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Minimal k-connected graphs with small number of vertices of degree k

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Abstract

Let $n = (2k-1)t + 2\ell$, where k, t, ℓ are positive integers, such that $k \geq 3$ and $2 \leq \ell < \frac{8k+1}{9}$. It is proved that any minimal k-connected graph on n vertices has at least

$$\left\lceil \frac{(k-1)v(G) + 2k}{2k-1} \right\rceil + 1$$

vertices of degree k. An infinite series of graphs, for which this bound is attained are constructed.

Keywords: graph, connectivity, minimal k-connected graph.

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1 Introduction

We consider undirected graphs without loops and multiple edges and use standard notation.

For a graph G we denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. We use notation $v(G)$ and $e(G)$ for the number of vertices and edges of G, respectively.

For disjoint sets $A, B \subset V(G)$ we denote by $E_G(A, B)$ the set of all edges incident to vertices of both sets A and B. Let $e_G(A, B) = |E_G(A, B)|$.

The *degree* of a vertex x in the graph G is denoted by $d_G(x)$. The *maximal* vertex degree of the graph G is denoted by $\Delta(G)$.

The *neighborhood* of a vertex x in the graph G (i.e., the set of all vertices adjacent to x) is denoted by $\mathrm{N}_G(x)$.

Definition 1. 1) A graph G is called *k-connected* if $v(G) \geq k + 1$ and G remains connected after deleting any its $k - 1$ vertices.

2) A k-connected graph is called *minimal*, if it becomes not k-connected after deleting any edge.

Clearly, all vertices of a k-connected graph have degree at least k. We denote by $v_k(G)$ the number of vertices of degree k of a graph G.

In 1967 minimal biconnected graphs were considered in the papers [1] and [2]. It can be deduced from the results of these papers that

$$v_2(G) \geq \frac{v(G) + 4}{3}$$

for a minimal biconnected graph G.

In 1979 W. Mader [6, 5] proved a very strong result that generalizes for arbitrary k the one written above:

$$v_k(G) \geq \frac{(k-1)v(G) + 2k}{2k - 1} \tag{1}$$

for a minimal k-connected graph G. This bound is tight: there is infinite series of graphs for which the inequality (1) turns to equality.

Definition 2. Let $k \geq 2$ and T be a tree with $\Delta(T) \leq k + 1$. The graph $G_{k,T}$ is constructed from k disjoint copies $T_1, \ldots, T_k$ of the tree T. For any vertex $a \in V(T)$ we denote by a_i the corresponding vertex of the copy T_i. If $d_G(a) = j$ then we add $k + 1 - j$ new vertices of degree k that are adjacent to $\{a_1, \ldots, a_k\}$.

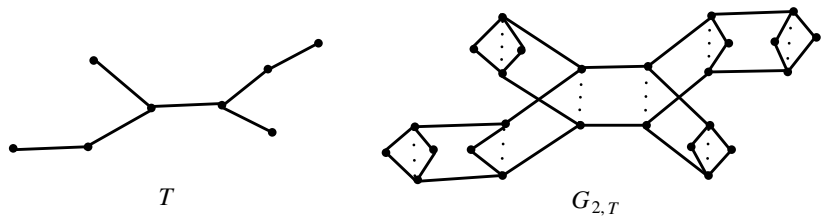

Рис. 1: A tree T and corresponding minimal biconnected graph $G_{2,T}$.

Clearly, if $v(T) = n$ then $v(G_{k,T}) = (2k-1)n + 2$. It is not difficult to verify that $G_{k,T}$ is a minimal k-connected graph and equality in (1) is attained for $G_{k,T}$. A tree T with $\Delta(T) = 3$ and corresponding graph $G_{2,T}$ are shown on the picture 1.

Definition 3. 1) Denote by $\mathcal{GM}_k(n)$ the set of all minimal k-connected graphs G on n vertices, such that

$$v_k(G) = \left\lceil \frac{(k-1)v(G) + 2k}{2k-1} \right\rceil. \tag{2}$$

2) The *defect* of a minimal k-connected graph G is

$$f(G) = (2k-1)v_k(G) - (k-1)v(G) - 2k.$$

Remark 1. Let G be a minimal k-connected graph.

1) It follows from inequality (1) that $f(G) \geq 0$.

2) It is easy to prove, that G satisfies (2) if and only if $f(G) < 2k-1$.

3) If G satisfies (2) then its defect $f(G)$ is equal to the residue of

$$-(k-1)v(G) - 2k$$

modulo $2k-1$. For $G \in \mathcal{GM}_k((2k-1)n + 2\ell)$, where $1 \leq \ell \leq k-1$, we have $f(G) = \ell - 1$.

Equality in (1) can be attained only for $v(G) \equiv 2 \pmod{2k-1}$. It was shown in [13] that $\mathcal{GM}_k((2k-1)t+2)$ consists of graphs of type $G_{k,T}$, where T is a tree with $\Delta(T) \leq k+1$ and $v(T) = t$.

In 1982 Oxley [7] presented an algorithm of constructing biconnected graphs of the sets $\mathcal{GM}_2(3t+1)$ and $\mathcal{GM}_2(3t)$. Both sets $\mathcal{GM}_2(3t+1)$ and $\mathcal{GM}_2(3t)$ are nonempty for every positive integer t. In [12] the author has described extremal biconnected graphs in terms of the graphs $G_{2,T}$.

The situation is quite different for $k \geq 3$. Let

$$n = (2k-1)t + p \geq 2k+1, \quad \text{where} \quad 0 \leq p \leq 2k-2 \quad \text{and} \quad t \geq 0.$$

In 1979 W. Mader [5] has constructed graphs satisfying equality (2) for $p \notin \{4, 6, \ldots, 2k-2\}$. Therefore, $\mathcal{GM}(n)$ is nonempty in this case. However, for $p \in \{4, 6, \ldots, 2k-2\}$ such graphs were not constructed. One can construct in this case a minimal k-connected graph G such that $v_k(G)$ is greater by 1 than the bound from (2). Examples of minimal k-connected graphs with small number of vertices of degree k are discussed in the end of this paper.

W. Mader [5] has conjectured that

$$v_k(G) \geq \left\lceil \frac{(k-1)v(G) + 2k}{2k-1} \right\rceil + 1 \tag{3}$$

$$\text{for} \quad v(G) = (2k-1)t + 2\ell, \quad \text{where} \quad 2 \leq \ell \leq k-1 \quad \text{and} \quad t > 0$$

(i.e., $\mathcal{GM}_k((2k-1)t + 2\ell) = \varnothing$). Mader has proved this conjecture for $\ell = 2$. We prove Mader's conjecture in some cases. The following theorem is the main result of this paper.

Theorem 1. *Let $k \geq 3$ and G be a minimal k-connected graph with $v(G) = (2k-1)t + 2\ell$, where $t \geq 1$ and $2 \leq \ell < \frac{8k+1}{9}$. Then*

$$v_k(G) \geq \left\lceil \frac{(k-1)v(G) + 2k}{2k-1} \right\rceil + 1.$$

Remark 2. If $k \leq 9$, then $k - 1 < \frac{8k+1}{9}$. Hence, Mader's conjecture for $k \leq 9$ is completely proved.

The following results are direct consequences of Theorem 1 and Remark 1.

Corollary 1. *Let integers k, ℓ, t satisfy the conditions of Theorem 1. Then the following statements hold.*

1) $\mathcal{GM}_k((2k-1)t + 2\ell) = \varnothing$.

2) *The defect of a minimal k-connected graph G with $v(G) = (2k-1)t + 2\ell$ satisfies the inequality $f(G) \geq \ell + 2k - 2$.*

At the end of the paper we construct minimal k-connected graphs with small number of vertices of degree k and prove the following simple result, which shows, that the bound from Theorem 1 is tight.

Theorem 2. *Let $n = (2k-1)t + p \geq 2k+1$. Then there exists a minimal k-connected graph G with $v(G) = n$, satisfying the following conditions.*

1) *If p is odd and $1 \leq p \leq 2k-1$ or $p = 2$, then*

$$v_k(G) = \left\lceil \frac{(k-1)n + 2k}{2k-1} \right\rceil$$

(i.e., $G \in \mathcal{GM}_k(n)$).

2) *If p is even and $4 \le p \le 2k-2$, then*

$$v_k(G) = \left\lceil \frac{(k-1)n + 2k}{2k-1} \right\rceil + 1.$$

In addition, we formulate a conjecture that complements Mader's one. Let G be a minimal k-connected graph. Item 3 of Remark 1 implies that $f(G) = \ell - 1$ (where $1 \le \ell \le k-1$) if and only if $G \in \mathcal{GM}_k((2k-1)t + 2\ell)$. Therefore, Mader's conjecture means that if $f(G) > 0$ then $f(G) \ge k-1$. By Remark 1 we have that $f(G) = k-1$ if and only if $G \in \mathcal{GM}_k((2k-1)t+1)$.

Conjecture. *For $t \ge 2$ the set $\mathcal{GM}_k((2k-1)t+1)$ consists of graphs of type $G_{k,T} \cdot xy$, where T is a tree with $\Delta(T) \le k+1$, $v(T) = t$, and x, y are two adjacent vertices of the graph $G_{k,T}$ with $d_{G_{k,T}}(x) = d_{G_{k,T}}(y) = k+1$.*

This statement is proved in [12] for $k = 2$.

1.1 The structure of the paper

In **Section 2** we define a cut and a decomposition of a graph by a cut or by a set of several cuts. We also list some properties of cuts proved in [14]. In **Section 3** we discuss properties of cuts of a minimal k-connected graph. In this section important Lemma 5 about wry cuts is proved. In **Section 4** we prove properties of the defect of a minimal k-connected graph. In **Section 5** we prove the main theorem of this paper — Theorem 1. Finally, in **Section 6** we construct extremal examples showing that the bound proved in Theorem 1 is tight.

2 Cuts

Note that in our paper a *connected component* of a graph is a vertex set of its maximal up to inclusion connected subgraph.

Definition 4. Let $R \subset V(G) \cup E(G)$.

1) We denote by $G - R$ the graph obtained from G upon deleting all vertices and edges of the set R and all edges incident to vertices of R.

2) The set R is called a *cutset*, if the graph $G - R$ is disconnected.

3) Let $X, Y \subset V(G)$, $X \not\subset R$, $Y \not\subset R$. We say that R *separates* the set X from Y, if any two vertices $v_x \in X \setminus R$ and $v_y \in Y \setminus R$ are disconnected in the graph $G - R$.

4) We say that R *splits* a set $X \subset V(G)$, if the set $X \setminus R$ is nonempty and disconnected in the graph $G - R$.

Definition 5. Let G be a k-connected graph.

1) A *cut* is a cutset consisting of k elements and containing at least one edge. The set of all cuts of the graph G is denoted by $\mathfrak{T}(G)$.

2) For a cut $T \in \mathfrak{T}$ we denote by $V(T)$ the set of all vertices that belong to T and by $W(T)$ the set of all vertices that belong to T or are incident to edges of the cut T.

Remark 3. 1) No vertex of the set $V(T)$ is adjacent to an edge of the cut T.

2) For any cut $T \in \mathfrak{T}(G)$ the graph $G-T$ has exactly two connected components, let it be U_1 and U_2. For any edge $e \in T$ each of the components U_1 and U_2 contains an end of e.

Definition 6. 1) Let $T \in \mathfrak{T}(G)$ and let U_1 and U_2 be connected components of the graph $G - T$. The sets $A_i = U_i \cup V(T)$ are *parts of decomposition* of the graph G by the cut T. We use the notation $\mathrm{Part}(T) = \{A_1, A_2\}$.

2) The sets $A_1 \cap W(T)$ and $A_2 \cap W(T)$ are *boundaries* of the cut T.

Remark 4. Let $T \in \mathfrak{T}(G)$, $\mathrm{Part}(T) = \{A_1, A_2\}$. The following statements hold.

1) $A_1 \cup A_2 = V(G)$, $A_1 \cap A_2 = V(T)$.

2) Both boundaries R_1 and R_2 of the cut T contain the set $V(T)$. For every edge $e \in T$ one of its ends belongs to R_1 and the other belongs to R_2.

3) If the set $A' = A_1 \setminus W(T)$ is nonempty, then $R_1 = A_1 \cap W(T)$ (a boundary of the cut T) separates A' from $V(G) \setminus A_1$ and every vertex $x \in R_1$ is adjacent to at least one vertex of A'.

Definition 7. Let $\mathfrak{S} \subset \mathfrak{T}(G)$.

1) *Parts of decomposition* of the graph G by the set of cuts $\mathfrak{S}$ are maximal up to inclusion sets of type

$$A = \bigcap_{S \in \mathfrak{S}} A_S, \quad \text{where} \quad A_S \in \mathrm{Part}(S). \tag{4}$$

We denote the set of all such parts by $\mathrm{Part}(\mathfrak{S})$.

2) The *boundary* of a part $A \in \mathrm{Part}(\mathfrak{S})$ is the set $\mathrm{Bound}(A)$ consisting of all vertices of the part A which belong to a cut of $\mathfrak{S}$ (i.e., to a set $V(S)$ for some cut $S \in \mathfrak{S}$).

3) The *interior* of a part $A \in \mathrm{Part}(\mathfrak{S})$ is the set $\mathrm{Int}(A) = A \setminus \mathrm{Bound}(A)$.

Definition 8. Cuts $S, T \in \mathfrak{T}(G)$ are *independent*, if the notation $\mathrm{Part}(S) = \{A_1, A_2\}$, $\mathrm{Part}(T) = \{B_1, B_2\}$ can be chosen such that $A_1 \supset B_2$ and $B_1 \supset A_2$. Otherwise, the cuts S and T are *dependent*.

Definition 9. Let G be a k-connected graph and $\mathfrak{S} \subset \mathfrak{T}(G)$ be a set consisting of pairwise independent cuts.

1) The bipartite graph $\mathrm{BT}(G;\mathfrak{S})$ is defined as follows. One partition of this graph is the set $\mathfrak{S}$ and the other partition is $\mathrm{Part}(\mathfrak{S})$. A cut $S \in \mathfrak{S}$ and a part $A \in \mathrm{Part}(\mathfrak{S})$ are adjacent if and only if A contains a boundary of the cut S.

2) A part $A \in \mathrm{Part}(\mathfrak{S})$ is called *pendant*, if it corresponds to a pendant vertex of the graph $\mathrm{BT}(G;\mathfrak{S})$.

3) The graph $\mathrm{T}(G;\mathfrak{S})$ is defined as follows. Its vertex set is $\mathrm{Part}(\mathfrak{S})$. Parts $A, B \in \mathrm{Part}(\mathfrak{S})$ are adjacent if and only if these parts contain boundaries of the same cut of the set $\mathfrak{S}$.

The following results are proved in [14, Theorem 1 and Corollary 1].

Theorem 3. *Let G be a k-connected graph and $\mathfrak{S} \subset \mathfrak{T}(G)$ be a set consisting of pairwise independent cuts, such that no cuts of $\mathfrak{S}$ contain a common edge. Then the following statements hold.*

1) *Both graphs $\mathrm{BT}(G;\mathfrak{S})$ and $\mathrm{T}(G;\mathfrak{S})$ are trees.*

2) *Any cut $S \in \mathfrak{S}$ is adjacent in $\mathrm{BT}(G;\mathfrak{S})$ to exactly two parts of $\mathrm{Part}(\mathfrak{S})$, these two parts are contained in different parts of $\mathrm{Part}(S)$ and contain different boundaries of the cut S.*

3) *If a pendant part A is adjacent in $\mathrm{Part}(\mathfrak{S})$ to the cut T, then $A \in \mathrm{Part}(T)$.*

4) *Let*

$$\mathrm{Part}'(\mathfrak{S}) = \bigcup_{S \in \mathfrak{S}} \mathrm{Part}(S).$$

Then pendant parts of $\mathrm{Part}(\mathfrak{S})$ are exactly minimal up to inclusion parts of $\mathrm{Part}'(\mathfrak{S})$.

5) *Assign to each edge AB of the tree $\mathrm{T}(G;\mathfrak{S})$ the cut of $\mathfrak{S}$, which boundaries are contained in the parts A and B. Then this is a one-to-one correspondence between edges of the tree $\mathrm{T}(G;\mathfrak{S})$ and cuts of the set $\mathfrak{S}$.*

6) $|\mathrm{Part}(\mathfrak{S})| = |\mathfrak{S}| + 1$.

7) *Let R be a boundary of a cut of the set $\mathfrak{S}$. Then there exists exactly one part of $\mathrm{Part}(\mathfrak{S})$, which contains R.*

3 Minimal k-connected graphs

3.1 Notation

In what follows we study a minimal k-connected graph G and use for this graph the following notation:

V_k is the set of all vertices of degree k of the graph G,

$$V_{k+1} = V(G) \setminus V_k, \quad G_k = G(V_k), \quad G_{k+1} = G(V_{k+1}),$$
$$E_k = E(G_k), \quad e_k = |E_k|, \quad E_{k+1} = E(G_{k+1}).$$

In the case where the name of a graph discussed is not clear we will write it in brackets ($V_k(G)$ and similarly for others).

Definition 10. Denote by c the number of connected components of the graph G_{k+1}.

For every edge $e \in E_{k+1}$ there exists a cut which contains e and $k-1$ vertices. Let $\mathfrak{R}$ be the set of all such cuts. We pass to studying properties of cuts of the set $\mathfrak{R}$.

3.2 Cuts in a minimal k-connected graphs

The following Lemma and Corollary are proved by Mader [4]. Proofs of these statements are also written in [13, Lemma 4 and Corollary 2].

Lemma 1. *Let G be a minimal k-connected graph, $xy, xz \in E_{k+1}$. Let $T_{xy} \ni xy$ and $T_{xz} \ni xz$ be cuts of the set $\mathfrak{R}$,*

$$x \in H_x \in \mathrm{Part}(T_{xy}) \quad \textit{and} \quad z \in F_z \in \mathrm{Part}(T_{xz}).$$

Then $|H_x| > |F_z|$.

Corollary 2. *If G is a minimal k-connected graph then G_{k+1} is a forest.*

Definition 11. A cut $S \in \mathfrak{R}$ is called *wry*, if there exists a part $A \in \mathrm{Part}(S)$, such that $|\mathrm{Int}(A)| < \frac{k}{2}$. Otherwise, this cut is called *normal*.

The first item of the following lemma is proved in Lemma 8 and the second — in Lemma 10 of the paper [13].

Lemma 2. *Let G be a minimal k-connected graph, and the set $E \subset E_{k+1}$ be such that every cut of $\mathfrak{R}$, which contains an edge of E, is normal. Then the following statements hold.*

1) *There exists a set $\mathfrak{S} = \{S_e\}_{e \in E} \subset \mathfrak{R}$, where $e \in S_e$, that consists of pairwise independent cuts.*

2) *Let R be a boundary of a cut $S \in \mathfrak{S}$. Then any simple path with ends in R contains an edge which does not belong to the set E.*

Lemma 3. *Let G be a minimal k-connected graph, $E \subset E_{k+1}$, and let the set $\mathfrak{S} = \{S_e\}_{e \in E} \subset \mathfrak{R}$, where $e \in S_e$, consist of pairwise independent cuts. Let $A \in \mathrm{Part}(\mathfrak{S})$ be a pendant part. Then the following statements hold.*

1) *There exists an edge $ab \in E$, such that $A \in \mathrm{Part}(S_{ab})$.*

2) *Let $xy \in E$ and $x \in \mathrm{Int}(A)$. Then $xy = ab$.*

Proof. 1) A direct consequence of item 3 of Theorem 3.

2) Let $xy \neq ab$. Then $y \in A$ and the cut $S_{xy} \in \mathfrak{S}$ splits the set $\{x, y\}$. Since $\{x, y\} \subset A \in \mathrm{Part}(\mathfrak{S})$, this is impossible. □

Before the main lemma of this section we formulate a simple generalization of Menger's theorem proved in [8].

Lemma 4. *Let $X, Y \subset V(G)$, $|X| \geq k$, $|Y| \geq k$ and let any set $R \subset V(G)$, separating X from Y, contain at least k vertices. Then there exist k nonintersecting XY-paths. (Any two of these paths have no common ends. A vertex of the set $X \cap Y$ is an XY-path.)*

Definition 12. 1) An edge $y_1y_2 \in E(G_{k+1})$ is *wry*, if there is a wry cut $S_{y_1y_2} \in \mathfrak{R}$, which contains y_1y_2. If

$$y_1 \in Y_1 \in \mathrm{Part}(S_{y_1y_2}) \quad \text{and} \quad |\mathrm{Int}(Y_1)| < \frac{k}{2},$$

then we call the arc (i.e. directed edge) y_2y_1 a *wry arc* and the part Y_1 — a *wry part* for the wry arc y_2y_1.

2) If y_2y_1 is a wry arc and y_1 is a leaf of G_{k+1} then y_1 is a *wry leaf* of G_{k+1}.

The following lemma strengthens the result proved in [13].

Lemma 5. *Let G be a minimal k-connected graph. Assume, that there is at least one wry cut in the set $\mathfrak{R}$. Then $e_k + c \geq \frac{3k-1}{2}$.*

Proof. The proof of Lemma is long and complicated. During the proof we formulate and prove several claims, which will be used only in the proof of Lemma. All these claims are formulated and proved under the condition of Lemma.

Claim 1. *Let a_1 be a wry leaf, a_2a_1 be a wry arc and A_1 be a wry part for a_2a_1. Then one of the two following statements holds.*

(1) $e_k + c \geq \frac{3k-1}{2}$, *i.e., Lemma is proved.*

(2) $\mathrm{Int}(A_1) = \{a_1, x\}$ *and $k-1$ edges of the set E_k connect x with vertices of S (see figure 2c). In particular, $e_k \geq k-1$.*

Proof. Let

$$a_1a_2 \in S_{a_1a_2} \in \mathfrak{R}, \quad a_1 \in A_1 \in \mathrm{Part}(S_{a_1a_2}), \quad S = V(S_{a_1a_2}),$$
$$|\mathrm{Int}(A_1)| = p, \quad M = \mathrm{Int}(A_1) \cap V_k, \quad m = |M|$$

(see figure 2a). Then

$$|S| = k-1, \quad p \le \frac{k-1}{2}, \quad m \le p-1 \le \frac{k-3}{2}. \tag{5}$$

Since $d_G(a_1) \ge k+1$ and $a_1 \in \mathrm{Int}(A_1)$, we have $p = |\mathrm{Int}(A_1)| \ge 2$. Therefore, $2 \le \frac{k-1}{2}$, whence

$$k \ge 5. \tag{6}$$

Clearly, the vertex a_1 is not adjacent to vertices of the set $V_{k+1} \cap A_1$. Hence, a_1 is adjacent to at most m vertices of the set $\mathrm{Int}(A_1)$. Since $d_G(a_1) \ge k+1$, the vertex a_1 is not adjacent to at most $m-1$ vertices of the set S. All vertices of the set S, that are adjacent to a_1, belong to V_k. Thus,

$$|S \cap V_{k+1}| \le m-1. \tag{7}$$

It follows from $|\mathrm{Int}(A_1) \cap V_{k+1}| = p-m$ and inequality (7), that

$$|A_1 \cap V_{k+1}| \le p-1. \tag{8}$$

Set the notation

$$Y = (\mathrm{Int}(A_1) \setminus \{a_1\}) \cap V_{k+1}, \quad Y' = (A_1 \setminus \{a_1\}) \cap V_{k+1}, \quad T' = G(Y').$$

By Corollary 1 the graph T' is a forest. Consider several cases.

1. $m \ge 3$.

A vertex of the set M is adjacent only to vertices of A_1. Then by inequality (8) any vertex of M is adjacent to at most $p-1$ vertices of the set V_{k+1}, and, therefore, it is adjacent to at least $k-p+1$ vertices of V_k. Summing edges from vertices of M to vertices of V_k, we obtain at least $m(k-p+1) \ge m \cdot \frac{k+3}{2}$ edges (by inequality (5)). However, edges between vertices of the set M were counted twice. Since the number of such edges does not exceed $\frac{m(m-1)}{2}$, we can write the following:

$$e_k \ge m \cdot \frac{k+3}{2} - \frac{m(m-1)}{2} \ge \frac{m}{2} \cdot (k+4-m) \ge \frac{3k+3}{2}. \tag{9}$$

(The last inequality in (9) follows from $m^2 - (k+4)m + (3k+3) \le 0$. This inequality is true, since $3 \le m < k+1$.) Hence, in this case $e_k + c \ge \frac{3k+5}{2}$ and Lemma is proved.

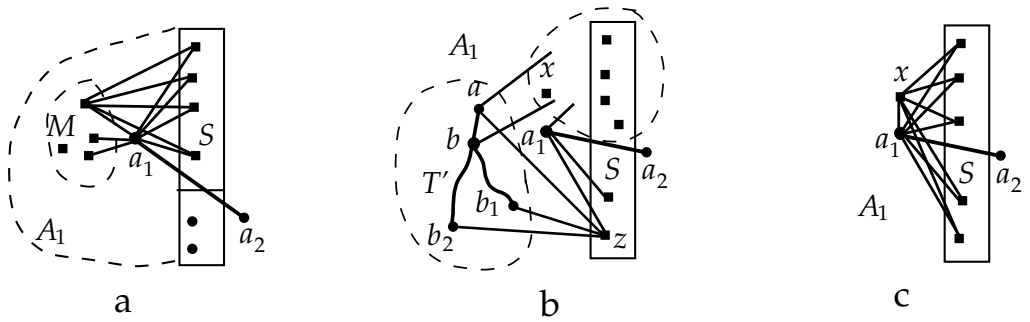

Рис. 2: The wry cut $S_{a_1a_2}$ and the part A_1.

2. $m = 1$.
Let $M = \{x\}$. In this case, a_1 is adjacent to exactly one vertex of $\mathrm{Int}(A_1)$ (the vertex x), hence, a_1 is adjacent to all vertices of S. Thus, $S \subset V_k$ and $Y = Y'$.

Assume that $Y \neq \varnothing$. Let $a \in Y$ and $d_{T'}(a) \leq 1$. Since $d_G(a) \geq k+1$, the vertex a must be adjacent to x and all $k-1$ vertices of S and $d_{T'}(a) = 1$. Hence, the forest T' has no isolated vertices.

Let a be a leaf of T', adjacent in T' to a vertex b of degree $d_{T'}(b) = \ell$ (see figure 2b). Then the forest T' contains $\ell - 1$ paths $P_1, \ldots, P_{\ell-1}$ from b to distinct leaves $b_1, \ldots, b_{\ell-1}$ which are different from a. Any two of these paths have no common vertex different from b.

Consider a cut $S_{ab} \in \mathfrak{R}$, such that $ab \in S_{ab}$. Note, that b is adjacent to at least $k - \ell + 1$ vertices of the set $S \cup \{x\}$, and all these vertices must belong to S_{ab}. The cut S_{ab} contains $k-1$ vertices, hence, it doesn't contain a certain vertex $z \in S \cup \{x\}$.

As it was proved above, a and each of vertices $b_1, \ldots, b_{\ell-1}$ is adjacent to z. Hence, the cut S_{ab} which separates a from b must contain a vertex on each of paths $P_1, \ldots, P_{\ell-1}$. However, then S_{ab} contains at least k vertices. We obtain a contradiction.

Thus, $\mathrm{Int}(A_1) = \{a_1, x\}$ (see figure 2c). In this case $k-1$ edges connect x with vertices of S and all these edges belong to E_k. Hence, $e_k \geq k-1$ and we have case (2) of Claim.

3. $m = 2$.
Then $|Y| = p - 3$. Inequality (7) implies $|S \cap V_{k+1}| \leq 1$. Set the notation

$$U = A_1 \cap V_k.$$

Consider three cases.

3.1. *$Y = \varnothing$ or Y' is an independent set of the graph G.*
In this case all $p-3$ vertices of the set Y are isolated in the graph G_{k+1}, therefore, $c \geq p-2$. Vertices of the set M are adjacent only to vertices of the

set A_1, which contains at most $p-1$ vertices of V_{k+1}. Summing edges from two vertices of the set M to vertices of V_k, we obtain at least $2(k-p+1)$. If two vertices of M are adjacent, the edge between them is counted twice. Hence, $e_k \geq 2k-2p+1$ and

$$e_k + c \geq (2k-2p+1) + (p-2) \geq 2k-p-1 \geq \frac{3k-1}{2} \qquad (10)$$

(the last inequality follows from (5)). We have case (1) of Claim.

3.2. *Y is an independent set and Y' is not an independent set of the graph G.*

By inequality (7) the set S contains exactly one vertex of V_{k+1}, denote it by w. Then

$$U = M \cup (S \setminus \{w\}), \quad |U| = k.$$

The vertex w is incident to all edges of the forest T'. Thus every vertex of the set Y must be adjacent to w and all vertices of the set U. Therefore, T' is a tree and all vertices of Y are its leaves, adjacent to w (see figure 3a).

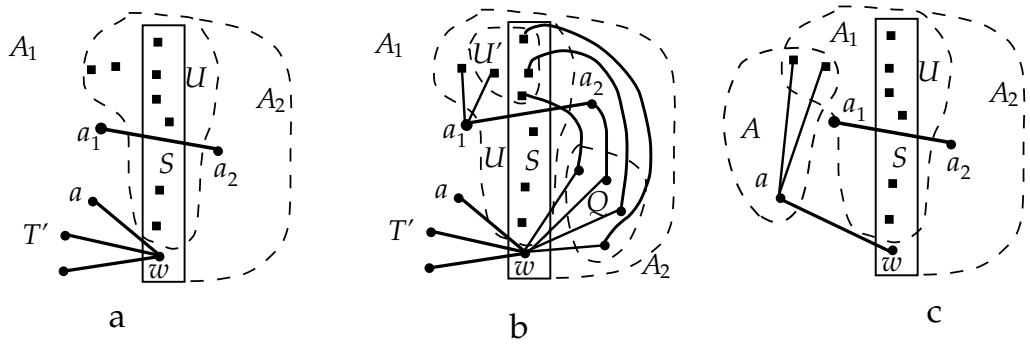

Рис. 3: The wry cut $S_{a_1a_2}$. Case 3.2.

Let $a \in Y$ and a cut $S_{aw} \in \mathfrak{R}$ contain aw. Assume that a vertex $a' \in Y$ is different from a and does not belong to the cut S_{aw}. Since $U \not\subset S_{aw}$, the vertices a' and a have a common adjacent vertex in the graph $G - S_{aw}$. At the same time, a' is adjacent to w. Hence, the graph $G - S_{aw}$ is connected. We obtain a contradiction. Thus, $Y \setminus \{a\} \subset S_{aw}$.

Every vertex of the set U which is adjacent to w is also adjacent to a and, therefore, belongs to S_{aw}. Denote by F the set of all vertices of $A_1 \setminus \{a\}$ which are adjacent to w, let $f = |F|$. We have proved that $F \subset S_{aw}$.

Recall, that $a_1a_2 \in E_{k+1}$, $w \in V_{k+1}$, $d_G(a_1) \geq k+1$ and $d_{G_{k+1}}(a_1) = 1$. Hence, the vertex a_1 is not adjacent to w and, therefore, $a_1 \notin F$. Moreover, $\mathrm{N}_{G(a_1)} = U \cup \{a_2\}$.

Let S' be the set of all vertices of $S \setminus \{w\}$ which are not adjacent to w. If w is adjacent to both vertices of M, then we set $U' = S'$. Otherwise, we set $U' = S' \cup \{v\}$, where $v \in M$ is a vertex not adjacent to w (if no vertex of M is adjacent to w we choose any of them). Let Q be the set of all vertices of the set $\mathrm{Int}(A_2)$, which are adjacent to w (see figure 3b). Set the notation $s' = |S'|$, $q = |Q|$.

Consider the graph $G' = G - (S \setminus S')$. This graph is $(s'+1)$-connected, and the set U' has s' or $s'+1$ vertices. Thus by Lemma 4 there exist at least $\ell = \min(q, |U'|)$ pairwise nonintersecting paths $P_1, \dots, P_\ell$ from Q to U' in the graph G', such that no inner vertex of these paths belongs to $Q \cup U'$.

At most one of the paths $P_1, \dots, P_\ell$ can contain the vertex a_1. If such a path exists, let it be P_1 and let $u \in U'$ be the end of P_1. Since $\mathrm{N}_{G(a_1)} \supset U' \ni u$, the vertex a_1 is adjacent to u. Therefore, we can replace the part of P_1 from a_1 to u by the edge a_1u. Denote by P the path obtained. Since $Q \subset \mathrm{N}_G(w)$ and $u \in \mathrm{N}_G(a)$, at least one vertex of the path P must belong to S_{aw}. This vertex belong to $\mathrm{Int}(A_2) \cup S' \cup \{a_1\}$ and, therefore, doesn't belong to F. Since all vertices of the path P belong to P_1, the path P intersects none of the paths $P_2, \dots, P_\ell$.

Consider a path P_i, which does not contain a_1. Since w is not adjacent to $\mathrm{Int}(A_2) \setminus Q$, we have $w \notin V(P_i)$. Recall, that $S' \cup \{a_1, w\}$ separates $\mathrm{Int}(A_2) \supset Q$ from $\mathrm{Int}(A_1) \supset M$ in the graph G'. Hence the path P_i connects Q to S' and $V(P_i) \subset A_2$. Since all vertices of the set S' are adjacent to a, the path P_i must have a vertex in S_{aw}. This vertex belongs to $\mathrm{Int}(A_2) \cup S'$ and, consequently, do not belong to F.

Thus, the cut S_{aw} contains at least ℓ vertices on the paths P_1 (or P), $P_2, \dots, P_\ell$. Hence,

$$k - 1 = |V(S_{aw})| \geq f + \ell = f + \min(q, |U'|).$$

Since $f + q = d_G(w) - 1 \geq k$, we have

$$\min(q, |U'|) = |U'| \leq k - 1 - f. \tag{11}$$

Recall, that w is adjacent to all $k - s' - 2$ vertices of the set $S \setminus (S' \cup \{w\})$. Consider two cases.

3.2.1. *The vertex w is adjacent to both vertices of M.*
Then

$$f \geq (k - s' - 2) + 2 = k - s' \quad \text{and} \quad |U'| = s'.$$

Therefore, $|U'| \geq k - f$, that contradicts (11).

3.2.2. *The vertex w is adjacent to at most one vertex of M.*

Then $|U'| = s' + 1$. Thus, inequality (11) implies, that $f \leq k - s' - 2$. Therefore, w is adjacent to no vertex of the set $\mathrm{Int}(A_1) \setminus \{a\}$. Thus, $Y = \{a\}$ and w is not adjacent to vertices of M. In this case

$$S'_{aw} = (S \setminus \{w\}) \cup \{a_1\} \cup \{aw\} \in \mathfrak{R}$$

is a cut which separates a part A with $\mathrm{Int}(A) = M \cup \{a\}$ (see figure 3c). Clearly, $|\mathrm{Int}(A)| < |\mathrm{Int}(A_1)|$, therefore the cut S'_{aw} is wry. Let's repeat reasonings of the proof for the cut S'_{aw} and the part A instead of the cut $S_{a_1a_2}$ and the part A_1. It is easy to see that for the part A we have $m = 2$ and Case 3.1, in this case Lemma is proved.

3.3. *Y is not an independent set of the graph G.*
Consider two subcases.

3.3.1. $S \cap V_{k+1} \neq \varnothing$.
By inequality (7) the set S contains exactly one vertex of V_{k+1}, denote it by w. Then

$$U = M \cup (S \setminus \{w\}), \quad |U| = k.$$

The forest T' has an edge which is not incident to w. Hence, T' has a leaf $a \neq w$, adjacent in T' to a vertex $b \neq w$. Let $d_{T'}(b) = \ell$. Consider a cut $S_{ab} \in \mathfrak{R}$, which contains ab. Note that the vertex $b \in \mathrm{Int}(A_1)$ is adjacent to at least $k - \ell + 1$ vertices of the set U and all these vertices must belong to S_{ab}. The cut S_{ab} contains $k - 1$ vertices, hence, it does not contain a certain vertex $z \in U$. Note, that a_1 is adjacent to all k vertices of the set U and, in particular, to z.

In the forest T' there are $\ell - 1$ paths $P_1, \ldots, P_{\ell-1}$ from b to distinct leaves $b_1, \ldots, b_{\ell-1}$, different from a. Any two of these paths have no common vertex different from b (see figure 4a).

One of these leaves can coincide with w. Let $b_1 = w$. Since G is a k-connected graph the vertex w is adjacent to $\mathrm{Int}(A_2)$. The graph $G(\mathrm{Int}(A_2))$ is connected. Hence, there is a wa_2-path Q, all inner vertices of which lie in $\mathrm{Int}(A_2)$. Recall, that a_1 is adjacent to z and is not a vertex of the tree T'. Hence, a_1 does not belong to paths $P_1, \ldots . P_{\ell-1}$. Therefore, $P'_1 = P_1 Q a_2 a_1 z$ is a simple bz-path that has no common inner vertex with $P_2, \ldots, P_{\ell-1}$. Since the vertex $z \notin S_{ab}$ is adjacent to a, the cut S_{ab} must contain at least one inner vertex of path P'_1.

Let $b_i \neq w$. Then both vertices a and b_i are adjacent to $z \notin S_{ab}$. Hence, the cut S_{ab} must contain a different from b vertex of the path P_i. Therefore, S_{ab} contains at least $k - \ell + 1$ vertices of the set U and $\ell - 1$ vertices on the paths P_1 (or P'_1), $P_2, \ldots, P_\ell$. Thus, S_{ab} contains at least k vertices. We obtain a contradiction. The case is impossible.

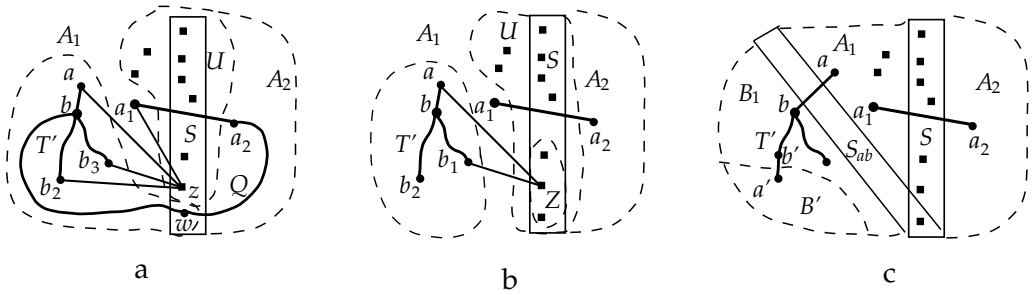

Рис. 4: The wry cut $S_{a_1a_2}$. Case 3.3.

3.3.2. $S \cap V_{k+1} = \varnothing$.
Then $U = M \cup S$ and $|U| = k+1$. Let a be a leaf of the forest T'. In our case, $a \in \mathrm{Int}(A_1)$. Let a be adjacent in T' to a vertex b, $d_{T'}(b) = \ell$.

The vertex a must be adjacent to at least k vertices of the set U. Therefore, a is not adjacent to at most one vertex of U. If this vertex exists we denote it by v.

Consider a cut $S_{ab} \in \mathfrak{R}$ which contains ab. We have two subcases.

3.3.2.1. $S_{ab} \cap \mathrm{Int}(A_2) \neq \varnothing$.
A vertex b is adjacent to at least $k-\ell+1$ vertices of U. At most one of them coincides with v, hence, at least $k-\ell$ vertices of the set U are adjacent to both vertices a and b. All these vertices must belong to S_{ab}.

Let $Z = U \setminus S_{ab}$ (see figure 4b). Recall, that S_{ab} contains $k-1$ vertices, at least one of them lies in $\mathrm{Int}(A_2)$ (i.e. does not belong to U) and $|U| = k+1$. Hence, $|Z| \geq 3$.

In the forest T' there exist $\ell - 1$ paths $P_1, \ldots, P_{\ell-1}$ from b to distinct leaves $b_1, \ldots, b_{\ell-1}$, different from a. Any two of these paths have no common vertex different from b. All vertices of these paths lie in $\mathrm{Int}(A_1)$.

Both a and any vertex b_i are adjacent to k vertices of the set U. Hence, each of the vertices a and b_i can be not adjacent to at most one vertex of the set Z. Thus, there exists a vertex $z \in Z$, adjacent to both vertices a and b_i. Since $z \notin S_{ab}$, the cut S_{ab} must contain at least one vertex different from b of each path P_i. Consequently, S_{ab} contains $k-\ell$ vertices of the set U (which are adjacent to b), at least one vertex of the set $\mathrm{Int}(A_2)$ and at least $\ell - 1$ vertices on the paths $P_1, \ldots, P_{\ell-1}$. Hence, S_{ab} contains at least k vertices. We obtain a contradiction. The case is impossible.

3.3.2.2. $S_{ab} \cap \mathrm{Int}(A_2) = \varnothing$.
Then S_{ab} does not split A_2, and, consequently, there exists a part $B_2 \in \mathrm{Part}(S_{ab})$, such that $B_2 \supset A_2$. Since $W(S_{ab}) \subset A_1$, the second part $B_1 \in \mathrm{Part}(S_{ab})$ is a subset of A_1 and does not contain one of the vertices a

and b. Hence,

$$|\mathrm{Int}(B_1)| \le |\mathrm{Int}(A_1)| - 1 = p - 1 \le \frac{k-3}{2}. \tag{12}$$

We will define a cut $S_{a'b'}$ and a part $B_1' \in \mathrm{Part}(S_{a'b'})$. Consider several cases.

a. If $a \in B_1$, we set

$$a' = a, \quad b' = b, \quad B_1' = B_1.$$

b. Assume, that $b \in B_1$. Consider two cases.

b.1. If $d_{T'}(b) = 1$, we set

$$a' = b, \quad b' = a, \quad B_1' = B_1.$$

b.2. Let $d_{T'}(b) \ne 1$. Then in the forest $T' - ab$ there is a path from b to a leaf a' different from a (see figure 4c). In this case we set

$$a'b' \in E(T'), \quad a'b' \in S_{a'b'} \in \mathfrak{R}, \quad a' \in B_1' \in \mathrm{Part}(S_{a'b'}).$$

Now the definition is finished. Note, that $a' \in B_1' \in \mathrm{Part}(S_{a'b'})$ Applying several times Lemma 1, we obtain

$$|\mathrm{Int}(B_1')| < |\mathrm{Int}(B_1)| < \frac{k-3}{2}.$$

Hence in all cases the cut $S_{a'b'}$ is wry and $|B_1'| < |A_1|$. We repeat the reasonings of the proof of our Claim for the cut $S_{a'b'}$ and the part B_1' instead of S_{ab} and A_1. Let $\mathrm{Int}(B_1')$ contain m' vertices of degree k. If $m' \ge 3$, then we have Case 1 and Lemma is proved.

If $m' = 1$, then we have Case 2. In this case statement (2) of our Claim holds: $\mathrm{Int}(B_1') \cap V_k = \{x'\}$ and this vertex is incident to $k-1$ edges of the set E_k. Let's return to the part A_1: there is a vertex $x^* \ne x'$ in the set M and this vertex is incident to

$$k - |A_1 \cap V_{k+1}| \ge k - p + 1 \ge \frac{k+3}{2}$$

edges of the set E_k (we used inequalities (5) and (8)). Since at most one of these edges is incident to x', we have

$$e_k \ge k - 1 + \frac{k+3}{2} - 1 \ge \frac{3k-1}{2}$$

and statement (1) of Claim holds (i.e., Lemma is proved).

Finally, let $m' = 2$. In this case we repeat the reasonings of Case 3 with the part B_1' and either prove Lemma or obtain Case 3.3.2.2 for the part B_1' (note, that $|B_1'| < |A_1|$). In the last case we find a part B_2' which is less than B_1', repeat the reasonings for this part, and so on. Since the size of the part we consider is decreased on each step, this process will be finished and we obtain statement (1) of our Claim (the proof of Lemma will be finished). $\square$

Claim 2. *Let $y_n y_{n-1} \dots y_1$ be a simple path in G_{k+1} and let $y_n y_{n-1}$ be a wry arc. Then $y_2 y_1$ is a wry arc.*

Proof. Induction by n. The **base** for $n = 2$ is clear.

The induction step. Assume, that Claim is proved for any path shorter than our one. Let $y_{n-1}y_{n-2} \in S_{y_{n-1}y_{n-2}} \in \mathfrak{R}$, $y_{n-2} \in Y_{n-2} \in \mathrm{Part}(S_{y_{n-1}y_{n-2}})$ and let Y_{n-1} be a wry part for the wry arc $y_n y_{n-1}$. By Lemma 1 we have $|\mathrm{Int}(Y_{n-2})| < |\mathrm{Int}(Y_{n-1})| < \frac{k}{2}$, hence $y_{n-1}y_{n-2}$ is a wry arc. Now by induction assumption for a shorter path $y_{n-1}y_{n-2} \dots y_1$ we have that $y_2 y_1$ is a wry arc. $\square$

Claim 3. *One of the two following statements hold.*

(1) $e_k + c \geq \frac{3k-1}{2}$, *i.e., Lemma is proved.*

(2) *All wry edges of the graph G_{k+1} are edges of a simple path $Q = a_n \dots a_2 a_1$, where $n \leq \frac{k-1}{2}$, a_1 is a wry leaf of G_{k+1} and all inner vertices of Q have degree 2 in the graph G_{k+1}.*

Proof. Let $y_2 y_1$ be a wry arc. Consider any path from y_1 to a leaf a_1 of the forest G_{k+1} which does not pass the edge $y_2 y_1$. By Claim 2 we have that a_1 is a wry leaf of G_{k+1}.

Let $a_2 a_1$ be a wry arc, and let A_1 be a wry part for this arc. By Claim 1 for the wry arc $a_2 a_1$ and the wry part A_1 we obtain that either Lemma is proved (i.e. statement (1) of our Claim holds), or statement (2) of Claim 1 holds: $\mathrm{Int}(A_1) = \{a_1, x\}$ and $k-1$ edges of the set E_k are incident to the vertex x. Consider two cases.

1. *There exists a wry leaf a_1', different from a_1.*
Let $a_2' a_1'$ be a wry arc and let A_1' be a wry part for this arc. By Claim 1 either statement (1) of our Claim holds, or $\mathrm{Int}(A_1') = \{a_1', x'\}$ and $k-1$ edges of the set E_k are incident to the vertex x'. Clearly, $a_1 x, a_1' x' \in E(G)$, whence it follows $x \neq x'$.

Since at most one edge can be incident to both vertices x and x', we obtain $e_k \geq 2k-3$. By inequality (6) we have

$$e_k + c \geq (2k-3) + 1 \geq \frac{3k+1}{2}$$

and statement (1) holds.

2. *There exists exactly one wry leaf* a_1.
Let Q be the subgraph of G_{k+1} induced by the set of wry edges (edges of Q are all wry edges and vertices of Q are ends of these edges).

Consider any wry arc y_2y_1 and any path from y_1 to a leaf a of the forest G_{k+1} which doesn't pass the edge y_2y_1. By Claim 2 the vertex a is a wry leaf, hence, $a = a_1$. Thus any path from an end of wry arc to a leaf of G_{k+1} ends at a_1. Therefore $d_{G_{k+1}}(y_1) = 2$ and the graph Q is connected.

If there is a vertex $y \in Q$ with $d_{G_{k+1}}(y) > 2$ then by proved above all wry arcs incident to y begin at y. If there are at least two such arcs, say yz_1 and yz_2, then there are two paths from y to different leaves of G_{k+1} starting at z_1 and z_2. That contradicts proved above. Hence, the graph Q is a simple path $Q = a_n \dots a_2a_1$, all inner vertices of which have degree 2 in G_{k+1}.

Let A_i be a wry part for the wry arc $a_{i+1}a_i$. By Lemma 1, then

$$2 = \mathrm{Int}(A_1) < \mathrm{Int}(A_2) < \dots < \mathrm{Int}(A_{n-1}) \leq \frac{k-1}{2}.$$

Since all these numbers are integers, $n \leq \frac{k-1}{2}$. □

Let's return to the proof of Lemma. In what follows we consider the case where Lemma is not proved. Hence, statement (2) of Claim 1 and statement (2) of Claim 3 hold. We formulate and prove some new claims.

Claim 4. *If* $c \geq \frac{k+1}{2}$, *then* $e_k + c \geq \frac{3k-1}{2}$, *i.e., Lemma is proved.*

Proof. By claim 1 we have $e_k \geq k-1$, whence our Claim follows. □

In what follows we assume that $c \leq \frac{k}{2}$. Recall, that $I(A_1) = \{a_1, x\}$ and the vertex x is incident to $k-1$ edges of the set E_k by Claim 1. We need to find at least $\frac{k+1}{2} - c$ edges in E_k, which are not incident to x.

Let E be the set of all edges of G_{k+1}, which are not wry. Consider two cases.

1. $E \neq \varnothing$.
By Lemma 2 one can choose a set $\mathfrak{S} = \{S_e\}_{e \in E} \subset \mathfrak{R}$, where $e \in S_e$, such that all these cuts are pairwise independent. By Theorem 3 there are at least two *pendant* parts in $\mathrm{Part}(\mathfrak{S})$.

Let $D_1, \dots, D_m$ be all pendant parts of $\mathrm{Part}(\mathfrak{S})$, $P_i \in \mathfrak{S}$ be the cut such that $D_i \in \mathrm{Part}(P_i)$, $f_i \in P_i$ be the edge of E with the end $d_i \in \mathrm{Int}(D_i)$, R_i be a boundary of the cut P_i which is contained in D_i,

$$D_i' = D_i \setminus R_i, \quad r_i = |R_i \cap V_{k+1}|.$$

Denote by W_i the set of all vertices of $V_k \cap \mathrm{Int}(D_i)$, adjacent to d_i (see figure 5).

Let's formulate two obvious properties.

Claim 5. $\mathrm{Int}(D_i) \cap \mathrm{Int}(D_j) = \varnothing$ *for* $i \neq j$.

Claim 6. *Either* d_i *is a leaf of the forest* G_{k+1}*, or* $d_i = a_n$ *(the end of the path* Q*) and in this case* $d_{G_{k+1}}(a_n) = 2$.

Proof. By Lemma 3, the vertex d_i is incident to exactly one edge of the set E — the edge f_i. Whence the claim follows. □

Set new notation:

$$U = V_{k+1} \setminus V(Q), \quad c_i = |U \cap D_i|, \quad q_i = |V(Q) \cap D_i|.$$

Continue studying pendant parts.

Claim 7. $c_i \leq c$. *If* $U \cap D'_i \neq \varnothing$*, then* $c_j \leq c - 1$ *for all* $j \neq i$.

Proof. Assume that a connected component of the forest G_{k+1} contains two vertices $y, y' \in U \cap D_i$. It follows from Claim 3, that the path P between y and y' in the graph G_{k+1} consists of edges of the set E. Let $y \in D'_i$. There exists an edge $xy \in E$ (an edge of the path P). Hence, by Lemma 3 we have $y = d_i$, but $d_i \notin D'_i$. We obtain a contradiction. Therefore, $y, y' \in R_i$, but this is impossible by item 2 of Lemma 2.

Thus, $c_i \leq c$. Assume, that $v \in U \cap D'_i$. Then by Lemma 3 the vertex v cannot be incident to an edge of E, hence, $\{v\}$ is a connected component of the forest G_{k+1}, and this component does not intersect any different from D_i pendant part. □

Claim 8. *If* $\mathrm{Int}(D_i) \cap V(Q) \neq \varnothing$*, then* $q_j \leq n - 1$ *for all* $j \neq i$.

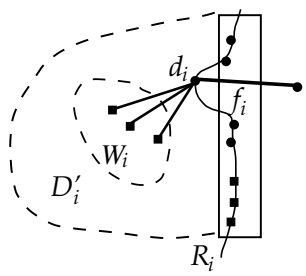

Рис. 5: A pendant part D_i

Proof. A vertex $v \in V(Q) \cap \mathrm{Int}(D_i)$ doesn't belong to another pendant part. $\square$

Claim 9. *A vertex $z \in W_i$ is incident to at least*

$$k - c_i - q_i > 0$$

edges of the set E_k. At most one of these edges is incident to the vertex x, and this edge can exist only if $x \in D_i$.

Proof. Note, that $\mathrm{N}_G(z) \subset D_i$ and $|D_i \cap V_{k+1}| = c_i + q_i$. It remains to write, that

$$c_i + q_i \leq c + n \leq \frac{k}{2} + \frac{k-1}{2} < k.$$

$\square$

Claim 10. $|W_i| \geq 1 - d_{G_{k+1}}(d_i) + r_i$ *and* $x \notin W_i$.

Proof. A vertex $d_i \in \mathrm{Int}(D_i)$ is adjacent to at least $k+1-d_{G_{k+1}}(d_i)$ vertices of the set V_k, denote the set of them by M_i. Clearly, $M_i \subset D_i$ and

$$M_i \setminus W_i \subset M_i \setminus \mathrm{Int}(D_i) = M_i \cap R_i \subset V_k \cap R_i.$$

Since $R_i \cap V_k \leq |R_i| - r_i = k - r_i$, we have

$$|W_i| \geq k + 1 - d_{G_{k+1}}(d_i) - (k - r_i) = 1 - d_{G_{k+1}}(d_i) + r_i.$$

All vertices of the set W_i are adjacent to $d_i \neq a_1$, hence they are different from x. $\square$

Claim 11. *If $c_i + q_i \leq c + \frac{k-3}{2}$, then $e_k + c \geq \frac{3k-1}{2}$, i.e., Lemma is proved.*

Proof. Let $w \in W_i$. By Claim 9, the vertex w is incident to at least

$$k - c_i - q_i \geq \frac{k+3}{2} - c$$

edges of the set E_k. At most one of these edges is incident to x. Hence, taking into account $k-1$ edges of E_k incident to x, we obtain

$$e_k + c \geq (k-1) + \left(\frac{k+3}{2} - c\right) - 1 + c \geq \frac{3k-1}{2}.$$

$\square$

We pass to finishing the proof of Lemma. Assume, that $d_1 = a_n$. Then $d_2 \neq a_n$ and $q_2 \leq n-1 \leq \frac{k-3}{2}$ by Claim 8. Since $c_2 \leq c$, the part D_2 satisfies the condition of Claim 11 and the proof is finished.

In what follows we assume that none of the vertices d_1 and d_2 coincides with a_n, hence,

$$d_{G_{k+1}}(d_1) = d_{G_{k+1}}(d_2) = 1.$$

Moreover,

$$c_1 = c_2 = c \quad \text{and} \quad q_1 = q_2 = n \geq \frac{k-2}{2},$$

otherwise one of the parts D_1 and D_2 satisfies the condition of Claim 11. Then, by Claims 7 and 8, each of the the boundaries R_1 and R_2 must contain c vertices of the set U,

$$V(Q) \subset V(P_1) \cap V(P_2) \quad \text{and} \quad D_1', D_2' \subset V_k.$$

Thus,

$$r_1 = r_2 = n + c \geq \frac{k}{2}.$$

Set $r = n + c$ and consider vertices of the sets W_1 and W_2. By Claim 10, we have $|W_1| \geq r$. By Claim 9, each vertex of the set W_1 is incident to at least $k - c_1 - q_1 = k - r$ edges of the set E_k. Similarly, each vertex of the set W_2 is incident to at least $k - r$ edges of the set E_k. Since each of these edges have two ends, we obtain

$$e_k \geq \frac{1}{2} \cdot 2r(k-r) = r(k-r).$$

If $k - r \geq 2$ we have

$$e_k \geq r(k-r) \geq 2k-4 \quad \text{and} \quad e_k + c \geq 2k-3 \geq \frac{3k-1}{2},$$

that is enough. In the remaining case $|W_1| \geq k-1$ and $|W_2| \geq k-1$. By Claim 9, each vertex of the set $W_1 \cup W_2$ is adjacent to at least one edge of the set E_k. Since the vertex x is adjacent to at most $k-1$ of these vertices and each edge has two ends, we have at least $\frac{k-1}{2}$ edges in E_k, which are not incident to x. Taking into account edges of the set E_k incident to x, we obtain

$$e_k \geq k - 1 + \frac{k-1}{2} \geq \frac{3k-1}{2} - 1 \quad \text{and} \quad e_k + c \geq \frac{3k-1}{2},$$

Lemma is proved.

2. $E = \varnothing$.

Then E_{k+1} consists of edges of the path Q, hence, a_n is a leaf of the forest G_{k+1}. Let $a_n a_{n-1} \in S_{n-1} \in \mathfrak{R}$ and $a_n \in D \in \mathrm{Part}(S_{n-1})$. The set $\{a_1, \ldots, a_n\}$ is a connected component of the forest G_{k+1} and other $c-1$ connected components of G_{k+1} are isolated vertices. Thus, $|V_{k+1}| = n+c-1$. Since $a_{n-1} \notin D$,

$$|D \cap V_{k+1}| \leq n + c - 2 \leq c + \frac{k-1}{2} - 2. \tag{13}$$

The vertex a_n is adjacent to k vertices of $V_k \cap D$, denote the set of them by W. Since $a_n \neq a_1$, we have $x \notin W$. It follows from $|V(S_{n-1})| = k-1$, that the set $\mathrm{Int}(D) \cap W$ contains at least one vertex, let it be w.

The vertex w is adjacent to k vertices of D. By inequality (13), among these vertices there are at least $\frac{k+5}{2} - c$ vertices of degree k. Therefore, w is incident to at least $\frac{k+5}{2} - c$ edges of E_k. At most one of these edges is incident to x. Consequently,

$$e_k \geq k - 1 + \frac{k+5}{2} - c - 1 \quad \text{and} \quad e_k + c \geq \frac{3k+1}{2}.$$

The proof of Lemma is completed. □

4 The defect of a graph

Recall that the *defect* of a minimal k-connected graph G is

$$f(G) = (2k-1)v_k(G) - (k-1)v(G) - 2k.$$

Definition 13. 1) The *decreased degree* of a vertex $v \in V_{k+1}$ is

$$d^-(v) = d_G(v) - k - 1.$$

Denote by $s(G)$ the sum of decreased degrees of all vertices of the set V_{k+1}.

2) The *semi-defect* of the graph G is $\varphi(G) = e_k + c - k$.

Lemma 6. *Let G be a minimal k-connected graph. Then the following statements hold.*

1) $v_k(G) = \frac{(k-1)v(G)+s(G)+2(c+e_k)}{2k-1}$.

2) $f(G) = 2(c + e_k - k) + s(G) = 2\varphi(G) + s(G)$.

Proof. 1) A vertex $x \in V_{k+1}$ is incident to $k+1+d_G^-(x)$ edges. The sum of degrees of vertices of the forest G_{k+1} is equal to $2v_{k+1}-2c$. Therefore, the number of edges going from V_{k+1} to V_k is $(k-1)v_{k+1}+2c+s(G)$. Exactly kv_k-2e_k edges go from V_k to V_{k+1}. Hence, $(k-1)v_{k+1}+2c+s(G)=kv_k-2e_k$, whence the statement we prove immediately follows.

2) A direct consequence of item 1 and the definitions of defect and semi-defect. □

Corollary 3. *Let $G \in \mathcal{GM}_k(n)$. Then there is no wry cut in $\mathfrak{R}$.*

Proof. If G has a wry cut, then by Lemma 5 we have $e_k+c \geq \frac{3k-1}{2}$. Thus, by item 2 of Lemma 6 we have $f(G) \geq k-1$, that contradicts remark 1. □

5 The proof of theorem 1

Let's write a plan of the proof of the main theorem. First, in **Subsection 5.1** we define a set of pairwise independent cuts $\mathfrak{S}$ and the tree $\mathcal{T}$. In **Subsection 5.2** we study properties of pendant parts of Part($\mathfrak{S}$).

In **Subsection 5.3** we define a rooted orientation of the tree $\mathcal{T}$, ranks of parts of Part($\mathfrak{S}$) and vertices of the set V_{k+1}. Several properties of the rank are proved.

In **Subsection 5.4** we divide vertices of V_k into two subsets: good and bad vertices (the latter consists of all vertices of V_k which belong to cuts of $\mathfrak{S}$). Several properties of bad vertices are proved.

In **Subsection 5.5** we prove a lemma about dividing the set V_{k+1} into k ordinary and one extra groups. In **Subsection 5.6** we finish the proof of the main theorem by counting vertices in the groups obtained in the previous section and edges between these groups.

5.1 The set of cuts $\mathfrak{S}$

Let G be a minimal k-connected graph, satisfying the condition $f(G) \leq k-2$. By Corollary 3 the set $\mathfrak{R}$ contains no wry cut. Then by Lemma 2 we can choose the set $\mathfrak{S}=\{S_e\}_{e\in E_{k+1}} \subset \mathfrak{R}$, where $e \in S_e$ and all these cuts are pairwise independent. In what follows we work only with cuts of the set $\mathfrak{S}$.

Since cuts of the set $\mathfrak{S}$ are pairwise independent, the following statement is a particular case of item 2 of Lemma 2.

Corollary 4. *Let G be a minimal k-connected graph and R be a boundary of a cut $S_e \in \mathfrak{S}$. Then any two vertices of the set $V_{k+1} \cap R$ belong to different connected components of the forest G_{k+1}.*

Let $\mathcal{T} = \mathrm{T}(G, \mathfrak{S})$ be the tree, which vertices are parts of $\mathrm{Part}(\mathfrak{S})$ and edges correspond to cuts of the set $\mathfrak{S}$ (see definition 9). For each edge $e \in E_{k+1}$ the edge of the tree $\mathcal{T}$, corresponding to the cut $S_e \ni e$, will be also denoted by e.

5.2 Properties of pendant parts

Let's study properties of pendant parts of $\mathrm{Part}(\mathfrak{S})$.

Definition 14. 1) The vertices of the set V_{k+1}, which are isolated vertices of the forest G_{k+1}, are called *single* vertices.

2) Let $A \in \mathrm{Part}(\mathfrak{S})$ be a pendant part. Denote by $e_{k,A}$ the number of edges of the set E_k, which are incident to vertices of $\mathrm{Int}(A)$. Denote by c_A the number of single vertices lying in $\mathrm{Int}(A)$.

Remark 5. Any single vertex is a connected component of the graph G_{k+1}.

Recall, that by item 3 of Theorem 3 for a pendant part $A \in \mathrm{Part}(\mathfrak{S})$ there exists an edge $ab \in E_{k+1}$, such that $A \in \mathrm{Part}(S_{ab})$. We reformulate item 3 of Lemma 10 of the paper [13] in our notation.

Lemma 7. *Let G be a minimal k-connected graph, such that no edge of the set E_{k+1} belongs to a wry cut. Let $A \in \mathrm{Part}(\mathfrak{S})$ be a pendant part, such that $\mathrm{Bound}(A)$ contains $p > 0$ vertices of degree k. Then $c_A + e_{k,A} \geq p$.*

We need a stronger statement.

Lemma 8. *Let G be a minimal k-connected graph, such that no edge of the set E_{k+1} belongs to a wry cut. Let $A \in \mathrm{Part}(\mathfrak{S})$ be a pendant part, such that $\mathrm{Bound}(A)$ contains $p > 0$ vertices of degree k. Then $c_A \geq p$.*

Proof. Let A' be a pendant part of $\mathrm{Part}(\mathfrak{S})$ different from A, $p' = |\mathrm{Bound}(A') \cap V_k|$. Then $c_{A'} + e_{k,A'} \geq p'$ by Lemma 7. Recall that $k - p'$ vertices of the set $\mathrm{Bound}(A') \cap V_{k+1}$ belong to $k - p'$ different connected components of the graph G_{k+1} by Corollary 4. Hence, the sum of $e_{k,A'}$ and the number of connected components of G_{k+1}, which intersect the part $A' \neq A$ is at least k. Therefore, $e_k + c - k \geq e_{k,A} + c_A$. Applying Lemmas 6 and 7, we obtain:

$$k - 2 \geq f(G) \geq 2(e_k + c - k) \geq 2(e_{k,A} + c_A) \geq 2p. \tag{14}$$

Let $ab \in E_{k+1}$, $A \in \mathrm{Part}(S_{ab})$, $a \in \mathrm{Int}(A)$. By Lemma 3 we have $d_{G_{k+1}}(a) = 1$. Then the vertex a is adjacent to at least k vertices of the part A and all these vertices belong to V_k. Since $\mathrm{Bound}(A)$ contains $p \leq \frac{k-2}{2}$ vertices of V_k, the set $Q = \mathrm{Int}(A) \cap V_k$ contains at least $k - p > 0$ vertices.

Let $W = \mathrm{Int}(A) \cap V_{k+1}$. By Lemma 3 all vertices of the set W are single, i.e., $|W| = c_A$. Assume, that $c_A = t < p$. Every vertex $x \in Q$ is adjacent to k vertices of the part A, and at most $k - p + t$ of these vertices have degree more than k. Then x is adjacent to at least $p - t$ vertices of degree k. Thus, $e_{k,A} \geq \frac{(k-p)(p-t)}{2}$. Continue the chain of inequalities (14) as follows:

$$k - 2 \geq 2(c_A + e_{k,A}) \geq 2t + (k-p)(p-t). \tag{15}$$

If $p - t \geq 2$, then by inequalities (15) and (14) it follows, that

$$k - 2 \geq 2t + (k-p)(p-t) \geq 2k - 2p \geq k + 2.$$

We obtain a contradiction. In the case where $t = p - 1$ we write down the following:

$$k - 2 \geq 2t + (k-p)(p-t) = 2p - 2 + k - p = k + p - 2 \geq k - 1,$$

that is also impossible. The contradictions obtained show us, that $c_A \geq p$. $\square$

5.3 A rooted orientation of the tree $\mathcal{T}$ and the forest G_{k+1}. Ranks of vertices and parts

Let us fix a pendant part $A_0 \in \mathrm{Part}(\mathfrak{S})$ as a root of the tree $\mathcal{T}$. We orient every edge of the tree $\mathcal{T}$ from the end that is closer to the root to the other end. Denote by $A(\mathcal{T})$ the set of all obtained arcs.

Definition 15. 1) The *rank* of a part $A \in \mathrm{Part}(\mathfrak{S})$ is the distance from the root to A in the tree $\mathcal{T}$. Denote it by $\mathrm{rank}(A)$.

2) For any part $A \in \mathrm{Part}(\mathfrak{S})$ different from A_0 there exists exactly one incoming arc in $A(\mathcal{T})$. Let the part B be the beginning of this arc. Then B is the *ancestor* of the part A and A is a *descendant* of the part B.

3) The *rank* of a vertex $a \in V_{k+1}$ is the minimal rank of a part of $\mathrm{Part}(\mathfrak{S})$, that contains a. Denote it by $\mathrm{rank}(a)$.

Remark 6. 1) Let $A, B \in \mathrm{Part}(\mathfrak{S})$ and $AB \in A(\mathcal{T})$. Then $\mathrm{rank}(A) = \mathrm{rank}(B) - 1$.

2) Let $a \in V_{k+1}$ and $A \in \mathrm{Part}(\mathfrak{S})$ be such that $a \in A$. Then $\mathrm{rank}(A) \geq \mathrm{rank}(a)$.

By construction of the tree $\mathcal{T}$, for any its edge e each of two incident to e parts of $\mathrm{Part}(\mathfrak{S})$ contains a boundary of the cut S_e and these two boundaries are different.

Definition 16. 1) Let $A \neq A_0$ and e be the edge of the tree $\mathcal{T}$, connecting A to its ancestor. Then the *upper boundary* of A is the boundary of the cut S_e which lies in A. Denote it by $\mathrm{UBound}(A)$.

2) Denote by A^- the set of all vertices of the part A, which do not belong to descendants of A.

Definition 17. 1) Consider an arbitrary edge $e = ab \in E_{k+1}$. Let the corresponding edge of $\mathcal{T}$ connect parts $A \ni a$ and $B \ni b$. If $AB \in A(\mathcal{T})$, then we orient the edge ab of the graph G_{k+1} from a to b. Denote by $A(G_{k+1})$ the set of all obtained arcs.

2) Let $a, b \in V_{k+1}$, $ab \in A(G_{k+1})$. Then b is a *descendant* of a and a is an *ancestor* of b.

3) Let U be a connected component of the forest G_{k+1}. Then a *root* of the component U is a vertex of minimal rank that lies in U.

Lemma 9. 1) *For any vertex $a \in V_{k+1}$ there exists exactly one part $A \in \mathrm{Part}(\mathfrak{S})$, such that $a \in A$ and $\mathrm{rank}(a) = \mathrm{rank}(A)$.*

2) *Let $ab \in A(G_{k+1})$. Then ab is the only arc entering b in $A(G_{k+1})$ and $\mathrm{rank}(a) < \mathrm{rank}(b)$.*

3) *Each connected component U of the forest G_{k+1} has unique root. Arcs of $A(G_{k+1})$ induce on U a rooted tree which root is the root of the component U.*

Proof. 1) Let $A, A' \in \mathrm{Part}(\mathfrak{S})$ be two such parts. Consider a cut $S_e \in \mathfrak{S}$, that separates A from A' and is incident to A in the tree $\mathrm{BT}(\mathfrak{S})$ and corresponding edge e of the tree $\mathcal{T}$. Note, that $a \in V(S_e)$. Let e connects A to its ancestor D. Then D separates $A \setminus D$ from $A' \setminus D$ in the graph G. Therefore, $a \in D$ and $\mathrm{rank}(D) < \mathrm{rank}(A) = \mathrm{rank}(a)$. This is impossible. Hence, e connects A to its descendant. In this case all parts separated by the cut S_e from the part A have rank less than $\mathrm{rank}(A)$, i.e. the part A' cannot be among them.

2) There exists a cut $S_{ab} \in \mathfrak{S}$ and two parts $A, B \in \mathrm{Part}(\mathfrak{S})$, such that $a \in A$, $b \in B$ and each of these parts contains a boundary of S_{ab}. Since $ab \in A(G_{k+1})$, we have $AB \in A(\mathcal{T})$. Hence, $\mathrm{rank}(B) = \mathrm{rank}(A) + 1$. The cut S_{ab} separates the part B from its ancestor A and, therefore, from all other parts which rank is at most $\mathrm{rank}(B)$. Thus,

$$\mathrm{rank}(b) = \mathrm{rank}(A) + 1 > \mathrm{rank}(A) \geq \mathrm{rank}(a).$$

Since $b \in \mathrm{Int}(B)$, all other edges of the forest G_{k+1}, which are incident to b, join b with vertices which do not belong to parts of rank less than $\mathrm{rank}(b)$. Hence, any different from ab arc of $A(G_{k+1})$, which is incident to b, begins at b.

3) Since for any vertex $b \in U$ there is at most one arc in $A(G_{k+1})$ that enters b, arcs of $A(G_{k+1})$ induce on U a rooted tree T_U. If $ab \in A(G_{k+1})$, then $\mathrm{rank}(a) < \mathrm{rank}(b)$. Therefore, the minimal rank over all vertices of U has the root of T_U and only it. $\square$

Definition 18. Let U be a connected component of the graph G_{k+1} and x be its root. Let $A \in \mathrm{Part}(\mathfrak{S})$ be the part of minimal rank, that contains x. Then we say, that *the component U begins* at the part A.

Corollary 5. *For each connected component U of G_{k+1} the part of* $\mathrm{Part}(\mathfrak{S})$*, at which U begins, is unique.*

Proof. Let x be the root of U. By Lemma 9 this root is unique and there exists unique part $A \in \mathrm{Part}(\mathfrak{S})$, such that $x \in A$ and $\mathrm{rank}(x) = \mathrm{rank}(A)$. Then a part at which the component U begins must coincide with A. $\square$

Definition 19. Let $S \in \mathfrak{S}$, $A_0 \in \mathrm{Part}(S)$ (such cut exists by Theorem 3). Let R be the bound of S contained in A_0. By Lemma 8, the part A_0 intersects $k_0 \geq k$ connected components of the forest G_{k+1}. If $k_0 > k$, then we choose k of these components, such that all components which contain vertices of R will be chosen. We call k chosen components *basic*, and all other connected components of the forest G_{k+1} — *additional*.

Remark 7. 1) All connected components of G_{k+1}, which begin at different from A_0 parts, are additional.

2) Let $A \neq A_0$ and x be a root of an additional component which begins at A. Then $x \notin \mathrm{UBound}(A)$.

3) The number of additional components is equal to $c - k$.

5.4 Good and bad vertices

Definition 20. Consider a part $A \in \mathrm{Part}(\mathfrak{S})$.

1) Denote by $\mathcal{T}[A]$ the subtree of a rooted tree $\mathcal{T}$ with the root A.

2) Set the notation $\mathcal{P}[A] = V(\mathcal{T}[A])$.

3) Denote by $\mathcal{V}[A]$ the union of parts which belong to $\mathrm{Part}(\mathfrak{S}) \cap \mathcal{P}[A]$.

Definition 21. 1) A vertex $x \in V_k$ is *bad*, if x belongs to a boundary of a cut of the set $\mathfrak{S}$. Otherwise, the vertex x is *good*. Denote by $b(G)$ the number of bad vertices of the graph G.

2) Denote by E'_k the set of all edges of E_k, both ends of which are good vertices, let $e'_k(G) = |E'_k|$.

Definition 22. 1) Set for a part $A \in \mathrm{Part}(\mathfrak{S})$ the following notation:

— $c(A)$ is the number of additional connected components of the forest G_{k+1}, that begin at the part A;

— $c^*(A) = \max(|A \cap V_{k+1}| - k, 0)$;

— $B(A)$ is the set of all bad vertices contained in A^- and $b(A) = |B(A)|$;

— $e'_k(A)$ is the number of edges of the set E'_k, both ends of which lie in the part A.

2) Let

$$c^*(G) = \sum_{A \in \mathrm{Part}(\mathfrak{S})} c^*(A), \quad c^*(\mathcal{P}[A]) = \sum_{B \in \mathcal{P}[A]} c^*(B).$$

Remark 8. Every bad vertex of the set V_k belongs to the set $B(A)$ for at least one part $A \in \mathrm{Part}(\mathfrak{S})$. By Corollary 5, each additional component of the forest G_{k+1} begins at exactly one part $A \in \mathrm{Part}(\mathfrak{S})$. Hence,

$$b(G) \leq \sum_{A \in \mathrm{Part}(\mathfrak{S})} b(A), \qquad c - k = \sum_{A \in \mathrm{Part}(\mathfrak{S})} c(A).$$

Lemma 10. *For a part $A \in \mathrm{Part}(\mathfrak{S})$ the following statements hold.*

1) $B(A) \subset \mathrm{UBound}(A)$ *and* $B(A_0) = \varnothing$.

2) $A \cap V_{k+1}$ *is an independent set of the graph* G.

3) *The part* A *intersects* $|A \cap V_{k+1}|$ *connected components of the graph* G_{k+1}.

4) $c^*(A) \leq c(A)$.

Proof. 1) Any bad vertex $x \in B(A)$ belongs to $\mathrm{Bound}(A)$ and belongs to no descendant of A. Hence, $B(A) \subset \mathrm{UBound}(A)$. Since $\mathrm{UBound}(A_0) = \varnothing$ we have $B(A_0) = \varnothing$.

2) Let two vertices of the set $A \cap V_{k+1}$ be connected by an edge e. Then the cut S_e separates one of them from the other, but that is impossible.

3) A direct consequence of item 2.

4) Let $A = A_0$. Then by the definition 19 we have $c(A) = c^*(A)$. In what follows assume, that $A \neq A_0$.

If $c^*(A) = 0$ the statement is clear. Let $c^*(A) \neq 0$. Then by item 3 we have at least $|A \cap V_{k+1}| - |\mathrm{UBound}(A)| = |A \cap V_{k+1}| - k = c^*(A)$ roots of different additional components of the graph G_{k+1}, which begin at A. Hence, $c^*(A) \leq c(A)$. □

Lemma 11. $e'_k(G) = \sum_{A \in \mathrm{Part}(\mathfrak{S})} e'_k(A)$

Proof. Let $xy \in E'_k$. Since vertices x and y are good, there is exactly one part $A \in \mathrm{Part}(\mathfrak{S})$, that contains both vertices x and y. Hence, the edge xy is counted in exactly one summand $e'_k(A)$. $\square$

Lemma 12. *Let $A \in \mathrm{Part}(\mathfrak{S})$ be such that $B(A) \neq \varnothing$. Then there exists a matching $L(A)$, which satisfies the following two conditions:*

1° *any vertex of the set $B(A)$ is incident to an edge of $L(A)$;*

2° *any edge of $L(A)$ connects $B(A)$ with $A \setminus \mathrm{UBound}(A)$.*

Proof. Let $P = B(A)$. Denote by Q the set of all vertices of $\mathcal{V}[A] \setminus \mathrm{UBound}(A)$, which are adjacent to P. Since $P \subset A^-$, vertices of the set P are not adjacent to $\mathcal{V}[A] \setminus A$. Hence, $Q \subset A$. For $X \subset P$ denote by $Q(X)$ the set of all vertices of Q, adjacent to X (see figure 6a).

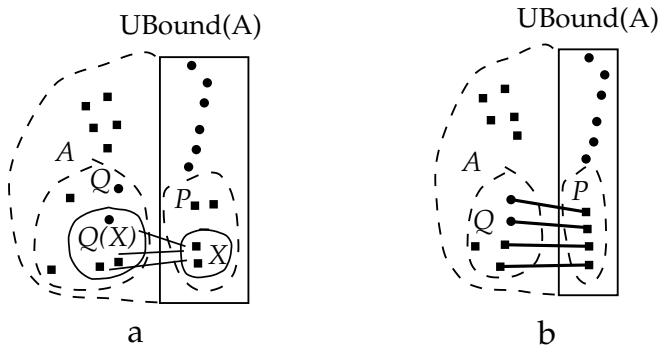

Рис. 6: Bad vertices in the part A and matching that covers them.

Let's prove, that for every $X \subseteq P$

$$|Q(X)| \geq |X|. \tag{16}$$

Recall, that $\mathrm{UBound}(A)$ is a k-vertex cutset in a k-connected graph G, that separates a nonempty set $\mathcal{V}[A] \setminus \mathrm{UBound}(A)$ from other vertices of G. Let

$$A_X = \mathcal{V}[A] \setminus (\mathrm{UBound}(A) \cup Q(X)) \quad \text{and} \quad R_X = (\mathrm{UBound}(A) \setminus X) \cup Q(X).$$

If $A_X \neq \varnothing$ then R_X separates A_X from other vertices of the graph G. Therefore, $|R_X| \geq k$, whence inequality (16) follows.

Let $A_X = \varnothing$. Since $Q \subset A$, we have $\mathcal{V}[A] = A$, i.e., A is a pendant part. Moreover, $\mathrm{Int}(A) = Q = Q(X)$. By Theorem 3 there is a cut $S_e \in \mathfrak{S}$, such that $A \in \mathrm{Part}(S_e)$. By Lemma 8 then $\mathrm{Int}(A)$ contains at least $b(A)$ single vertices and one end of the edge e. Thus,

$$|\mathrm{Int}(A)| \geq b(A) + 1 > |P| \quad \text{and} \quad |Q(X)| = |\mathrm{Int}(A)| > |P| \geq |X|.$$

Inequality (16) means, that by Hall's theorem there exists a desired matching $L(A)$ (see figure 6b). $\square$

Now let $G \in \mathcal{GM}_k((2k-1)t+2\ell)$, where $2 \le \ell \le k-1$ and $t > 0$ is an integer. We know, that $f(G) = \ell - 1$, whence it follows that

$$v_{k+1} = kt + \ell - 1, \quad v_k = (k-1)t + \ell + 1, \quad \varphi(G) = \frac{\ell - 1 - s(G)}{2}. \tag{17}$$

Lemma 13. $b(G) + c^*(G) + e'_k(G) \le \varphi(G) = \frac{\ell-1-s(G)}{2}$.

Proof. 1. Let $A \in \mathrm{Part}(\mathfrak{S})$. If $b(A) \ne 0$ then by Lemma 12 there exists a matching $L(A)$, which covers the set $B(A)$, such that edges of $L(A)$ connect $B(A)$ with $A \setminus \mathrm{UBound}(A)$.

Denote by $f_k(A)$ the number of edges of the matching $L(A)$, both ends of which belong to V_k (if $b(A) = 0$, then we set $L(A) = \varnothing$ and $f_k(A) = 0$).

Our first aim is to prove the following inequality for every part $A \in \mathrm{Part}(\mathfrak{S})$:

$$b(A) + c^*(A) \le c(A) + f_k(A). \tag{18}$$

If $b(A) = 0$, then item 4 of Lemma 10 implies inequality (18). In what follows $b(A) \ne 0$ and, consequently, $A \ne A_0$. Consider two cases.

1.1. $c^*(A) > 0$.
Then $|V_{k+1} \cap A| = k + c^*(A) > k$. Since $|\mathrm{UBound}(A)| = k$, the set $\mathrm{UBound}(A) \setminus B(A)$ consists of $k - b(A)$ vertices of the set V_{k+1}. All these vertices belong to connected components of G_{k+1} which intersect the ancestor of A. Hence, vertices of $\mathrm{UBound}(A) \setminus B(A)$ are not roots of additional components that begin at A. Therefore,

$$c(A) = |V_{k+1} \cap A| - |\mathrm{UBound}(A) \setminus B(A)| = \\ k + c^*(A) - (k - b(A)) = c^*(A) + b(A),$$

whence inequality (18) follows.

1.2. $c^*(A) = 0$.
Let $y_1, \ldots, y_m$ be all vertices of the set V_{k+1} which are incident to edges of the matching $L(A)$. Clearly, $f_k(A) = b(A) - m$. The vertices $y_1, \ldots, y_m$ are roots of additional connected components of G_{k+1}, which begin at A. Hence, $c(A) \ge m$ and

$$b(A) = f_k(A) + m \le f_k(A) + c(A).$$

2. Let

$$f_k = \sum_{A \in \mathrm{Part}(\mathfrak{S})} f_k(A). \tag{19}$$

If $A \neq A'$ then $L(A) \cap L(A') = \varnothing$. Hence, each edge was counted in the sum (19) at most once. Note, that edges, counted in f_k, have a bad end, and both ends of edges counted in e'_k are good. Therefore, $f_k + e'_k \leq e_k$. Sum up inequalities (18) for all parts of $\mathrm{Part}(\mathfrak{S})$. Applying the proved above, we obtain:

$$e'_k(G) + b(G) + c^*(G) \leq e'_k(G) + \sum_{A \in \mathrm{Part}(\mathfrak{S})} \big(b(A) + c^*(A)\big) \leq$$
$$e'_k(G) + \sum_{A \in \mathrm{Part}(\mathfrak{S})} \big(c(A) + f_k(A)\big) = c(G) - k + f_k(G) + e'_k(G) \leq$$
$$c(G) - k + e_k(G) = \varphi(G),$$

whence the statement of Lemma follows. □

5.5 Dividing the set V_{k+1} into $k+1$ groups

Lemma 14. *Vertices of the set V_{k+1} can be divided into sets $M_1, \ldots, M_k$ and W, satisfying the following conditions:*

(1) *vertices of the set W are roots of additional connected components, $|W| = c^*(G)$;*

(2) *if $B \in \mathrm{Part}(\mathfrak{S})$ then every set of $M_1, \ldots, M_k$ contains at most one vertex of $B \cap V_{k+1}$;*

(3) *if $a \in M_i$ for a certain $i \in \{1, \ldots, k\}$ then all descendants of a also belong to M_i.*

Proof. Let $A \in \mathrm{Part}(\mathfrak{S})$. We prove by induction on $\mathrm{rank}(A)$ the following statement.

Claim. *Vertices of the set $\mathcal{V}[A] \cap V_{k+1}$ can be divided into sets $M_1(A), \ldots, M_k(A)$ and $W(A)$, satisfying the following conditions:*

(1′) *vertices of the set $W(A)$ are roots of additional connected components, $|W(A)| = c^*(\mathcal{P}[A])$;*

(2′) *if $B \in \mathcal{P}[A]$ then every set of $M_1(A), \ldots, M_k(A)$ contains at most one vertex of $B \cap V_{k+1}$;*

(3′) *if $a \in M_i(A)$ for a certain $i \in \{1, \ldots, k\}$ then all descendants of a, which lie in $\mathcal{V}[A]$, also belong to $M_i(A)$;*

Proof. We begin with the **base of induction** for the case where $A \neq A_0$ is a pendant part. Let $S_{ab} \in \mathfrak{S}$ be the cut such that $a \in A \in \mathrm{Part}(S_{ab})$. Then $\mathrm{UBound}(A) = \mathrm{Bound}(A) \cup \{a\}$ and $a \in V_{k+1}$.

Let $|\mathrm{Bound}(A) \cap V_k| = p$. We put $k - p$ vertices of the set $\mathrm{UBound}(A) \cap V_{k+1}$ into the sets $M_1(A), \ldots, M_{k-p}(A)$ such that each set contains exactly

one vertex. Lemma 8 implies that $\mathrm{Int}(A)$ contains at least p single vertices. We put p of them into the sets $M_{k-p+1}(A)$, ..., $M_k(A)$, such that each set contains exactly one vertex. We put in $W(A)$ all other single vertices which lie in $\mathrm{Int}(A)$ (if such vertices remain, their number is equal to $c^*(A)$ and each of them is the root of an additional component). All conditions are satisfied.

Induction step. Let $B_1, \ldots, B_n \in \mathrm{Part}(\mathfrak{S})$ be all descendants of A. For these parts the Claim is proved. First, we set $W(A) = \cup_{\alpha=1}^{n} W(B_\alpha)$.

Let $\alpha \in \{1, \ldots, n\}$ and $S_\alpha \in \mathfrak{S}$ be the cut that separates B_α from A, $a_\alpha b_\alpha \in S_\alpha$, $a_\alpha \in A$, $b_\alpha \in B_\alpha$. Then $b_\alpha \in \mathrm{UBound}(B_\alpha)$, therefore, it is not a root of an additional component, which begins at B_α. Hence, by condition $(1')$ one of the sets $M_1(B_\alpha)$, ..., $M_k(B_\alpha)$ contains b_α, let it be $M_1(B_\alpha)$. We add the vertex a_α to $M_1(B_\alpha)$ (see figure 7). We perform such operations for all $\alpha \in \{1, \ldots, n\}$.

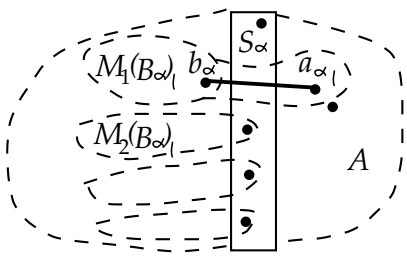

Рис. 7: Adding the vertex a_α.

Set the notation $\mathfrak{M}_\alpha = \{M_1(B_\alpha), \ldots, M_k(B_\alpha)\}$ and $\mathfrak{M} = \cup_{\alpha=1}^{n} \mathfrak{M}_\alpha$.

We will join some sets of $\mathfrak{M}$. Let $A \cap V_{k+1} = \{x_1, \ldots, x_p\}$. If $A \neq A_0$ and $p \geq k$ then let

$$\{x_1, \ldots, x_k\} \supset \mathrm{UBound}(A) \cap V_{k+1}.$$

If $A = A_0$ then let x_1, ..., x_k be all roots of basic components. Consider vertices $x_1, \ldots, x_k$ in ascending order. If x_i belongs to no set of $\mathfrak{M}$, then $M_{x_i} = \{x_i\}$. If some sets of $\mathfrak{M}$ contain x_i, we join all these sets into the set M_{x_i}. Let

$$M_1(A) = M_{x_1}, \ldots, M_k(A) = M_{x_k}$$

be *new sets* (if $p < k$, then the sets $M_{p+1}(A)$,..., $M_k(A)$ are empty). If $p > k$, then we add the vertices $x_{k+1}, \ldots, x_p$ to $W(A)$ and delete them from the sets of $\mathfrak{M}$. Note, that we have added to $W(A)$ exactly $c^*(A)$ vertices, i.e.,

$$|W(A)| = \sum_{B \in \mathcal{P}[A]} c^*(B) = c^*(\mathcal{P}[A]).$$

Moreover, all these vertices are roots of additional components, which begin at the part A, i.e. condition $(1')$ holds.

Consider all $\alpha \in \{1, \dots, n\}$ in ascending order. Let $M'_1, \dots, M'_s \in \mathfrak{M}_\alpha$ be all sets, that are not contained in new sets. Both the number of sets in $\mathfrak{M}_\alpha$ before joining and the number of new sets are equal to k. Hence, there exist s new sets, that contain no sets of $\mathfrak{M}_\alpha$, let it be new sets $M_1(A)$, ..., $M_s(A)$. Then we substitute $M_1(A)$, ..., $M_s(A)$ by $M_1(A) \cup M'_1$, ..., $M_s(A) \cup M'_s$. We perform such operations for each α and obtain after that sets $M_1(A), \dots,$ $M_k(A)$, that contain all vertices of $V_{k+1} \cap \mathcal{V}[A]$.

Let's prove that condition $(2')$ holds. If $B = A$ then it is clear. If $B \neq A$ then $B \in \mathcal{P}[B_\alpha]$ for unique $\alpha \in \{1, \dots, n\}$. Then each set of $\mathfrak{M}_\alpha$ contains at most one vertex of $V_{k+1} \cap B$. Each of the sets $M_1(A)$, ..., $M_k(A)$ intersects at most one set of $\mathfrak{M}_\alpha$, whence condition $(2')$ follows.

Let's prove, that condition $(3')$ holds. Consider a vertex $y \in M_i(A)$ and its descendant $z \in \mathcal{V}[A]$.

Assume, that one of the vertices y and z belongs to A. Since by Lemma 10 the set $A \cap V_{k+1}$ is independent, exactly one of these two vertices belong to A. By Lemma 9 we have $\mathrm{rank}(z) < \mathrm{rank}(y)$, therefore, $y \in A$ and $z \notin A$.

Thus, in any case $z \notin A$. Hence, $z \in \mathcal{V}[B_\alpha] \setminus A$ for a certain $\alpha \in \{1, \dots, n\}$. Let $z \in M_j(B_\alpha)$ for a certain $j \in \{1, \dots, k\}$.

Consider two cases.

1. $y \in A$.
In this case, $y \in A \cap M_i(A)$, hence $y = x_i$. Consider two subcases.

1.1. $y \in \mathcal{V}[B_\alpha]$.
By induction assumption (statement $(3')$ for the part B_α) we have $y, z \in M_j(B_\alpha)$. Since $y = x_i$, by constructing in this case $M_j(B_\alpha) \subset M_i(A)$. Hence, $y, z \in M_i(A)$.

1.2. $y \notin \mathcal{V}[B_\alpha]$.
Then the cut S_α separates y from $(\mathcal{V}[B_\alpha] \setminus A) \ni z$. Since $yz \in E(G)$ we have $y = a_\alpha$ and $z = b_\alpha$. During the constructing of our sets, we have added $y = a_\alpha$ to the set $M_j(B_\alpha)$. Since $y = x_i$, after that the whole set $M_j(B_\alpha) \cup \{y\}$ was included into $M_i(A)$. Thus, $y, z \in M_i(A)$.

2. $y \notin A$.
Assume, that $y \in \mathcal{V}[B_\beta] \setminus A$. Let $\beta \neq \alpha$. Then the cut S_α separates $(\mathcal{V}[B_\alpha] \setminus A) \ni z$ from $(\mathcal{V}[B_\beta] \setminus A) \ni y$ and $yz \notin S_\alpha$. Therefore, y and z cannot be adjacent. The contradiction obtained implies that $\beta = \alpha$.

Then by induction assumption for the part B_α we have $y, z \in M_j(B_\alpha) \setminus A$. By constructing $M_j(B_\alpha) \setminus A$ is included in one of the sets $M_1(A)$, ..., $M_k(A)$. Since $y \in M_j(B_\alpha) \setminus A$ this set is $M_i(A)$, i.e. $y, z \in M_i(A)$. $\square$

Recall that $V(G) = \mathcal{V}[A_0]$. Therefore, the sets $M_1 = M_1(A_0)$, ..., $M_k = M_k(A_0)$ and $W = W(A_0)$ satisfy conditions (1) – (3). The proof of Lemma is finished. □

Corollary 6. *Let $M_1, \ldots, M_k$ be the sets defined in Lemma 14. Then the following statements hold.*

1) *For distinct $i, j \in \{1, \ldots, k\}$ there is no edge connecting M_i and M_j.*

2) *For every $i \in \{1, \ldots, k\}$ any good vertex is adjacent to at most one vertex of the set M_i.*

3) *Each of the sets M_1, ..., M_k contains a certain basic component of the graph G_{k+1}.*

Proof. 1) A direct consequence of condition (3) of Lemma 14.

2) Consider a good vertex x. Clearly, $x \in \mathrm{Int}(B)$ for a certain part $B \in \mathrm{Part}(\mathfrak{S})$. Then $\mathrm{N}_G(x) \subset B$. Hence, $\mathrm{N}_G(x) \cap M_i \subset B \cap M_i$. By condition (2) of Lemma 14 we have $|B \cap M_i| \leq 1$, whence the statement we prove follows.

3) By condition (1) of Lemma 14 all roots of basic components belong to M_1, ..., M_k. By condition (2) of Lemma 14 for the part A_0, each set contains exactly one of these roots. Hence, condition (3) of Lemma 14 implies the statement we prove. □

5.6 The end of the proof of Theorem 1

Let $c^* = c^*(G)$, $b = b(G)$, $s = s(G)$, $e_k = e_k(G)$ and $e'_k = e'_k(G)$. Let X and Y be the sets of good and bad vertices of the graph G, respectively. By formula (17),

$$|Y| = b, \quad |X| = (k-1)t + \ell + 1 - b. \tag{20}$$

Let the set M_i contain t_i vertices and s_i be the sum of reduced degrees of vertices of the set M_i. We know that $G(M_i)$ is a forest, let it have $c_i + 1$ connected components. Hence, this forest has $t_i - 1 - c_i$ edges. For each vertex $x \in M_i$ we have $d_G(x) = k + 1 + d^-(x)$. Since by Lemma 14 a vertex of M_i cannot be adjacent to vertices of any set M_j (where $j \neq i$),

$$e_G(M_i, X \cup Y \cup W) = (k+1)t_i + s_i - 2e(G(M_i)) = (k-1)t_i + 2 + 2c_i + s_i. \tag{21}$$

Definition 23. 1) A set M_i is called *small*, if $t_i \leq t$ (see formula (17)). Otherwise the set M_i is called *big*.

2) Let M_1, ..., M_p be small sets and M_{p+1}, ..., M_k be big sets.

3) The *lack* of a set M_i is $t'_i = t + 1 - t_i$.

4) For a small set M_i denote by q_i the number of good vertices which are not adjacent to the set M_i.

5) Set the notation

$$M = \bigcup_{i=1}^{p} M_i, \quad M' = \bigcup_{i=p+1}^{k} M_i, \quad q = \sum_{i=1}^{p} q_i, \quad s_M = \sum_{i=1}^{p} s_i, \quad c_M = \sum_{i=1}^{p} c_i.$$

Remark 9. The lack of a small set is at least 1. The lack of a big set is at most 0.

We need two technical lemmas.

Lemma 15. $q + s_M + 2c_M - e_G(M, Y \cup W) \geq (k + 1 + c^* - \ell)(\ell - 1 - b)$.

Proof. It follows from formula (17) and Lemma 14, that the sets $M_1, \ldots,$ M_k have together $kt + \ell - 1 - c^*$ vertices. Then by Remark 9 we have the following inequality:

$$\sum_{i=1}^{p} t'_i \geq \sum_{i=1}^{k} t'_i = k(t+1) - (kt + \ell - 1 - c^*) = k + 1 + c^* - \ell. \tag{22}$$

By Corollary 6 and formulas (21) and (20),

$$q_i = |X| - e_G(M_i, X) = |X| - e_G(M_i, X \cup Y \cup W) + e_G(M_i, Y \cup W) = \\ \ell - 1 - b - s_i - 2c_i + (k-1)(t'_i - 1) + e_G(M_i, Y \cup W),$$

whence summing over all small sets $M_1, \ldots, M_p$ we obtain:

$$q = p(\ell - 1 - b) + (k-1)\sum_{i=1}^{p}(t'_i - 1) - s_M - 2c_M + e_G(M, Y \cup W). \tag{23}$$

Consider two cases.

1. $p \geq k + 1 + c^* - \ell$.
By Remark 9 we have $t'_i - 1 \geq 0$ for all $i \in \{1, \ldots, p\}$. By Lemma 13 we have $\ell - 1 - b \geq 0$. Hence, the equality (23) implies:

$$q + s_M + 2c_M - e_G(M, Y \cup W) = p(\ell - 1 - b) + (k-1)\sum_{i=1}^{p}(t'_i - 1) \geq \\ (k + 1 + c^* - \ell)(\ell - 1 - b).$$

2. $p < k + 1 + c^* - \ell$.
By formulas (23), inequality (22) and an obvious inequality $k - 1 > \ell - 1 - b$ we obtain:

$$q + s_M + 2c_M - e_G(M, Y \cup W) = p(\ell - 1 - b) + (k - 1)\sum_{i=1}^{p}(t'_i - 1) \geq$$
$$p(\ell - 1 - b) + (k + 1 + c^* - \ell - p)(k - 1) \geq (k + 1 + c^* - \ell)(\ell - 1 - b).$$

In both cases Lemma is proved. □

Lemma 16.

$$e_k + e'_k + 2c_M + s_M + e_G(W, X) - e_G(M, Y \cup W) \leq (k - 1)c^* + \ell - 1 - b.$$

Proof. By Corollary 6, each of the sets $M_1, \ldots, M_k$ contains a basic component. All other connected components of the forests $G(M_1), \ldots, G(M_p)$ are called *small* and all other connected components of the forests $G(M_{p+1}), \ldots, G(M_k)$ are called *big*. Then c_M is the number of small components. Denote by c_B the number of big components. Each small or big component is either an additional component of G_{k+1} or a part of an additional component, which root belongs to W.

Recall, that $|W| = c^*$. Let $W = \{w_1, \ldots, w_{c^*}\}$ and w_i be adjacent to f_i small components and f'_i big components. Since $G(V_{k+1})$ is a forest, $f_i = e_G(w_i, M)$ and $f'_i = e_G(w_i, M')$. Recall, that the number of additional components of the graph G_{k+1} is equal to $c - k$. Hence,

$$c_M + c_B = c - k + \sum_{i=1}^{c^*}\big(f_i + f'_i - 1\big) = c - k - c^* + e_G(W, M') + e_G(W, M)$$

and

$$c_M - e_G(W, M) = c - k - c^* - c_B + e_G(W, M'). \tag{24}$$

Every vertex of the set W is a root of an additional component of the graph G_{k+1}. Any big component U by Lemma 9 has unique root. Hence, U is adjacent to at most one vertex of the set W (the root of U if it belongs to W). Thus,

$$c_B \geq e_G(W, M'). \tag{25}$$

Consider a part A with $b(A) > 0$ and bad vertices of the set $B(A) = \{y_1, \ldots, y_{b(A)}\}$. By Lemma 12 these vertices are covered by a matching $L(A) = \{y_1v_1, \ldots, y_{b(A)}v_{b(A)}\}$. If $v_i \in V_k$, then $y_iv_i \in E_k \setminus E'_k$. If $v_i \in M \cup W$, then the edge $y_iv_i \in E_G(M \cup W, Y)$. All remaining vertices v_i

belong to M', denote the set of such vertices by M'_A (if $b(A) = 0$ we set $M'_A = \varnothing$).

Consider a vertex $v \in M'_A$. Clearly, v is a root of a big component, which begins at the part A. The vertex $v \in \mathrm{Int}(A)$ is not covered by a matching $L(A')$ for $A \neq A'$. Hence, summing $|M'_A|$ over all parts $A \in \mathrm{Part}(\mathfrak{S})$ we count each big component at most once and obtain at most c_B.

Taking into account Remark 8 we can write down the following:

$$b = b(G) \leq \sum_{A \in \mathrm{Part}(\mathfrak{S})} b(A) \leq |E_k \setminus E'_k| + |E_G(M \cup W, Y)| + \sum_{A \in \mathrm{Part}(\mathfrak{S})} |M'_A| \leq$$
$$e_k - e'_k + c_B + e_G(M \cup W, Y). \quad (26)$$

Let s_W be the sum of reduced degrees of all vertices of the set W. Then the sum of degrees of vertices of the set W is equal to $(k+1)c^* + s_W$. Since W is an independent set of the graph G,

$$e_G(W, X) = (k+1)c^* + s_W - e_G(W, Y \cup M' \cup M). \quad (27)$$

Applying formulas (24) – (27), an obvious inequality $s_W + s_M \leq s$ and Lemma 6, we can write the following chain of inequalities:

$$\begin{aligned}
&e_k + e'_k + 2c_M + s_M + e_G(W, X) - e_G(M, Y \cup W) = \\
&e_k + e'_k + 2c_M + s_M + \big((k+1)c^* + s_W - e_G(W, Y \cup M' \cup M)\big) - e_G(M, Y \cup W) = \\
&(k+1)c^* + e_k + e'_k + \big(s_W + s_M\big) + 2\big(c_M - e_G(W, M)\big) - \big(e_G(W, Y \cup M') + e_G(M, Y)\big) \leq \\
&(k+1)c^* + e_k + e'_k + s + 2\big(c - k - c^* - c_B + e_G(W, M')\big) - \big(e_G(W, M') + e_G(M \cup W, Y)\big) = \\
&(k-1)c^* + \big(s + 2(c-k) + 2e_k\big) - \big(e_k - e'_k + c_B + e_G(M \cup W, Y)\big) + \big(e_G(W, M') - c_B\big) \leq \\
&\qquad (k-1)c^* + f(G) - b \quad = \quad (k-1)c^* + \ell - 1 - b.
\end{aligned}$$

□

Consider edges, which are incident to good vertices (i.e., vertices of the set X) and not incident to vertices of the set $\cup_{i=1}^{k} M_i$. Let's divide each of these edges into two halves and mark all obtained halves that have an end at the set X. We have marked one half of each edge of the set $E_G(X, W \cup Y)$ and two halves of each edge of the set $E(G(X))$.

Note, that any vertex of the set X has degree k and by Corollary 6 is adjacent to at most one vertex of each of k sets $M_1, \ldots, M_k$. Recall, that for all $i \in \{1, \ldots, p\}$ there are q_i vertices in X which are not adjacent to the small set M_i. Hence, at least $q = \sum_{i=1}^{p} q_i$ halves of edges were marked and we can write the following inequality:

$$q \leq e_G(X, W) + 2e(G(X)) + e_G(X, Y) \leq$$
$$e_G(X, W) + 2e'_k + (e_k - e'_k) = e_G(W, X) + e_k + e'_k. \quad (28)$$

Inequality (28) together with Lemmas 15 and 16 imply, that:

$$(k+1+c^*-\ell)(\ell-1-b) \le q + s_M + 2c_M - e_G(M, Y\cup W) \le \\ e_G(W,X) + e_k + e'_k + s_M + 2c_M - e_G(M, Y\cup W) \le \\ (k-1)c^* + \ell - 1 - b. \quad (29)$$

Inequality (29) implies the following:

$$bc^* \ge (k-\ell)(\ell-1-b-c^*).$$

By Lemma 13 we know, that $b+c^* \le \frac{\ell-1}{2}$. Hence, $\left(\frac{\ell-1}{4}\right)^2 \ge bc^*$ and we obtain the following:

$$\left(\frac{\ell-1}{4}\right)^2 \ge (k-\ell)\cdot\frac{\ell-1}{2}. \quad (30)$$

Inequality (30) implies $\ell \ge \frac{8k+1}{9}$. That contradicts the condition of Theorem 1. The proof of Theorem 1 is finished. □

6 Extremal examples

Lemma 17. *Let $t_k \ge k+1$ and $t_{k+1} \ge k$ be integers, such that*

$$kt_k \ge (k+1)t_{k+1}. \quad (31)$$

Then there exists a bipartite minimal k-connected graph G with partitions V_k (which consists of vertices of degree k) and V_{k+1} (which consists of vertices of degree more than k), such that $|V_k| = t_k$ and $|V_{k+1}| = t_{k+1}$.

Proof. If $t_{k+1} = k$, then the statement is clear: the complete bipartite graph $K_{t_k,k}$ satisfies all conditions. Let $t_{k+1} > k$, $V_{k+1} = \{a_1, \dots, a_{t_{k+1}}\}$. Clearly, $|V_k| = t_k > t_{k+1}$. We choose in V_k vertices $b_1, \dots, b_{t_{k+1}}$ (the numeration is cyclic modulo t_{k+1}). The vertex $b_i \in V_k$ is adjacent to $a_i, \dots, a_{i+k-1}$ for all i. In the obtained graph the degree of every vertex of the set V_{k+1} is equal to k. It follows from inequality (31) that vertices of the set $V_k \setminus \{b_1, \dots, b_{t_{k+1}}\}$ can be connected with V_{k+1} such that every vertex of V_k will have degree k and every vertex of the set V_{k+1} will have degree at least $k+1$.

Let $R \subset V(G)$, $|R| = k-1$. We will prove, that vertices of the set $V_{k+1} \setminus R$ are connected in the graph $G - R$. Assume the converse. Then there exist two vertices $a_m, a_{m+i} \notin R$, that are disconnected in $G - R$ and $a_{m+1}, \dots, a_{m+i-1} \in R$. Hence, $i - 1 \le |R| = k-1$. The vertices a_m and a_{m+i} have $k-i$ common neighbors $(b_{m-k+i+1}, \dots, b_m)$ and all these neighbors must belong to R. Thus, all $k-1$ vertices of the set R are determined.

Clearly, there exists another similar pair of vertices $a_s, a_{s+j} \in V_{k+1}$ (the vertices a_s and a_{s+j} do not belong to R and are disconnected in $G - R$ and $a_{s+1}, \ldots, a_{s+j-1} \in R$). Since R separates a_s from a_{s+j}, the set R must contain $k-j$ common neighbors of a_s and a_{s+j} and vertices $a_{s+1}, \ldots, a_{s+j-1}$. Clearly, this set of $k-1$ vertices differs from the one found above. We obtain a contradiction which shows that all vertices of $V_{k+1} \setminus R$ are connected in the graph $G - R$. Hence, the graph $G - R$ is connected.

Since any edge of G is incident to a vertex of degree k, this graph is a minimal k-connected graph. □

Proof of theorem 2. The case $p = 2$ is completely analyzed in [13]: it was proved that $\mathcal{GM}_k((2k-1)t+2)$ consists of graphs of type $G_{k,T}$, where T is a tree with $\Delta(T) \leq k+1$ and $v(T) = t$.

For $p = 1$ we must construct a minimal k-connected graph with

$$v_k(G) = (k-1)t + 2 \quad \text{and} \quad v_{k+1}(G) = kt - 1.$$

Consider any graph $G_{k,T} \in \mathcal{GM}_k((2k-1)t+2)$. Since $n \geq 2k+1$, then $t \geq 2$ and the tree T has at least one edge. Hence, the graph $G_{k,T}$ has two adjacent vertices x, y of degree $k+1$. It is easy to verify, that $G_{k,T} \cdot xy$ (the graph, obtained from $G_{k,T}$ upon contracting the edge xy) is a minimal k-connected graph with desired parameters.

Let $p \geq 3$. We prove by induction on t that the desired graph exists. Moreover, we construct a graph G such that any edge of the set $E_{k+1}(G)$ is contained in a cut of the graph G, consisting of k edges.

The **base of induction** for $t = 1$ consists of two cases. In both cases $E_{k+1}(G) = \varnothing$.

a. $p = 2\ell - 1$.
It can be verified with the help of equality (2), that the graph G must have parameters $v_k(G) = k + \ell$ and $v_{k+1}(G) = k + \ell - 2$. These parameters satisfy inequality (31). Hence, the desired example exists by Lemma 17.

b. $p = 2\ell$.
Then the graph G must have parameters $v_k(G) = k + \ell + 1$ and $v_{k+1}(G) = k+\ell-2$. These parameters satisfy inequality (31). Hence, the desired example exists by Lemma 17.

Induction step $t \to t+1$. Let G_t be the graph constructed for t, $x \in V_k(G_t)$. We substitute the vertex x by a complete bipartite graph $K_{k,k}$, one partition of which is joined by a matching to k vertices of $\mathrm{N}_{G_t}(x)$ (see figure 8).

Clearly, the obtained graph G_{t+1} is k-connected. The number of vertices of degree k is increased by $k-1$ and the number of vertices of degree more

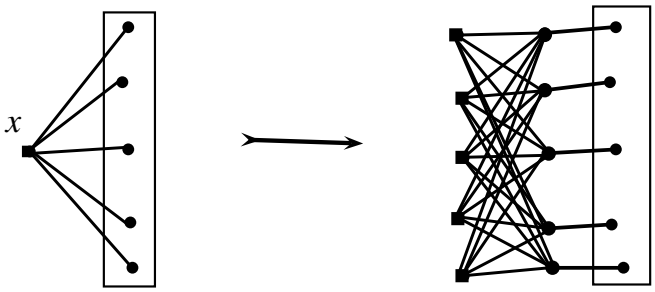

Рис. 8: Substituting the vertex x by a complete bipartite graph $K_{k,k}$.

than k is increased by k. Hence, parameters of the new graph are what we want. All edges of the set $E_{k+1}(G_t)$ are contained in the graph G_{t+1} in the same cuts as in G_t. The set $E_{k+1}(G_{t+1}) \setminus E_{k+1}(G_t)$ is a subset of the matching which join new vertices to G_t. Clearly, k edges of this matching form a cut of the graph G_{t+1}. Hence, G_{t+1} is a minimal k-connected graph. $\square$

Remark 10. 1) As it was shown in [13], examples for $p = 2$ also can be constructed by several operations of substituting a vertex of degree k by a graph $K_{k,k}$. The initial graph is $K_{k,k+1}$, which is the only extremal example for $t = 1$ and $p = 2$.

2) An example for $p = 1$ cannot be constructed with the help of Lemma 17, since for any t parameters of extremal graphs do not satisfy the conditions of this lemma.

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