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On the question of genericity of hyperbolic knots

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Abstract. A well-known conjecture in knot theory says that the percentage of hyperbolic knots amongst all of the prime knots of n or fewer crossings approaches 100 as n approaches infinity. In this article, it is proved that this conjecture contradicts several other plausible conjectures, including the 120-year-old conjecture on additivity of the crossing number of knots under connected sum and the conjecture that the crossing number of a satellite knot is not less than that of its companion.

Keywords: knot, tangle, crossing number, unknotting number, prime, composite, hyperbolic, torus, satellite.

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1. Introduction

William Thurston proved in 1978 that every non-torus non-satellite knot is a hyperbolic knot. Computations show that the overwhelming majority of prime knots with small crossing number are hyperbolic knots. Table 1 below gives the number of hyperbolic, prime satellite, and torus knots of n crossings for $n = 3, \dots, 16$ (see [HTW98] or the sequences A002863, A052408, A051765, and A051764 in the Sloane's encyclopedia of integer sequences).

type \ $n =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16
all prime	1	1	2	3	7	21	49	165	552	2 176	9 988	46 972	253 293	1 388 705
hyperbolic	0	1	1	3	6	20	48	164	551	2 176	9 985	46 969	253 285	1 388 694
prime satellite	0	0	0	0	0	0	0	0	0	0	2	2	6	10
torus	1	0	1	0	1	1	1	1	1	0	1	1	2	1

Table 1. Number of prime knots

(A part of) these data gave rise to the following conjecture (see [Ad94b, p. 119]).

Conjecture 1. *The percentage of hyperbolic knots amongst all of the prime knots of n or fewer crossings approaches* 100 *as n approaches infinity.*

In the present paper, we show that Conjecture 1 contradicts several other long standing conjectures, including the following one.

Conjecture 2. *The crossing number of knots is additive with respect to connected sum.*

See, e. g., [Ad94b, p. 69], [Kir97, Problem 1.65], and [La09] for comments and related results. Another related conjecture is as follows.

Conjecture 3. *The crossing number of a satellite knot is larger (a weaker variant: not less) than that of its companion.*

See [Ad94b, p. 118], [Kir97, Problem 1.67 (attributed to de Souza)], and [La14]. Concerning Conjecture 3, it is noticed in [Kir97, Problem 1.67] that 'Surely the answer is yes, so the problem indicates the difficulties of proving statements about the crossing number'. Since a composite knot is a connected sum of its factors and, at the same time, is a satellite of each of its factors, the 'intersection' of Conjectures 2 and 3 yields the following.

Conjecture 4. *The crossing number of a composite knot is larger (a weaker variant: not less) than that of each of its factors.*

We denote by $\mathrm{cr}(X)$ the crossing number of a knot X. If P is a prime knot and λ is a real number, we say that P is *λ-regular* if we have $\mathrm{cr}(K) \geq \lambda \cdot \mathrm{cr}(P)$ whenever P is a factor of a knot K. In this terminology, Conjecture 4 says that each prime knot is 1-regular. Lackenby [La09] proved that each knot is $\frac{1}{152}$-regular. Our considerations involve the following conjecture.

Conjecture 5. *Each prime knot is $\frac{2}{3}$-regular.*

We also consider the following weakening of Conjecture 5.

Conjecture 6. *There exist $\varepsilon > 0$ and $N > 0$ such that, for all $n > N$, the percentage of $\frac{2}{3}$-regular knots amongst all of the hyperbolic knots of n or fewer crossings is at least ε.*

We have the following obvious implications.

$$\begin{array}{ccccccc} \text{Conj. 2} & & & & & & \\ & \searrow & & & & & \\ & & \text{Conj. 4} & \Longrightarrow & \text{Conj. 5} & \Longrightarrow & \text{Conj. 6} \\ & \nearrow & & & & & \\ \text{Conj. 3} & & & & & & \end{array}$$

The main result of this paper is the following theorem.

Theorem 1. *Conjecture 1 contradicts Conjecture 6 and hence it also contradicts (each of) Conjectures 2, 3, 4, and 5.*

The paper is organised as follows. Section 2 contains remarks concerning Conjectures 1–6. In Section 3, we present the key idea of the proof of Theorem 1 and reduce Theorem 1 to Proposition 1 consisting of three assertions. Sections 4–8 contain the proof of Proposition 1. In Sections 4 and 5, we prove the first two assertions of Proposition 1. Section 6 contains a combinatorial lemma used in the proof of the last assertion of Proposition 1. Section 7 contains preliminaries on tangles. In Section 8, we prove the last assertion of Proposition 1. In Section 9, we introduce a new property of knots (strong property PT) and prove Theorem 3 strengthening Theorem 1. In Section 10, we show that an assumption that Conjectures 2–5 has many strong counterexamples contradicts Conjecture 1 as well. In Section 11, we show that certain assumptions concerning unknotting numbers of knots contradict Conjecture 1.

The paper should be interpreted as being in either the PL or smooth category. For standard definitions we mostly use the conventions of [BZ06] and [BZH14]. There will be a certain abuse of language in order to avoid complicating the notation. In particular, a *knot* K will be a circle embedded in the 3-sphere S^3, a pair (S^3, K), or a class of homeomorhic pairs (cf. [BZ06, p. 1]). No orientations on knots and spaces are placed if not otherwise stated.

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2. Remarks

We list certain results related to Conjectures 1–6.

Predominance of hyperbolic objects. In recent years, a number of results have been obtained showing the predominance of hyperbolic objects in various cases. We refer to the works of Ma [Ma14] and Ito [Ito15, Theorem 2] for results concerning genericity of hyperbolic knots and links. See also [Mah10a], [LMW14, Theorem 2], [LusMo12], [Riv14], and [Ito15, Theorem 1] for results on genericity of hyperbolic 3-manifolds. Related results show that pseudo-Anosovs prevail (in various senses) in mapping class groups of surfaces. We refer to [Riv08, Riv09, Riv10, Riv12, Riv14], [Kow08], [Mah10b, Mah11, Mah12], [AK10], [Sis11], [Mal12], [LubMe12], [MS13], and also [Car13, CW13, Wi14] (non-random approach) for precise statements and detailed discussions. See also [GTT16] for

the genericity of loxodromic isometries for actions of hyperbolic groups on hyperbolic spaces. Other examples of hyperbolicity predominance can be found in extensive literature on exceptional Dehn fillings (see [Thu79] etc.) and in [Gro87, 0.2.A], [GhH90, p. 20], [Ch91] and [Ch95], [Ols92], [Gro93], [Żuk03], [Oll04], [Oll05], where viewpoints are given from which it appears that a generic finitely presented group is word hyperbolic. Apparently, combining approaches developed by [Ito15] and [Ma14] with results of [Car13, CW13, Wi14] ([LusMo12]) one can obtain more viewpoints where generic knots will be hyperbolic.

Predominance of non-hyperbolic objects. As for natural models where it is proved that hyperbolic objects are rare, we have standard methods of generating knots as polygons in $\mathbb{R}^3$. Under this approach, composite knots prevail and prime knots (including hyperbolic ones) are asymptotically scarce. See Sumners and Whittington [SW88], Pippenger [Pip89], and also Soteros, Sumners, and Whittington [SSW92] for the case of self-avoiding random polygons on the simple cubic lattice; see [Oetal94] and [Sot98] for such polygons in specific subsets of the lattice; see [DPS94] and [Jun94] for local and global knotting in Gaussian random polygons; see [Di95] and [DNS01] for local and global knotting in equilateral random polygons; see also [Ken79] for knotting of Brownian motion and [Sum09, MMO11] for more references. An interesting idea appeared in [Ad05, p. 4] and [Cr04, p. 95] that prime satellite knots should prevail over hyperbolic ones when we consider Gaussian random polygons. However, in both [Ad05] and [Cr04], the idea was apparently inspired by a misinterpretation of results in [Jun94].

Crossing number additivity. Murasugi [Mur87, Corollary 6] proved that Conjecture 2 holds for alternating knots. (This follows from the proof of the Tait conjecture stating that reduced alternating projections are minimal; this Tait conjecture was also proved, independently, by Kauffman [Kau87] and Thistlethwaite [Thi87].) Conjecture 2 is valid for *adequate* knots (see [LT88]). Diao [Di04] and Gruber [Gru03] independently proved that Conjecture 2 holds for torus knots and certain other special classes of knots. Results of [Mur87, Kau87, Thi87] imply that alternating knots are 1-regular.[1] Diao [Di04, Theorem 3.8] showed that torus knots are 1-regular. In [PZ15], the authors introduce a telescopic family of conjectures concerning monotonic simplification of link diagrams and provide supporting evidence for (the strongest of) these conjectures.[2] Each of Petronio–Zanellati conjectures implies Conjecture 2.

Torus knots. Murasugi [Mur91, Proposition 7.5] proved that the torus link of type (p, q) with $2 \le p \le q$ has crossing number $(p-1)q$. Taking into account that the number of all prime knots of n crossings grows exponentially in n (see [ES87, Wel92]), we see that the percentage of torus knots amongst all of the prime knots

[1] In [Mur87, Kau87, Thi87], it is shown that (i) for each knot K we have $\operatorname{span} V_K(t) \le \operatorname{cr}(K)$, and (ii) for alternating K we have $\operatorname{span} V_K(t) = \operatorname{cr}(K)$, where $V_K(t)$ is the Jones polynomial of K and $\operatorname{span} V_K(t)$ is the difference between the maximal and minimal degrees of $V_K(t)$. (It is known that $V_K(t) \neq 0$, so that $\operatorname{span} V_K(t)$ is well-defined. See [Jon85, Theorem 15].) Then the 1-regularity of alternating knots follows because, for any two knots K_1 and K_2, we have [Jon85, Theorem 6]

$$\operatorname{span} V_{K_1 \sharp K_2}(t) = \operatorname{span} V_{K_1}(t) + \operatorname{span} V_{K_1}(t).$$

[2] Ivan Dynnikov found a counterexample to Petronio–Zanellati conjectures (a private communication).

of n or fewer crossings approaches 0 as n approaches infinity. Thus, only satellite knots 'pose a danger' to Conjecture 1.

Hyperbolic knots. Several interesting classes of knots consist of hyperbolic and torus knots only. In particular, amongst these classes are:
– prime alternating knots, including 2-bridge knots (see [Men84]),
– prime almost alternating knots (see [Aetal92]),
– prime toroidally alternating knots (see [Ad94a]),
– arborescent knots, including 2-bridge knots, pretzel knots, and Montesinos knots (see [BS10], Theorem 1.5 and subsequent discussion in [FG09]).

More families of hyperbolic knots, links, and tangles are listed in [Ad05]. See [Ito11], [IK12, Theorems 8.3, 8.4] for new examples of huge classes of hyberbolic knots, links, and 3-manifolds.

3. The idea of the proof of Theorem 1

Our proof of Theorem 1 involves a specific way of constructing satellite knots. For brevity, we use the term *γ-knots* for the satellite knots constructed in this way.

Definition. γ-Knots. Let K be a knot in the 3-sphere S^3, and let V be an unknotted solid torus in S^3 such that K is contained in the interior of V. Let $\psi\colon V \to W \subset S^3$ be a homeomorphism onto a tubular neighbourhood W of a hyperbolic knot. We recall that the *winding number* of K in V is the absolute value of the algebraic intersection number of K with a meridional disk in V. We assume that the winding number of K in V is at least 2 and that ψ maps a longitude[3] of V to a longitude of W. Then we say that the knot $\psi(K) \subset S^3$ is a *γ-knot over K*.

A method of constructing a γ-knot is presented in Fig. 1. We assume that a diagram D' of a knot K' is obtained from a diagram D of a knot K by a local move as in Fig. 1. (See Fig. 2 for an example.) Our definitions imply that if the two arrows on arcs in Fig. 1(a) determine the same orientation on K, then K' is a γ-knot over K. Here, the winding number is 2 while the companion hyperbolic knot is the figure-eight knot. (In order to check that the condition on longitudes is also fulfilled, we observe that each arc in Fig. 1 has zero total curvature.)

We deduce Theorem 1 from the following proposition on γ-knots.

Proposition 1. (i) *Each γ-knot is a satellite knot.*

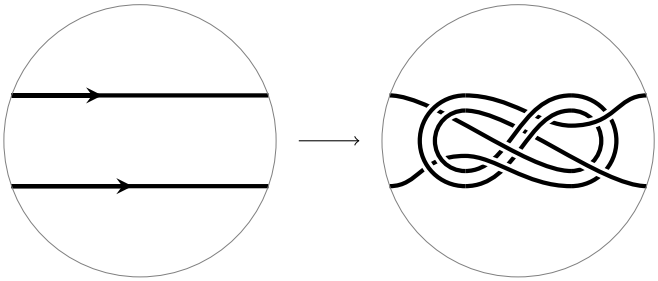
(ii) *The sets of γ-knots over distinct non-satellite knots are disjoint.*

(iii) *If P is a $\frac{2}{3}$-regular prime knot, then there exists a* <u>*prime*</u> *γ-knot P' over P with* $\mathrm{cr}(P') \leq \mathrm{cr}(P) + 17$.

Remark. Assertion (iii) of Proposition 1 is not obvious because a γ-knot over a prime knot is not necessarily prime (see Fig. 2).

Proposition 1 *implies Theorem* 1. We introduce the following notation. Let p_n (resp., h_n, s_n) denote the number of prime (resp., hyperbolic, prime satellite) knots with crossing number n. We set $P_n = \sum_{k=1}^{n} p_k$, $H_n = \sum_{k=1}^{n} h_k$, and $S_n = \sum_{k=1}^{n} s_k$.

[3] If a solid torus U is embedded in the 3-sphere S^3, then there exists an essential curve in ∂U that bounds a 2-sided surface in $S^3 \setminus \mathrm{int}(U)$ (a Seifert surface). This curve is unique up to isotopy on ∂U and is called a *longitude* of U in S^3 (see, e. g., [BZ06, Theorem 3.1]).



(a) (b)

FIGURE 1. Double figure-eight move

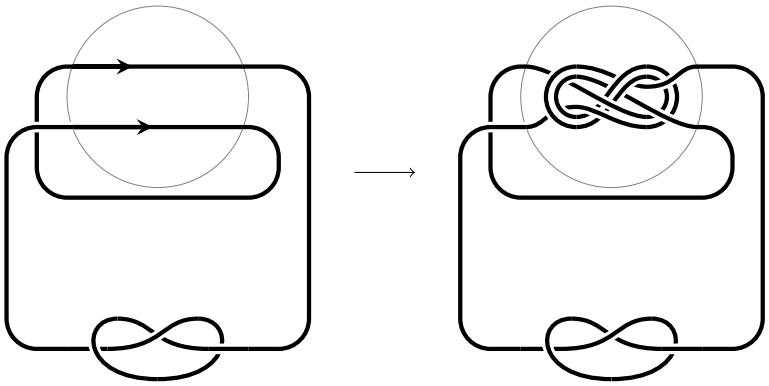


FIGURE 2. A composite γ-knot over the trefoil

Since each of Conjectures 2, 3, 4, and 5 implies Conjecture 6 (see the diagram before Theorem 1), it suffices to prove only that Conjectures 6 and 1 are incompatible. If Conjecture 6 is true, then there exist $\varepsilon_0 > 0$ and $N_0 > 0$ such that, for all $n > N_0$, the number of $\frac{2}{3}$-regular hyperbolic knots of n or fewer crossings is at least $\varepsilon_0 H_n$. Obviously, in this case assertions (i), (ii), and (iii) of Proposition 1 imply that (for all $n > N_0$) we have

$$S_{n+17} \ \geq \ \varepsilon_0 H_n.$$

Therefore, we have

$$P_{n+17} \ \geq \ H_{n+17} + \varepsilon_0 H_n.$$

This is equivalent to the following inequality

$$1 \ \geq \ \frac{H_{n+17}}{P_{n+17}} + \varepsilon_0 \frac{H_n}{P_n} \frac{P_n}{P_{n+17}}. \tag{1}$$

If Conjecture 1 is true, then both sequences $\frac{H_{n+17}}{P_{n+17}}$ and $\frac{H_n}{P_n}$ tend to 1. In this case, Eq. (1) implies that

$$\frac{P_{n+17}}{P_n} \xrightarrow{n\to+\infty} +\infty.$$

Consequently, for each $B > 0$ we have $P_n > B^n$ for all sufficiently large n. (We consider subsequences of the form P_{n_0+17i}, $i \in \mathbb{N}$.) In other words, we have

$$P_n^{1/n} \xrightarrow{n\to+\infty} +\infty. \tag{2}$$

However, it is shown in [Wel92] that

$$\limsup_{n\to\infty} p_n^{1/n} < +\infty,$$

which implies that there exists $B > 0$ such that $p_n < B^n$ for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$ we have $P_n < (B+1)^n$ whence it follows that

$$\limsup_{n\to\infty} P_n^{1/n} \leq B + 1 < +\infty.$$

This contradicts (2). The obtained contradiction completes the proof. □

4. Proof of assertion (i) of Proposition 1

We recall definitions of satellite knots. A knot K in S^3 is a *satellite knot* if S^3 contains a non-trivial knot C such that K lies in the interior of a regular neighbourhood V of C, V does not contain a 3-ball containing K, and K is not a core curve of the solid torus V. The knot K is a satellite knot if and only if K contains an incompressible, non-boundary parallel torus in its complement. (For a proof, see [BZH14, Remark 16.1, p. 335].)

Let K be a γ-knot in S^3. Then the definition of γ-knots implies that K lies in a knotted solid torus $W \subset S^3$ such that the winding number of K in W is at least 2. Since the winding number of K in W is at least 2, it follows that W does not contain a 3-ball containing K, and K is not a core curve of V. This means by the above definition that K is a satellite knot.

5. Proof of assertion (ii) of Proposition 1

We show that the sets of γ-knots over distinct non-satellite knots are disjoint. Suppose to the contrary that there exist a knot K and two distinct non-satellite knots H_1 and H_2 such that K is a γ-knot both over H_1 and over H_2. By the definition of γ-knots, this means that there exist embedded solid tori V_1 and V_2 in S^3 and re-embeddings $\phi_1 \colon V_1 \to S^3$ and $\phi_2 \colon V_2 \to S^3$ such that, for each $i \in \{1, 2\}$, the following conditions hold:

– V_i is a tubular neighbourhood of a hyperbolic knot,
– K lies in the interior of V_i and the winding number of K in V_i is at least 2,
– the solid torus $\phi_i(V_i)$ is unknotted,
– ϕ_i maps a longitude of V_i to a longitude of $\phi_i(V_i)$,
– we have $\phi_i(K) = H_i$.

Claim 1. *The tori ∂V_1 and ∂V_2 are both incompressible in $S^3 \setminus K$.*

Since the winding number of K in V_i is non-zero, it follows that no 3-ball in V_i contains K. If a knotted solid torus U in the 3-sphere S^3 contains a knot L in its interior while no 3-ball in U contains L, then ∂U is sometimes called a *companion torus* of L. It is well known that, in this case, ∂U is incompressible in $S^3 \setminus L$. (See, e. g., [BZH14, Propositions 3.10 and 3.12, and E 2.9].) This implies Claim 1.

Claim 2. *There exists an isotopy of ∂V_1 in $S^3 \setminus K$ that moves ∂V_1 to a position where $\partial V_1 \cap \partial V_2 = \emptyset$.*

It may be assumed that ∂V_1 intersects ∂V_2 transversely in simple closed curves. If the intersection $\partial V_1 \cap \partial V_2$ contains a curve that is inessential in ∂V_2, let C be an innermost of such curves and let d be the open disk in $\partial V_2 \setminus \partial V_1$ bounded by C. Then C is inessential in ∂V_1 because ∂V_1 is incompressible in $S^3 \setminus K$ (Claim 1). Let

δ be the open disk in ∂V_1 bounded by C (δ may intersect ∂V_2). Then the sphere $d \cup \delta \cup C$ bounds a ball (say, B) in $S^3 \setminus K$. We have $B \cap \partial V_1 = \delta \cup C$. It follows that we can eliminate C (together with $\delta \cap \partial V_2$, if nonempty) by an isotopy of ∂V_1 in a neighborhood of B. Therefore, we can eliminate all components of $\partial V_1 \cap \partial V_2$ that are inessential in ∂V_2. The remaining curves of $\partial V_1 \cap \partial V_2$ are essential in ∂V_1 as well. (For if C is an innermost of inessential curves from $\partial V_1 \cap \partial V_2$ on ∂V_1, then C is inessential in ∂V_2 because ∂V_2 is incompressible in $S^3 \setminus K$ by Claim 1.) Now, if $\partial V_1 \cap \partial V_2$ is still nonempty, the space $\partial V_1 \setminus \partial V_2$ is a collection of annuli. It is known that every incompressible properly embedded annulus in the closure of the complement of a hyperbolic knot is boundary parallel (see, e. g., [BZ06, Lemma 15.26]). Applying this to the space $S^3 \setminus \mathrm{int}(V_2)$, we see that there exists an isotopy of ∂V_1 in $S^3 \setminus K$ moving ∂V_1 in $S^3 \setminus \partial V_2$. Claim 2 is proved.

The classical Isotopy Extension Theorem (for smooth manifolds) says that if A is a compact submanifold of a manifold M and $F\colon A \times I \to M$ is an isotopy of A with $F(A \times I) \subset \mathrm{int}(M)$, then F extends to an ambient isotopy (i. e., a diffeotopy of M) having compact support (see, e. g., [Hir76, p. 179]). Applying this theorem to the isotopy of ∂V_1 in $S^3 \setminus K$ from Claim 2 yields the following.

Claim 3. *There exists an ambient isotopy of S^3, fixing K pointwise, that moves V_1 to a position in which $\partial V_1 \cap \partial V_2 = \emptyset$.*

Thus, we can assume without loss of generality that $\partial V_1 \cap \partial V_2 = \emptyset$ (while V_1 and V_2 satisfy all properties listed at the beginning of the proof). Now, let M_1 and M_2 denote the closures of the complements $S^3 \setminus V_1$ and $S^3 \setminus V_2$ respectively.

Claim 4. *M_1 and M_2 are disjoint.*

In order to prove Claim 4, we need the following assertion.

Claim 5. *There is no isotopy between ∂V_1 and ∂V_2 in $S^3 \setminus K$.*

Suppose to the contrary that such an isotopy exists. Then the Isotopy Extension Theorem (see above) implies that there exists an ambient isotopy of S^3, fixing K pointwise, that moves ∂V_1 to ∂V_2. This yields an isotopy between V_1 and V_2 that fixes K pointwise. Then the triples (V_1, K, ℓ_1) and (V_2, K, ℓ_2), where ℓ_i is a longitude of V_i, $i = 1, 2$, are homeomorphic, i. e., there exists a homeomorphism $\tau\colon V_1 \to V_2$ such that $\tau(K) = K$ and $\tau(\ell_1) = \ell_2$. This implies that the pairs $(S^3, \phi_1(K))$ and $(S^3, \phi_2(K))$ are homeomorphic. Indeed, we observe that $(S^3, \phi_i(K))$ is obtained from (V_i, K) by a *Dehn filling* along ℓ_i, that is, $(S^3, \phi_i(K))$ is obtained by attaching a solid torus V to V_i by a gluing homeomorphism $\sigma_i\colon \partial V \to \partial V_i$ such that $\sigma_i^{-1}(\ell_i)$ bounds a meridional disk of V. Thus, the homeomorphism $\tau\colon V_1 \to V_2$ extends to a homeomorphism $S^3 \to S^3$ that maps $\phi_1(K)$ to $\phi_2(K)$. This means that the knots H_1 and H_2 are equivalent because we have $\phi_i(K) = H_i$ by construction. This contradicts the assumption that H_1 and H_2 are distinct. Claim 5 is proved.

Now, we pass to the proof of Claim 4. Observe that neither M_1 contains $\partial M_2 = \partial V_2$ nor M_2 contains $\partial M_1 = \partial V_1$ because an incompressible torus in a hyperbolic knot complement is boundary parallel by Thurston's hyperbolization theorem, while $\partial M_1 = \partial V_1$ and $\partial M_2 = \partial V_2$ are not parallel by Claim 5. Obviously, this implies that M_1 and M_2 are disjoint.

Another fact that we need is implied by the following proposition.

Proposition 2. *Let C_1, C_2, ..., C_n be n disjoint submanifolds of S^3 such that for all $i \in \{1, 2, \ldots, n\}$, $K_i = \mathrm{clos}(S^3 \setminus C_i)$ is a non-trivially embedded solid-torus in S^3. Then there exists n disjointly embedded 3-balls B_1, B_2, ..., $B_n \subset S^3$ such that $C_i \subset B_i$ for all $i \in \{1, 2, \ldots, n\}$. Moreover, each B_i can be chosen to be C_i union a 2-handle which is a tubular neighbourhood of a meridional disk for K_i.*

Proof. See [Bud06, Proposition 2.1] and references therein for earlier proofs. □

Applying Proposition 2 to M_1 and M_2, we obtain the following claim.

Claim 6. *There exists a meridional disk D_2 for V_2 such that $D_2 \subset V_1$.*

Now, since we have $M_2 \subset V_1$ (Claim 4), the image $\phi_1(M_2)$ is well defined. We consider the complement $W := S^3 \setminus \phi_1(\mathrm{int}(M_2))$. Due to Alexander's theorem on embedded torus in S^3, we observe that W is a knotted solid torus because we know that the boundary $\partial W = \partial\phi_1(M_2) = \phi_1(\partial M_2) = \phi_1(\partial V_2)$ is a torus, while the complement $S^3 \setminus W = \phi_1(\mathrm{int}(M_2))$ is homeomorphic to $\mathrm{int}(M_2)$, which is the complement of the knotted solid torus V_2. (Of course, by the Gordon–Luecke theorem we know, moreover, that W is a tubular neighbourhood of a hyperbolic knot.) We see that W contains $\phi_1(K)$ by construction. Finally, we see that the winding number of $\phi_1(K)$ in W is equal to the winding number of K in V_2 because there exists a meridional disk D_2 for V_2 such that $D_2 \subset V_1$, so that ϕ_1 maps D_2 to a meridional disk of W. Therefore, $\phi_1(K)$ is contained in a knotted solid torus W and the winding number of $\phi_1(K)$ in W is at least 2. This means that $\phi_1(K)$ is a satellite knot. Since we have $H_1 = \phi_1(K)$, this contradicts the assumption that H_1 is not a satellite knot. This contradiction completes the proof of assertion (ii) of Proposition 1.

6. A combinatorial lemma

The present section contains a lemma which is used in the proof of assertion (iii) of Proposition 1.

Definitions. Let K be a knot in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$, and let $D \subset S^2$ be a projection of K on the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ in S^3. A knot projection is said to be *regular* if its only singularities are transversal double points. If D is a regular knot projection, an *edge* in D is the closure of a component of the set $D \setminus V$, where V is the set of double points of D. We say that two edges I and J of D are *neighboring edges* or *neighbors* if there exists a component Q of $S^2 \setminus D$ such that the boundary ∂Q contains both I and J. We say that two edges I and J of D are *consecutive* if the union $I \cup J$ is the image of a (connected) arc of the knot. We will denote by ρ the maximal metric on the set $E(D)$ of edges of D in the class of metrics satisfying the condition

$$\rho(I, J) = 1 \quad \text{if } I \text{ and } J \text{ are consecutive edges of } D.$$

Lemma 1. *Any regular knot projection with $n > 0$ double points has a pair of neighboring edges I and J with $\rho(I, J) \geq 2n/3$.*

Proof. Let $D \subset S^2$ be a regular knot projection with n double points. We consider the case with $n \geq 2$ (the case $n = 1$ is obvious). Observe that D has $2n$ edges. Put $k := \lfloor 2n/3 \rfloor$, the largest integer not greater than $2n/3$, and split the set $E(D)$ of edges of D in three parts, E_1, E_2, and E_3, such that each part is a chain of consecutive edges, two parts consist of k edges each, and the third part consists

of $2n-2k$ edges. (Note that $2n-2k \in \{k, k+1, k+2\}$; in particular, we have $2n-2k \geq k$, that is, each part consists of at least k edges. No part is empty since we assume $n \geq 2$.) Let $D_i \subset D$, $i = 1, 2, 3$, be the union of edges from E_i. Observe that each D_i is compact and connected and $D = D_1 \cup D_2 \cup D_3$. Let us smoothly embed S^2 in $\mathbb{R}^3$ as a sphere of radius 1 and let dist denote the metric on S^2 induced by the euclidean metric in $\mathbb{R}^3$. For each $i \in \{1, 2, 3\}$ we set

$$R_i := \{x \in S^2 : \operatorname{dist}(x, D_i) = \operatorname{dist}(x, D)\}.$$

Observe that $R_1 \cup R_2 \cup R_3 = S^2$ because $D = D_1 \cup D_2 \cup D_3$. We see that for each $i \in \{1, 2, 3\}$ the set R_i is closed because D_i is compact (consider a convergent sequence of points in R_i). Also, we see that for each $i \in \{1, 2, 3\}$ the set R_i is connected. Indeed, if $p \in R_i$, then due to compactness of D_i there exists a point $q \in D_i$ such that $\operatorname{dist}(p, q) = \operatorname{dist}(p, D)$. Then the geodesic segment between p and q is in R_i by the triangle inequality. Therefore, R_i is connected because D_i is connected. Finally, we see that for any $\{i, j\} \subset \{1, 2, 3\}$ the intersection $R_i \cap R_j$ is not empty because $D_i \subset R_i$ and $D_j \subset R_j$, while $D_i \cap D_j$ is not empty.

Thus, the sets R_1, R_2, and R_3 satisfy assumptions of Lemma 2 below. Lemma 2 implies that R_1, R_2, and R_3 have a common point x. Clearly, x is not an inner point of an edge of D, so we have two possible cases:

1) x is a double point of D,

2) $x \notin D$.

Suppose x is a double point of D. Then there exists a triple $\{J_1, J_2, J_3\}$ of edges of D incident to x such that $J_i \in E_i$ for all $i \in \{1, 2, 3\}$. Without loss of generality we can and will assume that J_1 and J_3 are consecutive. Then J_1 and J_2 are neighbors, and J_2 and J_3 are neighbors. It is easily seen that we have $\rho(J_1, J_2) \geq k$ and if $\rho(J_1, J_2) = k$ then $\rho(J_2, J_3) = k+1$, and the theorem follows.

Suppose $x \in S^2 \setminus D$. Let Q be the component of $S^2 \setminus D$ containing x. Observe that the set

$$\{y \in D : \operatorname{dist}(x, y) = \operatorname{dist}(x, D)\}$$

is contained in $\partial Q \subset D$ and contains no double points of D (due to smoothness of embedding $S^2 \to \mathbb{R}^3$). Therefore, since $x \in R_1 \cap R_2 \cap R_3$, for each $i \in \{1, 2, 3\}$ the set $\partial Q \cap D_i$ contains at least one edge of D. This means that there exists a triple $\{J_1, J_2, J_3\}$ of pairwise neighboring edges of D such that we have $J_i \in E_i$ for all $i \in \{1, 2, 3\}$. It is an easy exercise to check that this triple contains a pair $\{I, J\}$ with[4]

$$\rho(I, J) \geq \lceil 2n/3 \rceil \geq 2n/3. \quad \square$$

Lemma 2. *If a triple of pairwise intersecting closed connected sets cover a simply connected space, then these three sets have a common point.*

Proof. This follows, e. g., from Theorem 5 of [Bog02] in the case $m = 1$. $\square$

7. Tangles

Our proof of assertion (iii) of Proposition 1 uses tangles. The present section contains some preliminaries on tangles.

[4]We use notation $\lceil 2n/3 \rceil$ for the smallest integer not less than $2n/3$.

Definitions. A k-string *tangle*, where $k \in \mathbb{N}$, is a pair (B, t) where B is a 3-ball and t is the union of k disjoint arcs in B with $t \cap \partial B = \partial t$. We mostly interested in the cases where $k \in \{1, 2\}$. Two tangles, (B, t) and (A, s), are *equivalent* if there is a homeomorphism of pairs from (B, t) to (A, s). A tangle (B, t) is *trivial* if B contains a properly embedded disk containing t. A tangle (B, t) is *locally knotted* if B contains a ball B' such that $(B', B' \cap t)$ is a nontrivial 1-string tangle. A 2-string tangle (B, t) is *prime* if it is neither locally knotted nor trivial. If (B, t) and (A, s) are k-string tangles and $f\colon (\partial B, \partial t) \to (\partial A, \partial s)$ is a homeomorphism, a link in S^3 can be obtained by identifying the boundaries of the tangles using f. The result, $(B, t) \cup_f (A, s)$, is referred to as a *sum* of the two tangles. If (B, t) is a 1-string tangle and (A, s) is the trivial 1-string tangle, then there is a unique (up to a homeomorphism of pairs) knot which is a sum of (B, t) and (A, s). This knot is called the *closure* of (B, t). We say that a 2-string tangle (B, t) is a *cable* tangle if there exists an embedding $f\colon I \times I \to B$ such that $f(I \times I) \cap \partial B = I \times \partial I$ and $t = f(\partial I \times I)$, where $I := [0, 1]$. (We treat the trivial 2-string tangle as a cable tangle.) Clearly, each tangle (B, t) can be embedded in $\mathbb{R}^3$ in such a way that B becomes a Euclidean ball while the endpoints ∂t lie on a great circle of this ball and t is in general position with respect to the projection onto the flat disc bounded by the great circle. The projection, with additional information of over- and undercrossings, then gives us a *tangle diagram.* Examples of tangle diagrams are given in Figs. 1, 2, and 4.

Theorem 2 ([Lick81, Theorem 1])**.** *A sum of two* 2*-string prime tangles is a prime link.*

Lemma 3. *Each nontrivial cable* 2*-string tangle is prime.*

Proof. (See [Lick81, Examples (a) and (b)].) It is enough to observe that we can, in an obvious manner, add the trivial 2-string tangle to any cable 2-string tangle so as to create the trivial knot, which proves that the initial tangle has no local knots (this follows by the Unique Factorization Theorem by Schubert [Schu49]). □

Lemma 4. *No composite knot is a sum of a nontrivial cable* 2*-string tangle with the trivial* 2*-string tangle.*

Proof. Suppose that a knot K in S^3 is presented as a sum

$$(S^3, K) = (B, t) \cup_f (A, s), \quad f\colon (\partial A, \partial s) \to (\partial B, \partial t),$$

of a nontrivial cable 2-string tangle (B, t) with a trivial 2-string tangle (A, s). Let

$$f_0\colon (\partial A, \partial s) \to (\partial B, \partial t)$$

yields an obvious 'trivializing' sum for (B, t), that is, the sum $(B, t) \cup_{f_0} (A, s)$ is the trivial knot.[5] (See left side of Fig. 3.) Let M_0 denote the double cover of the 3-sphere $B \cup_{f_0} A$ branched over the trivial knot $t \cup_{f_0} s$, and let M_1 be the double cover of the 3-sphere $B \cup_f A$ branched over the knot $t \cup_f s = K$. Then M_0 is homeomorphic to the 3-sphere, while M_0 and M_1 are related by a Dehn surgery along the solid torus covering (A, s). We observe that the solid torus $V_A \subset M_0 = S^3$ that covers (A, s) is knotted as a composite knot. Indeed,

[5] In fact, the results of [BS86, BS88] imply that there is essentially unique way to create the trivial knot as a sum of a given prime 2-string tangle and a trivial 2-string tangle. In particular, if $\phi\colon (\partial A, \partial s) \to (\partial B, \partial t)$ is a homeomorphism such that $(B, t) \cup_\phi (A, s)$ is the trivial knot then the map $f_0^{-1} \circ \phi\colon (\partial A, \partial s) \to (\partial A, \partial s)$ extends to a map $F\colon (A, s) \to (A, s)$ such that $F(s) = s$.

the definition of cable tangles imply that there is an obvious ambient isotopy of $B \cup_{f_0} A$ that moves $t \cup_{f_0} s$ and A to a position in which $t \cup_{f_0} s$ is a geometric circle and A is a closed regular neigborhood of a 'knotted diameter' of this circle. See Fig. 3.

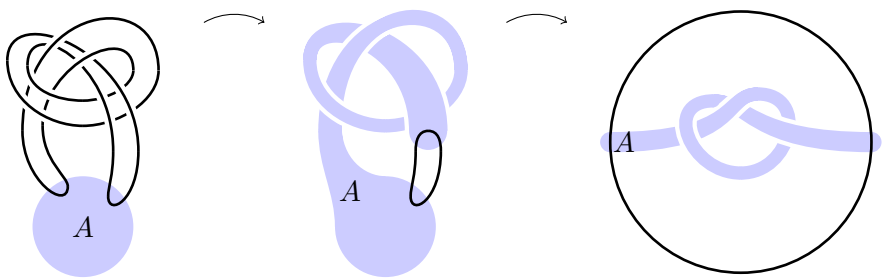


FIGURE 3. For the proof of Lemma 4

This clearly implies that V_A is a regular neigborhood of a composite knot. (This composite knot is a sum of two copies of the 1-string tangle (B, t_1), where t_1 is a component of t.) It is known that a nontrivial Dehn surgery on a composite knot in S^3 yields an irreducible (hence prime) manifold (see [Gor83, Theorem 7.1]). It is known that if the double cover of S^3 branched over a knot R is prime then R is prime (see [Wal69]; see also [KT80, Corollary 4] for the inverse implication). Consequently, K is a prime knot if nontrivial. □

Remarks. 1. Lemma 4 also follows from results of [E-M86] (see also [E-M88, Theorem 6]) or equivalently from the fact that *only integral Dehn surgeries can yield reducible manifolds* [GL87]. This way of proof uses the fact that cable knots are prime (see [Schu53, p. 250, Satz 4], [Gra91, Cor. 2]).

2. Lemma 4 is used in the proof of Proposition 4 (which in its turn is used in the proof of assertion (iii) of Proposition 1), where it covers the case of 2-bridge knots. It is known (see [Wel92]) that the percentage of 2-bridge knots amongst all of the prime knots of n or fewer crossings approaches 0 as n approaches infinity. Thus, in the proof of Theorem 1, we can discard 2-bridge knots together with Lemma 4. Nevertheless, we use Lemma 4 for the sake of completeness of Propositions 4 and 1.

Corollary 1. *No composite knot is a sum of two cable* 2*-string tangles.*

Proof. A cable 2-string tangle is either prime or trivial (Lemma 3). A sum of two prime 2-string tangles is a prime link by Theorem 2. No composite knot is a sum of a nontrivial cable 2-string tangle with the trivial 2-string tangle by Lemma 4. If a knot K is a sum of two trivial 2-string tangles, then the bridge number $\mathrm{b}(K)$ of K is at most 2. If $\mathrm{b}(K) = 1$ then K is a trivial knot. If $\mathrm{b}(K) = 2$ then K a prime knot by [Schu54]. □

8. PROOF OF ASSERTION (iii) OF PROPOSITION 1

Definition. Weak property PT. Let D be a knot diagram on the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$. We say that D has *weak property PT* (PT stands for 'tangle

primeness') if D is obtained by adding ears to a diagram of a tangle that is not locally knotted (that is, the tangle is either prime or trivial). In other words, D has weak property PT if there exists a 2-disk $d \subset S^2$ such that

– the boundary ∂d intersects D transversely in four points;

– the intersection $d \cap D$ consists of two simple non-intersecting arcs (as on the left side of Fig. 1);

– the complementary disk $\delta := S^2 \setminus \mathrm{int}(d)$ with the diagram $\delta \cap D$ represents a 2-string tangle (B, t) which is not locally knotted (that is, (B, t) is either prime or trivial).

We say that a knot has *weak property PT* if it has a minimal diagram with weak property PT.

Proposition 3. *Each minimal diagram of each $\frac{2}{3}$-regular prime knot has weak property PT. In particular, each $\frac{2}{3}$-regular prime knot has weak property PT.*

Proof. Let D_P be a minimal diagram, on the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$, of a $\frac{2}{3}$-regular prime knot P. By Lemma 1, D_P has a pair of neighboring edges I and J with $\rho(I, J) \geq \frac{2\,\mathrm{cr}(P)}{3}$. Since I and J are neighbors, there exists a disk $d \subset S^2$ such that the intersection $d \cap D_P$ consists of a subarc of I and a subarc of J, while ∂d intersects D_P transversely in four points. Let δ denote the disk $S^2 \setminus \mathrm{int}(d)$, and let (B, t) be the 2-string tangle represented by the diagram $\delta \cap D_P$. Let t_1 and t_2 be the components of t, and let K_1 and K_2 be the knots that are the closures of the 1-string tangles (B, t_1) and (B, t_2).

Claim 7. *We have* $\mathrm{cr}(K_i) \leq \frac{2}{3}\,\mathrm{cr}(P) - 1$ *for* $i \in \{1, 2\}$.

Proof. The diagram $\delta \cap D_P$ of the tangle (B, t) is formed by two curves, c_1 and c_2 say, corresponding to the components t_1 and t_2, respectively, of t. We denote by $\mathrm{cr}(c_i)$ the number of double points of c_i. Since a diagram of K_1 can be obtained from c_1 by adding a simple arc in d, it follows that we have

$$\mathrm{cr}(K_1) \;\leq\; \mathrm{cr}(c_1). \tag{3}$$

Observe that by construction we have

$$\mathrm{cr}(P) \;=\; \mathrm{cr}(D_P) \;=\; \mathrm{cr}(c_1) + \mathrm{cr}(c_2) + \mathrm{card}(c_1 \cap c_2). \tag{4}$$

By the definition of ρ (this definition is given at the beginning of Sec. 6) we have

$$\rho(I, J) = \min\{2\,\mathrm{cr}(c_1) + \mathrm{card}(c_1 \cap c_2), 2\,\mathrm{cr}(c_2) + \mathrm{card}(c_1 \cap c_2)\}. \tag{5}$$

Since $\rho(I, J) \geq \frac{2\,\mathrm{cr}(P)}{3}$, it follows from (3), (4), and (5) that

$$\mathrm{cr}(K_1) \;\leq\; \mathrm{cr}(c_1) \;\leq\; \frac{2}{3}\,\mathrm{cr}(P) - \frac{\mathrm{card}(c_1 \cap c_2)}{2}.$$

Since D_P is a minimal diagram of a prime knot and $I \neq J$, it follows that $c_1 \cap c_2 \neq \emptyset$. Assuming that c_1 intersects c_2 in a unique point (q, say) implies that q is a cutpoint[6] of D_P. However, no minimal diagram of a knot has a cutpoint. This implies that $\mathrm{card}(c_1 \cap c_2) \geq 2$ and $\mathrm{cr}(K_1) \leq \frac{2}{3}\,\mathrm{cr}(P) - 1$, as required. The case of K_2 is analogous. □

Claim 8. *The 2-string tangle (B, t) represented by the diagram $\delta \cap D_P$ is either prime or trivial.*

[6] A point x of a connected topological space X is a *cutpoint* if the set $X \setminus \{x\}$ is not connected.

Proof. Suppose on the contrary that (B,t) is neither prime nor trivial. Then (B,t) is locally knotted, that is, B contains a ball A such that the pair $(A, A\cap t)$ is a nontrivial 1-string tangle. Let t_i with $i \in \{1,2\}$ be the component of t that meets A. We denote by L the knot that is the closure of the 1-string tangle $(A, A\cap t)$. Then L is a factor of P. Since P is prime and L is nontrivial, it follows that L and P are equivalent. At the same time, L is a factor of K_i (as defined above, K_i is the closure of the 1-string tangle (B,t_i)). Since L and P are equivalent, while P is assumed to be $\frac{2}{3}$-regular, we have $\mathrm{cr}(K_i) \geq \frac{2}{3}\,\mathrm{cr}(P)$, which contradicts Claim 7. The obtained contradiction proves that (B,t) is either prime or trivial. □

Thus, all requirements from the definition of weak property PT are fulfilled. Consequently, D_P has weak property PT. Proposition 3 is proved. □

Proposition 4. *If P is a knot with weak property PT, then there exists a* <u>*prime*</u> *γ-knot P' over P with* $\mathrm{cr}(P') \leq \mathrm{cr}(P) + 17$.

Proof. By definition, P has a minimal crossing diagram D_P with weak property PT. This means that there exists a disk $d \subset S^2$ such that

– the boundary ∂d intersects D_P transversely in four points;

– the intersection $d \cap D_P$ consists of two simple non-intersecting arcs;

– the tangle diagram $\delta \cap D_P$, where $\delta := S^2 \setminus \mathrm{int}(d)$, represents either prime or trivial 2-string tangle (B,t).

Without loss of generality we can identify the pair $(d, d\cap D_P)$ with the tangle diagram in Fig. 1(a). We have the following two cases:

(α) the two arrows on the arcs in Fig. 1(a) induce the same orientation on P,

(β) the two arrows on the arcs in Fig. 1(a) induce opposite orientations on P.

In case (α), let D_α be the diagram obtained from D_P by a local move as in Fig. 1 and let P_α be the knot represented by D_α. Since the figure-eight knot is hyperbolic, an easy argument shows that P_α is a γ-knot over P. We check that P_α has all of the desired properties. First, the obtained diagram D_α of P_α has $\mathrm{cr}(P) + 16$ crossings. This means that $\mathrm{cr}(P_\alpha) \leq \mathrm{cr}(P) + 16$. Next, we prove that P_α is prime. We observe that, by construction, P_α is a sum of the cable tangle of Fig. 1(b) and the tangle (B,t), which is prime or trivial. Each nontrivial cable tangle is prime (see Lemma 3). If (B,t) is prime then P_α is prime by Theorem 2. If (B,t) is trivial then P_α is prime by Lemma 4 and assertion (i) of Proposition 1 (Lemma 4 implies that P_α is either prime or trivial if (B,t) is trivial; assertion (i) implies that P_α is a satellite knot and hence nontrivial). Thus, P_α is a prime γ-knot over P with $\mathrm{cr}(P_\alpha) \leq \mathrm{cr}(P) + 16$, as required.

In case (β), let D_β be the diagram obtained from D_P by a local move as in Fig. 4 and let P_β be the knot represented by D_β.

The local move in Fig. 4 is the composition of a type I Reidemeister move and the move shown in Fig. 1. This implies that P_β is a γ-knot over P. Obviously, D_β has $\mathrm{cr}(P) + 1 + 16$ crossings. This means that $\mathrm{cr}(P_\beta) \leq \mathrm{cr}(P) + 17$. The primeness of P_β follows by the same argument as in case (α) because P_β is a sum of a nontrivial cable tangle and the tangle (B,t). Thus, P_β is a prime γ-knot over P with $\mathrm{cr}(P_\beta) \leq \mathrm{cr}(P) + 17$, as required. □

Assertion (iii) of Proposition 1 readily follows from Proposition 4 by Proposition 3.

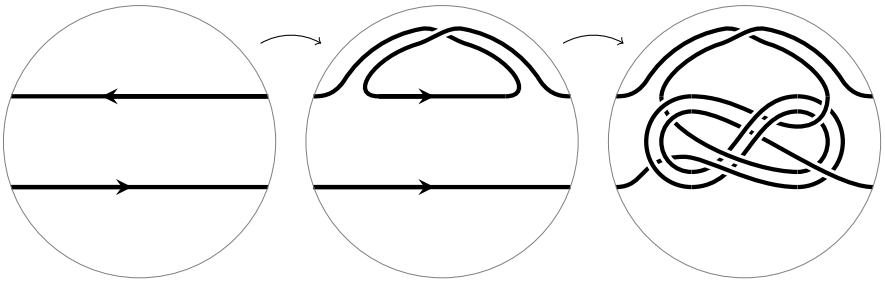


FIGURE 4. Type I Reidemeister move plus double figure-eight move

9. Addendum I: Strong property PT

In addition to weak property PT defined in Sec. 8, we introduce strong property PT.

Definition. Strong property PT. Let D be a knot diagram on the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$. We say that a tangle (B, t) is *represented by a connected subdiagram of* D if there exists a 2-disk $\delta \subset S^2$ such that the intersection $\delta \cap D$ is connected and the pair $(\delta, \delta \cap D)$, with information of under- and overcrossings inherited from D, is a diagram of (B, t). We say that D has *strong property PT* if every 2-string tangle represented by a connected subdiagram of D is either prime or trivial. We say that a knot has *strong property PT* if all of its minimal diagrams have strong property PT.

Proposition 5. *1. Each minimal diagram of each* 1*-regular prime knot has strong property PT. In particular, each* 1*-regular prime knot has strong property PT.*

2. Each minimal diagram with strong property PT has weak property PT. In particular, each knot with strong property PT has weak property PT.

Proof. 1. Assume to the contrary that a non-prime non-trivial 2-string tangle (B, t) is represented by a connected subdiagram $\delta \cap D_P$ in a minimal diagram D_P of a 1-regular prime knot P. This implies in particular that (B, t) is locally knotted, that is, B contains a ball B' such that $(B', B' \cap t)$ is a nontrivial 1-string tangle. Let K_1 denote the knot obtained by the closure of $(B', B' \cap t)$. Then K_1 is a factor of P, which is a prime knot, so that we have $K_1 = P$. (This follows by the Unique Factorization Theorem by Schubert [Schu49].) On the other hand, the knot $K_1 = P$ is a factor of the knot K_2 obtained as the closure of (B, t_1), where t_1 is the component of t that meets B'. Observe that we have $\mathrm{cr}(K_2) \leq \mathrm{cr}(P) - 1$ because, since the diagram $\delta \cap D_P$ representing (B, t) is connected, the projection of t_1 has at least one crossing with the projection of the second component of t. The inequality $\mathrm{cr}(K_2) \leq \mathrm{cr}(P) - 1$ implies that $K_2 \neq P$. Therefore, K_2 is a composite knot, P is a factor of K_2, and $\mathrm{cr}(K_2) \leq \mathrm{cr}(P) - 1$. This contradicts the assumption that P is a 1-regular knot.

2. Let D be a minimal diagram with strong property PT. If D is a circle with no double points then D has weak property PT (obvious). We assume that D has double points. We take a double point x of D and consider a disk $d \subset S^2$ in a small neighborhood of x such that the intersection $d \cap D$ consists of two non-intersecting arcs (as on the left side of Fig. 1). Since D is a minimal diagram, x is not a cutpoint of D. This easily implies that the intersection $\delta \cap D$, where

$\delta := S^2 \setminus \mathrm{int}(d)$, is connected. Since D has strong property PT, it follows that the 2-string tangle represented by the connected subdiagram $\delta \cap D$ is either prime or trivial. This means that D has weak property PT. $\square$

Propositions 5 and 3 give the following dependence for properties of prime knots.

$$\begin{array}{ccc} \text{1-regularity} & \Longrightarrow & \tfrac{2}{3}\text{-regularity} \\ \Downarrow & & \Downarrow \\ \text{strong property PT} & \Longrightarrow & \text{weak property PT} \end{array}$$

This implications can be treated in terms of conjectures. We consider the following conjectures.

Conjecture 7. *Each prime knot has strong property PT.*

Conjecture 8. *Each prime knot has weak property PT.*

Conjecture 9. *There exist $\varepsilon > 0$ and $N > 0$ such that, for all $n > N$, the percentage of knots with weak property PT amongst all of the hyperbolic knots of n or fewer crossings is at least ε.*

We have the following implications.

$$\begin{array}{ccccccccc} \text{Conj. 2} & & & & & & & & \\ & \searrow & & & & & & & \\ & & \text{Conj. 4} & \Longrightarrow & \text{Conj. 5} & \Longrightarrow & \text{Conj. 6} & & \\ & \nearrow & & \searrow & & \searrow & & \searrow & \\ \text{Conj. 3} & & & & \text{Conj. 7} & \Longrightarrow & \text{Conj. 8} & \Longrightarrow & \text{Conj. 9} \end{array}$$

The implication Conj. 4 $\Rightarrow$ Conj. 7 follows from assertion 1 of Proposition 5. The implication Conj. 7 $\Rightarrow$ Conj. 8 follows from assertion 2 of Proposition 5. The implications Conj. 5 $\Rightarrow$ Conj. 8 and Conj. 6 $\Rightarrow$ Conj. 9 follow from Proposition 3. The implication Conj. 8 $\Rightarrow$ Conj. 9 is obvious.

Theorem 1 can be strengthened in the following way.

Theorem 3. *Conjecture 1 contradicts (each of) Conjectures 2–9.*

Proof. Since each of Conjectures 2–8 implies Conjecture 9 (see the system of implications before Theorem 3), it suffices to show that Conjecture 1 contradicts Conjecture 9. In order to prove this, we repeat verbatim the reduction of Theorem 1 to Proposition 1 up to replacing $\frac{2}{3}$-regularity with weak property PT and assertion (iii) of Proposition 1 with Proposition 4. $\square$

10. Addendum II: Non-$\frac{1}{4}$-regular knots

The main theorem of the present paper states that Conjecture 1 concerning predominance of hyperbolic knots contradicts the conjecture on additivity of the crossing number (of knots under connected sum) as well as several weaker conjectures. In this section, we show that Conjecture 1 also contradicts an assumption that the conjecture on additivity has many strong counterexamples.

We say that a knot P is *non-λ-regular*, $\lambda \in \mathbb{R}$, if there exists a knot K such that P is a factor of K while $\mathrm{cr}(K) < \lambda \cdot \mathrm{cr}(P)$. In this section, we prove the following theorem.

Theorem 4. *If there exist $\varepsilon_0 > 0$ and $N_0 > 0$ such that, for all $n > N_0$, the number of non-$\frac{1}{4}$-regular knots of n or fewer crossings is at least $\varepsilon_0 H_n$, where H_n is the number of hyperbolic knots of n or fewer crossings, then Conjecture 1 does not hold.*

Proof. Suppose that the assumption of the theorem holds true, denote by $\mathscr{M}_{\frac{1}{4}}$ the set of all non-$\frac{1}{4}$-regular knots, and let f be a map with domain $\mathscr{M}_{\frac{1}{4}}$ sending $K \in \mathscr{M}_{\frac{1}{4}}$ to a composite knot $f(K)$ with factor K such that

$$\mathrm{cr}(f(K)) \;<\; \frac{1}{4}\,\mathrm{cr}(K). \tag{6}$$

Then the result of Lackenby [La09] stating that for any knots $K_1, \ldots, K_n$ in the 3-sphere we have

$$\frac{\mathrm{cr}(K_1) + \cdots + \mathrm{cr}(K_n)}{152} \;\leq\; \mathrm{cr}(K_1 \sharp \ldots \sharp K_n) \tag{7}$$

implies that for each knot L in the codomain $f(\mathscr{M}_{\frac{1}{4}})$ we have

$$\mathrm{card}(f^{-1}(L)) \;<\; 152/4 = 38. \tag{8}$$

Indeed, let K be a knot with $f(K) = L$ having the smallest crossing number among the elements of $f^{-1}(L)$. Then (7) implies that

$$\frac{\mathrm{card}(f^{-1}(L))\,\mathrm{cr}(K)}{152} \;\leq\; \mathrm{cr}(L). \tag{9}$$

Obviously, (6) and (9) imply (8).

Since all of the knots in $f(\mathscr{M}_{\frac{1}{4}})$ are composite, it follows by (6) and (8) that for all $n \in \mathbb{N}$ we have

$$\frac{\mathrm{card}\{K \in \mathscr{M}_{\frac{1}{4}} : \mathrm{cr}(K) \leq 4n\}}{38} \;<\; \mathrm{card}\{L \in f(\mathscr{M}_{\frac{1}{4}}) : \mathrm{cr}(L) \leq n\} \;\leq\; C_n, \tag{10}$$

where C_n is the number of composite knots of n or fewer crossings. At the other hand, by the assumption of the theorem, for all $m > N_0$ we have

$$H_m \varepsilon_0 \;\leq\; \mathrm{card}\{K \in \mathscr{M}_{\frac{1}{4}} : \mathrm{cr}(K) \leq m\}. \tag{11}$$

Then (10) and (11) imply that for all $n > N_0/4$ we have

$$\frac{\varepsilon_0}{38} H_{4n} \;<\; C_n. \tag{12}$$

Now, we observe that each knot K in the 3-sphere obviously has a two-strand cable knot J_K with $\mathrm{cr}(J_K) \leq 4\,\mathrm{cr}(K) + 1$. Since a cable knot over a nontrivial knot is a prime satellite knot (see [Schu53, p. 250, Satz 4], [Gra91, Cor. 2]), while cable knots over distinct knots are distinct (Lemma 5 below), it follows by (12) that for all $n > N_0/4$ we have

$$\frac{\varepsilon_0}{38} H_{4n} \;<\; C_n \;<\; S_{4n+1},$$

where S_m denotes the number of all prime satellite knots of m or fewer crossings. Consequently, since the sequences $(H_i)_{i\in\mathbb{N}}$ and $(S_i)_{i\in\mathbb{N}}$ are monotonically increasing, for all $m > N_0$ we have

$$\frac{\varepsilon_0}{38} H_m \;<\; S_{m+4}.$$

As is shown in Section 3 (see deduction of Theorem 1 from Proposition 1), conditions of this kind contradict Conjecture 1. □

Lemma 5. *Cable knots over distinct knots are distinct.*

Proof. By Corollary 2 of [FW78], the group of a cable knot $J(p, q; K)$ determines the numbers $|p|$ and $|q|$ and the topological type of K's complement. By the Gordon–Luecke theorem [GL89], the knot complement determines the knot. □

11. Addendum III: Weak property PT and unknotting numbers

This section deals with a relation between weak property PT and the unknotting number of knots. The unknotting number of a knot K is denoted by $u(K)$.

Definitions. Let us say that a knot P is *weakly U-regular* if we have $u(P) \leq u(K)$ whenever P is a factor of a knot K. We say that a knot P is *strictly U-regular* if we have $u(P) < u(K)$ whenever P is a factor of a knot $K \neq P$. We say that a knot P has *weak BJ-property* if by altering one of the crossings in a minimal diagram of P we obtain a knot $J \neq P$ with $u(J) \leq u(P)$. We say that a knot P has *strict BJ-property* if by altering one of the crossings in a minimal diagram of P we obtain a knot J with $u(J) < u(P)$.

Remarks. 1. The conjecture that all knots are strictly U-regular is weaker than the old conjecture on additivity of the unknotting number of knots under connected sum (see, e. g., [Ad94b, p. 61], [Kir97, Problem 1.69]). At the moment, no counterexample seems to be known to the latter conjecture. Thus, no examples of non-U-regular knots are known up to now. The theorem of Scharlemann [Scha85] saying that unknotting number one knots are prime (together with the Unique Factorization Theorem by Schubert [Schu49]) implies that all knots with unknotting number one are strictly U-regular, while all knots with unknotting number two are weakly U-regular.

2. The so-called Bernhard–Jablan conjecture (see [Be94], [Ja98], and [JS07]) is equivalent to the conjecture that all knots have strict BJ-property. Kohn's conjecture [Koh91, Conjecture 12] (which can be viewed as a particular case of the Bernhard–Jablan conjecture) is equivalent to the conjecture that all knots with unknotting number one have strict BJ-property. The set of knots with strict BJ-property contains the set of knots satisfying the Bernhard–Jablan conjecture. At the moment, no counterexample seems to be known to the Bernhard–Jablan conjecture. Available results concerning unknotting number shows that many small knots and some specific classes of knots satisfy the Bernhard–Jablan conjecture, hence have strict BJ-property. For example, results of [KrM93] and [Mur91] imply that all torus knots have strict BJ-property. Results of McCoy [McC13] imply that alternating knots with unknotting number one have strict BJ-property.

Proposition 6. *1. Each <u>weakly</u> U-regular prime knot with <u>strict</u> BJ-property has weak property PT.*

2. Each <u>strictly</u> U-regular prime knot with <u>weak</u> BJ-property has weak property PT.

Proof. If P is a weakly [resp., strictly] U-regular prime knot with strict [resp., weak] BJ-property, then there exists a minimal diagram D_P of P (on the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$) with a crossing X_1 such that the change of the crossing

yields a diagram of a knot J with $u(J) = u(P) - 1$ [resp., a knot $J \neq P$ with $u(J) \leq u(P)$]. Let d be a disk in S^2 containing x_1 such that the intersection $d \cap D_P$ is homeomorphic to $\times$ while ∂d intersects D_P transversally in four points. Let δ denote the disk $S^2 \setminus \mathrm{int}(d)$, and let (B, t) be the 2-string tangle represented by the diagram $\delta \cap D_P$.

We show that (B, t) has no local knots. Suppose on the contrary that (B, t) is locally knotted, that is, B contains a ball A such that the pair $(A, A \cap t)$ is a nontrivial 1-string tangle. We denote by L the knot that is the closure of the 1-string tangle $(A, A \cap t)$. Then L is a factor of P. Since P is prime and L is nontrivial, it follows that L and P are equivalent. At the same time, $L(= P)$ is a factor of J. Then we have $u(P) \leq u(J)$ because P is weakly U-regular [resp., $u(P) < u(J)$ because P is strictly U-regular while $J \neq P$]. However, $u(J) = u(P) - 1$ [resp., $u(J) \leq u(P)$]. The obtained contradiction proves that (B, t) has no local knots.

Now, we take a subdisk d' in d such that the intersection $d' \cap D_P$ consists of two subarcs on two distinct legs of $\times = d \cap D_P$ (while $\partial d'$ intersects D_P transversely in four points):

Let δ' denote the disk $S^2 \setminus \mathrm{int}(d')$. Obviously, the diagram $\delta' \cap D_P$ represents the same 2-string tangle (B, t), which has no local knots. Thus, the requirements from the definition of weak property PT are fulfilled. Consequently, P has weak property PT. □

Corollary 2. *If there exist $\varepsilon > 0$ and $N > 0$ such that, for all $n > N$, the percentage of* <u>*weakly*</u> *U-regular knots with* <u>*strict*</u> *BJ-property amongst all of the hyperbolic knots of n or fewer crossings is at least ε, then Conjecture 1 does not hold.*

Proof. By Proposition 6, the assumption of the corollary implies Conjecture 9 (which concerns the set of knots having weak property PT). By Theorem 3, Conjecture 9 contradicts Conjecture 1. □

Corollary 3. *If there exist $\varepsilon > 0$ and $N > 0$ such that, for all $n > N$, the percentage of* <u>*strictly*</u> *U-regular knots with* <u>*weak*</u> *BJ-property amongst all of the hyperbolic knots of n or fewer crossings is at least ε, then Conjecture 1 does not hold.*

Proof. See the proof of Corollary 2. □

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