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Some observations on a function vanishing at the non-trivial zeroes of Riemann's zeta function

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Abstract. The paper present a naturally arising function vanishing at the non-trivial zeroes of Riemann's zeta function and having one more zero lying on the critical line.

Key words: Riemann's zeta function, functional equation, non-trivial zeroes.

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1 The first definition of the function from the title

There are many ways to define a function that would vanish at the non-trivial zeroes of Riemann's zeta function. The author was interested to consider *natural* functions vanishing at zeta zeroes, and to examine their properties, in particular, the other zeroes of such functions. One particular example is considered below.

Riemann's zeta function satisfies the functional equation

$$\xi(s) = \xi(1-s) \tag{1}$$

where

$$\begin{aligned} \xi(s) &= g(s)\zeta(s), && (2) \\ g(s) &= \pi^{-\frac{s}{2}}(s-1)\Gamma(\tfrac{s}{2}+1). && (3) \end{aligned}$$

According to Hamburger theorem [2], the fuctional equation (together with some other mild restrictions) uniquely determines the zeta function.

The gamma function (entering in (3)) satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s) \tag{4}$$

According to Bohr–Mollerup theorem [1], this equation (again together with some other mild restrictions) uniquely determines the gamma function.

Combining copies of the two functional equations, (1) and (4), we can obtain functional equations for the zeta function not containing the gamma function. In particular, we can eliminate $\Gamma\left(\frac{s}{2}+2\right)$, $\Gamma\left(\frac{s}{2}+1\right)$, $\Gamma\left(-\frac{s}{2}+\frac{3}{2}\right)$, and $\Gamma\left(-\frac{s}{2}+\frac{1}{2}\right)$ from the system of 4 equations

$$\begin{aligned} \xi(s) &= \xi(1-s), && (5) \\ \xi(s+2) &= \xi(-1-s), && (6) \\ \Gamma\left(\tfrac{s}{2}+2\right) &= \left(\tfrac{s}{2}+1\right)\Gamma\left(\tfrac{s}{2}+1\right), && (7) \\ \Gamma\left(-\tfrac{s}{2}+\tfrac{3}{2}\right) &= \left(-\tfrac{s}{2}+\tfrac{1}{2}\right)\Gamma\left(-\tfrac{s}{2}+\tfrac{1}{2}\right) && (8) \end{aligned}$$

and get a first equation of the desired kind:

$$4\pi^2\zeta(-1-s)\zeta(s) + s(s+1)\zeta(1-s)\zeta(s+2) = 0. \tag{9}$$

Differentiation gives yet another identity:

$$\begin{aligned} &-s(s+1)\zeta(s+2)\zeta'(1-s) - 4\pi^2\zeta(s)\zeta'(-s-1) + 4\pi^2\zeta(-s-1)\zeta'(s)+ \\ &\quad + s(s+1)\zeta(1-s)\zeta'(s+2) + (2s+1)\zeta(1-s)\zeta(s+2) = 0. \quad (10) \end{aligned}$$

It would be interesting to see *to what extent functional equations* (9) *and* (10) *determine the zeta function?* In other words, *could the Humburger theorem be strengthen via replacing the functional equation* (1) *by its consequence* (9), *or even by yet weaker consequence* (10)*?*

The sum from the left-hand side of (10) vanishes identically. If s is a zeta zero, then certain individual summands vanish, namely, those containing either $\zeta(s)$ or $\zeta(1-s)$. We define function $\zeta_2(s)$ by dropping them:

$$\zeta_2(s) = 4\pi^2\zeta(-1-s)\zeta'(s) - s(1+s)\zeta(2+s)\zeta'(1-s). \tag{11}$$

Clearly, this function vanishes at each non-trivial zero of the zeta function.

2 Other zeroes and definitions of the function

Figures 1–2 depict the curves where the real part of $\zeta_2(s)$ vanishes (in blue), and where the imaginary part does (in yellow). We see that, besides the expected non-trivial zeta zeroes $\rho_k = \frac{1}{2} + \mathrm{i}\gamma_k$, function $\zeta_2(s)$ vanishes near $\rho_k - 2$. It is easy to see that indeed these points are "additional" zeroes of $\zeta_2(s)$:

$$\zeta_2(\rho_k - 2) = 4\pi^2\zeta\left(1-\rho_k\right)\zeta'(\rho_k - 2) - (\rho_k - 2)(\rho_k - 1)\zeta\left(\rho_k\right)\zeta'(3-\rho_k). \tag{12}$$

Thus vanishing of $\zeta_2(\rho_k - 2)$ follows directly from the definition (11) and vanishing of $\zeta_2(\rho_k)$ and $\zeta_2(1-\rho_k)$ (no information about the derivatives is required).

Figure 1 also shows that $\zeta_2(s)$ has yet another (different from ρ_k and $\rho_k - 2$) non-real zero $\rho_0 = \beta_0 + \mathrm{i}\gamma_0$, where

$$\gamma_0 = 6.28983598883690277966509010... \tag{13}$$

As a miracle, this zero do lay on the critical line. To see that $\beta_0 = \frac{1}{2}$, we apply the functional equation again in order to replace $\zeta(-1-s)$ and $\zeta'(1-s)$ in (11) by the values at the mirror points $s+2$ and s. In this way we get yet another definition of the same function:

$$\zeta_2(s) = h(s)\xi(s)\xi(s+2) \tag{14}$$

where

$$h(s) = \frac{\pi^{\frac{3}{2}}(2 - 2(s-1)s\log(\pi) + (s-1)s(\psi\left(\frac{3}{2} - \frac{s}{2}\right) + \psi\left(1 + \frac{s}{2}\right)))}{2(s-1)s\Gamma\left(\frac{3}{2} - \frac{s}{2}\right)\Gamma\left(2 + \frac{s}{2}\right)}, \tag{15}$$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{16}$$

In particular,

$$h\left(\tfrac{1}{2}+\mathrm{i}t\right)=\frac{\pi^{\frac{3}{2}}\left((4t^2+1)\left(2\log(\pi)+\psi\left(\frac{5}{4}+\frac{\mathrm{i}t}{2}\right)+\psi\left(\frac{5}{4}-\frac{\mathrm{i}t}{2}\right)\right)+8\right)}{2(4t^2+1)\Gamma\left(\frac{5}{4}-\frac{\mathrm{i}t}{2}\right)\Gamma\left(\frac{9}{4}+\frac{\mathrm{i}t}{2}\right)}. \tag{17}$$

On the critical line the numerator in (17) assumes real values and changes sign between $t=6$ and $t=7$.

The new definition of $\zeta_2(s)$, given by (14), suggests that the extra zero, $\frac{1}{2}+\mathrm{i}\gamma_0$, has nothing to do with the zeta function – it is just a zero of $h(s)$. On the other hand, definitions (12) and (14) do not indicate the extraordinary character of one of the non-real zeroes of the function $\zeta_2(s)$. So here we have an intruiging question: *can zero $\frac{1}{2}+\mathrm{i}\gamma_0$ (definable just via the gamma function) be of the same nature as all other non-trivial zeroes of the zeta function?* This question can be stated differently: *could all non-trivial zeroes of the zeta function be defined as zeroes of certain function similar to $h(s)$?*

Representation (14) shows that the real zeroes of $\zeta_2(s)$ are due to the real zeroes of $h(s)$.

3 Further observations

Figures 1–2 allow us to state a number of conjectures. Of course, it covers only a small part of the complex plane, and some of the guesses might be wrong.

The set

$$\{s : \mathrm{Re}(\zeta_2(s)) = 0\} \tag{18}$$

splits into connected components (blue curves on Figures 1–2). Each component contains exactly one zero of $\zeta_2(s)$. According to the position of this zero the components can be classified into 4 categories:

- *trivial*, if the zero is real;
- *non-trivial*, if the zero is a zero of the zeta function;
- *mirror*, if the zero is of the form $\rho_k - 2$;
- *extra*, if the zero is equal either to ρ_0 or to its conjugate;

The two components containing zeroes ρ_k and $\rho_k - 2$ are almost mirror images of one another with respect to the axis of symmetry $\mathrm{Re}(s) = -\frac{1}{2}$. Each of the two extra component are themselves almost symmetrical with respect to this line.

Similar to (18), the set

$$\{s : \mathrm{Im}(\zeta_2(s)) = 0\} \tag{19}$$

also splits into connected components (yellow curves on Figures 1–2). Now all real zeroes belong to the same *trivial* component. Each other component again contains just one zero of $\zeta_2(s)$ and respectively can be classified as *non-trivial*, *mirror*, or *extra*. The imaginary part of $\zeta_2(s)$ tends to vanish around line $\mathrm{Re}(s) = -\frac{1}{2}$, but not on the line itself. This line no longer is an axis of symmetry. Nevertheless, if we remove its neighbourhood, say, the strip $1 < \mathrm{Re}(\zeta_2(s)) < 0$, then each remaining connected part of a non-trivial component will be almost symmetrical to a remaining connected part of certain mirror component, not necessary corresponding to the mirror zero.

Let λ be the largest number such that the strip

$$-\tfrac{1}{2} - \lambda < \mathrm{Re}(s) < -\tfrac{1}{2} + \lambda \tag{20}$$

intersects neither with non-trivial nor with mirror components of the set (18). It seems that $\lambda > \frac{1}{2}$ and thus the strip (20) overlaps with the critical strip.

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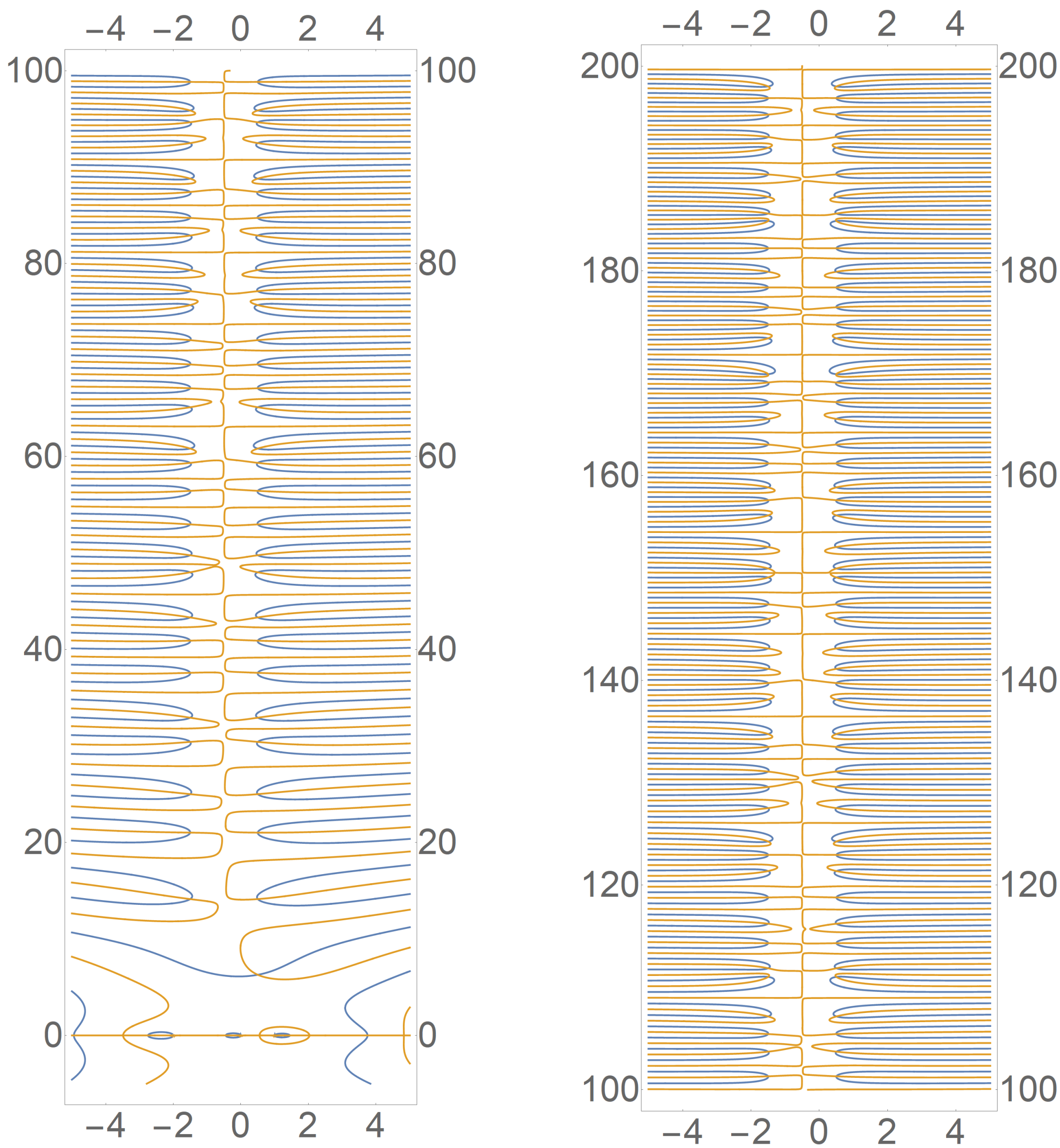

Figure 1: Real and imaginary parts of $\zeta_2(s)$ vanish on blue and yellow curves respectively

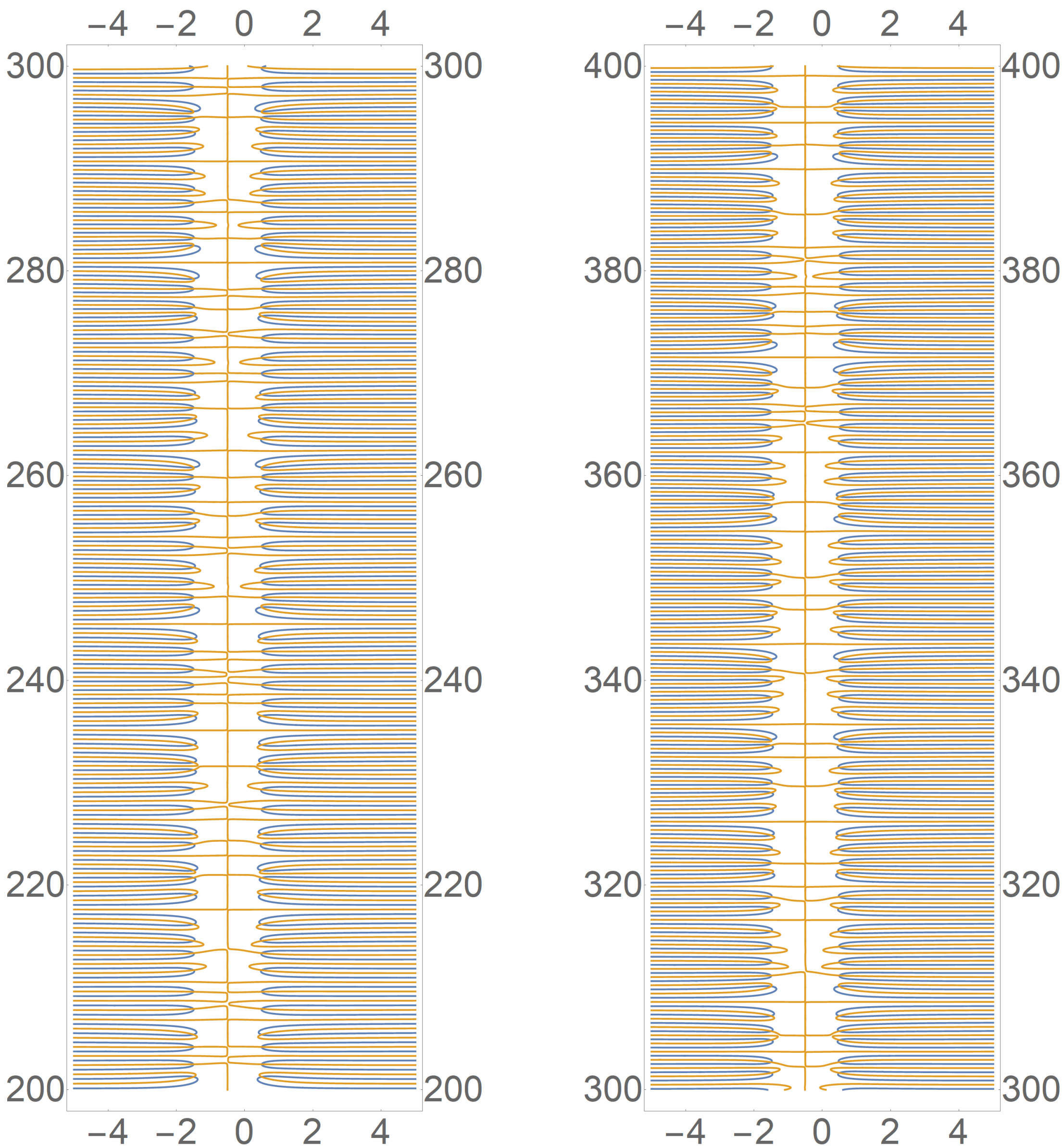

Figure 2: Real and imaginary parts of $\zeta_2(s)$ vanish on blue and yellow curves respectively