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Point distribution in compact metric spaces, III. Two-point homogeneous spaces

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We continue the investigation of point distributions in compact metric spaces started in the papers [28,29]. Our concern is with discrepancies of such distributions in metric balls and sums of pairwise distances between points of distributions. In the present paper we consider compact connected two-point homogeneous spaces (Riemannian symmetric spaces of rank one). All such spaces are known, they are spheres in the Euclidean spaces, the real, complex and quaternionic projective spaces and the octonionic projective plane.

Using the geometric features of two-point spaces, we show that Stolarsky's invariance principle, well-known for the Euclidean spheres, can be extended to all projective spaces and the octonionic projective plane. Relying on the theory of spherical functions, we obtain the best possible bounds for quadratic discrepancies and sums of distances for point distributions in the two-point homogeneous spaces. Applications to t-designs and Lévy-Schoenberg kernels in such spaces are also considered in the paper.

Key words: geometry of distances, uniform distributions, t-designs , two-point homogeneous spaces

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A. Main results

1. Discrepancies and metrics

Let $\mathcal{M}$ be a compact connected metric space with a fixed metric θ and a finite Borel measure μ, normalized by

$$\operatorname{diam}(\mathcal{M},\theta)=\pi, \quad \mu(\mathcal{M})=1, \tag{1.1}$$

where

$$\operatorname{diam}(\mathcal{E},\rho)=\sup\{\rho(x_1,x_2):x_1,x_2\in\mathcal{E}\} \tag{1.2}$$

denotes the diameter of a subset $\mathcal{E}\subseteq\mathcal{M}$ with respect to a metric ρ.

We write $B_r(y)=\{x:\theta(x,y)<r\}$ for the ball of radius $r\in[0,\pi]$ centered at $y\in\mathcal{M}$ and of volume $v_r(y)=\mu(B_r(y))$. Since the space $\mathcal{M}$ is connected, we have $\mathcal{R}=[0,\pi]$, where $\mathcal{R}=\{r=\rho(x_1,x_2):x_1,x_2\in\mathcal{M}\}$ is the set of all possible radii.

Let $\mathcal{D}_N \subset \mathcal{M}$ be a finite subset consisting of N points (not necessary different). The *local discrepancy* of the subset $\mathcal{D}_N$ in the ball $B_r(y)$ is defined by

$$\begin{aligned}\Lambda[B_r(y), \mathcal{D}_N] &= \#\{B_r(y) \cap \mathcal{D}_N\} - N v_r(y) \\ &= \sum_{x \in \mathcal{D}_N} \Lambda(B_r(y), x),\end{aligned} \tag{1.3}$$

where

$$\Lambda(B_r(y), x) = \chi(B_r(y), x) - v_r(y), \tag{1.4}$$

and $\chi(\mathcal{E}, x)$ denotes the characteristic function of s subset $\mathcal{E} \subset \mathcal{M}$.

The *quadratic discrepancies* are defined by

$$\lambda_r[\mathcal{D}_N] = \int_{\mathcal{M}} \Lambda[B_r(y), \mathcal{D}_N]^2 \, d\mu(y) = \sum_{x_1, x_2 \in \mathcal{D}_N} \lambda_r(x_1, x_2), \tag{1.5}$$

where

$$\lambda_r(x_1, x_2) = \int_{\mathcal{M}} \Lambda(B_r(y), x_1) \Lambda(B_r(y), x_2) \, d\mu(y), \tag{1.6}$$

and

$$\lambda[\eta, \mathcal{D}_N] = \int_0^\pi \lambda_r[\mathcal{D}_N] \eta(r) \, dr = \sum_{x_1, x_2 \in \mathcal{D}_N} \lambda(\eta, x_1, x_2), \tag{1.7}$$

where

$$\lambda(\eta, x_1, x_2) = \int_0^\pi \lambda_r(x_1, x_2) \eta(r) \, dr, \tag{1.8}$$

where $\eta(r)$, $r \in [0, \pi]$, is a non-negative weight function. The quantities $\lambda_r[\mathcal{D}_N]^{1/2}$ and $\lambda[\eta, \mathcal{D}_N]^{1/2}$ are known as L_2-discrepancies. In the present paper it is more convenient to deal with the quadratic discrepancies (1.5) and (1.7).

We introduce the following extremal characteristic

$$\lambda_N(\eta) = \inf_{\mathcal{D}_N} \lambda[\eta, \mathcal{D}_N], \tag{1.9}$$

where the infimum is taken over all N-point subsets $\mathcal{D}_N \subset \mathcal{M}$.

For any metric ρ on $\mathcal{M}$ we put

$$\rho[\mathcal{D}_N] = \sum_{x_1, x_2 \in \mathcal{D}_N} \rho(x_1, x_2), \tag{1.10}$$

and introduce yet another extremal characteristic

$$\rho_N = \sup_{\mathcal{D}_N} \rho[\mathcal{D}_N], \tag{1.11}$$

where the supremum is taken over all N-point subsets $\mathcal{D}_N \subset \mathcal{M}$.

We write $\langle \rho \rangle$ for the average value of a metric ρ,

$$\langle \rho \rangle = \iint\limits_{\mathcal{M} \times \mathcal{M}} \rho(y_1, y_2)\, d\mu(y_1)\, d\mu(y_2). \tag{1.12}$$

Introduce the folloving *symmetric difference metrics* on the space $\mathcal{M}$

$$\theta^{\Delta}(\eta, y_1, y_2) = \int\limits_0^{\pi} \theta_r^{\Delta}(y_1, y_2) \eta(r)\, dr, \tag{1.13}$$

where

$$\begin{aligned} \theta_r^{\Delta}(y_1, y_2) &= \frac{1}{2} \mu(B_r(y_1) \Delta B_r(y_2)) \\ &= \frac{1}{2} \Big(v_r(y_1) + v_r(y_2) - 2\mu(B_r(y_1) \cap B_r(y_2)) \Big). \end{aligned} \tag{1.14}$$

Here

$$B_r(y_1) \Delta B_r(y_2) = B_r(y_1) \cup B_r(y_2) \setminus B_2(y_1) \cap B_r(y_2) \tag{1.15}$$

is the symmetric difference of the balls $B_r(y_1)$ and $B_r(y_2)$.

The symmetry of the metric θ implies the following useful relation

$$\chi(B_r(y), x) = \chi(B_r(x), y) = \chi(r - \theta(x, y)) = \chi_r(\theta(x, y)) \tag{1.16}$$

where $\chi(z)$, $z \in \mathbb{R}$ is the characteristic function of the half-axis $[0, \infty)$, and $\chi_r(\cdot)$ is the characteristic function of the interval $[0, r]$, $0 \le r \le \pi$. Using

(1.14), (1.15), (1.16), we can write

$$\theta_r^\Delta(y_1,y_2) = \frac{1}{2}\int\limits_{\mathcal{M}} \chi(B_r(y_1)\Delta B_r(y_2))\,d\mu(y)$$
$$= \frac{1}{2}\int\limits_{\mathcal{M}} [\chi(B_r(y_1),y) + \chi(B_r(y_2),y) - 2\chi(B_r(y_1),y)\chi(B_r(y_2),y)]\,d\mu(y)$$
$$= \frac{1}{2}\int\limits_{\mathcal{M}} |\chi(B_r(y_1),y) - \chi(B_r(y_2),y)|\,d\mu(y) \tag{1.17}$$

For the average values (1.12) of metrics (1.14) and (1.13) we obtain

$$\langle\theta^\Delta(\eta)\rangle = \int\limits_0^\pi \langle\theta_r^\Delta\rangle\eta^{(r)}\,dr, \tag{1.18}$$

$$\langle\theta_r^\Delta\rangle = \frac{1}{2}\iint\limits_{\mathcal{M}\times\mathcal{M}} \theta_r^\Delta(y_1,y_2)\,d\mu(y_1)\,d\mu(y_2) = \int\limits_{\mathcal{M}} (v_r(y) - v_r(y)^2)\,d\mu(y) \tag{1.19}$$

The symmetric difference of any two subsets coincides with the symmetric difference of their complements. Hence

$$\theta_r^\Delta(y_1,y_2) = \frac{1}{2}\mu(B_r'(y_1)\Delta B_r'(y_2))$$
$$= \frac{1}{2}\Big(v_r'(y_1) + v_r'(y_2) - 2\mu(B_r'(y_1)\cap B_r'(y_2))\Big), \tag{1.20}$$

where $B_r'(y) = \mathcal{M}\setminus B_r(y)$ is the complement of the ball $B_r(y)$, and

$$v_r'(y) = \mu(B_r'(y)) = 1 - v_r(y). \tag{1.21}$$

Now the relation (1.19) takes the form

$$\langle\theta_r^\Delta\rangle = \int\limits_{\mathcal{M}} v_r(y)v_r'(y)\,d\mu(y) \tag{1.22}$$

In (1.17) the balls $B_r(y)$ can be also replaced by their complements $B_r'(y)$.

The study of the characteristics (1.9) and (1.11) falls within the subjects of the discrepancy theory and geometry of distances. An extensive literature

is devoted to such studies of point distributions on spheres in the Euclidean space, see, for examples, [1,2,5,6,12,27]. It was shown in our recent paper [28] that nontrivial results on the quantities (1.9) and (1.11) can be obtained for very general metric spaces. Some of these results, needed for the present work, are given below in Theorems 1.1 and 1.2.

A metric space $\mathcal{M}$ is called *distance-invariant*, if the volume of any ball $v_r = v_r(y)$ is independent of $y \in \mathcal{M}$, see [23]. For such spaces the formulas for the discrepancies (1.5) and the symmetric difference metrics (1.14) can be essentially simplified. Substituting (1.14) into (1.6), we obtain

$$\begin{aligned}\lambda_r(y_1, y_2) &= \int\limits_{\mathcal{M}} \chi(B_r(y_1), y)\chi(B_r(y_2), y)\, d\mu(y) - v_r^2 \\ &= \mu(B_r(y_1) \cap B_r(y_2) - v_r^2, \end{aligned} \tag{1.23}$$

and correspondingly,

$$\lambda_r[\mathcal{D}_N] = \sum_{y_1, y_2 \in \mathcal{D}_N} \mu(B_r(y_1) \cap B_r(y_2)) - v_r^2 N^2. \tag{1.24}$$

Similarly, the relations (1.14), (1.20) and (1.19), (1.22) take the form

$$\begin{aligned}\theta_r^{\Delta}(y_1, y_2) &= v_r - \int\limits_{\mathcal{M}} \chi(B_r(y_1), y)\chi(B_r(y_2), y)\, d\mu(y) \\ &= v_r - \mu(B_r(y_1) \cap B_r(y_2)), \\ &= v_r' - \mu(B_r'(y_1) \cap B_r'(y_2)) \end{aligned} \tag{1.25}$$

and

$$\langle \theta_r^{\Delta} \rangle = v_r - v_r^2 = v_r v_r', \tag{1.26}$$

and correspondingly,

$$\theta_r^{\Delta}[\mathcal{D}_N] = v_r N^2 - \sum_{y_1, y_2 \in \mathcal{D}_N} \mu(B_r(y_1) \cap B_r(y_2)). \tag{1.27}$$

Integrating these relations with $\eta(r)$, $r \in [0, \pi]$, one can obtain the corresponding formulas for the quantities (1.7), (1.8), (1.13), (1.18).

The typical examples of distance-invariant spaces are homogeneous spaces $\mathcal{M} = G/K$, where G is a compact group, $K \subset G$ is a closed subgroup, while θ

and μ are G-invariant metric and measure on $\mathcal{M}$. In this case, the quantities (1.6), (1.8) and (1.13), (1.14) are also G-invariant:

$$\left.\begin{array}{r}\lambda_r(gy_1,gy_2)=\lambda_r(y_1,y_2),\ \lambda(\eta,gy_1,gy_2)=\lambda(\eta,y_1,y_2),\\ \theta_r^{\Delta}(gy_1,gy_2)=\theta_r^{\Delta}(gy_1,gy_2),\theta^{\Delta}(\eta,gy_1,gy_2)=\theta^{\Delta}(\eta,y_1,y_2),\\ \mu(B_r(gy_1)\cap B_r(gy_2))=\mu(B_r(y_1)\cap B_r(y_2)),y_1,y_2\in\mu,g\in G.\end{array}\right\}\quad(1.28)$$

Comparing the relations (1.23)–(1.27), we arrive to the following result. This result and its generalizations were given in [28, Thms. 2.1, 3.1].

Theorem 1.1 (Weak Invariance Principles)**.** *Let a compact connected metric space $\mathcal{M}$ with a metric θ and a measure μ be distance-invariant. Then we have*

$$\lambda_r(y_1,y_2)+\theta_r^{\Delta}(y_1,y_2)=\langle\theta_r^{\Delta}\rangle,\tag{1.29}$$

$$\lambda(\eta,y_1,y_2)+\theta^{\Delta}(\eta,y_1,y_2)=\langle\theta^{\Delta}(\eta)\rangle,\tag{1.30}$$

$$\lambda(\eta,\mathcal{D}_N)+\theta^{\Delta}(\eta,\mathcal{D}_N)=\langle\theta^{\Delta}(\eta)\rangle N^2,\tag{1.31}$$

$$\lambda_N(\eta)+\theta_N^{\Delta}(\eta)=\langle\theta^{\Delta}(\eta)\rangle N^2.\tag{1.32}$$

Here $r\in[0,\pi]$ and $\mathcal{D}_N\subset\mathcal{M}$ is an arbitrary N-point subset. The equalities (1.30), (1.31) *and* (1.32) *hold with any weight function η such that the integrals* (1.7), (1.8) *and* (1.13), (1.18) *converge.*

Obviously, the integrals (1.7), (1.8) and (1.13), (1.18) converge for any wight function η summable on the interval $[0,\pi]$. More general conditions of convergence of these integrals for two-point homogeneous spaces are given in Lemma 2.1(i) below.

The strong invariance principle for two-point homogeneous spaces will be established in the next section in Theorem 2.2. Our terminology of strong and weak invariance principles is explained in comments to Theorem 2.2.

A compact metric space $\mathcal{M}$ with a metric θ and a measure μ is called d-rectifiable if there exist a measure ν on the d-dimensional unit cube $I^d=[0,1]^d$ absolutely continuous with respect to the d-dimensional Lebesgue measure on I^d, a measurable subset $\mathcal{O}\subset I^d$, and an injective Lipschitz map $f:\mathcal{O}\to\mathcal{M}$, such that $\mu(\mathcal{M}\setminus f(\mathcal{O}))=0$; and $\mu(\mathcal{E})=\nu(f^{-1}(\mathcal{E}\cap f(\mathcal{O}))$ for any μ-measurable subset $\mathcal{E}\subset\mathcal{M}$. Recall that a map $f:\mathcal{O}\subset\mathbb{R}^d\to\mathcal{M}$ is Lipschitz if

$$\theta(f(Z_1),f(Z_2))\le c\|Z_1-Z_2\|,\quad Z_1,Z_2\in\mathcal{O},\tag{1.33}$$

with a positive constant c, and the smallest such constant is called the Lipschitz constant of f and denoted by $\mathrm{Lip}(f)$; in (1.33) $\|\cdot\|$ denotes the Euclidean metric in $\mathbb{R}^d$, cf [26].

Notice that any smooth (or piece-wise smooth) compact d-dimensional manifold is d-rectifiable if in the local coordinates the metric satisfies (1.12), and the measure is absolutely continuous with respect to the d-dimensional Lebesgue measure. Particularly, any compact d-dimensional Riemannian manifold with the geodesic metric θ and the Riemannian measure μ is d-rectifiable. In this case, it is known that condition (1.12) holds; see [21, Chapter I, Proposition 9.10]. On the other hand, the condition on the Riemannian measure is obvious because the metric tensor is continuous.

The following result was established in [28, Thm.4.2].

Theorem 1.2. *Suppose that a compact metric space $\mathcal{M}$, with a metric θ and a measure μ, is d-rectifiable. Write $C = d2^{d-1}\,\mathrm{Lip}(f)$, where $\mathrm{Lip}(f)$ is the Lipschitz constant of the map f in the definition of d-rectifiability of the space $\mathcal{M}$. Then the following hold:*

(i) *If a metric ρ on $\mathcal{M}$ satisfies the inequality*

$$\rho(x_1, x_2) \le c_0\theta(x_1, x_2) \tag{1.34}$$

with a constant $c_0 > 0$, then

$$\rho_N \ge \langle\rho\rangle N^2 - c_0 C N^{1-1/d}. \tag{1.35}$$

(ii) *If the metric $\theta^\Delta(\eta)$ satisfies the inequality*

$$\theta^\Delta(\eta, x_1, x_2) \le c_0\theta(x_1, x_2) \tag{1.36}$$

with a constant $c_0 > 0$, then

$$\theta^\Delta_N(\eta) \ge \langle\theta^\Delta(\eta)\rangle N^2 - c_0 C N^{1-1/d} \tag{1.37}$$

and

$$\lambda_N(\eta) \le c_0 C N^{1-1/d}. \tag{1.38}$$

Under such general assumptions one cannot expect that the bounds (1.37) and (1.38) are best possible. The corresponding counterexample is given in the next section, see the relation (2.22). Theorem 1.2 guarantees the existence of well-distributed point subsets in all compact d-rectifiable spaces.

It should be emphasize that a very non-trivial and difficult problem is to construct such uniformly distributed subsets explicitly. For spheres in the Euclidean space a deep investigation of this problem has been given in [25].

In the present paper we will show that the bounds (1.37) and (1.38) are best possible for compact connected two-point spaces and general classes of weight functions η, see Theorem 2.2 below. Main results of this paper were stated previously in [29].

2. Strong invariance principle and best bounds for discrepancies and sums of distances

In this section we state and discuss our main results on strong invariance principles and best possible bounds for discrepancies and sums of distances on two-point homogeneous spaces.

Recall some necessary facts on two-point homogeneous spaces, see [7, 21, 22, 33, 34]. Additional facts on the geometry and harmonic analysis on such spaces will be given in sections 5 and 8.

Let $G = G(\mathcal{M})$ be the group of isometries of a metric space $\mathcal{M}$ with a metric θ, *i.e.* $\theta(gx_1, gx_2) = \theta(x_1, x_2)$ for all x_1, $x_2 \in \mathcal{M}$ and $g \in G$. The space $\mathcal{M}$ is called *two-point homogeneous*, if for any two pairs of points x_1, x_2 and y_1, y_2 with $\theta(x_1, x_2) = \theta(y_1, y_2)$ there exists an isometry $g \in G$, such that $y_1 = gx_1$, $y_2 = gx_2$. In this case, the group G is transitive on $\mathcal{M}$ and $\mathcal{M} = G/K$ is a homegeneous space, where the subgroup $K \subset G$ is the stabilizer of a point $x_0 \in \mathcal{M}$. Furthermore, the homogeneous space $\mathcal{M}$ is symmetric, *i.e.* for any two points y_1, $y_2 \in \mathcal{M}$ there exists an isometry $g \in G$, such that $gy_1 = y_2$, $gy_2 = y_1$.

We consider compact connected two-point homogeneous spaces $\mathcal{M} = G/K$. For such spaces G and $K \subset G$ are Lie groups and $\mathcal{M} = G/K$ are Riemannian symmetric spaces of rank one. All such spaces are classified completely, see [33, Sec. 8.12]. They are the following:

(i) The d-dimensional spheres in the Euclidean space $S^d \subset \mathbb{R}^{d+1}$, $S^d = SO(d+1)/SO(d) \times \{1\}$, $d \geq 2$, and $S^1 = O(2)/O(1) \times \{1\}$.

(ii) The real projective spaces $\mathbb{R}P^n = O(n+1)/O(n) \times O(1)$.

(iii) The complex projective spaces $\mathbb{C}P^n = U(n+1)/U(n) \times U(1)$.

(iv) The quaternionic projective spaces $\mathbb{H}P^n = Sp(n+1)/SP(n) \times Sp(1)$,

(v) The octonionic projective plane $\mathbb{O}P^2 = F_4/\operatorname{Spin}(9)$.

Here we use the standard notation from the theory of Lie groups; particularly, F_4 is one of the exceptional Lie groups in Cartan's classification. see [21, 22, 33, 34].

The indicated projective spaces $\mathbb{F}P^n$ as compact Riemannian manifolds have dimensions d,

$$d = \dim_{\mathbb{R}} \mathbb{F}P^n = nd_0, \ d_0 = \dim_{\mathbb{R}} \mathbb{F}, \tag{2.1}$$

where $d_0 = 1, 2, 4, 8$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, correspondingly.

For spheres S^d we put $d_0 = d$ by definition. Projective spaces of dimension d_0 ($n = 1$) are isomorphic to the spheres S^{d_0}: $\mathbb{R}P^1 \approx S^1, \mathbb{C}P^1 \approx S^2, \mathbb{H}P^1 \approx S^4, \mathbb{O}P^1 \approx S^8$. We can conveniently agree that $d > d_0$ ($n \geq 2$) for projective spaces, while the equality $d = d_0$ holds only for spheres. Under this convention, the dimensions $d = nd_0$ and d_0 define uniquely (up to isomorphism) the corresponding two-point homogeneous space which we denote by $Q = Q(d, d_0)$. We write θ for the geodesic distance and μ for the Riemannian mesure on $Q(d, d_0)$. The metric θ and the measure μ are invariant under the action of the corresponding group of isometries and normalized by (1.1). In what follows we always assume that $n = 2$ if $\mathbb{F} = \mathbb{O}$. Projective spaces $\mathbb{O}P^n$ do not exist for $n > 2$. In more detail the geometry of spaces $\mathbb{F}P^n$ will be outlined in section 5.

Any space $Q(d, d_0)$ is distance-invariant and the volume of balls is given by

$$\left.\begin{aligned} v_r &= \kappa(d, d_0) \int_0^r (\sin \frac{1}{2}u)^{d-1} (\cos \frac{1}{2}u)^{d_0-1} \, du, \quad r \in [0, \pi], \\ \kappa(d, d_0) &= B(d/2, d_0/2)^{-1} = \frac{\Gamma(d/2 + d_0/2)}{\Gamma(d/2)\Gamma(d_0/2)}. \end{aligned}\right\} \tag{2.2}$$

Here $B(\cdot, \cdot)$ and $\Gamma(\cdot)$ are beta and gamma functions, and we have $v_\pi = \mu(Q(d, d_0)) = 1$. Notice that the different equivalent forms of the relation (2.2) can be found in the literature, see [22, pp. 165–168], [19, pp. 177–178], [23, pp. 508–510].

From the formula (2.2) we obtain the following two-side bounds

$$v_r \simeq r^d, \ v_r' = 1 - v_r \simeq (\pi - r)^{d_0}, \ r \in [0, \pi]. \tag{2.3}$$

To simplify notation we write in some formulas $A \lesssim B$ instead of $B = O(A)$, $A \gtrsim B$ instead of $B = O(A)$, and $A \simeq B$ if $A = O(B)$ and $B = O(A)$.

The *chordal metric* on the spaces $Q(d, d_0)$ can be defined by

$$\tau(x_1, x_2) = \sin \frac{1}{2}\theta(x_1, x_2), \quad x_1, x_2 \in Q(d, d_0). \tag{2.4}$$

Notice that the expression (2.4) defines a metric because the function $\varphi(\theta) = \sin\theta/2$, $0 \le \theta \le \pi$, is concave, increasing and $\varphi(0) = 0$, that implies the triangle inequality. For the sphere $S^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ we have

$$\begin{aligned} \cos\theta(x_1, x_2) &= (x_1, x_2), x_1, x_2 \in S^d \\ \tau(x_1, x_2) &= \sin\theta(x_1, x_2) = \frac{1}{2}\|x_1 - x_2\|, \end{aligned} \tag{2.5}$$

where $(\cdot, \cdot)$ is the inner product and $\|\cdot\|$ is the Euclidean distance in $\mathbb{R}^{d+1}$.

Each projective space $\mathbb{F}P^n$ can be canonically imbedded into the unit sphere

$$\Pi : Q(d, d_0) \ni x \to \Pi(x) \in S^{m-1} \subset \mathbb{R}^m, \ m = \frac{1}{2}(n+1)(d+2), \tag{2.6}$$

such that

$$\tau(x_1, x_2) = \frac{1}{\sqrt{2}}\|\Pi(x_1) - \Pi(x_2)\|, \quad x_1, x_2 \in \mathbb{F}P^n, \tag{2.7}$$

where $\|\cdot\|$ is the Euclidean distance in $\mathbb{R}^{m+1}$. Hence, the metric $\tau(x_1, x_2)$ coincides with the Euclidean length of a segment joining the corresponding points $\Pi(x_1)$ and $\Pi(x_2)$ on the unit sphere. The metric is normalized by $\mathrm{diam}(Q(d, d_0), \tau) = 1$. The imbedding (2.6) will be described in Section 5, see (5.22).

Notice that the chordal metric τ on projective spaces $\mathbb{F}P^n$ coincides with the well-known Fubini-Study metric. In connection with special point configurations in two-point homogeneous spaces the chordal metric on projective spaces was discussed in the papers [13, 14], see also the paper [15], where the chordal metric was defined for Grassmannian manifolds.

Now we are in position to state our main results. First of all, we consider strong invariance principles. A careful analysis of the imbedding (2.6) leads to the following.

Theorem 2.1. *For any space $Q = Q(d, d_0)$ the chordal metric* (2.4) *and the symmetric difference metric* (1.13) *are related by*

$$\tau(x_1, x_2) = \gamma(Q)\theta^{\Delta}(\eta^{\natural}, x_1, x_2), \ x_1, x_2 \in Q, \tag{2.8}$$

where $\eta^\sharp(r) = \sin r$, $r \in [0, \pi]$, *and*

$$\gamma(Q) = \frac{\langle \tau \rangle}{\langle \theta^\Delta(\eta^*) \rangle} = \frac{\mathrm{diam}(Q, \tau)}{\mathrm{diam}(Q, \theta^\Delta(\eta^*))}. \tag{2.9}$$

The proof of Theorem 2.2 is given in Section 6. It is clear that the equalities (2.9) follow immediately from (2.8). It suffices to calculate the average value (1.12) of both metrics in (2.8) to obtain the first equality in (2.9). Similarly, it suffices to write (2.8) for any pair of antipodal points x_1, x_2, $\theta(x_1, x_2) = \pi$, to obtain the second equality in (2.9).

Comparing Theorems 1.1 and 2.1, we arrive at the following.

Corollary 2.1 (Strong Invariance Principle)**.** *For any space* $Q = Q(d, d_0)$ *we have the relation*

$$\gamma(Q)\lambda[\eta^\sharp, \mathcal{D}_N] + \tau[\mathcal{D}_N] = \langle \tau \rangle N^2, \tag{2.10}$$

where $\mathcal{D}_N \subset Q$ *is an arbitrary* N*-point subset.*

Particularly, for any N *we have the equality*

$$\gamma(Q)\lambda_N(\eta^\sharp) + \tau_N = \langle \tau \rangle N^2. \tag{2.11}$$

Notice that for the sphere S^d the discrepancy $\lambda[\eta^\sharp, \mathcal{D}_N]$ with the special weight function $\eta^\sharp(r) = \sin r$ can be written in the form

$$\lambda[\eta^\sharp, \mathcal{D}_N] = \int_{-1}^{1} dz \int_{\mathcal{M}} [\#\{B(y,z) \cap \mathcal{D}_N\} - N\mu(B(y,z))]^2 \, d\mu(y), \tag{2.12}$$

where μ is the standard normalized d-dimensional mesure on S^d, and

$$B(y,z) = \{x \in S^d : \cos\theta(x,y) \geq z\}, \ y \in S^d, z \in [-1, 1], \tag{2.13}$$

is the 'spherical cap', $B(y,z) = B_r(y)$, $z = \cos r$.

For spheres the strong invariance principle (2.10) was established by Stolarsky [30], see also the papers [8, 11], where the original proof of this relation has been essentially simplified. Corollary 2.1 can be thought of as an extension of Stolarsky's invariance principle to projective spaces.

Recall that a metric space $\mathcal{M}$ with a metric ρ is called isometrically L_q-embeddable, if there exists a map $\varphi : \mathcal{M} \ni x \to \varphi(x) \in L_q$, such that

$\rho(x_1, x_2) = \|\varphi(x_1) - \varphi(x_2)\|_{L_q}$ for all x_1, $x_2 \in \mathcal{M}$. A two-point homogeneous space Q is isometrically L_1-embeddable with respect to any metric $\theta^{\Delta}(\eta)$, see (1.17). At the same time, the space Q is isometrically L_2-embeddable with respect to the chordal metric τ, see (2.5) and (2.7). It is known, see [17, Sec. 6.3], that the L_2-imbeddability is stronger and implies the L_1-imbeddability. This explains our terminology of strong and week invariance principles. It would be very interesting to find out whether there are weight functions $\eta \neq \eta^{\natural}$ for which the spaces Q with the metric $\theta^{\Delta}(\eta)$ are also L_2-embeddable.

Now we consider best possible bounds for the extremal quantities (1.9) and (1.11). At first, we state in Lemma 2.1 some important auxiliary results. Introduce the following classes of weight functions $\eta(r)$, $r \in [0, \pi]$,

$$\left.\begin{aligned} W(a,b) &= \{\eta \geq 0 : \|\eta\|_{a,b} < \infty\},\ a \geq b \geq 1, \\ \|\eta\|_{a,b} &= \int\limits_0 (\sin\frac{1}{2}r)^{a-1}(\cos\frac{1}{2}r)^{b-1}\eta(r)\,dr. \end{aligned}\right\} \tag{2.14}$$

It is worth noting that weight functions in the classes (2.14) admit rather large singularities at points $r = 0$ and $r = \pi$.

Lemma 2.1. *For any space $Q(d, d_0)$ the following hold :*

(i) *The kernel* (1.6) *and the metric* (1.14) *satisfy the bounds*

$$\left.\begin{aligned} |\lambda_r(y_1, y_2)| &\leq C(\sin\frac{1}{2}r)^{d}(\cos\frac{1}{2}r)^{d_0}, \\ \theta_r^{\Delta}(y_1, y_2) &\leq C(\sin\frac{1}{2}r)^{d}(\cos\frac{1}{2}r)^{d_0}. \end{aligned}\right\} \tag{2.15}$$

If $\eta \in W(d+1, d_0+1)$, then the kernel (1.8) *and the metric* (1.13) *satisfy the bounds*

$$\left.\begin{aligned} |\lambda(\eta, y_1, y_2)| &\leq C\|\eta\|_{d+1,d_0+1}, \\ \theta^{\Delta}(\eta, y_1, y_2) &\leq C\|\eta\|_{d+1,d_0+1}. \end{aligned}\right\} \tag{2.16}$$

(ii) *The metric* (1.14) *satisfies the bound*

$$\theta_r^{\Delta}(y_1, y_2) \leq C(\sin\frac{1}{2}r)^{d-1}(\cos\frac{1}{2}r)^{d_0-1}\theta(y_1, y_2). \tag{2.17}$$

If $\eta \in W(d, d_0)$, then the metric (1.13) *satisfies the bound*

$$\theta^{\Delta}(\eta, y_1, y_2) \leq C\|\eta\|_{d,d_0}\theta(y_1, y_2). \tag{2.18}$$

Constants in the bounds (2.15), (2.16), (2.17) *and* (2.18) *depend only on* d *and* d_0.

The proof of Lemma 2.1 is given in Section 7. Particularly, it follows from Lemma 2.1 that the weak invariance principles (1.30) - (1.32) hold in spaces $Q(d, d_0)$ with weight functions $\eta \in W(d+1, d_0+1)$.

Our result on the extremal quantities (1.9) and (1.11) can be stated as follows

Theorem 2.2. *For any space* $Q(d, d_0)$ *the following hold* :

If $\eta \in W(d, d_0)$, $\eta \neq 0$, *then for any* N *we have*

$$\langle \theta^{\Delta}(\eta) \rangle N^2 - c(\eta) N^{1-\frac{1}{d}} > \theta_N^{\Delta}(\eta) > \langle \theta^{\Delta}(\eta) \rangle N^2 - C(\eta) N^{1-\frac{1}{d}}, \tag{2.19}$$

$$c_1(\eta) N^{1-\frac{1}{d}} < \lambda_N(\eta) < C_1(\eta) N^{1-\frac{1}{d}} \tag{2.20}$$

with positive constants independent of N.

Particularly, for the chordal metric τ *on* $Q(d, d_0)$ *we have*

$$\langle \tau \rangle N^2 - c N^{1-\frac{1}{d}} > \tau_N > \langle \tau \rangle N^2 - C N^{1-\frac{1}{d}} \tag{2.21}$$

with the constants $c = c(\eta^{\natural})$ *and* $C = C(\eta^{\natural})$.

The proof of Theorem 2.2 is given in Section 11. It is clear that the right bounds in (2.19) and (2.20) follow immediately from Theorem 1.2(ii) and Lemma 2.1(ii). In Section 11 we will prove the left bound in (2.20). This will imply immediately the left bound in (2.19) by the invariance principle (1.32). The proof of the left bound in (2.20) is relying on the theory of spherical functions on homogeneous spaces $Q(d, d_0)$.

For the chordal metric τ on the sphere S^d the bounds (2.21) were known earlier. The right bound in (2.21) was established by Alexander [1] and the left by Beck [5]. In [5], see also [6], the left bound (2.20) was proved for quadratic discrepancies on spheres with the special weight function $\eta^{\natural}$, see (2.12). Together with Stolarsky's invariance principle this implies the left bound (2.21) for the chordal metric on S^d.

The universal bound (1.35) of Theorem 1.2(i) holds for all metrics ρ on spaces Q Lipschitz continuous with respect to the geodesic distance θ. However, not all such metrics satisfy the two-side bounds of type (2.19). For example, for the geodesic distance θ on the sphere S^d we have

$$\theta_N = \langle \theta \rangle N^2 - \varepsilon_N, \quad \langle \theta \rangle = \pi/2, \tag{2.22}$$

where $\varepsilon_N = 0$ for even N and $0 \le \varepsilon_N \le \pi/2$ for odd N. Additional comments on this example and its relationships with invariance principles and spherical functions are given in Sections 4 and 9. Notice that very recently the exact equality $\varepsilon_N = \pi/2$ for odd N was proved in the paper [8].

In conclusion of this section we notice the following. Non-compact connected two-point spaces $\mathcal{M} = G/K$ are also classified completely as hyperbolic spaces over algebras $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, see [33, Sec. 8.12], and one could consider the spaces of double cosets $\mathcal{M} = \Gamma \setminus Q = \Gamma \setminus G/K$, where $\Gamma \subset G$ is a discrete subgroup in the group of isomerties, such that the invariant measure $\mu(\mathcal{M}) < \infty$. In this case, the extremal discrepancies (1.9) and sums of distances (1.11) for the symmetry difference metrics (1.13), (1.14) are well-defined and their study should be of much interest, especially for non-compact $\mathcal{M}$. A detailed study of these questions falls outside the scope of the present paper.

3. Applications to t-design

Many specific point configurations on spheres and other two-point homogeneous spaces are described in the literature, see, for example, [4, 8, 12–15, 23, 25, 27]. One can ask whether the points of such specific configurations are distributed uniformly in the corresponding spaces, and how the quadratic discrepancies (1.7) and the sums of distances (1.10) could be estimated precisely for such point subsets ?

In the present paper we consider these questions for t-designs. Recall that an N-point subset $\mathcal{D}_N \subset S^d$ is called a *spherical t-design*, if the exact quadrature formula

$$\sum_{x \in \mathcal{D}_N} F(x) = N \int\limits_{S^d} F(y)\, d\mu(y) \tag{3.1}$$

holds for all homogeneous polynomials $F(x), x \in \mathbb{R}^{d+1}$ of degree not exceeding t. The concept of t-design can be easily extended to compact two-point spaces, see [4, 23]. In this case, an N-point subset $D_N \subset Q(d, d_0)$ is called a *t-design*, if the exact quadrature formula

$$\sum_{x_1, x_2 \in \mathcal{D}_N} f(\cos\theta(x_1, x_2)) = N^2 \iint\limits_{Q \times Q} f(\cos\theta(t_1, y_2))\, d\mu(y_1)\, d\mu(y_2) \tag{3.2}$$

holds for all polynomials $f(z), z \in \mathbb{C}$, of degree not exceeding t. The condition (3.2) can be written in different equivalent forms, for example, as the following quadrature formula

$$\sum_{x \in \mathcal{D}_N} f(\cos\theta(x, y_1)) = N \int\limits_Q f(\cos\theta(y_1, y_2))\, d\mu(y), \tag{3.3}$$

which holds identically for all $y_1, y_2 \in Q$. The definition of t-designs can be also given in terms of spherical functions on the spaces $Q(d, d_0)$, see [4, 23]. We will return to these questions in Section 8, see (8.35).

The integrals in the right-hand sides in (3.2) and (3.3) are equal, that follows at once from the definition of two-point spaces. For brevity, we write

$$\langle f \rangle_Q = \int\limits_Q f(\cos\theta(y, y_2)\, d\mu(y) = \iint\limits_{Q \times Q} f(\cos(y_1, y_2))\, d\mu(y_1)\, d\mu(y_2). \tag{3.4}$$

From (2.2) we obtain

$$\langle f \rangle_Q = B(d/2, d_0/2)^{-1} \int\limits_0^\pi f(\cos\theta)(\sin\frac{1}{2}\theta)^{d-1}(\cos\frac{1}{2}\theta)^{d_0-1} d\theta. \tag{3.5}$$

It is known, see [23, p. 520], that any N-point t-design $\mathcal{D}_N \subset Q(d, d_0)$ satisfies the bound $N \geq ct^d$ with a constant $c > 0$ independent of N and t. An N-point t-design $\mathcal{D}_N \subset Q(d, d_0)$ is called an *optimum t-design*, if

$$c_+ t^d \geq N \geq c_- t^d \tag{3.6}$$

with some positive constants c_+ and c_- independent of N and t. Actually, in this definition we deal with sequences of N-point t-designs $\mathcal{D}_N$ as $N \to \infty$.

For any N-point subset $\mathcal{D}_N \subset Q(d, d_0)$ we put

$$\nu[\mathcal{D}_N, r] = \max_{y \in Q} \#\{B_r(y) \cap \mathcal{D}_N\}, \quad r \in [0, \pi], \tag{3.7}$$

and $\nu[\mathcal{D}_N, r] = N$ if $r > \pi$.

Our result on t-designs can be stated as follows.

Theorem 3.1. *Let the weight function $\eta \in W(d, d_0)$, then the following hold:*

(i) *There exists a constant $L \geq 1$ depending only on d and d_0, such that for any N-point t-design $\mathcal{D}_N \subset Q(d, d_0)$ with $t \geq 2L/\pi$ we have*

$$\lambda[\eta, \mathcal{D}_N] < Ct^{d-1}(\nu[\mathcal{D}_N, Lt^{-1}])^2. \tag{3.8}$$

(ii) *For optimum N-point t-designs $\mathcal{D}_N \subset Q(d, d_0)$ the bound* (3.8) *takes the form*

$$\lambda[\eta, \mathcal{D}_N] < CN^{1-\frac{1}{d}}(\nu[\mathcal{D}_N, c_+^{-1/d}LN^{-1/d}])^2, \tag{3.9}$$

where c_+ is the constant in the definition (3.6).

The constants C in the bounds (3.8) *я* (3.9) *depend only on d, d_0 and η.*

The proof of Theorem 3.1 is given in Section 11. It is clear that the equalities (3.9) follow immediately from (3.8) and the definition (3.6). The proof of the bound (3.8) is relying on the theory of spherical functions on homogeneous spaces $Q(d, d_0)$.

For an arbitrary N-point subset $\mathcal{D}_N \subset Q(d, d_0)$, we put

$$\delta[\mathcal{D}_N] = \frac{1}{2}\min\{\theta(x_1, x_2) : x_1, x_2 \in \mathcal{D}_N, x_1 \neq x_2\} \tag{3.10}$$

Hence, the balls $B_\delta(x)$, $\delta = \delta[\mathcal{D}_N]$, $x \in \mathcal{D}_N$, do not overlap. Therefore, $v_\delta N \leq 1$ and, in view of (2.3), we have $\delta \lesssim N^{-1/d}$. An N-point subset $\mathcal{D}_N \subset Q(d, d_0)$ is called *well-separated*, if $\delta[\mathcal{D}_N] \geq cN^{-1/d}$ with a constant $c > 0$ independent of N.

We conveniently agree that for $r > \pi$ the ball $B_r(y) = Q$ and $v_r = 1$. With this convention the following result is true.

Lemma 3.1. *Let an N-point subset $\mathcal{D}_N \subset Q(d, d_0)$ be well-separated. Then, for any constant $C > 0$ we have*

$$\nu[\mathcal{D}_N, CN^{-\frac{1}{d}}] \leq c_1 \tag{3.11}$$

with the constant $c_1 = c_1(C)$ independent of N.

Proof. For brevity, we write $a = CN^{-\frac{1}{d}}$. Consider the ball $B_a(y)$ centered at a point $y \in Q$ and put $\mathcal{E} = B_a(y) \cap \mathcal{D}_N$, $K = \#\{\mathcal{E}\}$. By the definition of a well-separated subset , the balls $B_\delta(x)$, $\delta = \delta[\mathcal{D}_N]$, $x \in \mathcal{E}$, do not overlap and all these balls are contained in the ball $B_{a+\delta}(y)$. Therefore, $v_\delta K \leq v_{a+\delta}$, and in view of (2.3), we have $K \leq v_{a+\delta}/v_\delta \simeq (1 + C/c)^d$. This proves the bound (3.11). □

Comparing Theorem 3.1 and Lemma 3.1, we arrive at the following.

Corollary 3.1. *Let the weight function $\eta \in W(d, d_0)$, $\eta \neq 0$. Suppose that an N-point subset $\mathcal{D}_N \subset Q(d, d_0)$ satisfies the following two conditions*:
(i) *$\mathcal{D}_N$ is an optimum t-design*
(ii) *$\mathcal{D}_N$ is a well-separated subset.*
Then, for all sufficiently large N we have

$$\langle \theta^\Delta(\eta) \rangle N^2 - cN^{1-\frac{1}{d}} > \theta^\Delta[\eta, \mathcal{D}_N] > \langle \theta^\Delta(\eta) \rangle N^2 - CN^{1-\frac{1}{d}}, \tag{3.12}$$

$$cN^{1-\frac{1}{d}} < \lambda[\eta, \mathcal{D}_N] < CN^{1-\frac{1}{d}} \tag{3.13}$$

Particularly, for the chordal metric τ on $Q(d, d_0)$ we have

$$\langle r \rangle N^2 - cN^{1-\frac{1}{d}} > \tau[\mathcal{D}_N] > \langle r \rangle N^2 - CN^{1-\frac{1}{d}} \tag{3.14}$$

The positive constants in (3.12) - (3.14) *are independent of N.*

The existence of optimum t-designs was a long standing open problem (the Korevaar-Meyers conjecture). In the recent papers by Bondarenko, Radchenko and Viazovska [9, 10] this problem was solved for spherical t-designs. The existence of optimum spherical t-designs was proved in [9] for all sufficiently large N, and it was proved in [10] that such optimum t-designs can be chosen as well-separated subsets on the sphere. Hence, Corollary 3.1 is applicable for this case, and we conclude that spherical t-designs constructed in [10] meet the best possible bounds for quadratic discrepancies (3.12) and sums of distances (3.13), (3.14).

Using spherical t-designs one can easily construct $[t/2]$-designs on the real projective space $\mathbb{R}P^d = Q(d, 1)$. Consider the canonical projection

$$p : S^d \ni x \to p(x) \in \mathbb{R}P^d, \tag{3.15}$$

where $p(x)$ denotes the one-dimensional subspace in $\mathbb{R}^{d+1}$ passing through the point $x \in S^d$, and $p(-x) = p(x)$. In the present discussion, we conveniently write θ° for the geodesic distance on S^d and θ for the geodesic distance on $\mathbb{R}P^d$. Both distances are normalized by (1.1). By definition, $\theta^\circ(x_1, x_2)$ is the angle between the vectors $x_1, x_2 \in S^d$, while $\frac{1}{2}\theta(x_1, x_2)$ is the angle between the subspaces $p(x_1), p(x_2) \in \mathbb{R}P^d$. Hence, $\cos\theta^\circ(x_1, x_2) = (x_1, x_2)$, see (2.5), and $\cos\frac{1}{2}\theta(x_1, x_2) = |\langle x_1, x_2 \rangle|$, яя. (5.6), $x_1, x_2 \in S^d$. Therefore,

$$\cos\theta(x_1, x_2) = 2(\cos\theta^\circ(x_1, x_2))^2 - 1 \tag{3.16}$$

and

$$\theta(x_1, x_2) = 2\min\{\theta^\circ(x_1, x_2), \pi - \theta^\circ(x_1, x_2)\}. \tag{3.17}$$

Particularly, $\theta = 2\theta^\circ$ if $0 \le \theta^\circ \le \pi/2$.

For an N-point subset $\mathcal{D}_N^\circ \subset S^d$ we define

$$\mathcal{D}_N = p(\mathcal{D}_N^\circ) = \{p(x) : x \in \mathcal{D}_N^\circ\} \subset \mathbb{R}P^d, \tag{3.18}$$

i.e. $\mathcal{D}_N$ is a collection of the one-dimensional subspaces in $\mathbb{R}^{d+1}$ passing through the points $x \in \mathcal{D}_N^\circ \subset S^d$. If $\mathcal{D}_N^\circ$ contains pairs of antipodal points x and $-x$, then the corresponding subspaces $p(x_1)$ and $p(x_2)$ coincide and are counted with the multiplicity 2 as points in $\mathbb{R}P^d$. It is obvious, that if a subset $\mathcal{D}_N^\circ \subset S^d$ is well-separated, then, in general, the subset $\mathcal{D}_N = p(\mathcal{D}_N^\circ) \subset \mathbb{R}P^d$ is not.

Lemma 3.2. (i) *If a subset $\mathcal{D}_N^\circ \subset S^d$ is an optimum t-design, then the subset $\mathcal{D}_N = p(\mathcal{D}_N^\circ) \subset \mathbb{R}P^d$ is an optimum $[t/2]$-design.*

(ii) *If a subset $\mathcal{D}_N^\circ \subset S^d$ is well-separated, then the subset $\mathcal{D}_N = p(\mathcal{D}_N^\circ) \subset \mathbb{R}P^d$ satisfies the bound* (3.11) *with an arbitrary constant $C > 0$ and the constant $c_1 = c_1(C)$ independent of N.*

Proof. (i) If f is a polynomial of degree m, then, in view of (3.16), $f(\cos\theta) = f(2\cos^2\theta^\circ - 1) = f_1(\cos\theta^\circ)$, where f_1 is a polynomial of degree $2m$, and $f_1(z) = f_1(-z)$. Furthermore,

$$\kappa(d,1)\int_0^\pi f(\cos\theta)(\sin\frac{1}{2}\theta)^{d-1}d\theta$$

$$= \kappa(d,d)\int_0^\pi f_1(\cos\theta^\circ)(\sin\frac{1}{2}\theta^\circ)^{d-1}(\cos\frac{1}{2}\theta^\circ)^{d-1}d\theta^\circ, \tag{3.19}$$

where the known identity for the beta function $B(z,z) = 2^{1-2z}B(z,1/2)$ has been taken into account. From (3.5) and (3.19), we obtain

$$\langle f\rangle_{\mathbb{R}P^d} = \langle f_1\rangle_{S^d}. \tag{3.20}$$

If $\mathcal{D}_N^\circ \subset S^d$ is a spherical optimum t-design and $2m \le t$, then the definition (3.3) together with (3.20) implies

$$\sum_{x\in\mathcal{D}_N} f(\cos\theta(x,y)) = \sum_{x\in\mathcal{D}_N^\circ} f_1(\cos\theta^\circ(x,y)) = \langle f_1\rangle_{S^d} = \langle f\rangle_{\mathbb{R}P^d},$$

Hence, the subset $\mathcal{D}_N \subset \mathbb{R}P^d$ is an optimum $[t/2]$-design.

(ii) Write $B_r(y) \subset \mathbb{R}P^d$ for the ball of radius r centered at $p(y)$, and $B_r^\circ(y) \subset S^d$ for the ball of radius r centered at $y \in S^d$. From (3.15) we conclude that a point $p(x) \in \mathbb{R}P^d$ belongs to the ball $B_r(y)$ if and only if the point $x \in S^d$ belongs to either the ball $B_{r/2}^\circ(y)$ or the ball $B_{r/2}^\circ(-y)$. Therefore,

$$\#\{B_r(y) \cap \mathcal{D}_n\} = \#\{B_{r/2}^\circ(y) \cap \mathcal{D}_N^0\} + \#\{B_{r/2}^\circ(-y) \cap \mathcal{D}_N^0\}$$

From this equality and the definition (3.7), we obtain the bound

$$\nu[\mathcal{D}_N, r] \leq 2\nu[\mathcal{D}_N^\circ, r/2]. \tag{3.21}$$

If a subset $\mathcal{D}_N^\circ \subset S^d$ is well-separated, then by Lemma 3.1 for any constant C the bound $\nu[\mathcal{D}_N^\circ, CN^{-1/d}] \leq c_1^\circ$ holds with a constant $c_1^\circ = c_1^\circ(C)$ independent of N. Together with (3.21) this implies that the bound (3.11) holds for the subset $\mathcal{D}_N \subset \mathbb{R}P^d$ with an arbitrary constant C and the constant $c_1 = 2c_1^\circ(C/2, d)$ independent of N. □

Comparing Theorem 3.1 and Lemma 3.2, we arrive at the following.

Corollary 3.2. *Let the weight function $\eta \in W(d,1)$, $\eta \neq 0$, and suppose that an N-point subset $\mathcal{D}_N^\circ \subset S^d$ satisfies the conditions* (i) *a* (ii) *of Corollary* 3.1.

Then, the N-point subset $\mathcal{D}_N = p(\mathcal{D}_N^\circ) \subset \mathbb{R}P^d$ satisfies the bounds (3.12), (3.13), (3.14) *of Corollary* 3.1.

The corresponding generalizations to the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{Q}P^2$ is not straightforward and involve the methods of cited papers [9, 10].

4. Remarks on Lévy-Schoenberg kernels

The Lévy-Schoenberg kernels occur as covariances of random processes parametrized by points of a homogeneous space. For details we refer to the paper by Gangolli [19]. In this section we discuss very briefly some topics related to such kernels in the context of the present paper.

As before, we consider compact homogeneous spaces $Q = G/K$. A real-valued symmetric kernel $f(y_1, y_2)$, $y_1, y_2 \in Q$, is called a *Lévy-Schoenberg kernel*, if the following conditions are satisfied:

(i) There exists a point $y_0 \in Q$ such that $f(y, y_0) = 0$ for all $y \in Q$,

(ii) The kernel f is positive definite, i.e. for any points $x_1, \dots, x_N \in Q$ and any complex numbers $z_1, \dots, z_N$

$$\sum_{1 \le i, y \le N} \bar{z}_i z_j f(x_i, x_j) \ge 0. \tag{4.1}$$

(iii) The polarization $\rho(y_1, y_2)$ of the kernel $f(y_1, y_2)$ defined by

$$\rho(y_1, y_2) = f(y_1, y_1) + f(y_2, y_2) - 2f(y_1, y_2), \tag{4.2}$$

is G-invariant, i.e. $\rho(gy_1, gy_2) = \rho(y_1, y_2)$ for all $y_1, y_2 \in Q$ and $g \in G$.

Notice that the kernel f can be recovered from its polarization ρ by

$$f(y_1, y_2) = \frac{1}{2}(\rho(y_1, y_0) + \rho(y_2, y_0) - \rho(y_1, y_2)) \tag{4.3}$$

It is known, see [19, Sec. 7], that if a Lévy-Schoenberg kernel f and its polarization ρ are given, then the standard methods of probability theory enable us to construct a Gaussian process as the mapping $Y : Q \ni x \to Y_x = Y_x(\omega) \in L_2(\Omega, d\omega)$, such that $\mathbb{E}Y_{x_1} = 0$, $\mathbb{E}Y_{x_1}Y_{x_2} = f(x_1, x_2)$ and $\mathbb{E}(Y_{x_1} - Y_{x_2})^2 = \rho(x_1, x_2)$, for all $x_1, x_2 \in Q$. Here $L_2(\Omega, d\omega)$ is the Hilbert space of real-valued square-integrable random variables on a probability space Ω with a probability measure $d\omega$ and $\mathbb{E}$ denotes the expectation on $L_2(\Omega, d\omega)$. Furthermore, if the homogeneous space Q is a Riemannian manifold and the polarization ρ is Hölder continuous with respect to the geodesic distance θ, i.e. $\rho(y_1, y_2) < c\theta(y_1, y_2)^\beta$ with some constants c and $\beta > 0$, then for almost all $\omega \in \Omega$ the trajectories of the process $Y_x(\omega)$ are continuous functions of $x \in Q$.

In terms of the present paper one can easily describe a large number of explicit examples of Lévy-Schoenberg kernels on homogeneous spaces. Fix arbitrary a point $y_0 \in Q$ and consider the kernel

$$f_r(y_1, y_2) = \int_Q F_r(y_1, y) F_r(y_2, y)\, d\mu(y), \quad r \in [0, \pi], \tag{4.4}$$

where

$$F_r(x, y) = \chi(B_r(x), y) - \chi(B_r(y_0), y) \tag{4.5}$$

and $\chi(B_r(x), \cdot)$ is the characteristic function of the ball $B_r(x)$. Put also

$$f(\eta, y_1, y_2) = \int_0^\pi f_r(y_1, y_2)\eta(r)\, dr, \tag{4.6}$$

where η is a weight function such that the indegral (4.6) converges. For $Q = Q(d, d_0)$ the integral (4.6) converges if $\eta \in W(d+1, d_0+1)$, see Lemma 2.1 (i).

Theorem 4.1. (i) *For any compact homogeneous space Q the kernels* (4.4) *and* (4.6) *are Lévy-Schoenberg kernels and their polarizations ρ_r and $\rho(\eta)$ are of the form*

$$\rho_r(y_1, y_2) = 2\theta_r^{\Delta}(y_1, y_2), \tag{4.7}$$

$$\rho(\eta, y_1, y_2) = 2\theta^{\Delta}(\eta, y_1, y_2), \tag{4.8}$$

where θ_r^{Δ} and $\theta^{\Delta}(\eta)$ are symmetric difference metrics (1.14) *and* (1.13). *The inversion formulas are the following*

$$\theta_r^{\Delta}(y_1, y_0) + \theta_r^{\Delta}(y_2, y_0) - \theta_r^{\Delta}(y_1, y_2) = 2f_r(y_1, y_2), \tag{4.9}$$

$$\theta^{\Delta}(\eta, y_1, y_0) + \theta^{\Delta}(\eta, y_2, y_0) - \theta^{\Delta}(\eta, y_1, y_2) = 2f(\eta, y_1, y_2), \tag{4.10}$$

Particularly, for any symmetric difference metrics (1.14) *and* (1.13) *the expressions in the left-hand side of* (4.9) *and* (4.10) *are positive definite kernels.*

(ii) *For the two-point homogeneous spaces $Q = Q(d, d_0)$ the polarizations* (4.7) *and* (4.8) *satisfy the bounds*

$$\rho_r(y_1, y_2) \le C(\sin\frac{1}{2}r)^{d-1}(\cos\frac{1}{2}r)^{d_0-1}\theta(y_1, y_2), \tag{4.11}$$

$$\rho(\eta, y_1, y_2) \le C\|\eta\|_{d,d_0}\theta(y_1, y_2), \tag{4.12}$$

with constants depending only on d and d_0.

Proof. (i) Substituting (4.5) into (4.4), we obtain

$$\begin{aligned} f_r(y_1, y_2) = \mu(B_r(y_1) \cap B_r(y_2)) - \mu(B_r(y_1) \cap B_r(y_0) \\ - \mu(B_r(y_2) \cap B_r(y_0)) + v_r. \end{aligned}$$

Therefore, $f(y, y) = 2v_r - 2\mu(B_r(y) \cap B_r(y_0))$. With the help of these formulas, the polarization (4.2) can be written as $\rho(y_1, y_2) = 2v_r - 2\mu(B_r(y_1) \cap B_r(y_2))$. Comparing this expression with (1.25), we obtain (4.7). Integrating (4.7) with $\eta(r)$ and using (1.13), we obtain (4.8). Substituting (4.7) and (4.8) into (4.3), we obtain (4.9) я (4.10). It is obvious that the kernels (4.9) and (4.10) are positive definite.

(ii) The bounds (4.11), (4.12) follow immediately from (4.7), (4.8) and Lemma 2.1 (ii). □

One simple and instructive example can be adduced. Consider the sphere S^d with the geodesic metric θ and the standard Lebesgue measure μ normalized by (1.1). It is known, see [17, Sec. 6.4], that

$$\theta(x_1, x_2) = \pi\mu(B_{\pi/2}(x_1)\Delta B_{\pi/2}(x_2)), \quad x_1, x_2 \in S^d, \tag{4.13}$$

where $B_{\pi/2}(x) = \{y \in S^d : \theta(y, x) \leq \pi/2\} = \{y \in S^d : (y, x) \geq 0\}$ is the hemisphere centered at $x \in S^d$. Using (1.14), we can write (4.13) in the form

$$\theta(x_1, x_2) = \pi(1 - 2\mu(B_{\pi/2}(x_1) \cap B_{\pi/2}(x_2)) \tag{4.14}$$

In this form, this equality is almost obvious: it suffers to notice that the measure of the intersection of two hemispheres in (4.14) is a linear function of $\theta(x_1, x_2)$. Comparing (4.13) and (1.14), we can write

$$\theta(x_1, x_2) = 2\pi\theta^{\Delta}_{\pi/2}(x_1, x_2), \tag{4.15}$$

and so, the geodesic metric θ on the sphere S^d is a symmetric difference metric. Using the formulas (4.15) and (4.5), we obtain

$$\theta(x_1, x_0) + \theta(x_2, x_0) - \theta(x_1, x_2) = 4\pi \int\limits_{S^d} F_{\pi/2}(x_1, y)F_{\pi/2}(x_2, y)\, d\mu(y), \tag{4.16}$$

with $F_{\pi/2}(x, y) = \chi(B_{\pi/2}(x), y) - \chi(B_{\pi/2}(y_0), y)$.

From the formula (4.16) we immediately conclude that the kernel

$$f(x_1, x_2) = \theta(x_1, x_0) + \theta(x_2, x_0) - \theta(x_1, x_2) \tag{4.17}$$

is positive definite. This is the well-known theorem of Lévy. Its original proof was obtained in terms of 'white noise' integrals for random processes on S^d, see [24, Chap. 8; Appen. Chap. 3]. A direct proof was given in [19, Sec. 4] on the base of Gegenbauer polynomial expansion for the metric θ. The proof of Lévy's theorem given above is likely to be the simplest.

It should be emphasized that the geodesic metric θ for the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{Q}P^2$ is not a symmetric difference metric and for these spaces analogs of Lévy's theorem are not true. This follows from [19, Sec. 4, pp. 225–226]. At the same time, for the chordal metric $\tau(x_1, x_2) = \sin\theta(x_1, x_2)$ the kernel $f(x_1, x_2) = \tau(x_1, x_0) + \tau(x_2, x_0) - \tau(x_1, x_2)$ is positive definite for all two-point homogeneous spaces $Q(d, d_0)$. This follows from Theorems 2.1 and 4.1.

In conclusion of this section we wish to explain the appearance of anomalous small error terms in the relation (2.22). Using the formula (4.15) and the invariance principle (1.29) for the sphere S^d, we find that

$$\theta[\mathcal{D}_N] = \langle\theta\rangle N^2 - 2\pi\lambda_{\pi/2}[\mathcal{D}_N],$$

where

$$\lambda_{\pi/2}[\mathcal{D}_N] = \int\limits_{S^d} \Lambda[B_{\pi/2}(y), \mathcal{D}_N]^2 \, d\mu(y)$$

and $\Lambda[B_{\pi/2}(y), \mathcal{D}_N] = \#\{B_{\pi/2}(y) \cap \mathcal{D}_N\} - Nv_{\pi/2}$. Since $v_{\pi/2} = 1/2$, we derive from (1.26) that $\langle\theta\rangle = \pi/2$. Any N-point subset $\mathcal{D}_N \subset S^d$ can be represented as a disjoint union of two subsets $\mathcal{D}_N = \mathcal{D}_{2a}^{(0)} \cup \mathcal{D}_b^{(1)}, N = 2a + b$, where $\mathcal{D}_{2a}^{(0)} = \{x \in \mathcal{D}_N : -x \in \mathcal{D}_N\}$, $\mathcal{D}_b^{(1)} = \{x \in \mathcal{D}_N : -x \notin \mathcal{D}_N\}$. We have

$$\Lambda[B_{\pi/2}(y), \mathcal{D}_N] = \Lambda[B_{\pi/2}(y), \mathcal{D}_{2a}^{(0)}] + \Lambda[B_{\pi/2}(y), \mathcal{D}_b^{(1)}].$$

It is clear that $\Lambda[B_{\pi/2}(y), \mathcal{D}_{2a}^{(0)}] = 0$ for all $y \in S^d$ excepting the hyperplanes $\langle y, x\rangle = 0$, $x \in \mathcal{D}_{2a}^{(0)}$. Hence, $\lambda_{\pi/2}[\mathcal{D}_N] = \lambda_{\pi/2}[\mathcal{D}_b^{(1)}]$.

Let $N = 2a$ be even and $\mathcal{D}_N = \mathcal{D}_{2a}^{(0)}$, then $\lambda_{\pi/2}[\mathcal{D}_N] = 0$. Let $N = 2a + 1$ be odd and $\mathcal{D}_N = \mathcal{D}_{2a}^{(0)} \cup \mathcal{D}_1^{(0)}$, where $\mathcal{D}_1^{(1)} = \{x_0\}$ is a one-point subset. A simple calculation shows that $\lambda_{\pi/2}[\{x_0\}] = \pi/2$. Therefore, $\lambda_{\pi/2}[\mathcal{D}_N] = \pi/2$, and the relation (2.22) follows.

B. Geometry of two-point homogeneous spaces and strong invariance principles

5. Models of projective spaces and chordal metrics

In this section we define the chordal metrics on the projective spaces $\mathbb{F}P^n$, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}, \mathbb{H}$, $n \geq 2$, and the octonionic projective plane $\mathbb{O}P^2$ in terms of special models for these spaces. For the sake of convenience, we describe such models in sufficient detail and give the necessary references.

Recall the general facts on the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ over the field of real numbers. We have the natural inclusions

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}. \tag{5.1}$$

where the octonions $\mathbb{O}$ are a nonassociative and noncommutative algebra of dimension 8 with a basis $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ (their multiplication table can be found in [3, p. 150] and [7, p. 90]), the quaternions $\mathbb{H}$ are an associative but noncommutative subalgebra of dimension 4 spanned by $1, e_1, e_2, e_3$, finally, $\mathbb{C}$ and $\mathbb{R}$ are associative and commutative subalgebras of dimensions 2 and 1 spanned by $1, e_1$ and 1, correspondingly. From the multiplication table one can easily see that for any two indexes $7 \geq i, j \geq 1, i \neq j$, there exists an index $7 \geq k \geq 1$, such that

$$e_i e_j = -e_j e_i = e_k, \quad i \neq j, \quad e_i^2 = -1. \tag{5.2}$$

Let $a = \alpha_0 + \sum_{i=1}^7 \alpha_i e_i \in \mathbb{O}$, $\alpha_i \in \mathbb{R}$, $0 \leq i \leq 7$, be a typical octonion. We write $\operatorname{Re} a = \alpha_0$ for the real part, $\bar{a} = \alpha_0 - \sum_{i=1}^7 \alpha_i e_i$ for the conjugation, $|a| = \left(\alpha_0^2 + \sum_{i-1}^7 \alpha_i^2\right)^{1/2}$ fot the norm. Using (5.2), one can easily check that

$$\operatorname{Re} ab = \operatorname{Re} ba, \quad \overline{ab} = \overline{ba}, \quad |a|^2 = a\bar{a} = \bar{a}a, \quad |ab| = |a|\,|b|.$$

It follows from the last equality that all algebras (5.1) are division algebras. Notice also that by a theorem of Artin a subalgebra generated in $\mathbb{O}$ by any two octonions is associative and isomorphic to either $\mathbb{H}$, or $\mathbb{C}$, or $\mathbb{R}$, see [3].

First of all, we recall the standard model of projective spaces over the associative algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Let $\mathbb{F}^{n+1}$ be a linear space of vectors $\mathbf{a} = (a_0, \ldots, a_n)$, $a_i \in \mathbb{F}$, $1 \leq i \leq n$ with the right multiplication by scalars $a \in \mathbb{F}$, the Hermitian inner product

$$(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^n \bar{a}_i b_i, \quad \mathbf{a}, \mathbf{b} \in \mathbb{F}^{n+1}, \tag{5.3}$$

and the norm $|\mathbf{a}|$,

$$|\mathbf{a}|^2 = (\mathbf{a}, \mathbf{a}) = \sum_{i=0}^n |a_i|^2. \tag{5.4}$$

In this case, in view of associativity of the algebras $\mathbb{F} = \mathbb{R}$, $\mathbb{C}, \mathbb{H}$, a projective space $\mathbb{F}P^n$ can be defined as a set of one-dimensional (over $\mathbb{F}$) subspaces in $\mathbb{F}^{n+1}$:

$$\mathbb{F}P^n = \{p(\mathbf{a}) = \mathbf{a}\mathbb{F} : \mathbf{a} \in \mathbb{F}^{n+1},\ |\mathbf{a}| = 1\}. \tag{5.5}$$

The metric θ on $\mathbb{F}P^n$ is defined by

$$\cos \frac{1}{2}\theta(\mathbf{a}, \mathbf{b}) = |(\mathbf{a}, \mathbf{b})|, \quad \mathbf{a}, \mathbf{b} \in \mathbb{F}^{n+1}, \quad |\mathbf{a}| = |\mathbf{b}| = 1,\ 0 \leq \theta(\mathbf{a}, \mathbf{b}) \leq \pi, \tag{5.6}$$

i.e. $\frac{1}{2}\theta(\mathbf{a},\mathbf{b})$ is the angle between the subspaces $p(\mathbf{a})$ and $p(\mathbf{b})$. The transitive group of isometries $U(n+1,\mathbb{F})$ for the metric θ consists of nondegenerate linear transformations of the space $\mathbb{F}^{n+1}$, preserving the inner product (5.3), and the stabilizer of a point is isomorphic to the subgroup $U(n,\mathbb{F})\times U(1,\mathbb{F})$. Hence,

$$\mathbb{F}P^n = U(n+1,\mathbb{F})/U(n,\mathbb{F})\times U(1,\mathbb{F}). \tag{5.7}$$

The groups $U(n+1,\mathbb{F})$ can be easily determined (they have been indicated in section 2 in the list of compact connected two-point homogeneous spaces). A Riemannian $U(n+1,\mathbb{F})$-invariant structure corresponding to the metric θ can be also defined on the projective space (5.5), and one can easily check that these spaces are two-point homogeneous spaces.

There is another model where a projective space $\mathbb{F}P^n$, $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$, is identified with the set of orthogonal projectors onto the one-dimensional subspaces in $\mathbb{F}^{n+1}$. This model admits a generalization to the octonionic projective plane $\mathbb{O}P^2$ and in its terms the chordal metric can be naturally defined for all projective spaces.

Let $\mathcal{H}(\mathbb{F}^{n+1})$ denote the set of all Hermitian $(n+1)\times(n+1)$ matrices with the entries in $\mathbb{F}$, $\mathbb{F}=\mathbb{R}$, $\mathbb{C},\mathbb{H},\mathbb{O}$,

$$\mathcal{H}(\mathbb{F}^{n+1}) = \{A=((a_{ij})) : a_{ij}=\overline{a}_{ji},\ a_{ij}\in\mathbb{F},\ 0\le i,j\le n\}. \tag{5.8}$$

It is clear that $\mathcal{H}(\mathbb{F}^{n+1})$ is a linear space over $\mathbb{R}$ of dimension

$$m = \dim_{\mathbb{R}}\mathcal{H}(\mathbb{F}^{n+1}) = \frac{1}{2}(n+1)(d+2), \quad d=nd_0. \tag{5.9}$$

The linear space $\mathcal{H}(\mathbb{F}^{n+1})$ is equipped with the symmetric real-valued inner product

$$\langle A,B\rangle = \frac{1}{2}\operatorname{Tr}(AB+BA) = \operatorname{Re}\operatorname{Tr}AB = \operatorname{Re}\sum_{i,j=0}^{n} a_{ij}\overline{b_{ij}} \tag{5.10}$$

and the norm

$$\|A\| = (\operatorname{Tr}A^2)^{1/2} = \left(\sum_{i,j=0}^{n}|a_{ij}|^2\right)^{1/2}, \tag{5.11}$$

here $\operatorname{Tr}A=\sum_{i=0}^{n}a_{ii}$ denotes the trace of a matrix A. For the distance $\|A-B\|$ between matrices $A,B\in\mathcal{H}(\mathbb{F}^{n+1})$, we have

$$\|A-B\|^2 = \|A\|^2+\|B\|^2-2\langle A,B\rangle. \tag{5.12}$$

Thus, $\mathcal{H}(\mathbb{F}^{n+1})$ can be thought of as the m-dimensional Euclidean space.

If $\mathbb{F} \neq \mathbb{O}$, the orthogonal projector $\Pi_{\mathbf{a}} \in \mathcal{H}(\mathbb{F}^{n+1})$ onto a one-dimensional subspace $p(\mathbf{a}) = \mathbf{a}\mathbb{F}$, $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{F}^{n+1}$, $|\mathbf{a}| = 1$, can be given by $\Pi_{\mathbf{a}} = \mathbf{a}(\mathbf{a}, \cdot)$ or in the matrix form $\Pi_{\mathbf{a}} = ((a_i \bar{a}_j))$, $0 \leq i, j \leq n$. Therefore, the projective space (5.5) can be written as follows

$$\mathbb{F}P^n = \{\Pi \in \mathcal{H}(\mathbb{F}^{n+1}) : \Pi^2 = \Pi,\ \operatorname{Tr}\Pi = 1\}. \tag{5.13}$$

The group of isometries $U(n+1, \mathbb{F})$ acts on such projectors by the formula $g(\Pi) = g\Pi g^{-1}$, $g \in U(n+1, \mathbb{F})$.

For the octonionic projective plane $\mathbb{O}P^2$ the similar model (due to Freudenthal and Jordan) is also known. A detailed discussion of this model can be found in [3, 7, 18, 20] including an explanation why octonionic projective spaces $\mathbb{O}P^n$ do not exist if $n > 2$. In this model one puts by definition

$$\mathbb{O}P^2 = \{\Pi \in \mathcal{H}(\mathbb{O}^3) : \Pi^2 = \Pi,\ \operatorname{Tr}\Pi = 1\}. \tag{5.14}$$

Thus, the formulas (5.13) and (5.14) are quite similar. One can check that each matrix in (5.14) can be written as $\Pi_{\mathbf{a}} \in \mathbb{O}P^2$ for a vector $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{O}^3$, where $\Pi_{\mathbf{a}} = ((a_i \bar{a}_j))$, $0 \leq i, j \leq 2$, $|\mathbf{a}|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 = 1$, and additionally $(a_0 a_1) a_2 = a_0 (a_1 a_2)$, see [20, Lemma 14.90]. The additional condition means that the subalgebra in $\mathbb{O}$ generated by the coordinates a_0, a_1, a_2 is associative. Using this fact, one can easily show that $\mathbb{O}P^2$ is a 16-dimensional compact connected Riemannian manifold, see [20, p. 290].

The group of nondegenerate linear transformations g of the space $\mathcal{H}(\mathbb{O}^3)$ preserving the squares $g(A^2) = g(A)^2$, $A \in \mathcal{H}(\mathbb{O}^3)$, is isomorphic to the 52-dimensional exceptional Lie group F_4. This group also preserves the trace, inner product (5.10) and norm (5.11) of matrices $A \in \mathcal{H}(\mathbb{O}^3)$. The group F_4 is transitive on $\mathbb{O}P^2$, and the stabilizer of a point is isomorphic to the spinor group Spin(9), see [20, Lemma 14.96 and Theorem 14.99]. Hence, $\mathbb{O}P^2 = F_4/\operatorname{Spin}(9)$ is a homogeneous space, and one can prove that $\mathbb{O}P^2$ is a two-point homogeneous space.

For our discussion we need to describe the structure of geodesics in projective spaces. Such a description can be easily done in terms of models (5.13) and (5.14). It is known, see [7,21,33], that all geodesics on a two-point homogeneous space $Q(d, d_0)$ are closed and homeomorphic to the unit circle. The group of isometries is transitive on the set of geodesics and the the stabilizer of a point is transitive on the set of geodesics passing through this point.

Therefore, all geodesics have the same length 2π (under the normalization (1.1) for the invariant Riemannian distance).

The inclusions (5.1) induce the following inclusions of the corresponding projective spaces

$$\mathbb{F}_1 P^{n_1} \subseteq \mathbb{F}P^n, \quad \mathbb{F}_1 \subseteq \mathbb{F}, \quad n_1 \leq n, \tag{5.15}$$

moreover, the subspace $\mathbb{F}_1 P^{n_1}$ is a geodesic submanifold in $\mathbb{F}P^n$, see [7, Sec. 3.24]. Particularly, the real projective line $\mathbb{R}P^1$, homeomorphic to the unit circle S^1, is embedded as a geodesic into all projective spaces $\mathbb{F}P^n$,

$$S^1 \approx \mathbb{R}P^1 \subset \mathbb{F}P^n, \tag{5.16}$$

see [7, Proposition 3.32]. In (5.16) $n = 2$ if $\mathbb{F} = \mathbb{O}$.

Using the models (5.13) and (5.14), we can write the real projective line $\mathbb{R}P^1$ as the following set of 2×2 matrices:

$$\mathbb{R}P^1 = \{\zeta(u), u \in \mathbb{R}/\pi\mathbb{Z}\}, \tag{5.17}$$

$$\zeta(u) = \begin{pmatrix} \cos^2 u & \sin u \cos u \\ \sin u \cos u & \sin^2 u \end{pmatrix} = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos u & \sin u \\ \sin u & \cos u \end{pmatrix}.$$

For each $u \in \mathbb{R}$ the matrix $\zeta(u)$ is an orthogonal projector onto the one-dimensional subspace $x\mathbb{R}$, $x = (\cos u, \sin u) \in S^1$. The embedding $\mathbb{R}P^1$ into $\mathbb{F}P^n$ can be written as the following set of $(n+1) \times (n+1)$ matrices:

$$Z = \{Z(u), u \in \mathbb{R}/\pi\mathbb{Z}\} \subset \mathbb{F}P^n, \tag{5.18}$$

$$Z(u) = \begin{pmatrix} \zeta(u) & 0_{n-1,2} \\ 0_{2,n-1} & 0_{n-1,n-1} \end{pmatrix},$$

where $0_{k,l}$ denotes the zero matrix of size $k \times l$. The set of matrices (5.18) is a geodesic in $\mathbb{F}P^n$. All other geodesics are of the form $g(Z)$, where $g \in G$ is an isometry of the space $\mathbb{F}P^n$. The parameter u in (5.18) and the geodesic distance θ on the space $\mathbb{F}P^n$ are related by

$$\theta(Z(u), Z(0)) = 2|u|, \quad -\pi/2 < u \leq \pi/2, \tag{5.19}$$

and for all $u \in \mathbb{R}$ this formula can be extended by periodicity. Particularly, we have

$$\theta(Z(u/2), Z(-u/2)) = \begin{cases} 2\min\{u, \pi - u\} & \text{if} \quad 0 \leq u \leq \pi, \\ 2u & \text{if} \quad 0 \leq u \leq \pi/2. \end{cases}$$

Therefore,

$$\theta(Z(v), Z(-v)) = 4v, \quad 0 \le v \le \pi/4. \tag{5.20}$$

In such a form, this relation will be needed in the next section.

Now, we define the chordal distance on projective spaces. The formulas (5.13), (5.14) я (5.11) imply

$$\|\Pi\|^2 = \operatorname{Tr}\Pi^2 = \operatorname{Tr}\Pi = 1. \tag{5.21}$$

for any $\Pi \in \mathbb{F}P^n$. Therefore, the projective spaces $\mathbb{F}P^n$, defined by (5.13) and (5.14), are submanifolds in the unit sphere

$$\mathbb{F}P^n \subset S^{m-1} = \{A \in \mathcal{H}(\mathbb{F}^{n+1}) : \|A\| = 1\} \subset \mathcal{H}(\mathbb{F}^{n+1}) \approx \mathbb{R}^m. \tag{5.22}$$

It fact, this is an embedding of $\mathbb{F}P^n$ into the $(m-2)$-dimensional sphere, the intersection of the sphere S^{m-1} with the hyperplane in $\mathcal{H}(\mathbb{F}^{n+1})$ defined by $\operatorname{Tr} A = 1$, see (5.21).

The *chordal distance* $\tau(\Pi_1, \Pi_2)$ between $\Pi_1, \Pi_2 \in \mathbb{F}P^n$ is defined as the Euclidean distance (5.12):

$$\tau(\Pi_1, \Pi_2) = \frac{1}{\sqrt{2}}\|\Pi_1 - \Pi_2\| = (1 - \langle \Pi_1, \Pi_2 \rangle)^{1/2}. \tag{5.23}$$

The coefficient $1/\sqrt{2}$ is chosen to satisfy $\operatorname{diam}(\tau, \mathbb{F}P^n) = 1$.

It is clear from (5.23) that $\tau(g(\Pi_1),\, g(\Pi_2)) = \tau(\Pi_1, \Pi_2)$ for all isometries $g \in G$ of the space $\mathbb{F}P^n$. Since $\mathbb{F}P^n$ is a two-point homogeneous space, for any $\Pi_1, \Pi_2 \in \mathbb{F}P^n$ with $\theta(\Pi_1, \Pi_2) = 2u$, $0 \le u \le \frac{1}{2}\pi$, there exists $g \in G$, such that $g(\Pi_1) = Z(u)$, $g(\Pi_2) = Z(0)$. From (5.23), (5.18) and (5.17), we obtain $\tau(Z(u), Z(0)) = \sin u = \sin \frac{1}{2}\theta(\Pi(u), \Pi(0))$. Therefore,

$$\tau(\Pi_1, \Pi_2) = \sin \frac{1}{2}\theta(\Pi_1, \Pi_2), \tag{5.24}$$

as it was claimed before in (2.4).

Notice also that antipodal points $\Pi_+, \Pi_- \in \mathbb{F}P^n$, i.e. $\theta(\Pi_+, \Pi_-) = \pi$ and $\tau(\Pi_+, \Pi_-) = 1$, can be characterized by the orthogonality condition $\langle \Pi_+, \Pi_- \rangle = 0$, see (5.23), (5.24).

6. Proof of Theorem 2.1

The proof of Theorem 2.1 is relying on the following special representation of the symmetric difference metric (1.13) on a compact metric space $\mathcal{M}$, see [28, Lemma 2.1]. Here this representation is given in a form adapted for the chordal metric (5.23) on the spaces $Q(d, d_0)$.

Lemma 6.1. *Let the weight function η be summarized on the interval $[0, \pi]$, then*

$$\theta^{\Delta}(\eta, y_1, y_2) = \frac{1}{2} \int\limits_{\mathcal{M}} |\sigma(\theta(y_1, y)) - \sigma(\theta(y_2, y))| \, d\mu(y) \tag{6.1}$$

with the nonincreasing function

$$\sigma(r) = \int\limits_{r}^{\pi} \eta(u) \, du. \tag{6.2}$$

Particularly, if $\mathcal{M}$ is a two-point homogeneous space $Q = Q(d, d_0)$ and the weight function $\eta^{\natural}(r) = \sin r$, then

$$\theta^{\Delta}(\eta^{\natural}, y_1, y_2) = \int\limits_{Q} |\tau(y_1, y)^2 - \tau(y_2, y)^2| \, d\mu(y), \tag{6.3}$$

where $\tau(\cdot, \cdot)$ is the chordal metric (5.23) *on $Q(d, d_0)$.*

Proof. For brevity, we write $\theta(y_1, y) = \theta_1$ and $\theta(y_2, y) = \theta_2$. Using (1.13), (1.17) and (1.16), we obtain

$$\begin{aligned} &\theta^{\Delta}(\eta, y_1, y_2) \\ &= \frac{1}{2} \int\limits_{\mathcal{M}} \left(\int\limits_{0}^{\pi} (\chi(r - \theta_1) + \chi(r - \theta_2) - 2\chi(r - \theta_1)\chi(r - \theta_2))\eta(r) \, dr \right) d\mu(y) \\ &= \frac{1}{2} \int\limits_{\mathcal{M}} (\sigma(\theta_1) + \sigma(\theta_2) - 2\sigma(\max\{\theta_1, \theta_2\})) \, d\mu(y). \end{aligned} \tag{6.4}$$

Since σ is a nonincreasing function, we have

$$2\sigma(\max\{\theta_1, \theta_2\}) = 2\min\{\sigma(\theta_1), \sigma(\theta_2)\} = \sigma(\theta_1) + \sigma(\theta_2) - |\sigma(\theta_1) - \sigma(\theta_2)|. \tag{6.5}$$

Substituting (6.5) into (6.4), we obtain (6.1).

If $\eta^{\natural}(r) = \sin r$, then $\sigma^{\natural}(r) = 2 - 2\sin^2 r/2$. Substituting this expression into (6.1) and using (5.24), we obtain (6.3). $\square$

For completeness we give at first the proof of Theorem 2.1 in the case of spheres S^d.

Proof of Theorem 2.1 for spheres. For the sphere S^d the chordal metric τ is defined (2.5). We have

$$\begin{aligned}\tau(y_1,y)^2-\tau(y_2,y)^2&=\frac{1}{4}(\|y_1-y\|^2-\|y_2-y\|^2)\\&=-\frac{1}{2}(y_1-y_2,y)=-\tau(y_1,y_2)(x,y),\ y_1,y_2\in S^d,\end{aligned}\tag{6.6}$$

where $x=\|y_1-y_2\|^{-1}(y_1-y_2)\in S^d$. Substituting (6.6) into (6.3), we obtain

$$\theta^{\Delta}(\eta^{\natural},y_1,y_2)=\tau(y_1,y_2)\int\limits_{S^d}|(x,y)|\,d\mu(y).\tag{6.7}$$

It is clear that the integral in (6.7) is independent of $x\in S^d$. This proves the equality (2.8) for S^d with the constant

$$\gamma(S^d)=\left(\int\limits_{S^d}|(x,y)|\,d\mu(y)\right)^{-1}.$$

□

Proof of Theorem 2.1 for projective spaces. We write Π_1,Π_2,Π for points in the models of projective spaces (5.13) and (5.14). With this notation, the relation (6.3) takes the form

$$\theta^{\Delta}(\eta^{\natural},\Pi_1,\Pi_2)=\int\limits_{\mathbb{F}P^n}|\tau(\Pi_1,\Pi)^2-\tau(\Pi_2,\Pi)^2|\,d\mu(\Pi).\tag{6.8}$$

Since $\mathbb{F}P^n$ is a two-point homogeneous space, for $\Pi_1,\Pi_2\in\mathbb{F}P^n$ with $\theta(\Pi_1,\Pi_2)=4v$, $0\le v\le\pi/4$, there exists an isometry $g\in G$, such that $g(\Pi_1)=Z(v)$, $g(\Pi_2)=Z(-v)$, see (5.20). Therefore,

$$\begin{aligned}&\int\limits_{\mathbb{F}P^n}|\tau(\Pi_1,\Pi)^2-\tau(\Pi_2,\Pi)^2|\,d\mu(\Pi)\\&\qquad=\int\limits_{\mathbb{F}P^n}|\tau(Z(v),\Pi)^2-\tau(Z(-v),\Pi)^2|\,d\mu(\Pi).\end{aligned}\tag{6.9}$$

From the definition (5.23), we obtain

$$\tau(Z(v),\Pi)^2-\tau(Z(-v),\Pi)^2=\frac{1}{2}(\|Z(v)-\Pi\|^2-\|Z(-v)-\Pi\|^2)$$
$$=\langle Z(v)-Z(-v),\Pi\rangle. \tag{6.10}$$

The formulas (5.17) and (5.18) imply

$$Z(v)-Z(-v)=\begin{pmatrix}\zeta(v)-\zeta(-v) & 0_{n-1,2}\\ 0_{2,n-1} & 0_{n-1,n-1}\end{pmatrix}$$

and

$$\zeta(v)-\zeta(-v)=\begin{pmatrix}0 & \sin 2u\\ \sin 2u & 0\end{pmatrix}=\sin 2u(\zeta_+-\zeta_-),$$

where

$$\zeta_+=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},\quad \zeta_-=\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

Therefore,

$$Z(v)-Z(-v)=\sin 2v(Z_+-Z_-), \tag{6.11}$$

where

$$Z_\pm=\begin{pmatrix}\zeta_\pm & 0_{n-1,2}\\ 0_{2,n-1} & 0_{n-1,n-1}\end{pmatrix}.$$

We have $Z_\pm^*=Z_\pm$, $Z_\pm^2=Z_\pm$, $\operatorname{Tr} Z_\pm=1$, i.e. $Z_\pm\in\mathbb{F}P^n$, and $\langle Z_+,Z_-\rangle=0$, i.e. Z_+ and Z_- are antipodal points. Using (5.24), we can write

$$\tau(\Pi_1,\Pi_2)=\tau(Z(v),Z(-v))=\sin 2v,$$

and the equality (6.11) takes the form

$$Z(v)-Z(-v)=\tau(\Pi_1,\Pi_2)(Z_+-Z_-). \tag{6.12}$$

Substituting (6.12) into (6.10), we find that

$$\tau(Z(v),\Pi)^2-\tau(Z(-v),\Pi)^2=\tau(\Pi_1,\Pi_2)\langle Z_+-Z_-,\Pi\rangle. \tag{6.13}$$

Substituting (6.13) into (6.9) and using (6.8), we obtain

$$\theta^\Delta(\eta^\natural,\Pi_1,\Pi_2)=\tau(\Pi_1,\Pi_2)\theta^\Delta(\eta^\natural,Z_+,Z_-). \tag{6.14}$$

where

$$\theta^{\Delta}(\eta^{\natural}, Z_+, Z_-) = \int\limits_{\mathbb{F}P^n} |\langle Z_+ - Z_-, \Pi\rangle| \, d\mu(\Pi). \tag{6.15}$$

The integral (6.15) is independent of Π_1, Π_2, This proves the equality (2.8) for $\mathbb{F}P^n$ with the constant

$$\gamma(\mathbb{F}P^n) = \left(\int\limits_{\mathbb{F}P^n} |\langle Z_+ - Z_-, \Pi\rangle| \, d\mu(\Pi) \right)^{-1}.$$

In this formula any pair of antipodal points in $\mathbb{F}P^n$ can be taken instead of Z_+, Z_-. The proof of Theorem 2.1 is complete. □

7. Proof of Lemma 2.1

(i) In (1.23) we put $y_1 = y_2 = y$ to obtain

$$\lambda_r(y, y) = v_r - v_r^2 = v_r v_r'. \tag{7.1}$$

Applying the Cauchy–Schwarz inequality to (1.6), we obtain

$$|\lambda_r(y_1, y_2)| \le (\lambda_r(y_1, y_2)\lambda_r(y_2, y_2))^{1/2} = v_r v_r'. \tag{7.2}$$

Using the weak invariance principl (1.29), the formula (1.26) and the bound (7.2), we obtain

$$\theta_r^{\Delta}(y_1, y_2) \le 2 v_r v_r'. \tag{7.3}$$

Substituting the bounds (2.3) for the volumes v_r and v_r' into (7.2) and (7.3), we obtain the bounds (2.15). Integrating (2.15) with $\eta \in W(d+1, d_0+1)$, we obtain the bounds (2.16).

(ii) We can assume that $0 < r < \pi$, since $\theta_r^{\Delta}(y_1, y_2) = 0$ identically, if $r = 0$ or $r = \pi$. For brevity, we write $\delta = \theta(y_1, y_2)/2$. The parameters r and δ vary in the region $0 < r < \pi$, $0 \le \delta \le \pi/2$. This rectangular region can be represented as a disjoint union of three triangular regions:

(a) $0 < r < \delta$, $0 \le \delta \le \frac{1}{2}\pi$,
(b) $\pi > r \ge \pi - \delta$, $0 \le \delta \le \frac{1}{2}\pi$,
(c) $r > \delta$, $0 < r < \pi - \delta$, $0 \le \delta < \frac{1}{2}\pi$.

In each of these triangular regions we will prove the bound (2.17). For $r \in [0, \pi]$, the function $\sin \frac{1}{2} r$ is increasing while $\cos \frac{1}{2} r$ is decreasing, and the bounds are satisfied

$$\sin \frac{1}{2} r \simeq r, \quad \cos \frac{1}{2} r \simeq \pi - r. \tag{7.4}$$

Case (a). Using the relations (1.25), (2.2), (2.3) and (7.4), we obtain

$$\begin{aligned} \theta_r^{\Delta}(y_1, y_2) \leq v_r &\simeq \int_0^r (\sin \frac{1}{2} u)^{d-1} (\cos \frac{1}{2} u)^{d_0 - 1} \, du \\ &\lesssim \int_0^r (\sin \frac{1}{2} u)^{d-1} \, du \simeq (\sin \frac{1}{2} r)^{d-1} r \\ &\lesssim (\sin \frac{1}{2} r)^{d-1} (\cos \frac{1}{2} r)^{d_0 - 1} \delta. \end{aligned} \tag{7.5}$$

Case (b). Similarly, from (1.25), (2.2), (2.3) and (7.4), we obtain

$$\begin{aligned} \theta_r^{\Delta}(y_1, y_2) \leq v_r' &\simeq \int_r^{\pi} (\sin \frac{1}{2} u)^{d-1} (\cos \frac{1}{2} u)^{d_0 - 1} \, du \\ &\lesssim \int_r^{\pi} (\cos \frac{1}{2} u)^{d_0 - 1} \, du \simeq (\cos \frac{1}{2} r)^{d_0 - 1} (\pi - r) \\ &\lesssim (\sin \frac{1}{2} r)^{d-1} (\cos \frac{1}{2} r)^{d_0 - 1} \delta \end{aligned} \tag{7.6}$$

Case (c). Since $\theta(y_1, y_2) < \pi$, there exists the unique geodesic $\gamma \subset Q(d, d_0)$ of shortest length $\theta(y_1, y_2)$ joining points y_1, y_2, яя. [21, Chap. VII, Sec. 10]. Let y_0 denote its midpoint, i.e. $y_0 \in \gamma$, $\theta(y_1, y_0) = \theta(y_2, y_0) = \delta$. The triangle inequality for the metric θ implies that the ball $B_{r-\delta}(y_0)$ is contained in the intersection $B_r(y_1) \cap B_r(y_2)$. Hence

$$\mu(B_r(y_1) \cap B_r(y_2)) \geq v_{r-\delta}. \tag{7.7}$$

Using again the relations (1.25), (2.2), (2.3) and (7.7), we obtain

$$\theta_r^{\Delta}(y_1, y_2) \le v_r - v_{r-\delta} \simeq \int\limits_{r-\delta}^{r} (\sin\frac{1}{2}u)^{d-1}(\cos\frac{1}{2}u)^{d_0-1}\,du$$
$$\lesssim (\sin\frac{1}{2}r)^{d-1}(\cos\frac{1}{2}(r-\delta))^{d_0-1} \simeq (\sin\frac{1}{2}r)^{d-1}(\pi - r + \delta)^{d_0-1}$$
$$\simeq (\sin\frac{1}{2}r)^{d-1}(\pi - r)^{d_0-1}\left(1 + \frac{\delta}{\pi - r}\right)^{d_0-1} \lesssim (\sin\frac{1}{2}r)^{d-1}(\pi - r)^{d_0-1}\delta$$
$$\simeq (\sin\frac{1}{2}r)^{d-1}(\cos\frac{1}{2}r)^{d_0-1}\delta. \tag{7.8}$$

Now, the bound (2.17) follows from the bounds (7.6), (7.7), (7.8). Integrating (2.17) with $\eta \in W(d, d_0)$, we obtain the bound (2.18). The proof of Lemma 2.1 is complete.

C. Spherical functions and bounds for discrepancies and sums of distances

8. Commutative spaces and spherical functions

In this section we outline general facts on harmonic analysis on commutative spaces. The two-point homogeneous spaces $Q(d, d_0)$ are an important subclass of such spaces. The general theory of commutative spaces is given in [34], see also [22, 32]. For compact groups this theory is rather simple.

Let G be a compact group and $K \subset G$ a closed subgroup. Denote by μ_G and μ_K Haar measures on the groups G and K, correspondingly, $\mu_G(G) = \mu_K(K) = 1$. As before, μ denotes the invariant measure on the homogeneous space $Q = G/K$, and $\mu_G = \mu_K \times \mu$. We write $L_q(G)$, $q = 1, 2$, for the space of functions on G integrable with the power q with respect to the Haar measure, $L_q(G/K)$ and $L_q(K\backslash G/K)$ for the subspaces of functions in $L_q(G)$ satisfying $f(gk) = f(g)$, $k \in K$, and, correspondingly, $f(k_1gk_2) = f(g), k_1, k_2 \in K$. Obviously, functions in these subspaces can be thought of as functions on $Q = G/K$. The spaces $L_1(K \setminus G/K) \subset L_1(G/K) \subset L_1(G)$ are Banach

algebras with respect to a multiplication defined as the convolution

$$f_1 * f_2(g) = \int\limits_G f_1(gh^{-1}) f_2(h)\, d\mu_G(h). \tag{8.1}$$

These algebras are associative but, in general, they are not commutative. For example, the algebra $L_1(G)$ is commutative if and only if G is commutative. If the algebra $L_1(K\backslash G/K)$ is commutative, the pair of groups $K \subset G$ is called a *Gelfand pair* and the corresponding homogeneous space $Q = G/K$ is called a *commutative space*, see [34]. The subgroup K in a Gelfand pair is called *massive*, see [32]. Two large classes of commutative spaces are Riemannian symmetric spaces and two-point homogeneous spaces, see [22,34]. The spaces $Q(d, d_0)$ belong to both of these classes.

Consider the following unitary representation of a group G in the space $L_2(G/K)$

$$T(g)f(h) = f(g^{-1}h), \quad f \in L_2(G/K), \quad g, h \in G. \tag{8.2}$$

and its decomposition into the orthogonal sum

$$T = \widehat{\bigoplus_{l\geq 0}} T_l, \qquad L_2(G/K) = \widehat{\bigoplus_{l\geq 0}} V_l \tag{8.3}$$

of unitary irreducible representations T_l in finite-dimensional spaces V_l. Let $m_l = \dim V_l$, and $(\cdot,\cdot)$ denote the inner product in V_l.

If $Q = G/K$ is a commutative space, then the irreducible representations T_l occurring in (8.3) are pair-wise nonequivalent and each subspace V_l in (8.3) contains a single K-invariant unit vector $\mathbf{e}^{(l)}$, i.e. $T_l(k)\mathbf{e}^{(l)} = \mathbf{e}^{(l)}$ for all $k \in K$.

Fix an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_{m_l}$ in the space V_l, such that $\mathbf{e}_1 = \mathbf{e}^{(l)}$ and define the matrix elements $t_{ij}^{(l)}(g) = (T_l(g)\mathbf{e}_i, \mathbf{e}_j)$. It is clear that

$$\left.\begin{aligned} t_{ij}^{(l)}(g_1 g_2) &= \sum_{p=1}^{m_l} t_{ip}^{(l)}(g_1) t_{pj}^{(l)}(g_2), \\ t_{ij}^{(l)}(g^{-1}) &= \overline{t_{ji}^{(l)}(g)}. \end{aligned}\right\} \tag{8.4}$$

We also have the orthogonality relations

$$\int\limits_G t_{ij}^{(l)}(g)\overline{t_{ij}^{(l')}(g)}\, d\mu_G(g) = m_l^{-1}\delta_{ll'}\delta_{ii'}\delta_{jj'}. \tag{8.5}$$

The sets of functions $\{m_l^{1/2} t_{1j}^{(l)}(g),\ j = 1, \ldots, m_l,\ l \geq 0\}$ and $\{m_l^{1/2} t_{11}^{(l)}(g), l \geq 0\}$ are orthonormal bases in the spaces $L_2(G/K)$ and $L_2(K \setminus G/K)$, correspondingly, see [32].

The matrix elements $\varphi_l(g) = t_{11}^{(l)}(g)$ are called (zonal) *spherical functions.* Notice that the matrix elements $t_{1j}^{(l)}(g)$, $j = 2, \ldots, m_l$ are called associated spherical functions. The definition and the formula (8.4) imply immediately that all spherical functions are continuous, $\varphi_l(\mathbb{1}) = 1$, where $\mathbb{1}$ is the unit element of G, $|\varphi_l(g)| \leq 1$ for all $g \in G$, and

$$\left.\begin{aligned} \varphi_l(g_1 g_2^{-1}) &= \sum_{j=1}^{m_l} t_{1j}^{(l)}(g_1) \overline{t_{1j}^{(l)}(g_2)}, \\ \varphi_l(g) &= \overline{\varphi_l(g^{-1})}. \end{aligned}\right\} \tag{8.6}$$

It follows from (8.6) that φ_l is positive definite:

$$\sum_{1 \leq i,j \leq N} \overline{c_i} c_j \varphi_l(g_i^{-1} g_j) \geq 0 \tag{8.7}$$

for any $g_1, \ldots, g_N \in G$ and any complex numbers $c_1, \ldots, c_N$. From (8.1), (8.5) and (8.6), we obtain the following 'orthogonality relations' with respect to the convolution

$$(\varphi_l * \varphi_{l'})(g) = \delta_{ll'} m_l^{-1} \varphi_l(g). \tag{8.8}$$

Functions $f \in L_2(K \setminus G/K)$ have the following expansions

$$f(g) \sim \sum_{l \geq 0} m_l c_l(f) \varphi_l(g), \tag{8.9}$$

where $\sim$ denotes the L_2-convergence. The Fourier coefficients are given by

$$c_l(f) = \int_G f(g) \overline{\varphi_l(g)}\, d\mu_G(g), \tag{8.10}$$

and

$$\int_G |f(g)|^2\, d\mu_G(g) = \sum_{l \geq 0} m_l |c_l(f)|^2 \tag{8.11}$$

Substituting the expansion (8.9) for two functions $f_1, f_2 \in L_2(K \setminus G/K)$ into the definition of convolution (8.1) and using the relation (8.8), we obtain

$$f_1 * f_2(g) = \sum_{l \geq 0} m_l c_l(f_1) c_l(f_2) \varphi_l(g). \tag{8.12}$$

Applying the Cauchy–Schwarz inequality to (8.12) and using (8.11), we observe that the series (8.12) converges absolutely. Since the spherical functions φ_l are continuous and $|\varphi_l(g)| \leq 1$, we conclude that the convolution $f_1 * f_2$ is a continuous function.

The facts given above are true for all compact commutative spaces. Now we wish to specify the expansions (8.9)–(8.12) for two-point homogeneous spaces.

Let $K \subset G$ be compact groups and $Q = G/K$ a two-point homogeneous space with a G-invariant metric θ. Suppose that K is the stabilizer of a point $y_0 \in Q$. It follows from the definition, see section 2, that the subgroup K is transitive on each sphere $\Sigma_r(y_0) = \{y : \theta(y, y_0) = r\} \subset Q$, $r \in \mathcal{R}$, where $\mathcal{R} = \{\theta(y, y_0) : y \in Q\}$ is the set of radii. Thus, any function $f \in L_q(K \setminus G/K)$, as a function on Q, is constant on each sphere $\Sigma_r(y_0)$. Therefore, we can write

$$f(g) = F(\theta(gy_0, y_0)) \tag{8.13}$$

with a function $F(r)$, $r \in \mathcal{R}$. In other words, double cosets $K \setminus G/K$ are parametrized by radii $r \in \mathcal{R}$.

Since the metric θ is G-invariant and symmetric, we have the relations

$$\left.\begin{aligned} \theta(gy_0, y_0) &= \theta(y_0, g^{-1}y_0) = \theta(g^{-1}y_0, y_0), \\ \theta(g_1y_0, g_2y_0) &= \theta(y_0, g_1^{-1}g_2y_0) = \theta(g_1^{-1}g_2y_{0,0}\,). \end{aligned}\right\} \tag{8.14}$$

Comparing (8.13) and (8.14), we obtain

$$f(g) = f(g^{-1}). \tag{8.15}$$

Using (8.13), the convolution (8.1) of functions $f_1, f_2 \in L_2(K \setminus G/K)$ can be written in the form

$$\begin{aligned} (f_1 * f_2)(g_1^{-1}g_2) &= \int\limits_G F_1(\theta(g_1y_0, gy_0))F_2(\theta(gy_0, g_2y_0))\, d\mu(g) \\ &= \int\limits_Q F_1(\theta(y_1, y))F_2(\theta(y, y_2))\, d\mu(y), \end{aligned} \tag{8.16}$$

where $y_1 = g_1y_0$, $y_2 = g_2y_0$.

For a function of the form (8.13) we have

$$\int\limits_G f(g)\, d\mu_G(g) = \int\limits_Q F(\theta(y, y_0))\, d\mu(y) = \int\limits_{\mathcal{R}} F(r)\, dv_r, \tag{8.17}$$

where the last integral is thought of as a Stieltjes integral with the nondecreasing function $v_r = \mu(B_r(y_0))$, $r \in \mathcal{R}$.

It follows from (8.13) and (8.17) that the mapping $f \to F$ is an isometry of the space $L_2(K \setminus G/K)$ onto the space $L_2(\mathcal{R}, v_r)$ of functions $F(r)$, $r \in \mathcal{R}$, with the norm

$$\|F\| = \left(\int\limits_{\mathcal{R}} |F(r)|^2 \, dv_r \right)^{1/2}. \tag{8.18}$$

Since the zonal spherical functions $\varphi_l \in L_2(K \setminus G/K)$, they can be written in the form (8.13):

$$\varphi_l(g) = \Phi_l(\theta(gy_0, y_0)), \tag{8.19}$$

where $\Phi_l \in L_2(\mathcal{R}, v_r)$. Putting $y_1 = g_1 y_0$, $y_2 = g_2 y_2$, $g_1, g_2 \in G$, we can write (8.19) as follows

$$\varphi_l(g_1^{-1} g_2) = \Phi_l(\theta(g_1 y_0, g_2 y_0)) = \Phi_l(\theta(y_1, y_2)). \tag{8.20}$$

It follows from the properties of φ_l that Φ_l are continuous, $\Phi_l(0) = 1$, $|\Phi_l(r)| \le 1$, $r \in \mathcal{R}$, moreover, Φ_l are real-valued, in view of (8.6) and (8.15). The set of functions $\{m_l^{1/2} \Phi_l, l \ge 0\}$ is an orthonormal basis in the space $L_2(\mathcal{R}, v_r)$ and the expansion (8.9) for $F \in L_2(\mathcal{R}, v_r)$ takes the form

$$F(r) \sim \sum_{l \ge 0} m_l c_l(F) \, \Phi_l(r) \tag{8.21}$$

with the Fourier coefficients

$$c_l(F) = \int\limits_{\mathcal{R}} F(r) \Phi_l(r) \, dv_r. \tag{8.22}$$

Comparing the formulas (8.12), (8.16), (8.21), we arrive at the following relation

$$\int\limits_{Q} F_1(\theta(y_1, y)) F_2(\theta(y, y_2)) \, d\mu(y) = \sum_{l \ge 0} m_l c_l(F_1) c_l(F_2) \Phi_l(\theta(y_1, y_2)). \tag{8.23}$$

For all spaces $Q = Q(d, d_0)$ spherical functions are known, see яя. [16, Chp. 9, Sec. 2], [19, p. 178], [22, Chp. V, Thm. 4.5], [23, pp. 514–512, 543–544], [34, Thm. 11.4.21]. The functions Φ_l in (8.19) are explicitly given by

$$\Phi_l(r) = \Phi_l^{(\alpha,\beta)}(r) = \frac{P_l^{(\alpha,\beta)}(\cos r)}{P_l^{(\alpha,\beta)}(1)}, \quad r \in \mathcal{R} = [0, \pi], \tag{8.24}$$

where $P_l^{(\alpha,\beta)}(z)$ are the standard Jacobi polynomials of degree l normalized by

$$P_l^{(\alpha,\beta)}(1) = \binom{\alpha+l}{l} = \frac{(\alpha+1)\dots(\alpha+l)}{l!} \simeq l^\alpha, \tag{8.25}$$

see [31]. The parameters α, β in (8.24) and the dimensions d, d_0 in $Q(d, d_0)$ are related by

$$\alpha = \frac{d}{2} - 1, \quad \beta = \frac{d_0}{2} - 1 \tag{8.26}$$

In what follows, we will use dimensions d, d_0 as well as parameters α, β, assuming that they are related by (8.26). Notice that in this case, we have $\alpha \geq \beta \geq -1/2$ always.

We have the following orthogonality relations for Jacobi polynomials, see [31, Eq. (4.3.3)],

$$\int_0^\pi P_l^{(\alpha,\beta)}(\cos u) P_{l'}^{(\alpha,\beta)}(\cos u)(\sin\frac{1}{2}u)^{d-1}(\cos\frac{1}{2}u)^{d_0-1}\,du$$

$$= (\frac{1}{2})^{\alpha+\beta+1}\int_{-1}^{1} P_l^{(\alpha,\beta)}(z) P_{l'}^{(\alpha,\beta)}(z)(1-z)^\alpha(1+z)^\beta\,dz = M_l^{-1}\delta_{ll'}, \tag{8.27}$$

where $M_0 = \kappa(d, d_0)$ and

$$M_l = (2l+\alpha+\beta+1)\frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)} \simeq l, \quad l \geq 1. \tag{8.28}$$

Comparing the orthogonality relations (8.5) and (8.28), we obtain the explicit formula for dimensions m_l of irreducible representations T_l in (8.3):

$$m_l = M_l B(d/2, d_0/2)\binom{\alpha+l}{l}^2 \simeq l^{d-1}. \tag{8.29}$$

For functions $F \in L_2([0,\pi], v_r)$ the expansion (8.21) takes the form

$$F(r) \sim \sum_{l\geq 0} M_l C_l(F) P_l^{(\alpha,\beta)}(\cos r), \tag{8.30}$$

with the Fourier-Jacobi coefficients

$$C_l(F) = \int_0^\pi F(u) P_l^{(\alpha,\beta)}(\cos u)(\sin\frac{1}{2}u)^{d-1}(\cos\frac{1}{2}u)^{d_0-1}\,du. \tag{8.31}$$

The Fourier coefficients (8.22) and Fourier-Jacobi coefficients (8.31) are related by

$$c_l(F) = C_l(F)\frac{\kappa(d,d_0)}{P_l^{(\alpha,\beta)}(1)}, \quad l \geq 0. \tag{8.32}$$

Using the relations (8.24), (8.31) and (8.32), we can write the relation (8.23) in the form

$$\int\limits_Q F_1(\theta(y_1,y))F_2(\theta(y,y_2))\,d\mu(y)$$

$$= \kappa(d,d_0)\sum_{l\geq 0} M_l C_l(F_1)C_l(F_2)\frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1,y_2))}{P_l^{(\alpha,\beta)}(1)} \tag{8.33}$$

This relation will be used in section 9 to obtain zonal spherical function expansions for discrepancies and metrics .

The condition of positive definiteness (8.7) for the zonal spherical functions (8.24) will be used in section 11 in the following special form

$$\varphi_l[D_N] = \sum_{x_1,x_2\in D_N} \frac{P_l^{(\alpha,\beta)}(\cos\theta(x_1,x_2))}{P_l^{(\alpha,\beta)}(1)} \geq 0 \tag{8.34}$$

for an arbitrary N-point subset $D_N \subset Q(d,d_0)$.

Obviously, the conditions (3.2), (3.3) in the definition of t-designs $D_N \subset Q(d,d_0)$ are equivalent to the following equalities, see also [4, 23],

$$\varphi_l[D_N] = 0, \quad l = 0,1,\ldots,t. \tag{8.35}$$

9. Spherical function expansions for discrepancies and metrics

In this section we obtain explicit zonal spherical function expansions for the kernels (1.6), (1.8) and the symmetric difference metrics (1.33), (1.14) on the spaces $Q(d,d_0)$. In the next sections we will estimate the coefficients of these expansions.

First of all, we recall the main facts on asymptotic behavior of Jacobi polynomials $P_l^{(\alpha,\beta)}(z)$, $z \in [-1,1]$, $\alpha \geq -1/2$, $\beta \geq -1/2$, as $l \to \infty$. The behavior is extremely irregular on the interval $z \in [-1,1]$: Inside the interval

Jacobi polynomials oscillate and are of order $l^{-1/2}$, while in neighborhoods of the end points $z = 1$ and $z = -1$ they increase sharply up to the quantities of order l^α and l^β, correspondingly. We describe such irregularities in terms of the following weighted bounds. We put

$$J_l^{(\alpha,\beta)}(r) = (\sin\frac{1}{2}r)^{\alpha+\frac{1}{2}}(\cos\frac{1}{2}r)^{\beta+\frac{1}{2}}P_l^{(\alpha,\beta)}(\cos r), \quad r \in [0,\pi] \tag{9.1}$$

For $r \in [c_0 l^{-1}, \pi - c_0 l^{-1}]$, where $c_0 > 0$ is an arbitrary constant, we have the asymptotic formula

$$J_l^{(\alpha,\beta)}(r) = (\pi l)^{-1/2}\{\cos[(l+l_0)r + r_0] + O((l\sin r)^{-1})\}, \tag{9.2}$$

where $l_0 = (\alpha+\beta+1)/2$, $r_0 = -\pi(2\alpha+1)/4$, see [31, Thm. 8.21.3].

For $r \in [0, c_0 l^{-1}]$ or $r \in [\pi - c_0 l^{-1}, \pi]$, we have the bound $J_l^{(\alpha,\beta)}(r) = O(l^{-1/2})$, see [31, Thm. 7.32.2]. This bound together with (9.2) implies the following bound

$$|J_l^{(\alpha,\beta)}(r)| < c(l+1)^{-1/2}, \quad l \geq 0, \tag{9.3}$$

uniformly for all $r \in [0,\pi]$ with a constant depending only on α and β.

Now, we consider the measure of the intersection of two balls $B_r(y_1)$ and $B_r(y_2)$ in the space $Q = Q(d, d_0)$

$$\mu_r(y_1,y_2) = \mu(B_r(y_1)\cap B_r(y_2)) = \int\limits_Q \chi_r(\theta(y_1,y))\chi_r(\theta(y,y_2))\,d\mu(y), \tag{9.4}$$

where $\chi_r(\cdot)$ is the characteristic function of the interval $[0,r]$, $0 \leq r \leq \pi$, see (1.16).

Lemma 9.1. *The kernel* (9.4) *has the following zonal spherical function expansion*

$$\mu_r(y_1,y_2) = v_r^2 + \kappa(d,d_0)\sum_{l\geq 1} l^{-2} M_l a_l(r)\frac{P^{(\alpha,\beta)}(\cos\theta(y_1,y_2))}{P_l^{(\alpha,\beta)}(1)}, \tag{9.5}$$

where $v_r = \mu(B_r(y))$ *and*

$$\begin{aligned} a_l(r) &= (\sin\frac{1}{2}r)^{2d}(\cos\frac{1}{2}r)^{2d_0}\left\{P_{l-1}^{(\alpha+1,\beta+1)}(\cos r)\right\}^2 \\ &= (\sin\frac{1}{2}r)^{d-1}(\cos\frac{1}{2}r)^{d_0-1}\left\{J_{l-1}^{(\alpha+1,\beta+1)}(r)\right\}^2. \end{aligned} \tag{9.6}$$

The coefficients in (9.5) *satisfy*

$$M_l a_l(r) \le c(\sin\frac{1}{2}r)^{d-1}(\cos\frac{1}{2}r)^{d_0-1} \tag{9.7}$$

with a constant depending only on d *and* d_0*. Furthermore, we have the equality*

$$\kappa(d,d_0)\sum_{l\ge 1} l^{-2} M_l a_l(r) = v_r - v_r^2 = v_r v_r'. \tag{9.8}$$

Proof. Applying the expansion (8.33) to the integral (9.4), we obtain

$$\mu_r(y_1,y_2) = \kappa(d,d_0)\sum_{l\ge 0} M_l\{C_l(\chi_r)\}^2\,\frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1,y_2))}{P_l^{(\alpha,\beta)}(1)}, \tag{9.9}$$

where $C_l(\chi_l)$ are Fourier-Jacobi coefficients (8.31) of the characteristic function χ_r. We have

$$\begin{aligned} C_l(\chi_r) &= \int_0^r P_l^{(\alpha,\beta)}(\cos u)(\sin\frac{1}{2}u)^{d-1}(\cos\frac{1}{2}u)^{d_0-1}\,du \\ &= (\frac{1}{2})^{\frac{d-1}{2}+\frac{d_0-1}{2}}\int_{\cos r}^1 (1-z)^\alpha(a+z)^\beta P_l^{(\alpha,\beta)}(z)\,dz. \end{aligned} \tag{9.10}$$

In view of (2.2), we have $C_0(\chi_r) = \kappa(d,d_0)^{-1}v_r$. For $l \ge 1$ we use Rodrigues' formula for Jacobi polynomials, see [31, Eq. (4.3.1)],

$$P_l^{(\alpha,\beta)}(z) = \frac{(-1)^l}{2^l l!}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^l}{dz^l}\left\{(1-z)^{l+\alpha}(1+z)^{l+\beta}\right\}. \tag{9.11}$$

Substituting (9.11) into (9.10), we obtain

$$\begin{aligned} &\int_{\cos r}^1 (1-z)^\alpha(1+z)^\beta P_l^{(\alpha,\beta)}(z)\,dz \\ &= (2l)^{-1}(1-\cos r)^{\alpha+1}(1+\cos r)^{\beta+1}P_{l-1}^{(\alpha+1,\beta+1)}(\cos r) \\ &= 2^{\alpha+\beta+1}l^{-1}(\sin\frac{1}{2}r)^{2\alpha+2}(\cos\frac{1}{2})^{2\beta+2}P_{l-1}^{(\alpha+1,\beta+1)}(\cos r). \end{aligned}$$

In view of the definitions (8.26) and (9.1), we have

$$C_l(\chi_r) = l^{-1}(\sin\frac{1}{2}r)^d(\cos\frac{1}{2}r)^{d_0}P_{l-1}^{(\alpha+1,\beta+1)}(\cos r)$$
$$= l^{-1}(\sin\frac{1}{2}r)^{\frac{d-1}{2}}(\cos\frac{1}{2}r)^{\frac{d_0-1}{2}}J_{l-1}^{(\alpha+1,\beta+1)}(r). \tag{9.12}$$

Substituting (9.12) into (9.9), we obtain the formulas (9.5) and (9.6).

The bound (9.7) follows from (9.6), since $M_l \simeq l$, see (8.28), and $J_{l-1}^{(\alpha+1,\beta+1)}(r) \lesssim l^{-1/2}$, see (9.3).

From (9.4), we obtain $\mu_r(y,y) = v_r$. Putting in (9.5) $y_1 = y_2 = y$, we obtain (9.8). □

An immediate corollary of Lemma 3.1 is the following.

Theorem 9.1. *For any space $Q(d,d_0)$ the following spherical function expansions hold:*

(i) *For the kernels $\lambda_r(y_1,y_2)$, see* (1.6), *and the metrics $\theta_r^\Delta(y_1,y_2)$, see* (1.14), *we have the expansions*

$$\lambda_r(y_1,y_2) = \kappa(d,d_0)\sum_{l\geq 1} l^{-2}M_l a_l(r)\frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1,y_2))}{P_l^{(\alpha,\beta)}(1)}, \tag{9.13}$$

$$\theta_r^\Delta(y_1,y_2) = \langle\theta_r^\Delta\rangle - \kappa(d,d_0)\sum_{l\geq 1} l^{-2}M_l a_l(r)\frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1,y_2))}{P_l^{(\alpha,\beta)}(1)},$$
$$= \sum_{l\geq 1} l^{-2}M_l a_l(r)\left[1 - \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1,y_2))}{P_l^{(\alpha,\beta)}(1)}\right], \tag{9.14}$$

where $\langle\theta_r^\Delta\rangle = v_r v_r'$ is the average value of metric θ_r^Δ, see (1.26), *and the coefficients $a_l(r)$ are defined in* (9.6).

(ii) *If the weight function $\eta \in W(d,d_0)$, then for the kernels $\lambda(\eta,y_1,y_2)$, see* (1.8), *and the metrics $\theta^\Delta(\eta,y_1,y_2)$, see* (1.13), *we have the expansions*

$$\lambda(\eta,y_1,y_2) = \kappa(d,d_0)\sum_{l\geq 1} l^{-2}M_l A_l(\eta)\frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1,y_2))}{P_l^{(\alpha,\beta)}(1)}, \tag{9.15}$$

$$\theta^{\Delta}(\eta, y_1, y_2) = \langle\theta^{\Delta}(\eta)\rangle - \kappa(d, d_0)\sum_{l\geq 1} l^{-2} M_l A_l(\eta)\frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)},$$

$$= \kappa(d, d_0)\sum_{l\geq 1} l^{-2} M_l A_l(\eta)\left[1 - \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)}\right], \quad (9.16)$$

where $\langle\theta^{\Delta}(\eta)\rangle$ *is the average value of metric* $\theta^{\Delta}(\eta)$*, see* (1.18)*, and the coefficients* $A_l(\eta)$ *are defined by*

$$A_l(\eta) = \int_0^{\pi} \eta(a) a_l(u)\, du. \quad (9.17)$$

Proof. (i) Substituting the expansion (9.5) into (1.23) and (1.25), we obtain the expansions (9.13) and (9.14). Notice that in the second equality in (9.14), the formula (9.8) has been taken into account.

(ii) In view of the bound (9.7), the series (9.13) and (9.14) can be integrated term by term with $\eta \in W(d, d_0)$. This gives the expansions (9.15) and (9.16). □

Notice that by Theorem 2.1 the chordal metric τ is a symmetric difference metric (1.13) with the weight function $\eta^{\natural}(r) = \sin r$ and, therefore, it has the expansion (9.16). At the same time, the chordal metric can be also written as follows

$$\tau(y_1, y_2) = c(\alpha, \beta)\left[1 - \frac{P_1^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_1^{(\alpha,\beta)}(1)}\right]^{1/2}, \quad (9.18)$$

with the constant

$$c(\alpha, \beta) = \left(\frac{\alpha + 1}{\alpha + \beta + 2}\right)^{1/2} = \left(\frac{d}{d + d_0}\right)^{1/2}. \quad (9.19)$$

Indeed, Rodrigues' formula (9.11) gives $P_1^{(\alpha,\beta)}(z) = \frac{1}{2}(\alpha + \beta + 2)z + \frac{1}{2}(\alpha - \beta)$, and so

$$\frac{1}{2}(1 - z) = \frac{\alpha + 1}{\alpha + \beta + 2}\left[1 - \frac{P_1^{(\alpha,\beta)}(z)}{P_1^{(\alpha,\beta)}(1)}\right]. \quad (9.20)$$

On the other hand, by the definitions (2.4), (5.24)

$$\tau(y_1, y_2) = \sin\frac{1}{2}\theta(y_1, y_2) = \left[\frac{1}{2}\left(1 - \cos\theta(y_1, y_2)\right)\right]^{1/2}. \quad (9.21)$$

Comparing (9.20) and (9.21), we obtain (9.18), (9.19).

For the sphere S^d, we have $d_0 = d$, $\beta = \alpha = d/2-1$, and Jacobi polynomials $P_l^{(\alpha,\alpha)}(z)$ coincide, up to constant factors, with Gegenbauer polynomials. Furthermore, $P_l^{(\alpha,\alpha)}(z)$ for even and odd l are even and, correspondingly, odd functions of z, see [31, Sec. 4.7]. Comparing the formula (4.5) and the expansion (9.14) for $r = \pi/2$, we obtain the following expansion for the geodesic distance on S^d

$$\theta(y_1, y_2) = 2\pi \left[\frac{1}{4} - (\frac{1}{4})^d \sum_{odd\, l \geq 1} l^{-2} M_l \left\{ P_{l-1}^{(\alpha+1,\alpha+1)}(0) \right\}^2 \frac{P_l^{(\alpha,\alpha)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\alpha)}(1)} \right]$$

$$= 2\pi (\frac{1}{4})^d \sum_{odd\, l \geq 1} l^{-2} M_l \left\{ P_{l-1}^{(\alpha+1,\alpha+1)}(0) \right\}^2 \left[1 - \frac{P_l^{(\alpha,\alpha)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\alpha)}(1)} \right]. \quad (9.22)$$

We emphasize that the expansion contains zonal spherical functions only with odd numbers. For the sums (8.34), one can easily check the formula

$$\varphi_l[D_N] = \begin{cases} 0 & \text{if } D_N = D_{2a}, \\ 1 & \text{if } D_N = D_{2a+1}, \end{cases} \quad (9.23)$$

where l is odd, the subset $D_{2a} \subset S^d$ consists of a pairs of antipodal points and $D_{2a+1} = D_{2a} \cup \{x_0\}$, where $x_0 \in S^d$ is an arbitrary point. Substituting (9.22) into (9.23) and using (9.8), we immediately obtain a further proof of the relation (2.22).

10. Bounds for Fourier-Jacobi coefficients

In this section we estimate the following coefficients

$$a_l(r) = (\sin\frac{1}{2}r)^{d-1} (\cos\frac{1}{2}r)^{d_0-1} \left\{ J_{l-1}^{(\alpha+1,\beta+1)}(r) \right\}^2, \quad (10.1)$$

$$A_l(\eta) = \int_0^\pi \eta(u) a_l(u)\, du, \quad (10.2)$$

$$A_l(\chi_r) = \int_0^\pi \chi_r(u) a_l(u)\, du = \int_0^r a_l(u)\, du, \quad (10.3)$$

where $J_l^{(\alpha,\beta)}(\cdot)$ is defined in (9.1). In fact, we prove special weighted bounds for Jacobi polynomials.

Lemma 10.1. *Let the weight function $\eta \in W(d, d_0)$, $\eta \neq 0$, then the following bounds hold:*

(i) *For $0 < r \leq \pi$ and $l \geq 1$, we have*

$$A_l(\eta) > cr^{-d+1} a_l(r). \tag{10.4}$$

(ii) *There exists a constant $L \geq 1$, depending only on α and β, such that for $0 < r \leq \pi/2$ and $lr > L$, we have*

$$A_l(\eta) < Cr^{-d} A_l(\chi_r). \tag{10.5}$$

The positive constants c and C in (10.4) *я* (10.5) *depend only on α, β and η.*

Proof. The asymptotic formula (9.2) implies the following relations

$$J_{l-1}^{(\alpha+1,\beta+1)}(r) = (\pi l)^{-1} \left\{ \sin[(l + l_0)r + r_0] + O((l \sin r)^{-1}) \right\}, \tag{10.6}$$

$$\{J_{l-1}^{(\alpha+1,\beta+1)}(r)\}^2 = \left\{ \frac{1}{2} - \frac{1}{2} \cos 2[(l + l_0)r + r_0] + R_l(r) \right\}, \tag{10.7}$$

where the error term $R_l(r)$ satisfies

$$R_l(r) = \begin{cases} O(l^{-1}) & \text{for} \quad 0 < c_0 \leq r \leq \pi - c_0, \\ O((lr)^{-1}) & \text{for} \quad l^{-1} \leq r \leq \pi/2, \end{cases} \tag{10.8}$$

where $0 < c_0 < \pi/2$ is an arbitrary constant.

(i) Since $\eta \in W(d, d_0)$, $\eta \neq 0$, a sufficiently small constant $0 < c_0 < \pi/2$ can be chosen to satisfy

$$\int_{c_0}^{\pi - c_0} \eta(u) (\sin \frac{1}{2} u)^{d-1} (\cos \frac{1}{2} u)^{d_0 - 1} \, du$$

$$\geq \frac{1}{2} \int_0^{\pi} \eta(u) (\sin \frac{1}{2} u)^{d-1} (\cos \frac{1}{2} u)^{d_0 - 1} \, du = \frac{1}{2} \|\eta\|_{d,d_0} > 0. \tag{10.9}$$

Using (10.9), (10.7) and the first bound in (10.8), we obtain

$$A_l(\eta) \geq \int\limits_{c_0}^{\pi-c_0} \eta(u)(\sin\frac{1}{2}u)^{d-1}(\cos\frac{1}{2}u)^{d_0-1}\left\{J_{l-1}^{(\alpha+1,\beta+1)}(u)\right\}^2 du$$

$$\geq (\pi l)^{-1}\Big\{\frac{1}{4}\|\eta\|_{d,d_0} - \frac{1}{2}\int\limits_{c_0}^{\pi-c_0} \eta(u)(\sin\frac{1}{2}u)^{d-1}(\cos\frac{1}{2}u)^{d_0-1}\cos 2[(l+l_0)u+r_0]\,du$$

$$+ O(l^{-1})\Big\} = (\pi l)^{-1}\frac{1}{4}\|\eta\|_{d,d_0} + \circ(1), \tag{10.10}$$

here in the last equality, the Riemann-Lebesgue lemma has been used. Hence

$$A_l(\eta) \geq (\pi l)^{-1}\frac{1}{8}\|\eta\|_{d,d_0} \tag{10.11}$$

for all sufficiently large $l > l_1$. We have

$$\min_{1\leq l\leq l_1} lA_l(\eta) > 0, \tag{10.12}$$

since, $A_l(\eta) > 0$ for all $l \geq 1$. From (10.11) and (10.12), we conclude that the bound

$$A_l(\eta) \geq c_1 l^{-1} \tag{10.13}$$

holds for all $l \geq 1$ with a constant $c_1 > 0$ depending only on α and β.

From the other hand, the bound (9.3) implies

$$r^{-d+1}a_l(r) = r^{-d+1}(\sin\frac{1}{2}r)^{d-1}(\cos\frac{1}{2}r)^{d_0-1}\left\{J_{l-1}^{(\alpha+1,\beta+1)}(r)\right\}^2 \leq c_2 l^{-1} \tag{10.14}$$

Comparing the bounds (10.13) and (10.14), we obtain the bound (10.14) with $c = c_1 c_2^{-1}$.

(ii) Let $0 < r \leq \pi/2$ and $lr \geq L$, where $L \geq 1$ is a constant. From the

definition (10.3), we obtain

$$
\begin{aligned}
r^{-d}A_l(\chi_r) &\geq r^{-d}\int\limits_{r/2}^{r} a_l(u)\,du \\
&\geq r^{-d}(\sin\frac{1}{4}r)^{d-1}(\cos\frac{1}{2}r)^{d_0-1}\int\limits_{r/2}^{r}\left\{J_{l-1}^{(\alpha+1,\beta+1)}(u)\right\}^2\,du \\
&> c_1 r^{-1}\int\limits_{r/2}^{r}\left\{J_{l-1}^{(\alpha+1,\beta+1)}\right\}^2\,du, \qquad (10.15)
\end{aligned}
$$

where we can put $c_1 = (1/8)^{d-1}(1/2)^{d_0-1}$. Using the asymptotic formula (10.7) and the second bound in (10.8), we obtain

$$
\begin{aligned}
&r^{-1}\int\limits_{r/2}^{r}\left\{J_{l-1}^{(\alpha+1,\beta+1)}(u)\right\}^2\,du \\
&= (\pi l)^{-1}\left\{\frac{1}{4}-\frac{1}{2}r^{-1}\int\limits_{r/2}^{r}\cos 2[(l+l_0)u+r_0]\,du + O(L^{-1})\right\}. \qquad (10.16)
\end{aligned}
$$

It is clear that the integral on the right-hand side in (10.16) is of order $O((rl)^{-1} \lesssim O(L^{-1})$. Substituting (10.16) into (10.15), we obtain

$$
r^{-d}A_l(\chi_r) > c_1(\pi l)^{-1}\left\{\frac{1}{4}+O(L^{-1})\right\}. \qquad (10.17)
$$

In view of (10.17), we can chose and fix a sufficiently large constant L, depending only on α and β, to satisfy

$$
r^{-d}A_l(\chi_r) > \frac{1}{8}c_1(\pi l)^{-1} = c_2 l^{-1}. \qquad (10.18)
$$

From the other hand, using the bound (9.3) and the definition (10.2), we obtain

$$
A_l(\eta) \leq C_2\|\eta\|_{d,d_0}l^{-1} = C_3 l^{-1}. \qquad (10.19)
$$

Comparing the bounds (10.18) and (10.19), we obtain the bound (10.5) with $C = C_3 c_2^{-1}$. □

11. Proof of Theorems 2.2 and 3.1

In this section we complete the proof of Theorems 2.2 and 3.1. These results will be obtained as immediate corollaries of a more general result on discrepancies given below in Theorem 11.1.

By application of Theorem 9.1 we can write the discrepancies (1.5), (1.7) in the following form

$$\lambda_r[D_N] = \kappa(d,d_0)\sum_{l\geq 1} l^{-2} M_l a_l(r)\varphi_l[D_N], \tag{11.1}$$

$$\lambda[\eta, D_N] = \kappa(d,d_0)\sum_{l\geq 1} l^{-2} M_l A_l(\eta)\varphi_l[D_N], \tag{11.2}$$

$$\lambda[\chi_r, D_N] = \kappa(d,d_0)\sum_{l\geq 1} l^{-2} M_l A_l(\chi_r)\varphi_l[D_N], \tag{11.3}$$

here $D_N \subset Q(d,d_0)$ is an arbitrary N-point subset and the quantities $\varphi_l[D_N] \geq 0$ are defined in (8.34). The series (11.1)–(11.3) converge and all their terms are nonnegative.

Theorem 11.1. *Let the weight function $\eta \in W(d,d_0)$, $\eta \neq 0$, then the following bounds hold:*

(i) *For any N-point subset $D_N \subset Q(d,d_0)$ and an arbitrary r, $0 < r \leq \pi$, we have*

$$\lambda[\eta, D_N] > cr^{-d+1}\lambda_r[D_N], \tag{11.4}$$

(ii) *There exists a constant $L \geq 1$, depending only d and d_0, such that for any N-point t-design $D_N \subset Q(d,d_0)$ with $t \geq 2L/\pi$, we have*

$$\lambda[\eta, D_N] < Cr^{-d}\lambda[\chi_r, D_N], \quad r = Lt^{-1}. \tag{11.5}$$

The positive constants c and C in (11.4) *and* (11.5) *depend only on d, d_0 and η.*

Proof. (i) Using the bound (10.4) and comparing the series (11.1) and (11.2), we obtain the bound (11.4).

(ii) If $D_N \subset Q(d,d_0)$ is a t-design, then $\varphi[D_N] = 0$ яяя $l = 0, 1, \ldots, t$, see (8.35), and summation in all series (11.1)–(11.3) is taken over $l > t$.

For L we chose the constant indicated in Lemma 10.1 (ii). If $r = Lt^{-1}$, then we have $0 < r \leq \pi/2$ for $t \geq 2L/\pi$ and $lr > L$ for $l > t$. Using the bound (10.5) and comparing the series (11.2) and (11.3), we obtain the bound (11.5). □

Now we are in position to prove Theorems 2.2 and 3.1.

Proof Theorem 2.2. As it was explained in comments to Theorem 2.2 we have to prove only the left bound in (2.20). From the definitions of discrepancies (1.3), (1.5), we conclude that

$$\lambda_r[D_N] \ \geq \ \langle\langle N v_r \rangle\rangle^2,$$

where $\langle\langle z \rangle\rangle = \min\{|z-n|, n \in \mathbb{Z}\}$ is the distance of $z \in \mathbb{R}$ from the nearest integer. Define r by $N v_r = 1/2$, then $\lambda_r[D] \geq 1/2$. In view of (2.3), $r \simeq N^{-1/d}$ and the bound (10.4) implies the left bound in (2.20). $\square$

Proof of Theorem 3.1. First of all, we notice that

$$\begin{aligned}\int\limits_Q (\#\{B_r(y) \cap D_N\})^2 \, d\mu(y) &= \int\limits_Q \left(\sum_{x \in D_N} \chi(B_r(y), x) \right)^2 d\mu(y) \\ &= \sum_{y_1, y_2 \in D_N} \mu(B_r(y_1) \cap B_r(y_2)). \qquad (11.6)\end{aligned}$$

Forom the formulas (11.6), (1.24), (3.7), we obtain

$$\lambda_r[D_N] < \int\limits_Q (\#\{B_r(y) \cap D_N\})^2 \, d\mu(y) \leq (\nu[D_N, r])^2. \qquad (11.7)$$

and

$$\lambda[\chi_r, D_N] = \int\limits_0^r \lambda_u[D_N] \, du < r(\nu[D_N, r])^2, \qquad (11.8)$$

since $\nu[D_N, r]$ is a nondecreasing function of r. Substituting (11.8) into (11.5), we obtain

$$\lambda[\eta, D_N] < C r^{-d+1} (\nu[D_N, r])^2. \qquad (11.9)$$

If $r = L t^{-1}$, then the bound (11.9) coincides with the bound (3.8). $\square$

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