+) \nabla (\nabla \cdot \mathbf{u}^+) + \frac{p_1}{\rho_m} \nabla \sigma^+ = \mathbf{f}^+, \\ \mathcal{D}_t \sigma^+ + \rho_m^+ \nabla \cdot \mathbf{u}^+ = h^+ & \text{in } Q_\infty^+, & \int_{\Omega'} \mathbf{u}^+ dy' = 0, & \int_{\Omega'} \sigma^+ dy' = 0, \\ \mathbf{u}^+|_{y_3=0} = \mathbf{u}^-|_{y_3=0}, & \mathbf{u}^+|_{t=0} = 0, & \sigma^+|_{t=0} = 0, & \mathbf{u}^+, \sigma^+ \xrightarrow{y_3 \rightarrow +\infty} 0. \end{cases} \quad (2.43)$$

We recall that \mathbf{f}^\pm and h^\pm satisfy the condition (2.3). We set $\mathbf{u}' = \nabla \Phi$ where Φ is a periodic solution of the problem

$$\nabla^2 \Phi = h^- \quad \text{in } Q^-, \quad \Phi|_{y_3=0} = 0, \quad \Phi \xrightarrow{y_3 \rightarrow -\infty} 0, \quad \int_{\Omega'} \frac{\partial \Phi}{\partial y_3} dy' = 0.$$

It is clear that $\nabla \cdot \mathbf{u}' = h^-$, $\int_{\Omega'} \mathbf{u}' dy' = 0$. Taking the Fourier-Laplace transform, we obtain

$$\frac{d^2 \tilde{\Phi}}{dy_3^2} - |\xi'|^2 \tilde{\Phi} = \tilde{h}^- \quad \text{in } \Omega', \quad \tilde{\Phi}|_{y_3=0} = 0, \quad \tilde{\Phi} \xrightarrow{y_3 \rightarrow -\infty} 0, \quad (2.44)$$

hence

$$\tilde{\Phi}(\xi', y_3, s) = \int_{-\infty}^0 G(y_3, z_3) \tilde{h}^-(\xi', z_3, s) dz_3,$$

where

$$G(y_3, z_3) = \frac{1}{2|\xi'|} \begin{cases} e^{|\xi'|(z_3+y_3)} - e^{-|\xi'|(z_3-y_3)} & \text{for } z_3 < y_3, \\ e^{+|\xi'|(z_3+y_3)} - e^{-|\xi'|(z_3-y_3)} & \text{for } z_3 > y_3 \end{cases}$$

is the Green function for problem (2.44), $\xi' = \frac{x}{d_0} \mathbf{k}'$, $|\mathbf{k}'| > 0$.

Since $|\xi'| \geq c > 0$, the function $\tilde{\Phi}$ satisfies the inequalities

$$|\xi'|^2 \|\tilde{\Phi}\|_{L_2(\mathbb{R}^-)} \leq c \|\tilde{h}^-\|_{L_2(\mathbb{R}^-)},$$

and

$$\|\tilde{\nabla} \tilde{\Phi}\|_{W_2^{2+l}(\mathbb{R}^-)} + |\xi'|^{2+l} \|\tilde{\nabla} \tilde{\Phi}\|_{L_2(\mathbb{R}^-)} \leq c (\|\tilde{h}^-\|_{W_2^{1+l}(\mathbb{R}^-)} + |\xi'|^{1+l} \|\tilde{h}^-\|_{L_2(\mathbb{R}^-)}),$$

where $\tilde{\nabla} = (i\xi_1, i\xi_2, \frac{d}{dy_3})$. In addition, since $h^- = \nabla \mathbf{H} + H_0$, we have

$$\tilde{\Phi} = \int_{-\infty}^0 G(y_3, z_3) (\tilde{\nabla} \cdot \tilde{\mathbf{H}} + \tilde{H}_0) dz_3 = \int_0^\infty (G(y_3, z_3) (\sum_{\alpha=1}^2 i\xi_\alpha \tilde{H}_\alpha + \tilde{H}_0) - \frac{\partial G(y_3, z_3)}{\partial z_3} \tilde{H}_3) dz_3$$

and

$$(1 + |s|^{1+l/2}) \|\tilde{\nabla} \tilde{\Phi}\|_{L_2(\mathbb{R}^-)} \leq c (1 + |s|^{1+l/2}) (\|\tilde{\mathbf{H}}\|_{L_2(\mathbb{R}^-)} + \|\tilde{H}_0\|_{L_2(\mathbb{R}^-)}).$$

It follows that

$$\|e^{\beta t} \mathbf{u}'\|_{W_2^{2+l,1+l/2}(Q_\infty^-)} \leq c(\|e^{\beta t} h^-\|_{W_2^{1+l,0}(Q_\infty^-)} + \|e^{\beta t} \mathbf{H}\|_{W_2^{0,1+l/2}(Q_\infty^-)} + \|e^{\beta t} H_0\|_{W_2^{0,1+l/2}(Q_\infty^-)}) \quad (2.45)$$

and $\nabla \cdot \mathbf{u}' \rightarrow h(y, 0)$ as $t \rightarrow 0$.

For the functions $\mathbf{w} = \mathbf{u}^- - \mathbf{u}'$ and σ^- we obtain the relations

$$\mathcal{D}_t \mathbf{w} - \nu^- \nabla^2 \mathbf{w} + \frac{1}{\rho^-} \nabla \sigma^- = \mathbf{f}^- - \mathcal{D}_t \mathbf{u}' + \nu^- \nabla^2 \mathbf{u}' \equiv \mathbf{f}_1, \quad \nabla \cdot \mathbf{w} = 0, \quad \mathbf{w}|_{t=0} = 0$$

in Q^- . We extend \mathbf{f}_1 from Q_∞^- into Q_∞ so that

$$\|e^{\beta t} \mathbf{f}_1^*\|_{W_2^{l,1/2}(Q_\infty)} \leq c \|e^{\beta t} \mathbf{f}_1\|_{W_2^{l,1/2}(Q_\infty^-)}, \quad \int_{\Omega'} \mathbf{f}_1^* dy' = 0$$

(\mathbf{f}_1^* is the extension of \mathbf{f}_1) and then take the Fourier-Laplace transform in all the variables (y_1, y_2, y_3, t) given by

$$\tilde{u}(\xi, t) = \int_0^\infty e^{-st} dt \int_0^\infty e^{-i\xi_3 y_3} dy_3 \int_{\Omega'} e^{-i\xi' \cdot y'} u(y, t) dy', \quad \xi' = \frac{\pi}{d_0} \mathbf{k}', \quad \xi_3 \in \mathbb{R},$$

$\text{Res} = s_1$ is a small negative number and $|\mathbf{k}'| > 0$. We seek $(\tilde{\mathbf{w}}, \tilde{\sigma}^-)$ as the solution of the system

$$(s + \nu^- |\xi|^2) \tilde{\mathbf{w}} + \frac{1}{\rho^-} i \xi \tilde{\sigma}^- = \tilde{\mathbf{f}}_1^*, \quad i \xi \cdot \tilde{\mathbf{w}} = 0,$$

so that

$$\tilde{\sigma}^- = -\frac{\rho^- i \xi \cdot \tilde{\mathbf{f}}_1^*}{|\xi|^2}, \quad \tilde{\mathbf{w}} = \frac{\tilde{\mathbf{f}}_1^* - i \xi \tilde{\sigma}^- / \rho^-}{s + \nu^- |\xi|^2},$$

Hence

$$\begin{aligned} & \|e^{\beta t} \mathbf{w}\|_{W_2^{2+l,1+l/2}(Q_\infty)} + \|e^{\beta t} \nabla \sigma^-\|_{W_2^{l,1/2}(Q_\infty)} + \|e^{\beta t} \sigma^-\|_{W_2^{l+1,0}(Q_\infty)} \\ & \leq c \|e^{\beta t} \mathbf{f}_1^*\|_{W_2^{l,1/2}(Q_\infty)} \leq c \|e^{\beta t} \mathbf{f}_1\|_{W_2^{l,1/2}(Q_\infty^-)} \end{aligned}$$

and

$$\begin{aligned} & \|e^{\beta t} \mathbf{u}^-\|_{W_2^{2+l,1+l/2}(Q_\infty^-)} + \|e^{\beta t} \nabla \sigma^-\|_{W_2^{l,1/2}(Q_\infty^-)} + \|e^{\beta t} \sigma^-\|_{W_2^{l+1,0}(Q_\infty^*)} \\ & \leq c(\|e^{\beta t} \mathbf{f}^-\|_{W_2^{l,1/2}(Q_\infty^-)} + \|e^{\beta t} h^-\|_{W_2^{1+l,0}(Q_\infty^-)} + \|e^{\beta t} \mathbf{H}\|_{W_2^{0,1+l/2}(Q_\infty^-)} + \|e^{\beta t} H_0\|_{W_2^{0,1+l/2}(Q_\infty^-)}). \end{aligned} \quad (2.46)$$

The functions \mathbf{u}^+, σ^+ satisfying (2.43) are sought in the form $\mathbf{u}^+ = \mathbf{u}_1 + \mathbf{u}_2$, $\sigma^+ = \sigma_1 + \sigma_2$ with (\mathbf{u}_i, σ_i) defined as solutions of

$$\begin{cases} \mathcal{D}_t \mathbf{u}_1 - \nu^+ \nabla^2 \mathbf{u}_1 - (\nu^+ + \nu_1^+) \nabla (\nabla \cdot \mathbf{u}_1) + \frac{p_1}{\rho_m} \nabla \sigma_1 = \mathbf{f}^+, \\ \mathcal{D}_t \sigma_1 + \rho_m^+ \nabla \cdot \mathbf{u}_1 = h_*^+ \quad \text{in } Q_\infty, \quad \mathbf{u}_1|_{t=0} = 0, \quad \sigma_1|_{t=0} = 0 \end{cases} \quad (2.47)$$

and

$$\begin{cases} \mathcal{D}_t \mathbf{u}_2 - \nu^+ \nabla^2 \mathbf{u}_2 - (\nu^+ + \nu_1^+) \nabla (\nabla \cdot \mathbf{u}_2) + \frac{p_1}{\rho_m} \nabla \sigma_2 = 0, \\ \mathcal{D}_t \sigma_2 + \rho_m^+ \nabla \cdot \mathbf{u}_2 = 0 \quad \text{in } Q_\infty^+, \\ \mathbf{u}_2|_{y_3=0} = (\mathbf{u}^- - \mathbf{u}_1)|_{y_3=0} \equiv \mathbf{a}, \quad \mathbf{u}_1|_{t=0} = 0, \quad \sigma_1|_{t=0} = 0. \end{cases} \quad (2.48)$$

where \mathbf{f}_*^+ and h_*^+ are extensions of \mathbf{f}^+ and h^+ , respectively, into $Q = \Omega' \times \mathbb{R}$ with preservation of class; it is assumed that $\int_{\Omega'} \mathbf{f}_*^+ dy' = 0$, $\int_{\Omega'} h_*^+ dy' = 0$.

The Fourier-Laplace transform with respect to (y_1, y_2, y_3, t) converts (2.47) into the algebraic system

$$\begin{cases} (s + \nu^+ |\xi|^2) \tilde{\mathbf{u}}_1 + (\nu^+ + \nu_1^+) \boldsymbol{\xi} (\boldsymbol{\xi} \cdot \tilde{\mathbf{u}}_1) + \frac{p_1}{\rho_m^+} i \boldsymbol{\xi} \tilde{\sigma}_1 = \tilde{\mathbf{f}}_*^+, \\ s \tilde{\sigma}_1 + \rho_m^+ (i \boldsymbol{\xi} \cdot \tilde{\mathbf{u}}_1) = \tilde{h}_*^+, \quad |\xi|^2 = |\xi'|^2 + \xi_3^2, \end{cases}$$

where $\xi' = (\pi k_1/d_0, \pi k_2/d_0)$, $|k_1| + |k_2| \geq 1$, $\xi_3 \in \mathbb{R}$. Elimination of $\tilde{\sigma}_1$ leads to the equation for $\tilde{\mathbf{u}}_1$:

$$(s + \nu^+ |\xi|^2) \tilde{\mathbf{u}}_1 + (\nu^+ + \nu_1^+(s)) \boldsymbol{\xi} (\boldsymbol{\xi} \cdot \tilde{\mathbf{u}}_1) = \tilde{\mathbf{f}}_*^+ - \frac{p_1 i \boldsymbol{\xi} \tilde{h}_*^+}{\rho_m^+ s} \equiv \tilde{\mathbf{g}}, \quad (2.49)$$

where $\nu_1^+(s) = \nu_1^+ + p_1/s$. The solution of (2.49) is given by

$$\tilde{\mathbf{u}}_1 = \mathbb{H}^+ \tilde{\mathbf{g}} / (s + \nu^+ |\xi|^2), \quad (2.50)$$

where \mathbb{H}^+ is the matrix with the elements

$$H_{jk}^+ = \delta_{jk} - \frac{(\nu^+ + \nu_1^+(s)) \xi_j \xi_k}{s + (2\nu^+ + \nu_1^+(s)) |\xi|^2} = \delta_{jk} - \frac{\xi_j \xi_k}{\frac{s}{bs+p_1} (s + \nu^+ |\xi|^2) + |\xi|^2},$$

$b = \nu^+ + \nu_1^+ > 0$. From (2.50) it follows that

$$\widetilde{\nabla \cdot \mathbf{u}}_1 = i \boldsymbol{\xi} \cdot \tilde{\mathbf{u}}_1 = \frac{i \boldsymbol{\xi} \cdot \tilde{\mathbf{g}}}{s + (2\nu^+ + \nu_1^+(s)) |\xi|^2} = \frac{s}{bs + p_2} \frac{i \boldsymbol{\xi} \cdot \tilde{\mathbf{g}}}{\frac{s}{bs+p_2} (s + \nu^+ |\xi|^2) + |\xi|^2} \equiv \chi. \quad (2.51)$$

We notice that the expression

$$P^+(\xi, s) = \frac{s}{bs + p_1} (s + \nu^+ |\xi|^2) + |\xi|^2 = \frac{sb|s|^2 + p_1 s^2}{|bs + p_1|^2} + \frac{s\nu^+}{bs + p_1} |\xi|^2 + |\xi|^2$$

is an analytic function of s if $\text{Res} \geq -p_1/b$ and satisfies the inequality

$$|P^+| \geq c(|s| + |\xi|^2), \quad (2.52)$$

since $|\xi| \geq \pi/d_0$ with small d_0 . Hence formulas (2.51) and (2.50) imply

$$\begin{aligned} & \|\nabla \cdot \hat{\mathbf{u}}_1\|_{W_2^{l+1}(Q)} + |s|^l \|\nabla \cdot \hat{\mathbf{u}}_1\|_{W_2^1(Q)} \\ & \leq c(\|\hat{\mathbf{f}}_*^+\|_{W_2^l(Q)} + |s|^l \|\hat{\mathbf{f}}_*^+\|_{L_2(Q)} + \|\hat{h}_*^+\|_{W_2^{l+1}(Q)} + |s|^l \|\hat{h}_*^+\|_{W_2^1(Q)}), \\ & |s| \|\hat{\mathbf{u}}_1\|_{W_2^2(Q)} + |s|^{1+l} \|\hat{\mathbf{u}}_1\|_{L_2(Q)} \leq c(\|\hat{\mathbf{f}}_*^+\|_{W_2^2(Q)} \\ & + |s|^l \|\hat{\mathbf{f}}_*^+\|_{L_2(Q)} + \|\nabla \hat{h}_*^+\|_{W_2^2(Q)} + |s|^l \|\nabla \hat{h}_*^+\|_{L_2(Q)}), \end{aligned} \quad (2.53)$$

where \hat{w} denotes the Laplace transform of w .

Now, we consider $\tilde{\mathbf{u}}_1$ as the solution of the transformed Stokes system

$$\begin{cases} \nu^+ |\xi|^2 \tilde{\mathbf{u}}_1 + \frac{p_1}{\rho_m^+} i \boldsymbol{\xi} \tilde{\sigma}_1 = \mathbf{f}_*^+ - s \tilde{\mathbf{u}}_1 - (\nu^+ + \nu_1^+) \boldsymbol{\xi} (\boldsymbol{\xi} \cdot \tilde{\mathbf{u}}_1) \equiv \mathbf{f}_3, \\ i \boldsymbol{\xi} \cdot \tilde{\mathbf{u}}_1 = \chi, \end{cases}$$

that is given explicitly by

$$\tilde{\sigma}_1 = \frac{\rho_m^+}{p_1}(\nu^+ \chi - \frac{i\xi \cdot \mathbf{f}_3}{|\xi|^2}), \quad \tilde{\mathbf{u}}_1 = -\frac{i\xi \chi}{|\xi|^2}.$$

From these representation formulas, as well as from $s\tilde{\sigma}_1 = \tilde{h}_*^+ - \rho_m^+ \chi$ and inequalities (2.53) we easily deduce

$$\begin{aligned} & \|\widehat{\mathbf{u}}_1\|_{W_2^{2+l}(Q)} + |s|^{1+l/2} \|\widehat{\mathbf{u}}_1\|_{L_2(Q)} + (1+|s|) \|\widehat{\sigma}_1\|_{W_2^{l+1}(Q)} + |s|^{1+l/2} \|\widehat{\sigma}_1\|_{W_2^1(Q)} \\ & \leq c(\|\widehat{\mathbf{f}}_3\|_{W_2^1(Q)} + |s|^l \|\widehat{\mathbf{f}}_3\|_{L_2(Q)} + \|\nabla \widehat{h}_*^+\|_{W_2^l(Q)} + |s|^l \|\nabla \widehat{h}_*^+\|_{L_2(Q)}), \end{aligned}$$

which implies

$$\begin{aligned} & \|e^{\beta t} \mathbf{u}_1\|_{W_2^{2+l,1+l/2}(Q_\infty^+)} + |e^{\beta t} \sigma_1|_{Q_\infty^+}^{(l+1,l/2)} + |e^{\beta t} \mathcal{D}_t \sigma_1|_{Q_\infty^+}^{(l+1,l/2)} \\ & \leq c(\|e^{\beta t} \mathbf{f}^+\|_{W_2^{l,1/2}(Q_\infty^+)} + |e^{\beta t} h^+|_{Q_\infty^+}^{(l+1,l/2)}), \end{aligned} \quad (2.54)$$

in view of Proposition 1 and condition (2.3) for \mathbf{u}_1 and σ_1 .

The estimate of solution of (2.48) is preceded by the analysis of a similar problem for the transformed Lamé system:

$$\begin{cases} (s + \nu^+ |\xi'|^2) \tilde{\mathbf{v}} - (\nu^+ + \nu_1^+) \tilde{\nabla}(\tilde{\nabla} \cdot \tilde{\mathbf{v}}) = 0, & \text{in } Q^+, \\ \tilde{\mathbf{v}}|_{y_3=0} = \tilde{\mathbf{a}}, \quad \tilde{\mathbf{v}} \xrightarrow{y_3 \rightarrow +\infty} 0, \end{cases}$$

where $\xi' = \frac{\pi}{d_0} \mathbf{k}'$, $|\mathbf{k}'| > 0$. The solution of this problem is given by

$$\tilde{\mathbf{v}} = \tilde{\mathbf{a}} e_0^+(y_3) + C'(r^+ - r_1^+) \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1^+ \end{pmatrix} e_1^+(y_3),$$

where $e_0(y_3) = e^{-r^+ y_3}$, $e_1(y_3) = \frac{e^{-r_1^+ y_3} - e^{-r^+ y_3}}{r_1^+ - r^+}$, $C'(r^+ - r_1^+) = \frac{\nu^+ + \nu_1^+}{2\nu^+ + \nu_1^+} \frac{\tilde{A} - r^+ \tilde{a}_3}{r_1^+ + \varkappa^+ r^+}$, $\tilde{A} = \sum_{\beta=1}^2 i\xi_\beta \tilde{a}_\beta$. By the same argument as in the proof of Proposition 5 it can be shown that

$$\|e^{\beta t} \mathbf{v}_2\|_{W_2^{2+l,1+l/2}(Q_\infty^+)} \leq c \|e^{\beta t} \mathbf{a}\|_{W_2^{3/2+l,3/4+l/2}(Q_\infty^+)}. \quad (2.55)$$

By replacing ν_1^+ with $\nu_1^+(s)$ we arrive at the solution of the transformed problem (2.48):

$$\tilde{\mathbf{u}}_2 = \tilde{\mathbf{a}} e^{-r^+ y_3} + C'(r^+ - r_{11}^+) \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_{11}^+ \end{pmatrix} \frac{e^{-r^+ y_3} - e^{-r_{11}^+ y_3}}{r^+ - r_{11}^+},$$

where

$$C'(r^+ - r_{11}^+) = -\frac{(bs + p_1)(\tilde{A} - r^+ \tilde{a}_3)}{(as + p_1)(r_{11}^+ + \varkappa^+(s)r^+)}, \quad \varkappa^+(s) = \frac{\nu^+ s}{as + p_1}, \quad b = \nu^+ + \nu_1^+.$$

The function $r_{11}^+ + \varkappa^+(s)r^+$ is analytic with respect to s if $\text{Res} > -\min\{\frac{p_1}{a}, \frac{\pi^2}{d_0^2}\}$; let us show that

$$|r_{11}^+ + \varkappa^+(s)r^+| \geq ct.$$

If $s_1 \geq 0$, $s_2 \geq 0$, then $\arg \varkappa^+(s) \in (0, \pi/2]$, and the angle between the vectors r_{11}^+ and $\varkappa^+(s)r^+$ on the complex plane is less than $3\pi/4$, hence, inequality

$$|r_{11}^+ + \varkappa^+(s)r^+| \geq c(|r_{11}^+| + |\varkappa(s)r^+|) \geq c\tau$$

holds.

Now let $s_1 < 0$. Since

$$\varkappa^+(s) - \varkappa^+(is_2) = \frac{\nu^+}{a} \left(\frac{s_1 + is_2}{s_1 + is_2 + p_1/a} - \frac{is_2}{is_2 + p_1/a} \right) = \frac{\nu^+}{a^2} \frac{s_1 p_1}{(s_1 + is_2 + p_1/a)(is_2 + p_1/a)}$$

and $s_1 + p_1/a > 0$, it is clear that $|\varkappa(s) - \varkappa(is_2)| \leq c_1 |s_1|$. Hence

$$|r_{11}^+ + \varkappa(s)r^+| \geq |r_{11}^+ + \varkappa(is_2)r| - |\varkappa(s) - \varkappa(is_2)||r| \geq c_2\tau,$$

if s_1 is small.

From this estimate and (2.55) it follows that

$$\begin{aligned} & \|e^{\beta t} \mathbf{u}_2\|_{W_2^{2+l, 1+l/2}(Q_\infty^\pm)} + |e^{\beta t} \sigma_2|_{Q_+^{(1+l/2, l/2)}} + |e^{\beta t} \mathcal{D}_t \sigma_2|_{Q_+^{(1+l/2, l/2)}} \\ & \leq c \|e^{\beta t} (\mathbf{u}^- - \mathbf{u}_1)\|_{W_2^{3/2+l, 3/4+l/2}(Q_\infty^*)}. \end{aligned} \quad (2.56)$$

Thus, we have constructed periodic functions $(\mathbf{u}^\pm, \sigma^\pm)$ satisfying equations (2.42), (2.43). Collecting estimates (2.45), (2.47), (2.54), (2.56), we obtain

$$\begin{aligned} & \|e^{\beta t} \mathbf{v}\|_{W_2^{2+l, 1+l/2}(\cup Q_T^\pm)} + \|e^{\beta t} \nabla \theta^-\|_{W_2^{l, l/2}(Q_T^-)} + |e^{\beta t} \theta^-|_{Q_T^{(l+1/2, l/2)}} + |e^{\beta t} \theta^+|_{Q_T^{(1+l, l/2)}} \\ & + |e^{\beta t} \mathcal{D}_t \theta^+|_{Q_T^{(1+l, l/2)}} \leq c \left(\|e^{\beta t} \mathbf{f}\|_{W_2^{l, l/2}(\cup Q_T^\pm)} + \|e^{\beta t} h^-\|_{W_2^{l+1, 0}(Q_T^-)} + \|e^{\beta t} \mathbf{H}\|_{W_2^{0, 1+l/2}(Q_T^-)} \right. \\ & \left. + \|e^{\beta t} H_0\|_{W_2^{0, 1+l/2}(Q_T^-)} + |e^{\beta t} h^+|_{Q_T^{(1+l, l/2)}} \right) \end{aligned} \quad (2.57)$$

For $\mathbf{v}' = \mathbf{v} - \mathbf{u}$, $\theta' = \theta - \sigma$ we have the relations (2.1) with the data $\mathbf{f} = 0$, $h = 0$,

$$\begin{aligned} b'_\alpha &= b_\alpha - [\mu^\pm (\frac{\partial u_\alpha}{\partial y_3} + \frac{\partial u_3}{\partial y_\alpha})] \Big|_{y_3=0}, \quad \alpha = 1, 2, \\ b'_3 &= b_3 + p_1 \sigma^+ - \sigma^- - [2\mu^\pm \frac{\partial u_3}{\partial y_3}] - \mu_1^+ \nabla \cdot \mathbf{u}^+ - \sigma \int_0^t \nabla'^2 \mathbf{u}(y, \tau) d\tau \Big|_{y_3=0}. \end{aligned}$$

These functions satisfy inequality (2.37), hence (\mathbf{v}, θ) satisfy (2.4), q.e.d.

2.3. Proof of Theorem 2 for $T = \infty$.

Taking the Laplace transformation, we convert problems (2.5) and (2.6) into

$$\begin{cases} s\tilde{v}_\alpha - \nu^\pm \frac{d^2\tilde{v}_\alpha}{dy_3^2} = \tilde{f}_\alpha, & y_3 \in I^\pm, \\ [\tilde{v}_\alpha]_{y_3=0} = 0, & \left[\mu \frac{d\tilde{v}_\alpha}{dy_3}\right]_{y_3=0} = \tilde{b}_\alpha, \quad \tilde{v}_\alpha|_{y_3=\pm d_1} = 0, \quad \alpha = 1, 2, \end{cases} \quad (2.58)$$

and

$$\begin{cases} s\tilde{v}_3^+ - (2\nu^+ + \nu_1^+) \frac{d^2\tilde{v}_3^+}{dy_3^2} + \frac{p_1}{\rho_m^+} \frac{d\tilde{\theta}^+}{dy_3} = \tilde{f}_3^+, & s\tilde{\theta}^+ + \rho^+ \frac{d\tilde{v}_3^+}{dy_3} = \tilde{h}^+, \quad y_3 \in I^+, \\ s\tilde{v}_3^- - \nu^- \frac{d^2\tilde{v}_3^-}{dy_3^2} + \frac{1}{\rho^-} \frac{d\tilde{\theta}^-}{dy_3} = \tilde{f}_3^-, & \frac{d\tilde{v}_3^-}{dy_3} = \tilde{h}^-, \quad y_3 \in I^-, \\ [\tilde{v}_3]_{y_3=0} = 0, & \tilde{v}_3|_{y_3=\pm d_1} = 0, \\ -p_1\tilde{\theta}^+ + \tilde{\theta}^- + (2\mu^+ + \mu_1^+) \frac{d\tilde{v}_3^+}{dy_3} - 2\mu^- \frac{d\tilde{v}_3^-}{dy_3} \Big|_{y_3=0} = \tilde{b}_3, \end{cases} \quad (2.59)$$

respectively. The solution of (2.58) can be estimated by the energy methods (the existence of the solution is evident). We multiply the first equation by $\rho^\pm \tilde{v}_\alpha^\pm$, integrate over $I^+ \cup I^-$ and take the real part, which leads to

$$\text{Res} \|\sqrt{\rho^\pm} \tilde{v}_\alpha\|_{L_2(I^+ \cup I^-)}^2 + \|\sqrt{\mu^\pm} \frac{d\tilde{v}_\alpha}{dy_3}\|_{L_2(I^+ \cup I^-)}^2 = \text{Re} \int_{I^+ \cup I^-} \rho^\pm \tilde{f}_\alpha \tilde{v}_\alpha dy_3 + \text{Re} \sum_{\alpha=1}^2 \tilde{b}_\alpha \tilde{v}_\alpha(0).$$

After easy calculations we obtain

$$(\text{Res} + \gamma) \|\tilde{v}_\alpha\|_{L_2(I^+ \cup I^-)}^2 + \|\tilde{v}_\alpha\|_{W_2^1(I^+ \cup I^-)}^2 \leq c(\|\tilde{f}_\alpha\|_{L_2(I^+ \cup I^-)}^2 + |\tilde{b}_\alpha| |\tilde{v}_\alpha(0)|)$$

with $\gamma > 0$; we assume that $\text{Res} > -\gamma$.

Multiplying the same equation by $\bar{s} \rho^\pm \tilde{v}_\alpha$ and integrating, we obtain

$$|s|^2 \|\sqrt{\rho^\pm} \tilde{v}_\alpha\|_{L_2(I^+ \cup I^-)}^2 + \text{Res} \|\sqrt{\mu^\pm} \frac{d\tilde{v}_\alpha}{dy_3}\|_{L_2(I^+ \cup I^-)}^2 = \text{Re}(\bar{s} \int_{I^+ \cup I^-} \rho^\pm \tilde{f}_\alpha \tilde{v}_\alpha dy_3 + \bar{s} \sum_{\alpha=1}^2 \tilde{b}_\alpha \tilde{v}_\alpha(0)).$$

From this and preceding relation we deduce

$$|s|^2 \|\tilde{v}_\alpha\|_{L_2(I^+ \cup I^-)}^2 + \|\tilde{v}_\alpha\|_{W_2^2(I^+ \cup I^-)}^2 \leq c(\|\tilde{f}_\alpha\|_{L_2(I^+ \cup I^-)}^2 + (1 + |s|^{1/2}) |\tilde{b}_\alpha|^2), \quad (2.60)$$

because

$$\begin{aligned} (1 + |s|) |\tilde{b}_\alpha| |\tilde{v}_\alpha(0)| &\leq (1 + |s|)^{1/4} |\tilde{b}_\alpha| (1 + |s|)^{3/4} |\tilde{v}_\alpha(0)| \\ &\leq c(1 + |s|)^{1/4} |\tilde{b}_\alpha| (|s| \|\tilde{v}_\alpha\|_{L_2(I^+ \cup I^-)} + \|\tilde{v}_\alpha\|_{W_2^2(I^+ \cup I^-)}). \end{aligned}$$

Next, we multiply (2.60) by $|s|^l$, estimate the $W_2^l(I^+ \cup I^-)$ -norm of the second derivative of \tilde{v}_α :

$$\|\frac{d^2\tilde{v}_\alpha}{dy_3^2}\|_{W_2^l(I^+ \cup I^-)} \leq c(|s| \|\tilde{v}_\alpha\|_{W_2^l(I^+ \cup I^0-)} + \|\tilde{f}_\alpha\|_{W_2^l(I^+ \cup I^-)})$$

and make use of the interpolation inequality

$$|s| \|\tilde{v}_\alpha\|_{W_2^l(I^+ \cup I^-)} \leq \epsilon \left\| \frac{d^2 \tilde{v}_\alpha}{dy_3^2} \right\|_{W_2^l(I^+ \cup I^-)} + c(\epsilon) |s|^{1+l/2} \|\tilde{v}_\alpha\|_{L_2(I^+ \cup I^-)}$$

with small ϵ . Collecting the above estimates, we arrive at

$$\|\tilde{v}_\alpha\|_{2+l, I^+ \cup I^-} \leq c(\|\tilde{f}_\alpha\|_{l, I^+ \cup I^-} + (1 + |s|)^{1/4+l/2} |\tilde{b}_\alpha|), \quad (2.61)$$

where

$$\|u\|_{l_1, \cup I^\pm} = \|u\|_{W_2^{l_1}(\cup I^\pm)} + |s|^{l_1/2} \|u\|_{L_2(\cup I^\pm)}.$$

This inequality implies (2.7).

Now we turn to the problem (2.59). Since $\tilde{h}^-(y_3, s) = \frac{d}{dy_3} \mathfrak{H}_3 + \mathfrak{H}_0$, $\mathfrak{H}_3|_{y_3=-d_1} = 0$, we have

$$\tilde{v}_3^-(y_3) = \int_{-d_1}^{y_3} \tilde{h}^-(z, s) dz = \mathfrak{H}_3(y_3, s) + \int_{-d_1}^{y_3} \mathfrak{H}_0(z, s) dz,$$

from which it follows that

$$\begin{aligned} \|\tilde{v}_3^-\|_{W_2^{2+l}(I^-)} &\leq c \|\tilde{h}^-\|_{W_2^{l+1}(I^-)}, \quad |s|^{1+l/2} \|\tilde{v}_3^-\|_{L_2(I^-)} \leq c |s|^{1+l/2} (\|\mathfrak{H}_3\|_{L_2(I^-)} + \|\mathfrak{H}_0\|_{L_2(I^-)}), \\ \|\tilde{v}_3^-\|_{2+l, \cup I^\pm} &\leq c (\|\tilde{h}^-\|_{W_2^{l+1}(I^-)} + |s|^{1+l/2} (\|\mathfrak{H}_3\|_{L_2(I^-)} + \|\mathfrak{H}_0\|_{L_2(I^-)})). \end{aligned} \quad (2.62)$$

The functions \tilde{v}_3^+ , $\tilde{\theta}^+$ are found as a solution to the problem

$$\begin{cases} s \tilde{v}_3^+ - (2\nu^+ + \nu_1^+) \frac{d^2 \tilde{v}_3^+}{dy_3^2} + \frac{p_1}{\rho^+} \frac{d\tilde{\theta}^+}{dy_3} = \tilde{f}_3^+, \\ s \tilde{\theta}^+ + \rho_m^+ \frac{d\tilde{v}_3^+}{dy_3} = \tilde{h}^+ \text{ in } I^+, \\ \tilde{v}_3^+|_{y_3=d_1} = 0, \quad \tilde{v}_3^+ - \tilde{v}_3^-|_{y_3=0} = 0, \end{cases} \quad (2.63)$$

By eliminating $\tilde{\theta}^+$ we obtain a problem for \tilde{v}_3^+ :

$$\begin{cases} R(s) \tilde{v}_3^+ - \frac{d^2 \tilde{v}_3^+}{dy_3^2} = \frac{s}{as + p_1} (\tilde{f}_3^+ - \frac{p_1}{s\rho^+} \frac{d\tilde{h}^+}{dy_3}) \equiv \tilde{g}_3 \text{ in } I^+, \\ \tilde{v}_3^+|_{y_3=d_1} = 0, \quad \tilde{v}_3^+|_{y_3=0} = \tilde{v}_3^-|_{y_3=0}, \end{cases} \quad (2.64)$$

where $R(s) = \frac{s^2}{as + p_1}$.

We set $\tilde{v}_3^+ = \tilde{w}_3 + \tilde{v}_-$ where $\tilde{v}_- = \tilde{v}_3^-(-y_3, s)$ and reduce problem (2.64) to

$$\begin{cases} R(s) \tilde{w}_3 - \frac{d^2 \tilde{w}_3}{dy_3^2} = \tilde{g}_3 - R(s) \tilde{v}_- + \frac{d^2 \tilde{v}_-}{dy_3^2} \equiv \tilde{g} \text{ in } I^+, \\ \tilde{w}_3|_{y_3=d_1} = 0, \quad \tilde{w}_3|_{y_3=0} = 0. \end{cases}$$

We expand \tilde{g} into the Fourier series $\tilde{g}(y_3, s) = \sum_{k=1}^{\infty} \tilde{g}(\xi_3, s) \sin \xi_3 y_3$, where $\xi_3 = \frac{k\pi}{d_1}$. Then the Fourier coefficients of \tilde{w}_3 are given by

$$\tilde{w}_3(\xi_3, s) = \frac{\tilde{g}(\xi_3, s)}{R(s) + \xi_3^2}. \quad (2.65)$$

Since the difference $R(s) - s/a = -\frac{p_1 s}{a(as+p_1)}$ is bounded by a constant independent of s (if $as_1 + p_1 > 0$), we have

$$c_1(|s| + a\xi_3^2) \leq |R(s) + \xi_3^2| \leq c_2(|s| + a\xi^2),$$

provided d_1 is small; hence (2.65) implies $\|\tilde{w}_3\|_{l,I^+} \leq c\|\tilde{g}\|_{l,I^+}$ and

$$\|\tilde{v}_3^+\|_{2+l,I^+} \leq c(\|\tilde{g}_3\|_{l,I^+} + \|\tilde{v}_3^-\|_{l+2,I^-}). \quad (2.66)$$

Taking (2.66) and (2.61) into account, we obtain

$$\begin{aligned} \|e^{\beta t} \mathbf{v}\|_{W_2^{2+l,1+l/2}(\cup I_\infty^\pm)} &\leq c(\|e^{\beta t} \mathbf{f}\|_{W_2^{l,l/2}(\cup I_\infty^\pm)} + \|e^{\beta t} \frac{dh^+}{dy_3}\|_{W_2^{l,l/2}(I_\infty^+)}) \\ &+ \|e^{\beta t} h^-\|_{W_2^{l+1,0}(I_\infty^-)} + \|\mathfrak{H}_3\|_{W_2^{0,1+l/2}(I_\infty^-)} + \|\mathfrak{H}_0\|_{W_2^{0,1+l/2}(I_\infty^-)}. \end{aligned} \quad (2.67)$$

Using the first two equations in (2.6) we estimate θ^\pm as follows:

$$|e^{\beta t} \mathcal{D}_t \theta^+|_{I_\infty^+}^{(l+1,l/2)} \leq c(|e^{\beta t} h^+|_{I_\infty^+}^{(1+l,l/2)} + \|e^{\beta t} v_3^+\|_{W_2^{2+l,1+l/2}(Q_\infty^+)}), \quad (2.68)$$

$$\|e^{\beta t} \frac{d\theta^+}{dy_3}\|_{W_2^{l,l/2}(I_\infty^+)} \leq c\|e^{\beta t} v_3^+\|_{W_2^{2+l,1+l/2}(I_\infty^+)} \quad (2.69)$$

(if θ^+ satisfies the condition $\int_0^{d_1} \theta^+(y_3, t) dy_3 = 0$, then a better estimate

$$|e^{\beta t} \theta^+|_{I_\infty^+}^{(l+1,l/2)} \leq c\|e^{\beta t} v_3^+\|_{W_2^{2+l,1+l/2}(I_\infty^+)}$$

holds).

Finally, the function θ^- defined by the second and the fourth lines in (2.59), satisfies the inequality

$$\begin{aligned} \|e^{\beta t} \theta^-\|_{W_2^{l+1,0}(I_\infty^-)} + \|e^{\beta t} \theta^-\|_{W_2^{1/4+l/2}(0,\infty)} &\leq c(\|e^{\beta t} v_3\|_{W_2^{2+l,1+l/2}(\cup I_\infty^\pm)} \\ &+ \|e^{\beta t} \theta^+|_{y_3=0}\|_{W_2^{1/4+l/2}(0,\infty)} + \|e^{\beta t} f_3^-\|_{W_2^{l,l/2}(Q_\infty^-)} + \|e^{\beta t} b_3\|_{W_2^{1/4+l/2}(0,\infty)}). \end{aligned} \quad (2.70)$$

Inequalities (2.68),(2.69),(2.70) yield (2.8). Theorem 2 is proved.

To treat the case of finite T , we construct extension of the data of problems (2.1), (2.5), (2.6) into the half-axis $t > T$ by using a standard formula

$$u(T + \tau) = \sum_{k=1}^m \lambda_k u(T - k\tau), \quad \tau > 0, \quad m > 1 \quad (2.71)$$

with $\tau > 0$, $u(t) = 0$ for $t \leq 0$ and with λ_k satisfying the equations

$$1 = \sum_{k=1}^m \lambda_k (-k)^j, \quad j = 0, \dots, m-1$$

This formula is applied to the functions of the form $f(t) = e^{\beta t} u(t)$ which yields

$$f(T + \tau) = \sum_{k=1}^m \lambda_k e^{-\beta(k-1)\tau} f(T - k\tau).$$

According to [10], formula (2.71) yields the extension of functions from $W_2^l(-\infty, T)$, $l \leq m + 1/2$ into \mathbb{R} with preservation of class. Hence the solutions of problems (2.1), (2.5), (2.6) can be defined as restrictions to the interval $t \in (0, T)$ of solutions of the same problems with the data extended as indicated above. Uniqueness of these solution follows from the energy relation for the difference of two possible solutions. For instance, in the case of Theorem 1 this relation has the form

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho_m^+} \mathbf{v}\|_{L_2(Q)}^2 + \int_{\cup Q^\pm} \mathbb{T}(\mathbf{v}) : \nabla \mathbf{v} \, dy - p_1 \int_{Q^+} \theta^+ \nabla \cdot \mathbf{v}^+ \, dy + \sigma \int_{\Omega'} (\nabla' v_3 \cdot \nabla' \int_0^t v_3|_{y_3=0} \, d\tau) \, dy' \\ &= \frac{1}{2} \frac{d}{dt} \left(\|\sqrt{\rho_m^+} \mathbf{v}\|_{L_2(Q)}^2 + \frac{p_1}{\rho_m^+} \|\theta^+\|_{L_2(Q^+)}^2 + \frac{\sigma}{2} \int_{\Omega'} \left(\int_0^t \nabla' v_3 \, d\tau \right)^2 \, dy' \right) + \int_{\cup Q^\pm} \mathbb{T}(\mathbf{v}) : \nabla \mathbf{v} \, dy = 0. \end{aligned}$$

Since the form $\int_{\cup Q^\pm} \mathbb{T}(\mathbf{v}) : \nabla \mathbf{v} \, dy$ is positive, we conclude, upon integrating this relation over the interval $t \in (0, T')$ that $\mathbf{v} = 0$ and $\theta^+ = 0$, but then also $\theta^- = 0$, q.e.d. The same arguments apply to problems (2.5), (2.6).

3 Appendix.

We need to mention model problems arising in the analysis of the solution of Problem (1.9) near the exterior boundary Σ , namely,

$$\left\{ \begin{array}{l} \mathcal{D}_t \mathbf{v}^+ - \nu^+ \nabla^2 \mathbf{v}^+ - (\nu^+ + \nu_1^+) \nabla (\nabla \cdot \mathbf{v}^+) + \frac{p_1}{\rho_m^+} \nabla \theta^+ = \mathbf{f}^+, \quad \mathcal{D}_t \theta^+ + \rho_m^+ \nabla \cdot \mathbf{v}^+ = h^+, \\ \mathbf{v}^+|_{t=0} = 0 \quad \theta^+|_{t=0} = 0 \quad \text{in } Q^+, \quad \int_{Q'} \mathbf{u}^+ \, dy' = 0, \quad \int_{Q'} \theta^+ \, dy' = 0, \\ \mathbf{v}^+|_{y_3=0} = 0, \quad \mathbf{v}^+ \xrightarrow[y_3 \rightarrow \infty]{} 0, \quad \theta^+ \xrightarrow[|y_3| \rightarrow \infty]{} 0, \end{array} \right. \quad (3.1)$$

$$\left\{ \begin{array}{l} \mathcal{D}_t v_\alpha^+ - \nu^+ \mathcal{D}_{y_3}^2 v_\alpha^+ = f_\alpha^+, \quad \alpha = 1, 2, \\ \mathcal{D}_t v_3^+ - (2\nu^+ + \nu_1^+) \mathcal{D}_{y_3}^2 v_3^+ + \frac{p_1}{\rho_m^+} \mathcal{D}_{y_3} \theta^+ = f_3^+, \quad \mathcal{D}_t \theta^+ + \rho_m^+ \mathcal{D}_{y_3} v_3^+ = h^+ \quad \text{in } \Omega^+, \\ \mathbf{v}|_{t=0} = 0, \quad \theta^+|_{t=0} = 0 \quad \text{in } \Omega^+, \quad \mathbf{v}|_{y_3=0, d_1} = 0. \end{array} \right. \quad (3.2)$$

These problems are treated in the same way as problems (2.1), (2.5), (2.6) above, and details are omitted. Estimates of solutions analogous to (2.4), (2.8), (2.9) have the form

$$\begin{aligned} & \|e^{\beta t} \mathbf{v}^+\|_{W_2^{2+l, 1+l/2}(Q_T^+)} + |e^{\beta t} \theta^+|_{Q_T^+}^{(1+l, l/2)} \\ & + |e^{\beta t} \mathcal{D}_t \theta^+|_{Q_T^+}^{(1+l, l/2)} \leq c(\|e^{\beta t} \mathbf{f}^+\|_{W_2^{l, l/2}(Q_T^+)} + |e^{\beta t} h^+|_{Q_T^+}^{(1+l, l/2)}), \end{aligned} \quad (3.3)$$

$$\begin{aligned} & \|e^{\beta t} \mathbf{v}\|_{W_2^{2+l, 1+l/2}(I_T^+)} + \|\mathcal{D}_{y_3} \theta^+\|_{W_2^{l, l/2}(I_T)} + |e^{\beta t} \mathcal{D}_t \theta^+|_{I_T^+}^{(1+l, l/2)} \\ & \leq c(\|\mathbf{f}\|_{W_2^{l, l/2}(I^+)} + |h^+|_{I_T^+}^{(1+l, l/2)}) \end{aligned} \quad (3.4)$$

Another addition that we would like to make concerns the solvability of Problem (2.1) in weighted Sobolev spaces with the exponential weight $e^{-\gamma t}$, $\gamma \gg 1$ in the domains \mathbb{R}_\pm^3 . The following theorem holds true.

Theorem 4. Assume that $e^{-\gamma t} \mathbf{f} \in W_2^{l,l/2}(\cup R_T^\pm)$, $e^{-\gamma t} h^+ \in W_2^{l+1,0}(R_T^+) \cap W_2^{l/2}((0,T); W_2^1(\mathbb{R}_+^3))$, $e^{-\gamma t} \nabla h^- \in W_2^{l+1,l/2+1/2}(R_T^-)$, $h^- = \nabla \cdot \mathbf{H} + H_0$, $e^{\beta t} \mathbf{H}$, $e^{-\gamma t} H_0 \in W_2^{0,1+1/2}(R_T^-)$, h^- , \mathbf{H} , H_0 are compactly supported, $e^{-\gamma t} b_\alpha \in W_2^{l+1/2,l/2+1/4}(\mathbb{R}_T^2)$, $\alpha = 1, 2$, $e^{-\gamma t} b_3 \in \dot{W}_2^{l+1/2,0}(\mathbb{R}_T^2) \cap W_2^{l/2}((0,T); \dot{W}_2^{1/2}(\mathbb{R}^2))$, $e^{-\gamma t} B \in W_2^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)$, where $\gamma \gg 1$, $\mathbb{R}_T^2 = \mathbb{R}^2 \times (0,T)$. Assume also that these functions satisfy the compatibility conditions

$$h^-|_{t=0} = 0, \quad \mathbf{H}|_{t=0} = 0, \quad H_0|_{t=0} = 0, \quad b_\alpha|_{t=0} = 0, \quad \alpha = 1, 2.$$

Then problem (2.1) has a unique solution such that $e^{-\gamma t} \mathbf{v} \in W_2^{2+l,1+l/2}(R_T^\pm)$, $e^{\beta t} \nabla \theta^- \in W_2^{l,l/2}(R_T^-)$, $e^{-\gamma t} \theta^- \in W_2^{l/2}((0,T); W_2^{1/2}(\mathbb{R}^2))$, $e^{-\gamma t} \theta^+, e^{-\gamma t} \mathcal{D}_t \theta^+ \in W_2^{l+1,0}(R_T^+) \cap W_2^{l/2}((0,T); W_2^1(\mathbb{R}_+^3))$. It is subject to the inequality

$$\begin{aligned} & \|e^{-\gamma t} \mathbf{v}\|_{W_2^{2+l,1+l/2}(R_T^\pm)} + \|e^{-\gamma t} \nabla \theta^-\|_{W_2^{l,l/2}(Q_T^-)} + |e^{-\gamma t} \theta^-|_{\mathbb{R}_T^2}^{(l+1/2,l/2)} + |e^{-\gamma t} \theta^+|_{\mathbb{R}_T^2}^{(1+l,l/2)} \\ & + |e^{-\gamma t} \mathcal{D}_t \theta^+|_{R_T^+}^{(1+l,l/2)} \leq c(\|e^{-\gamma t} \mathbf{f}\|_{W_2^{l,l/2}(\cup R_T^\pm)} + \|e^{-\gamma t} h^-\|_{W_2^{l+1,0}(R_T^-)} + \|e^{-\gamma t} \mathbf{H}\|_{W_2^{0,1+1/2}(R_T^-)} \\ & + \|e^{-\gamma t} H_0\|_{W_2^{0,1+1/2}(R_T^-)} + |e^{-\gamma t} h^+|_{R_T^+}^{(1+l,l/2)} + \sum_{\alpha=1,2} \|e^{-\gamma t} b_\alpha\|_{W_2^{l+1/2,l/2+1/4}(\mathbb{R}_T^2)} \\ & + \|e^{-\gamma t} b_3\|_{\dot{W}_2^{l+1/2,0}(\mathbb{R}_T^2)} + \|e^{-\gamma t} b_3\|_{W_2^{l/2}((0,T); \dot{W}_2^{1/2}(\mathbb{R}^2))} + \|e^{-\gamma t} B\|_{W_2^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)}) \end{aligned} \quad (3.5)$$

with the constant independent of T .

By $\|u\|_{\dot{W}_2^l(\mathbb{R}^2)}$ we mean the norm equivalent to $\|\xi'|\tilde{u}\|_{L_2(\mathbb{R}^2)}$.

As above, it is enough to consider the case $T = \infty$. The Fourier-laplace transform

$$\tilde{u}(\xi', s, y_3) = \int_0^\infty e^{-st} dt \int_{\mathbb{R}^2} e^{-i\xi' \cdot y'} u(y, t) dy'$$

where $\text{Res} = \gamma$, $\xi' \in \mathbb{R}^2$, converts (2.1) into

$$\left\{ \begin{array}{l} (s + \nu^- |\xi'|^2) \tilde{v}_\alpha^- - \nu^- \frac{d^2}{dy_3^2} \tilde{v}_\alpha^- + \frac{1}{\rho^-} i \xi_\alpha \tilde{\theta}^- = \tilde{f}_\alpha^-, \quad \alpha = 1, 2, \\ (s + \nu^- |\xi'|^2) \tilde{v}_3^- - \nu^- \frac{d^2}{dy_3^2} \tilde{v}_3^- + \frac{1}{\rho^-} \frac{d\tilde{\theta}^-}{dy_3} = \tilde{f}_3^-, \quad \tilde{\nabla} \cdot \tilde{\mathbf{v}}^- = h^- \text{ for } y_3 < 0, \\ (s + \nu^+ |\xi'|^2) \tilde{v}_\alpha^+ - \nu^+ \frac{d^2}{dy_3^2} \tilde{v}_\alpha^+ - (\nu^+ + \nu_1^+) i \xi_\alpha \tilde{\nabla} \cdot \tilde{\mathbf{v}}^+ + \frac{p_1}{\rho_m^+} i \xi_\alpha \tilde{\theta}^+ = \tilde{f}_\alpha^+, \quad \alpha = 1, 2, \\ (s + \nu^+ |\xi'|^2) \tilde{v}_3^+ - \nu^+ \frac{d^2}{dy_3^2} \tilde{v}_3^+ - (\nu_1 + \nu_1^+) \frac{d}{dy_3} \tilde{\nabla} \cdot \tilde{\mathbf{v}}^+ + \frac{p_1}{\rho_m^+} \frac{d\tilde{\theta}^+}{dy_3} = \tilde{f}_3^+, \\ s\tilde{\theta}^+ + \rho_m^+ \tilde{\nabla} \cdot \tilde{\mathbf{v}}^+ = \tilde{h}^+ \text{ for } y_3 > 0, \\ [\tilde{\mathbf{v}}]_{y_3=0} = 0, \quad \left[\mu^\pm \left(\frac{d\tilde{v}_\alpha}{dy_3} + i \xi_\alpha \tilde{v}_3 \right) \right]_{y_3=0} = \tilde{b}_\alpha, \quad \alpha = 1, 2, \\ -p_1 \tilde{\theta}^+ + \tilde{\theta}^- + \mu_1^+ \tilde{\nabla} \cdot \tilde{\mathbf{v}}^+ + \left[2\mu^\pm \frac{d\tilde{v}_3}{dy_3} \right]_{y_3=0} - \sigma \frac{|\xi'|^2}{s} \tilde{v}_3|_{y_3=0} = \tilde{b}_3 + \frac{\sigma}{s} \tilde{B}. \end{array} \right. \quad (3.6)$$

If $\mathbf{f} = 0$, $h = 0$, then, as shown in Sec. 2, the solution is given by (2.19), (2.20), (2.21), with ν_1^+ replaced by $\nu_1^+(s)$. the functions \mathbb{M} and \mathbb{P} satisfy (2.26), (2.29) and the solution satisfies inequalities

(2.17) and (2.41). In construction of auxiliary functions \mathbf{u}^\pm and σ^\pm , as in subsection 2.2, the vector field \mathbf{u}^- satisfying the relation $\nabla \cdot \mathbf{u}^- = h^-$, should be taken in the form $\mathbf{u}^- = \nabla \Phi$ where Φ solves the problem

$$\nabla^2 \Phi = h^- = \nabla \cdot \mathbf{H} + H_0, \quad \text{in } \mathbb{R}_+^3, \quad \Phi|_{y_3=0} = 0.$$

It satisfies inequality (2.46) in \mathbb{R}_-^3 , if h^-, \mathbf{H}, H_0 are compactly supported. Moreover, since the inequality (2.52) holds, if $\text{Res} \geq \gamma \gg 1$, the formula (2.50) for $\tilde{\mathbf{u}}_1$ and equation (2.52) for $\tilde{\sigma}_1$ yield estimate (2.54) in the domain R_∞^+ , and \mathbf{u}_2, σ satisfy (2.55) in this domain (the proof is the same as that of (2.55)). Putting the above estimates together we obtain (3.5).

It is convenient to use Theorem 4 for the proof of solvability of the problem (1.1) in a finite time interval.

References

1. I. V. Denisova, *Evolution of compressible and incompressible fluids separated by a closed interface*. Interface Free Bound. **2**(3) (2000), 283–312.
2. I. V. Denisova, *On energy inequality for the problem on the evolution of two fluids of different types without surface tension*, J. Math. Fluid. Mech. **17**(1) (2015), 183–198.
3. T. Kubo, Y. Shibata, K. Soga, *On the R-boundedness for the two phase problem with phase transition: compressible-incompressible model problem*, Funkcial. Ekvac. **59**, (2016), 243–287.
4. T. Kubo, Y. Shibata, *On the evolution of compressible and incompressible fluids with a sharp interface*, Preprint.
5. I. V. Denisova, V. A. Solonnikov, *Local and global solvability of free boundary problems for compressible Navier-Stokes equations near equilibria*, in: Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Y.Giga and A.Novotny (eds.), Springer, to appear.
6. V. A. Solonnikov, *On the initial-boundary problem for the Stokes system arising in the study of a free boundary problem*, Proc. Steklov Math. Inst., **188** (1990), 150–188.
7. T. Nishida, Y. Teramoto, H. Yoshihara, *Global in time behavior of viscous surface waves: horizontally periodic motion*, J.Meth. Kyoto Univ. **44**, No 2 (2004), 271–323.
8. M. S. Agranovich, V. I. Vishik, *Elliptic problems with a parameter and parabolic problems of general type*, Usp. Mat. Nauk **19**(3), 53–161 (1964) (English: Russian Math. Surveys **19**, 33–159).
9. V. A. Solonnikov, *On the solvability of free boundary problem for viscous compressible fluids in an infinite time interval*, in: Mathematical fluid mechanics, present and future, Tokyo, Japan, November 2014, Y.Shibata, Y.Suzuki (Eds.) Springer, Japan, 287–315.
10. V. M. Babich, *Concerning the problem of extension of functions*, Uspehi Mat. Nauk **8** (1953), 111–113.