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The Correlation Functions of Strongly Correlated Bosons and Random Walks over Simplicial Lattices

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Abstract

In the present paper we consider random walks over the multi-dimensional simplicial lattices. Our approach is based on the analysis of the dynamical correlation functions of the integrable phase model describing strongly correlated bosons on a chain. Random walks with the reflecting boundary conditions correspond to the Hermitian Hamiltonian of this model, while the directed random walks with the retaining boundary conditions are described by the non-Hermitian modification of this model. The algebraic Bethe Ansatz method allows expressing an appropriate dynamical correlation functions through the symmetric functions.

Key words: non-Hermitian model, correlation function, symmetric functions, partitions, random walks

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1 Introduction

Random walks [1, 2] as well as quantum walks, [3–5], are of considerable recent interest due to their role in quantum computations, [6–8], and in quantum information processing [9–11]. The walks on multi-dimensional lattices were studied by many authors [12–16]. In the present paper we consider random walks over the multi-dimensional simplicial lattices. Our approach is based on the analysis of the dynamical correlation functions of the integrable phase model. Certain quantum integrable models solvable by the Quantum Inverse Scattering Method, [17, 18], demonstrate close relationship [19–21] with the different objects of the enumerative combinatorics [22] and the theory of the symmetric functions [23].

In the present paper we consider the exactly solvable model describing the so-called *phase operators* governed by the non-Hermitian Hamiltonian on a chain with $M+1$ nodes. The phase operators were introduced in [24] and are connected with the quantum optics and the quantum phase problem [25]. The model considered in [24] is related to the exactly solvable model describing strongly correlated bosons [26, 27]. We shall demonstrate that the non-Hermitian model in question provides a natural description of the random walks on M-dimensional simplicial lattices.

The paper is organised as follows. Section 1 is introductory. In Section 2 random walks over M-dimensional simplicial lattice with free and retaining boundary conditions are introduced. The quantum generalized model and its solution in the approach of the Bethe Ansatz are presented in Sections 3 and 4, respectively. The Totally Asymmetric Zero Range Model and the Phase Model are correspondingly considered in Sections 5 and 6.

2 The walks on a simplicial lattice in general dimensionality

Starting from $(M+1)$-dimensional hypercubical lattice with unit spacing $\mathbb{Z}^{M+1} \ni \mathbf{m} \equiv (m_0, m_1, \ldots, m_M)$, let us consider the non-negative orthant $\mathbb{N}_0^{M+1} \equiv \{\mathbf{m} \mid 0 \le m_i, i \in \mathcal{M}\}$ as a subspace of $\mathbb{Z}^{M+1}$ (hereafter $\mathcal{M} \equiv \{0, 1, \ldots M\}$). Consider a set of points with coordinates constrained by the requirement $m_0 + m_1 + \ldots + m_M = N$:

$$\mathrm{Hyp}_{(N)}(\mathbb{Z}^{M+1}) \equiv \Big\{(m_0, m_1, \ldots, m_M) \in \mathbb{Z}^{M+1} \Big| \sum_{i\in\mathcal{M}} m_i = N\Big\} .$$

We call *simplicial lattice* the compact M-dimensional set of points belonging to the intersection of two sets

$$\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1}) = \mathrm{Hyp}_{(N)}(\mathbb{Z}^{M+1}) \bigcap \mathbb{N}_0^{M+1} .$$

Random walks of a particle (a walker) over sites of $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ are defined by a set of admissible steps Ω_M such that at each step an i^{th} coordinate m_i increases by unity while a nearest neighboring coordinate decreases by unity. Namely, Ω_M is the set of steps with coordinates $(e_0, e_2, \ldots, e_M)$ such that for all pairs $(i, i+1)$ with

$0 \leq i \leq M$ and $M+1 = 0(\mod 2)$, $e_i = \pm 1$, $e_{i+1} = \mp 1$ and $e_j = 0$ for all $j \in \mathcal{M}$ and $j \neq i, i+1$. The step-set $\Omega_M \equiv \Omega_M(\mathbf{m}_0)$ ensures that trajectory of a random walk determined by the starting point $\mathbf{m}_0$ lies in M-dimensional set $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$.

Directed random walks on M-dimensional *orientated simplicial lattice* are defined by a step-set $\Gamma_M = (k_0, k_1, \ldots, k_M)$ such that for all pairs $(i, i+1)$ with $i \in \mathcal{M}$ and $M+1 = 0(\mod 2)$, $k_i = -1$, $k_{i+1} = 1$, and $k_j = 0$ for all $j \in \mathcal{M}\backslash\{i, i+1\}$.

It might occur that some points on the boundary of the simplicial lattice belong also to a random walk trajectory. Therefore the walker's movements should be supplied with appropriate boundary conditions. The *reflecting boundary conditions* are defined by the requirement that when certain points of the trajectory and of the boundary are coinciding the corresponding admissible steps are still taken from the step-sets Ω_M or Γ_M. The *retaining boundary conditions* are the conditions under which the walker on the boundary continues to move in accordance with the elements of the step-sets Ω_M or Γ_M either stays on the boundary. The boundary of the simplicial lattice consists of $M+1$ faces of highest dimensionality $M-1$. To each component of the boundary a weight g_s, $s \in \mathcal{M}$, is assigned.

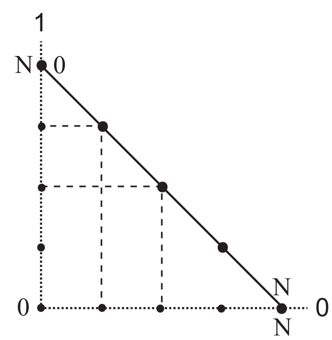

Figure 1: The hopping processes for the one-dimensional nearest-neighbor random walk on a segment [0,N].

As an example, the walks on $\mathbb{Z}^2$ are defined by a step-set $\Omega_1 = \{(1, -1), (-1, 1)\}$ that ensures that the walks lie on lines $\{(n_0, n_1) \in \mathbb{Z}^2 \mid n_0 + n_1 = N\}$. The step-set $\Omega_2 = \{(-1, 1, 0), (1, -1, 0), (0, -1, 1), (0, 1, -1), (1, 0, -1), (-1, 0, 1)\}$ of the random walks, or $\Omega_2 = \{(-1, 1, 0), (0, -1, 1), (1, 0, -1)\}$ of the directed walks, ensures that trajectories of random walks belong to $\mathrm{Simp}_{(N)}(\mathbb{Z}^3)$.

Consider the case of the retaining boundary conditions. The exponential generating function of lattice walks is defined as a formal series

$$F^{(N)}(\mathbf{l}, \mathbf{j}|t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G_k^{(N)}(\mathbf{l}, \mathbf{j}) \,, \tag{1}$$

where the coefficients $G_k^{(N)}(\mathbf{l}, \mathbf{j})$ characterize the k-step walks at a node $\mathbf{l} = (l_0, l_1, \ldots, l_M) \in \mathrm{Hyp}_{(N)}(\mathbb{Z}^{M+1})$ when starting at $\mathbf{j} = (j_0, j_1, \ldots, j_M) \in \mathrm{Hyp}_{(N)}(\mathbb{Z}^{M+1})$. The generating

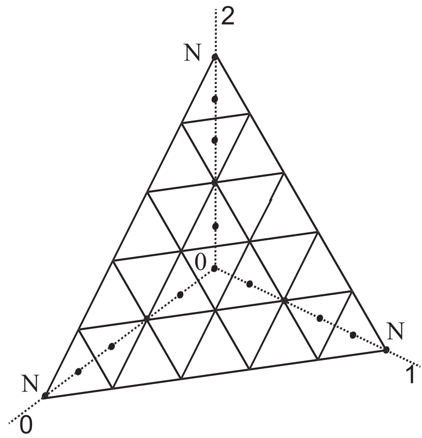

Figure 2: A two-dimensional triangular simplicial lattice.

function (1) satisfies the master equation

$$\begin{aligned}
\partial_t F^{(N)}(\mathbf{l},\mathbf{j}|t) = &\sum_{s=0}^{M} F^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_s - 1, j_{s+1} + 1, \ldots j_M) \\
&+\sum_{s=0}^{M} F^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_s + 1, j_{s+1} - 1, \ldots j_M) \\
+\sum_{s=0}^{M} g_s F^{(N)}(\mathbf{l}, j_0, j_1, &\ldots, j_{s-1}, 0, j_{s+1}, \ldots j_M)\, \delta(N, {\sum_{0\leq k\leq M}}' j_k)\,,
\end{aligned} \tag{2}$$

where $\delta(n,m)$ is the Kronecker symbol, and $\sum'$ implies that $k = s$ is omitted. The equation for the directed walks is obtained by removing the second sum in right-hand side of (2). Substituting (1) into (2), we obtain the system of equations for $G_k^{(N)}(\mathbf{l},\mathbf{j})$:

$$\begin{aligned}
G_k^{(N)}(\mathbf{l},\mathbf{j}) \;=\; &\sum_{s=0}^{M} G_{k-1}^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_s - 1, j_{s+1} + 1, \ldots j_M) \\
&+\sum_{s=0}^{M} G_{k-1}^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_s + 1, j_{s+1} - 1, \ldots j_M) \\
+\sum_{s=0}^{M} g_s G_{k-1}^{(N)}(\mathbf{l}, j_0, j_1, &\ldots, j_{s-1}, 0, j_{s+1}, \ldots j_M)\, \delta(N, {\sum_{0\leq k\leq M}}' j_k)\,,
\end{aligned} \tag{3}$$

where $k \geq 1$, while it is natural to impose the condition $G_0^{(N)}(\mathbf{l},\mathbf{j}) = \delta_{l_0 j_0}\delta_{l_1 j_1}\cdots\delta_{l_M j_M}$.

3 The quantum generalized phase model

To give the problem a quantum flavour we shall interpret the coordinates n_j of a particle $\mathbf{n} = (n_0, n_1, \ldots, n_M) \in \mathbb{Z}^{M+1}$ as the occupation numbers of $(M+1)$-component

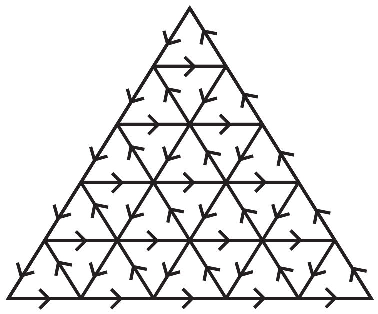

Figure 3: The orientated two-dimensional bounded simplicial lattice defined by the step-set Γ_2. The walks are allowed only in the direction of arrows.

Fock space and describe the dynamics of a particle with the help of the Fock state-vectors $\mid \mathbf{n}\rangle \equiv\mid n_0, n_1, \ldots, n_M\rangle$. To this end, let us introduce a description in terms of N bosonic particles on a cyclic chain consisting of $M+1$ nodes. The number of particles on any site is arbitrary, and each configuration of the particles is characterized by a collection of the occupation numbers $(n_M, \ldots, n_1, n_0)$, $\sum_{l\in\mathcal{M}} n_l = N$. Each particle is hopping with probability $\frac{1}{2}$ to one of the nearest sites.

To describe the hoppings of the particles let us introduce the so-called phase operators $\phi_n, \phi_n^\dagger$, [24], which satisfy the commutation relations

$$[\hat{N}_i, \phi_j] = -\phi_i\delta_{ij}\,, \quad [\hat{N}_i, \phi_j^\dagger] = \phi_i^\dagger\delta_{ij}\,, \quad [\phi_i, \phi_j^\dagger] = \pi_i\delta_{ij}\,, \tag{4}$$

where $\hat{N}_j$ is the number operator, and $\pi_i = 1 - \phi_i^\dagger\phi_i$ is the vacuum projector: $\phi_j\pi_j = \pi_j\phi_j^\dagger = 0$. We introduce the Fock state-vectors $|n_l\rangle_l = (\phi_l^\dagger)^{n_l}|0\rangle_l$, where $|0\rangle_l$ is the vacuum state $|0\rangle$ at l^{th} site defined by the relation $\phi_l|0\rangle = 0$, $l \in \mathcal{M}$. The representation of the algebra of the phase operators is given by the relations:

$$\phi_l^\dagger\,|n_l\rangle_l = |n_l+1\rangle_l\,, \qquad \phi_l\,|n_l\rangle_l = |n_l-1\rangle_l\,, \qquad \hat{N}_l\,|n_l\rangle_l = n_l\,|n_l\rangle_l\,. \tag{5}$$

We introduce the $(M+1)$-dimensional vacuum vector

$$|\mathbf{0}\rangle \equiv \bigotimes_{l=0}^{M} |0\rangle_l \tag{6}$$

and define, taking into account (5), an appropriate state-vector $|\mathbf{n}\rangle$:

$$|\mathbf{n}\rangle \equiv \bigotimes_{l=0}^{M} |n_l\rangle_l\,, \tag{7}$$

where $\mathbf{n} \in \mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$. The Fock state-vectors $|\mathbf{n}\rangle$ are generated from the vacuum state $|\mathbf{0}\rangle$ by action of the rising operators $\phi_j^\dagger$:

$$|\mathbf{n}\rangle = \Big(\prod_{j=0}^{M} \big(\phi_j^\dagger\big)^{n_j}\Big)|\mathbf{0}\rangle\,. \tag{8}$$

This algebra has a representation on the Fock space:

$$\begin{array}{rcl}\phi_j \mid n_0, \dots, 0_j, \dots, n_M\rangle & = & 0, \\ \phi_j^\dagger \mid n_0, \dots, n_j, \dots, n_M\rangle & = & \mid n_0, \dots, n_j + 1, \dots, n_M\rangle, \\ \phi_j| \mid n_0, \dots, n_j, \dots, n_M\rangle & = & |n_0, \dots, n_j - 1, \dots, n_M\rangle, \\ \hat{N}_j \mid n_0, \dots, n_j, \dots, n_M\rangle & = & n_j|n_0, \dots, n_j, \dots, n_M\rangle \\ \pi_j \mid n_0, \dots, 0_j, \dots, n_M\rangle & = & \mid n_0, \dots, 0_j, \dots, n_M\rangle. \end{array} \tag{9}$$

The states $|n_0, \dots, n_M\rangle$ are orthogonal, $\langle p_0, \dots, p_M|n_0, \dots, n_M\rangle = \delta_{p_0 n_0} \cdots \delta_{p_M k_M}$.

As the generator of the directed walks with the retaining boundary conditions we can consider the following non-Hermitian Hamiltonian, [24, 28]:

$$H = \sum_{m=0}^{M} \left(\phi_m \phi_{m+1}^\dagger + g_m \pi_m \right), \tag{10}$$

where $M + 1 = 0(\mod 2)$ and the periodic boundary conditions are imposed. The Hamiltonian (10) commutes with the number operator

$$[H, \hat{N}] = 0\,, \qquad \hat{N} = \sum_{j=0}^{M} \hat{N}_j\,. \tag{11}$$

This ensures that walks themselves lie in the hyperplanes $\mathrm{Hyp}_{(N)}(\mathbb{Z}^{M+1})$.

The exponential generating function of the directed walks (1) is expressed as the dynamical correlation function:

$$F^{(N)}(\mathbf{l}, \mathbf{j}|t) = \langle \mathbf{l} \mid e^{tH} \mid \mathbf{j}\rangle\,, \tag{12}$$

where H is the Hamiltonian (10), the coordinates of a particle moving over sites of $\mathrm{Hyp}_{(N)}(\mathbb{Z}^{M+1})$ coincide with the occupation numbers of the Fock states. Differentiating (12) by t and taking into account (9) we obtain Eq. (2) for the directed walks. Expanding the correlator in powers of t we obtain expression for the coefficients $G_k^{(N)}(\mathbf{l}, \mathbf{j})$ characterizing the lattice walks from the node $\mathbf{j}$ to the node $\mathbf{l}$ in k steps:

$$G_k^{(N)}(\mathbf{l}, \mathbf{j}) = \langle \mathbf{l} \mid H^k \mid \mathbf{j}\rangle \tag{13}$$

To find the analytical answers for $F^{(N)}$ and $G_k^{(N)}$, we shall apply the Quantum Inverse Scattering Method.

4 Solution of the model

4.1 Generalities

To apply the scheme of the Quantum Inverse Scattering Method to the solution of the Hamiltonian (10) we define L-operator [24] which is 2×2 matrix with the operator-valued entries acting on the Fock states according to (9):

$$L(n|u) \equiv \begin{pmatrix} u^{-1} + u g_n \pi_n & \phi_n^\dagger \\ \phi_n & u \end{pmatrix}, \tag{14}$$

where $u \in \mathbb{C}$ is a parameter and $g_n \in \mathbb{R}$. This L-operator satisfies the intertwining relation

$$R(u,v)\,(L(n|u) \otimes L(n|v)) = (L(n|v) \otimes L(n|u))\,R(u,v)\,, \tag{15}$$

in which $R(u,v)$ is the R-matrix

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}, \tag{16}$$

where

$$f(v,u) = \frac{u^2}{u^2 - v^2}, \qquad g(v,u) = \frac{uv}{u^2 - v^2}\,, \qquad u,v \in \mathbb{C}\,. \tag{17}$$

The monodromy matrix is the matrix product of L-operators

$$T(u) = L(M|u)L(M-1|u)\cdots L(0|u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{18}$$

The commutation relations of the matrix elements of the monodromy matrix are given by the same R-matrix (16):

$$R(u,v)\,(T(u) \otimes T(v)) = (T(v) \otimes T(u))\,R(u,v)\,. \tag{19}$$

The transfer matrix $\tau(u)$ is the trace of the monodromy matrix in the auxiliary space:

$$\tau(u) = \operatorname{tr} T(u) = A(u) + D(u)\,. \tag{20}$$

The relation (19) means that $[\tau(u), \tau(v)] = 0$ for arbitrary $u, v \in \mathbb{C}$.

The definitions of the L-operator (14) and the monodromy matrix (18) enable to obtain by direct calculation that the entries of the monodromy matrix $T(u)$ are characterized by the relations:

$$\begin{aligned} u^{M+1}A(u) &= 1 + u^2\Big(\sum_{m=0}^{M-1} \phi_m \phi_{m+1}^\dagger + \sum_{m=0}^{M} g_m \pi_m\Big) + \ldots \\ &\quad + u^{2(M+1)} \prod_{m=0}^{M} g_m \pi_m\,, \\ u^{M+1}D(u) &= u^2 \phi_0^\dagger \phi_M + \ldots + u^{2(M+1)}\,, \end{aligned} \tag{21}$$

and

$$u^M B(u) = \phi_0^\dagger + \ldots + u^{2M}\mathcal{P}_R \equiv \widetilde{B}(u)\,, \tag{22}$$

$$u^M C(u) = \phi_M + \ldots + u^{2M}\mathcal{P}_L \equiv \widetilde{C}(u)\,, \tag{23}$$

where

$$\mathcal{P}_R = \sum_{k=0}^{M} \phi_k^\dagger g_{k+1}\pi_{k+1} \ldots g_M \pi_M\,, \tag{24}$$

$$\mathcal{P}_L = \sum_{k=0}^{M} g_0 \pi_0 \ldots g_{k-1}\pi_{k-1}\phi_k\,. \tag{25}$$

The representation (21) allows to express the Hamiltonian (10) through the transfer matrix (20):

$$H = \frac{\partial}{\partial u^2}\, u^{M+1}\tau(u)\Big|_{u=0} = \frac{\partial}{\partial u^2} u^{M+1}(A(u) + D(u))\Big|_{u=0}. \tag{26}$$

By construction this Hamiltonian commutes with the transfer matrix:

$$[H, \tau(u)] = 0.$$

Since the Hamiltonian (10) is non-Hermitian we have to distinguish between its right and left eigen-vectors. The N-particle right state-vectors are taken in the form

$$|\Psi_N(\mathbf{u})\rangle = \Big(\prod_{j=1}^{N} \widetilde{B}(u_j)\Big)|\mathbf{0}\rangle, \tag{27}$$

where $\widetilde{B}(u)$ is defined in (22), and $\mathbf{u}$ implies a collection of arbitrary complex parameters $u_j \in \mathbb{C}$: $\mathbf{u} = (u_0, u_1, \ldots, u_N)$. The left state-vectors are equal to

$$\langle\Psi_N(\mathbf{u})| = \langle\mathbf{0}|\Big(\prod_{j=1}^{N} \widetilde{C}(u_j)\Big), \tag{28}$$

where $\widetilde{C}(u)$ is given by (23). The vacuum state $\mid \mathbf{0}\rangle$(6) is an eigen-vector of $A(u)$ and $D(u)$,

$$A(u)|\mathbf{0}\rangle = \alpha(u)|\mathbf{0}\rangle, \qquad D(u)|\mathbf{0}\rangle = \delta(u)|\mathbf{0}\rangle \tag{29}$$

with the eigen-values

$$\alpha(u) = \prod_{j=0}^{M}(u^{-1} + g_j u), \qquad \delta(u) = u^{M+1}. \tag{30}$$

The state-vectors (27) and (28) are the eigen-vectors both of the Hamiltonian (10) and of the transfer matrix $\tau(u)$ (20), if and only if the variables u_j satisfy the Bethe equations:

$$u_n^{-2N} \prod_{j=0}^{M} \left(g_j + u_n^{-2}\right) = (-1)^{N-1} \prod_{j=1}^{N} u_j^{-2}\,. \tag{31}$$

The eigen-values $\Theta_N(v)$ of $\tau(v)$ are equal to

$$v^M \Theta_N(v) = \prod_{j=0}^{M}(1 + g_j v^2) \prod_{m=1}^{N} \frac{u_m^2}{u_m^2 - v^2} + v^{2(M+1)} \prod_{m=1}^{N} \frac{v^2}{v^2 - u_m^2}. \tag{32}$$

Equation (26) enables to obtain the spectrum of the Hamiltonian (10). The N-particle eigen-energies are equal to

$$E_N(\mathbf{u}) = \frac{\partial}{\partial v^2} v^M \Theta_N(v)\Big|_{v=0} = \sum_{m=0}^{M} g_m + \sum_{m=1}^{N} u_m^{-2}. \tag{33}$$

4.2 Correlation functions and their calculation

For the models associated with the R-matrix (16) the scalar product of the state-vectors (27) and (28) is given by the formula [29]:

$$\langle\Psi_N(\mathbf{v})|\Psi_N(\mathbf{u})\rangle = \mathcal{V}_N^{-1}(\mathbf{v}^2)\mathcal{V}_N^{-1}(\mathbf{u}^{-2})\prod_{j=1}^{N}\Big(\frac{v_j}{u_j}\Big)^{M+N-1}\det Q\,, \tag{34}$$

where $\mathcal{V}_N(\mathbf{x})$ is the Vandermonde determinant,

$$\mathcal{V}_N(\mathbf{x}) \equiv \mathcal{V}_N(x_1, x_2, \ldots, x_N) = \prod_{1\le i<k\le N}(x_k - x_i)\,, \tag{35}$$

and the matrix Q is characterized by the entries Q_{jk}, $1 \le j, k \le N$:

$$Q_{jk} = \frac{\alpha(v_j)\delta(u_k)\Big(\dfrac{u_k}{v_j}\Big)^{N-1} - \alpha(u_k)\delta(v_j)\Big(\dfrac{u_k}{v_j}\Big)^{-N+1}}{\dfrac{u_k}{v_j} - \dfrac{v_j}{u_k}}\,, \tag{36}$$

with $\alpha(u)$ and $\delta(u)$ given by (30).

There is $\Omega = \frac{(N+M)!}{N!M!}$ sets of solutions of the Bethe equations (31), and the state-vectors belonging to the different sets of the solutions of the Bethe equations are orthogonal. The eigen-vectors (27) and (28) provide the resolution of the identity operator

$$I = \sum_{\{\mathbf{u}\}}\frac{|\Psi_N(\mathbf{u})\rangle\langle\Psi_N(\mathbf{u})|}{\mathcal{N}^2(\mathbf{u})}, \tag{37}$$

where the summation $\sum_{\{\mathbf{u}\}}$ is over all independent solutions of the Bethe equations (31).

To calculate the norms of the eigen-vectors, we put $v_k = u_k$ equal to the solutions of Bethe equations in (34). Non-diagonal entries of the matrix $\widetilde{Q}$ are equal to

$$\widetilde{Q}_{jk} = (-1)^N\delta(u_j)\delta(u_k)u_k^N u_j^N U^{-2}(1-\delta_{jk})\,, \tag{38}$$

where $U^2 \equiv \prod_{n=1}^N u_n^2$. The diagonal entries of this matrix should be understood in the sense of L'Hôspital rule:

$$\widetilde{Q}_{jj} = \frac{u_j}{2}\left[\alpha(u_j)\delta'(u_j) - \alpha'(u_j)\delta(u_j) + 2(N-1)u_j^{-1}\alpha(u_j)\delta(u_j)\right]. \tag{39}$$

The norm of the Bethe state-vectors is equal to

$$\mathcal{N}^2(\mathbf{u}) = \langle\Psi_N(\mathbf{u})|\Psi_N(\mathbf{u})\rangle = \frac{\det\widetilde{Q}}{\mathcal{V}_N(\mathbf{u}^2)\mathcal{V}_N(\mathbf{u}^{-2})}\,. \tag{40}$$

We put, as a special case, $g_s = \gamma$, $\forall i$, and eventually obtain from (34), (36), (38), and (39):

$$\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle = \mathcal{V}_N^{-1}(\mathbf{v}^2) \mathcal{V}_N^{-1}(\mathbf{u}^{-2}) \prod_{j=1}^{N} u_j^{2(1-M-N)} \tag{41}$$

$$\times \det \left(\frac{(1+\gamma v_j^2)^{M+1} u_k^{2(M+N)} - (1+\gamma u_k^2)^{M+1} v_j^{2(M+N)}}{u_k^2 - v_j^2} \right)_{1 \le k,j \le N} . \tag{42}$$

The determinant (42) is a polynomial in powers of γ. It is appropriate to use this to obtain several particular cases for the correlator $\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle$ (34). First, we obtain for $M = 1$ and arbitrary N:

$$\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle \; = \; \sum_{k=0}^{N} (k+1) \gamma^k \left(\sum_{r=0}^{N-k} e_{r+k}(\mathbf{v}^2)\, e_r(\mathbf{u}^{-2}) \right) , \tag{43}$$

where $e_r(\cdot)$ are the elementary symmetric functions [23]. Also we obtain for $N = 1$ and arbitrary M:

$$\begin{aligned} \langle \Psi_1(\mathbf{v}) | \Psi_1(\mathbf{u}) \rangle \; = \; & \sum_{k=0}^{M} \binom{M+1}{k} \gamma^k \, \Big(\big(e_1(\mathbf{v}^2)\big)^k \, e_0(\mathbf{u}^{-2}) \\ & + \sum_{r=1}^{M-k} \big(e_1(\mathbf{v}^2)\big)^{r+k} \big(e_1(\mathbf{u}^{-2})\big)^r \Big) , \end{aligned} \tag{44}$$

where $e_r(\cdot)$ are the elementary symmetric functions. *Appendix* provides the formula for the correlator $\langle \Psi_2(\mathbf{v}) | \Psi_2(\mathbf{u}) \rangle$ (34) at $M = N = 2$.

Let us consider the walker on $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ under the condition that the starting point is located at the node $(N, 0, \dots, 0)$, and the walk terminates at $(0, 0, \dots, N)$. The generating function (12) of these walks is specified as follows:

$$\mathcal{F}^{(N)}(t) \equiv \langle 0, 0, \dots, N \mid e^{tH} \mid N, 0, \dots, 0 \rangle = \langle \mathbf{0} \mid (\phi_M)^N e^{tH} (\phi_M^\dagger)^N \mid \mathbf{0} \rangle , \tag{45}$$

where (6) and (8) have been used. Inserting the resolution of the identity operator (37) into (45) we obtain:

$$\mathcal{F}^{(N)}(t) = \sum_{\{\mathbf{u}\}} \frac{e^{tE_N(\mathbf{u})}}{\mathcal{N}^2(\mathbf{u})} \langle \mathbf{0} \mid (\phi_M)^N \mid \Psi_N(\mathbf{u}) \rangle \langle \Psi_N(\mathbf{u}) \mid (\phi_M^\dagger)^N \mid \mathbf{0} \rangle , \tag{46}$$

where the summation is over all independent solutions of Eqs. (31). Applying the decomposition (23) for $B(u)$ and $C(u)$, we obtain:

$$\langle \mathbf{0} \mid (\phi_M)^N \mid \Psi_N(\mathbf{u}) \rangle = \lim_{\mathbf{v} \to 0} \langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle = 1 , \tag{47}$$

$$\langle \Psi_N(\mathbf{u}) \mid (\phi_M^\dagger)^N \mid \mathbf{0} \rangle = \lim_{\mathbf{v} \to 0} \langle \Psi_N(\mathbf{u}) | \Psi_N(\mathbf{v}) \rangle = 1 . \tag{48}$$

The unities appear as the limiting values in (47) and (48). The point is as follows. The determinant (42) is a polynomial in γ. Its coefficients (expressible in terms of the symmetric functions) at non-trivial powers of γ are vanishing in the limit $\mathbf{v} \to 0$. The free term of the polynomial in powers of γ is given by the series in products of the symmetric functions [20]. The free term is just responsible for the unities in right-hand sides of (47) and (48). Eventually we obtain:

$$\mathcal{F}^{(N)}(t) = \sum_{\{\mathbf{u}\}} \frac{e^{2t\sum_{k=1}^{N}(\gamma+u_k^{-2})}}{\mathcal{N}^2(\mathbf{u})} \,. \tag{49}$$

For instance, at $N = 2$ and $M = 1$ we obtain:

$$\mathcal{F}^{(2)}(t) = e^{4t\gamma} \sum_{\{u_1^2, u_2^2\}} \frac{e^{2t(u_1^{-2}+u_2^{-2})}}{\mathcal{N}^2(u_1^2, u_2^2)} \,, \tag{50}$$

$$\mathcal{N}^2(u_1^2, u_2^2) = 4 + \frac{u_1^2}{u_2^2} + \frac{u_2^2}{u_1^2} + 4\gamma(u_2^2 + u_1^2) + 3\gamma^2 u_2^2 u_1^2 \,. \tag{51}$$

5 Totaly Asymmetric Zero Range Model

When all g_i are equal to $\gamma = 1$, the Hamiltonian (10) is the Hamiltonian of Totally Asymmetric Zero Range Model [24, 28]:

$$H_{\text{zr}} = \sum_{m=0}^{M} \left(\phi_m \phi_{m+1}^{\dagger} + \pi_m \right) . \tag{52}$$

The state-vector (27) enables the representation in the form

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} \chi_{\boldsymbol{\lambda}}^R(\mathbf{u}) \Big(\prod_{j=0}^{M} (\phi_j^{\dagger})^{n_j} \Big) |\mathbf{0}\rangle \,, \tag{53}$$

where the function $\chi_{\boldsymbol{\lambda}}^R$ is equal, up to a multiplicative pre-factor, to

$$\chi_{\boldsymbol{\lambda}}^R(\mathbf{x}) = \chi_{\boldsymbol{\lambda}}^R(x_1, x_2, \ldots, x_N) = \mathcal{V}_N^{-1}(\mathbf{x}) \det \left\{ \left(\frac{x_i}{x_i + x_i^{-1}} \right)^{\lambda_j} x_i^{2(N-j)} \right\} . \tag{54}$$

Here $\boldsymbol{\lambda}$ denotes the partition $(\lambda_1, \ldots, \lambda_N)$ of non-increasing non-negative integers,

$$M \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0,$$

and $\mathcal{V}_N(\mathbf{x})$ is the Vandermonde determinant (35). There is a one-to-one correspondence between a sequence of the occupation numbers $(n_M, \ldots, n_1, n_0)$, $\sum_{j\in\mathcal{M}} n_j = N$, and the partition

$$\boldsymbol{\lambda} = (M^{n_M}, (M-1)^{n_{M-1}}, \ldots, 1^{n_1}, 0^{n_0}) \,,$$

where each number S appears n_S times (see Fig. 4). The sum in Eq. (53) is taken over all partitions $\boldsymbol{\lambda}$ into at most N parts with $N \leq M$.

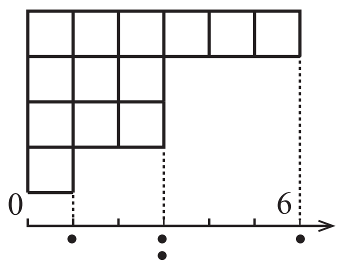

Figure 4: A configuration of particles ($N = 4$) on a lattice ($M = 6$), the corresponding partition $\boldsymbol{\lambda} = (6^1, 5^0, 4^0, 3^2, 2^0, 1^1, 0^0) \equiv (6, 3, 3, 1)$ and its Young diagram.

Acting by the Hamiltonian (52) on the state-vector (53), we find that the wave function (54) satisfies the equation:

$$\sum_{k=1}^{N} \chi^{R}_{\lambda_1,\dots,\lambda_k+1,\dots,\lambda_N}(\mathbf{u}) = E_N^{\mathrm{zr}}(\mathbf{u}) \chi^{R}_{\lambda_1,\dots,\lambda_N}(\mathbf{u}) \,, \tag{55}$$

together with the exclusion condition

$$\chi^{R}_{\lambda_1,\dots,\lambda_{l-1}=\lambda_l-1,\lambda_l,\dots,\lambda_N}(\mathbf{u}) = \chi^{R}_{\lambda_1,\dots,\lambda_{l-1}=\lambda_l,\lambda_l,\dots,\lambda_N}(\mathbf{u}) \,, \qquad 1 \le l \le N. \tag{56}$$

The energy E_N^{zr} is given by (33) with all $g_i = 1$. The state-vector (53) is the eigenvector of the Hamiltonian (52) with the periodic boundary conditions if the parameters u_j satisfy the appropriate Bethe equations.

The relations (53), (55) and (56) can be viwed as an implementation of the coordinate Bethe ansatz [18] which is an alternative to the approach of the algebraic Bethe ansatz considered in Section 4 to solve the present bosonic model (10). Although the model is solved by the algebraic Bethe ansatz, representations of the type of (53) are especially useful to discuss the combinatorial implementations of the quantum integrable models, [19–21].

Expanding the left state-vector (28), we obtain:

$$\langle \Psi_N(\mathbf{u}) \,| = \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} \chi^{L}_{\boldsymbol{\lambda}}(\mathbf{u}) \, \langle \mathbf{0} \,| \left(\prod_{i=0}^{M} \phi_i^{n_i} \right) , \tag{57}$$

where the wave function is given by

$$\chi^{L}_{\boldsymbol{\lambda}}(\mathbf{x}) = \mathcal{V}_N^{-1}(\mathbf{x}) \det \left\{ \left(1 + x_i^{-2}\right)^{\lambda_j} x_i^{2(N-j)} \right\} . \tag{58}$$

It satisfies the equations:

$$\sum_{k=1}^{N} \chi^{L}_{\lambda_1,\dots,\lambda_k-1,\dots,\lambda_N}(\mathbf{u}) = E_N^{\mathrm{zr}}(\mathbf{u}) \chi^{L}_{\lambda_1,\dots,\lambda_N}(\mathbf{u}) \,, \tag{59}$$

$$\chi^{L}_{\lambda_1,\dots,\lambda_l,\lambda_{l+1}=\lambda_l+1,\dots,\lambda_N}(\mathbf{u}) = \chi^{L}_{\lambda_1,\dots,\lambda_l,\lambda_{l+1}=\lambda_l,\dots,\lambda_N}(\mathbf{u}) \,, \qquad 1 \le l \le N. \tag{60}$$

From Eqs. (53), (57) one obtains:

$$\langle l_0, l_1, \ldots, l_M \mid \Psi_N(\mathbf{u})\rangle = \chi^R_{\boldsymbol{\lambda}_R}(\mathbf{u}) \tag{61}$$

$$\langle \Psi_N(\mathbf{u}) \mid j_0, j_1, \ldots, j_M\rangle = \chi^L_{\boldsymbol{\lambda}_L}(\mathbf{u})\,, \tag{62}$$

where $\boldsymbol{\lambda}_R = \left(M^{l_M}, (M-1)^{l_{M-1}}, \ldots, 1^{l_1}, 0^{l_0}\right)$, and $\boldsymbol{\lambda}_L = (M^{j_M}, (M-1)^{j_{M-1}}, \ldots, 1^{j_1}, 0^{j_0})$.

The exponential generating function (12) in the considered special case is equal to

$$F^{(N)}_{\mathrm{zr}}(\mathbf{l}, \mathbf{j}|t) = \sum_{\{\mathbf{u}\}} \frac{e^{tE^{\mathrm{zr}}_N(\mathbf{u})}}{\mathcal{N}^2_{\mathrm{zr}}(\mathbf{u})} \chi^R_{\boldsymbol{\lambda}_R}(\mathbf{u}) \chi^L_{\boldsymbol{\lambda}_L}(\mathbf{u})\,. \tag{63}$$

Here parameters u_j satisfy Bethe equations (31) with $g_i = 1$, and $\mathcal{N}^2_{\mathrm{zr}}(\mathbf{u})$ is the squared norm (40) in the same limit.

Let us consider the case $M = 1$. The trajectory of a particle on a bounded strip form a *modified Dyck* path. A Dyck path in a strip is a path that can go up and down but can not cross the border. The *modified Dyck* paths are the paths that can have a horizontal steps along the borders.

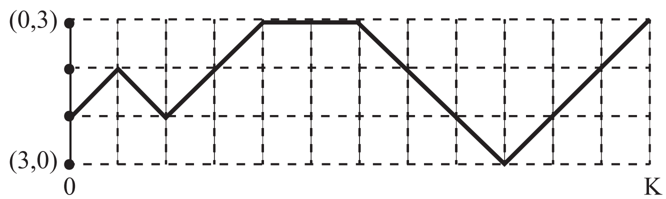

Figure 5: Modified Dyck path of the length $K = 12$.

The Schmidt decomposition of a quantum states associated with the lattice paths plays an important role in the quantum information theory [6, 7]. In the considered case each step of a particle is associated with a spin state in $\mathbb{C}^3$: the step up of a particle is identified with $\mid +\rangle$, the step down with $\mid -\rangle$, and the horizontal steps along the borders with $\mid 0\rangle$. The modified Dyck path of K steps on a strip of a length N is then a spin state in $(\mathbb{C}^3)^{\otimes K}$. The spin state that corresponds to a modified Dyck path in Fig. 5 is $\mid + - + + 00 - - - + + +\rangle$. The Schmidt decomposition of the normalized uniform superposition of all modified Dyck paths starting from the node $(p, N-p)$ and terminating at $(q, N-q)$ after K steps on a strip of the length N can be written as

$$\mid \mathcal{D}^N_{K,p,q}\rangle = \sum_{m=0}^{N} \sqrt{p^N_m} \mid \mathcal{D}^N_{k,p,m}\rangle \otimes \mid \mathcal{D}^N_{K-k,m,q}\rangle\,. \tag{64}$$

The parameters p^N_m in (64) are expressed through the averages $G^{(N)}_k$ (13):

$$p^N_m = \frac{G^{(N)}_k(m, N-m; p, N-p)\, G^{(N)}_{K-k}(q, N-q; m, N-m)}{G^{(N)}_K(q, N-q; p, N-p)}\,, \tag{65}$$

where $G_k^{(N)}$ are of the form:

$$G_k^{(N)}(x, N-x; y, N-y) = \langle x, N-x \mid H_{\rm zr}^k \mid y, N-y\rangle \tag{66}$$

$$= \sum_{\{\mathbf{u}\}} \frac{(E_N^{\rm zr}(\mathbf{u}))^k}{\mathcal{N}_{\rm zr}^2(\mathbf{u})} \chi^R_{\boldsymbol{\lambda}_R}(\mathbf{u}) \chi^L_{\boldsymbol{\lambda}_L}(\mathbf{u})\,. \tag{67}$$

The partitions $\boldsymbol{\lambda}_R = (1^y, 0^{N-y})$ and $\boldsymbol{\lambda}_L = (1^x, 0^{N-x})$ are used in (67), and the parameters u_j satisfy the Bethe equations

$$u_n^{-2N}\left(1+u_n^{-2}\right)^2 = (-1)^{N-1}\prod_{j=1}^N u_j^{-2}\,. \tag{68}$$

The condition $\sum_{m=0}^N p_m^N = 1$ is following from

$$\langle x, N-x \mid H_{\rm zr}^K \mid y, N-y\rangle$$
$$= \sum_{m=0}^N \langle x, N-x \mid H_{\rm zr}^k \mid m, M-m\rangle\langle m, M-m \mid H_{\rm zr}^{K-k} \mid y, N-y\rangle\,.$$

When $K > N$, the Schmidt rank is $r = 1+N$. The entanglement entropy of the modified Dyck paths is

$$S = -\sum_{m=0}^N p_m^N \log_2 p_m^N\,. \tag{69}$$

6 The Phase Model

In this section we shall consider the random walks on $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ which are defined by a step-set Ω_M. The Hermitian Hamiltonian $H_{\rm ph}$ of the phase model may be considered as the generator of the walks :

$$H_{\rm ph} = \sum_{m=0}^M \left(\phi_m\phi_{m+1}^\dagger + \phi_m^\dagger\phi_{m+1}\right). \tag{70}$$

The phase model is defined by the L-operator (14) with $g_i = 0$, and the Bethe equations are:

$$u_n^{-2(N+M+1)} = (-1)^{N-1}\prod_{j=1}^N u_j^{-2}\,. \tag{71}$$

The state-vectors of the phase model are expressed in the form [29, 30]:

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda}\subseteq\{M^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2)\left(\prod_{l=0}^M (\phi_l^\dagger)^{n_l}\right)|\mathbf{0}\rangle\,. \tag{72}$$

The coefficients of the expansion (72) are the Schur functions $S_{\boldsymbol{\lambda}}(\mathbf{u}^2)$ given by the Jacobi-Trudi identity:

$$S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \equiv \frac{\det(u_j^{2(\lambda_k+N-k)})_{1\le j,k\le N}}{\mathcal{V}_N(\mathbf{u}^2)}\,, \tag{73}$$

where the non-strict partition $\boldsymbol{\lambda}$ is connected with the coordinates of the Bose particles as it was explained above. Besides, $\mathcal{V}_N(\mathbf{u}^2)$ is the Vandermonde determinant (35) (subscript N is equal to the "length" of sequence $\mathbf{u}$). On the solutions of the Bethe equations the state-vectors are the eigen-vectors of the Hamiltonian (70) with the eigen-energies

$$E_N^{\mathrm{ph}}(\mathbf{u}) = \sum_{k=1}^{N}\left(u_k^2 + u_k^{-2}\right)\,. \tag{74}$$

The generating function of random walks on a $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ is similar to that of the Totally Asymmetric Zero Range Model, (63):

$$F_{\mathrm{ph}}^{(N)}(\mathbf{l},\mathbf{j}|t) = \sum_{\{\mathbf{u}\}} \frac{e^{tE_N^{\mathrm{ph}}(\mathbf{u})}}{\mathcal{N}_{\mathrm{ph}}^2(\mathbf{u})} S_{\boldsymbol{\lambda}_R}(\mathbf{u}^2) S_{\boldsymbol{\lambda}_L}(\mathbf{u}^2) \tag{75}$$

with the squared norm $\mathcal{N}_{\mathrm{ph}}^2(\mathbf{u})$:

$$\mathcal{N}_{\mathrm{ph}}^2(\mathbf{u}) = \sum_{\boldsymbol{\lambda}\subseteq\{M^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) S_{\boldsymbol{\lambda}}(\mathbf{u}^2)\,.$$

In the case $M=1$ the trajectory of a particle on a strip form an ordinary *modified Dyck* path. The detailed analyses of Bethe equations [31, 32] allows to rewrite the answer for the generating function of Dyck paths on a strip in a conventional form:

$$F^{(N)}\big((l,0),(0,j)|t\big) = \frac{2}{N+1}\sum_{k=1}^{N} e^{2t\cos(\pi k/(N+1))} \sin\left[\frac{\pi k(j+1)}{N+1}\right] \sin\left[\frac{\pi k(l+1)}{N+1}\right]\,. \tag{76}$$

With the help of this formula it is possible find the coefficients of the Schmidt decomposition and to calculate the entanglement entropy of the Dyck paths on a strip.

Appendix

The correlator $\langle \Psi_2(\mathbf{v})|\Psi_2(\mathbf{u})\rangle$ at $M = N = 2$ is of the form:

$$\begin{aligned}
\langle \Psi_2(\mathbf{v})|\Psi_2(\mathbf{u})\rangle \;=\; & e_0(\mathbf{v}^2)e_0(\mathbf{u}^{-2}) + e_1(\mathbf{v}^2)e_1(\mathbf{u}^{-2}) + e_2(\mathbf{v}^2)e_2(\mathbf{u}^{-2}) \\
& + \big(e_1^2(\mathbf{v}^2) - e_2(\mathbf{v}^2)\big)\big(e_1^2(\mathbf{u}^{-2}) - e_2(\mathbf{u}^{-2})\big) + e_2^2(\mathbf{v}^2)e_2^2(\mathbf{u}^{-2}) \\
& + 3\gamma\big(e_1(\mathbf{v}^2)e_0(\mathbf{u}^{-2}) + e_1^2(\mathbf{v}^2)e_1(\mathbf{u}^{-2}) + e_2(\mathbf{v}^2)e_1(\mathbf{v}^2)e_1^2(\mathbf{u}^{-2}) \\
& + e_2^2(\mathbf{v}^2)e_2(\mathbf{u}^{-2})e_1(\mathbf{u}^{-2})\big) \\
& + 3\gamma^2\Big(\big(e_1^2(\mathbf{v}^2) + e_2(\mathbf{v}^2)\big)e_0(\mathbf{u}^{-2}) + 3e_2(\mathbf{v}^2)e_1(\mathbf{v}^2)e_1(\mathbf{u}^{-2}) \\
& + e_2^2(\mathbf{v}^2)\big(e_1^2(\mathbf{u}^{-2}) + e_2(\mathbf{u}^{-2})\big)\Big) \\
& + 8\gamma^3\big(e_2(\mathbf{v}^2)e_1(\mathbf{v}^2)e_0(\mathbf{u}^{-2}) + e_2^2(\mathbf{v}^2)e_1(\mathbf{u}^{-2})\big) \\
& + 6\gamma^4 e_2^2(\mathbf{v}^2)e_0(\mathbf{u}^{-2})\,.
\end{aligned}$$

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