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Point distributions in compact metric spaces

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We consider finite point subsets (distributions) in compact metric spaces. Non-trivial bounds for sums of distances between points of distributions and for discrepancies of distributions in metric balls are given in the case of general rectifiable metric spaces (Theorem 1.1)

We generalize Stolarsky's invariance principle to distance-invariant spaces (Theorem 2.1), and for arbitrary metric spaces we prove a probabilistic invariance principle (Theorem 3.1).

Furthermore, we construct partitions of general rectifiable compact metric spaces into subsets of equal measure with minimum average diameter (Theorem 4.1).

Key words and phrases: geometry of distances, uniform distributions, rectifiable metric spaces

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1. Introduction

Let \mathcal{M} be a compact separable metric space with a metric ρ and a finite non-negative Borel measure μ normalized by $\mu(\mathcal{M}) = 1$.

For any N -point subset (distribution) $\mathcal{D}_N = \{x_1, \dots, x_N\} \subset \mathcal{M}$ we put

$$\rho[\mathcal{D}_N] = \sum_{i,j=1}^N \rho(x_i, x_j) \quad (1.1)$$

and denote by $\langle \rho \rangle$ the mean value of the metric ρ ,

$$\langle \rho \rangle = \iint_{\mathcal{M} \times \mathcal{M}} \rho(y_1, y_2) d\mu(y_1) d\mu(y_2). \quad (1.2)$$

We write $B_r(y) = \{x : \rho(x, y) \leq r\}$, $r \in T$, $y \in \mathcal{M}$, for the ball of radius r centered at y , and of volume $\mu(B_r(y))$. Here $T = \{r : r = \rho(y_1, y_2), y_1, y_2 \in \mathcal{M}\}$ is the set of radii, $T \subset [0, L]$, where $L = \sup\{r = \rho(y_1, y_2) : y_1, y_2 \in \mathcal{M}\}$ is the diameter of \mathcal{M} .

The local discrepancy \mathcal{D}_N is defined by

$$\begin{aligned} \Lambda[B_r(y), \mathcal{D}_N] &= \#\{B_r(y) \cap \mathcal{D}_N\} - N\mu(B_r(y)) \\ &= \sum_{x \in \mathcal{D}_N} \Lambda(B_r(y), x), \end{aligned} \quad (1.3)$$

where

$$\Lambda(B_r(y), x) = \chi(B_r(y), x) - \mu(B_r(y)) \quad (1.4)$$

and $\chi(\mathcal{E}, x)$ is the characteristic function of a subset $\mathcal{E} \subset \mathcal{M}$.

We put

$$\lambda_r[\mathcal{D}_N] = \int_{\mathcal{M}} \Lambda[B_r(y), \mathcal{D}_N]^2 d\mu(y) \quad (1.5)$$

This formula can be written as

$$\lambda_r[\mathcal{D}_N] = \sum_{y_1, y_2 \in \mathcal{D}_N} \lambda_r(y_1, y_2), \quad (1.6)$$

where

$$\lambda_r(y_1, y_2) = \int_{\mathcal{M}} \Lambda(B_r(y), y_1) \Lambda(B_r(y), y_2) d\mu(y). \quad (1.7)$$

Let ξ be a finite non-negative measure on the set of radii T . We put

$$\lambda[\xi, \mathcal{D}_N] = \int_T \lambda_r[\mathcal{D}_N] d\xi(r) = \sum_{y_1, y_2 \in \mathcal{D}_N} \lambda(\xi, y_1, y_2), \quad (1.8)$$

where

$$\lambda(\xi, y_1, y_2) = \int_T \lambda_r(y_1, y_2) d\xi(r). \quad (1.9)$$

The quantity $\lambda[\xi, \mathcal{D}_N]^{1/2}$ is known as the L_2 -discrepancy of a distribution \mathcal{D}_N in bolls $B_r(y)$, $r \in T$, $y \in \mathcal{M}$ with respect to the measures μ and ξ .

Introduce the following extremal characteristics

$$\rho_N(\mathcal{M}) = \sup_{\mathcal{D}_N} \rho[\mathcal{D}_N], \quad (1.10)$$

$$\lambda_N(\xi, \mathcal{M}) = \inf_{\mathcal{D}_N} \lambda[\xi, \mathcal{D}_N], \quad (1.11)$$

where the supremum and infimum are taken over all N -point distributions $\mathcal{D}_N \subset \mathcal{M}$.

The study of the characteristics (1.10) and (1.11) is a subject of the geometry of distances and the discrepancy theory, see [2, 5, 10].

In the present paper we will show that non-trivial bounds for the quantities (1.10) and (1.11) can be obtained under very general conditions on spaces \mathcal{M} , metrics ρ and measures μ and ξ .

It is convenient to introduce the concept of d -rectifiable spaces, that will allow us to compare the metric and measure on \mathcal{M} with Euclidean metric and Lebesgue measure on \mathbb{R}^d . The concept of rectifiability is well known in

the geometric measure theory, see [11]. Here this terminology is adapted to our purposes.

Recall that a map $f : \mathcal{O} \subset \mathbb{R}^d \rightarrow \mathcal{M}$ is a Lipschitz map if

$$\rho(f(Z_1), f(Z_2)) \leq c \|Z_1 - Z_2\|, Z_1, Z_2 \in \mathcal{O}, \quad (1.12)$$

with a constant c , and the smallest such constant is called the Lipschitz constant of f and denoted by $\text{Lip}(f)$; in (1.12) $\|\cdot\|$ is Euclidean metric in \mathbb{R}^d .

Definition 1.1. *A compact metric space \mathcal{M} with a metric ρ and a measure μ is called d -rectifiable if there exist a measure ν on the d -dimensional unite cube $I^d = [0, 1]^d$ absolutely continuous with respect to Lebesgue measure, a measurable subset $\mathcal{O} \subset I^d$, and an injective Lipschitz map $f : \mathcal{O} \rightarrow \mathcal{M}$, such that*

- (i) $\mu(\mathcal{M} \setminus f(\mathcal{O})) = 0$,
- (ii) $\mu(\mathcal{E}) = \nu(f^{-1}(\mathcal{E} \cap f(\mathcal{O})))$ for any μ -measurable subset $\mathcal{E} \subset \mathcal{M}$.

Since the map f is injective, the formula

$$\nu(\mathcal{K} \cap \mathcal{O}) = \mu(f(\mathcal{K} \cap \mathcal{O})) \quad (1.13)$$

is well defined for any measurable subset $\mathcal{K} \subset I^d$. Also, we can assume that the measure ν is concentrated on \mathcal{O} and $\nu(\mathcal{O}) = \mu(f(\mathcal{O})) = \mu(\mathcal{M}) = 1$.

Simple examples of d -rectifiable spaces can be easily given. Any smooth (or piece-wise smooth) compact d -dimensional manifold is d -rectifiable, if in the local coordinates the metric satisfies (1.12), and the measure is absolutely continuous with respect to Lebesgue measure. Particularly, any compact d -dimensional Riemannian manifold with Riemannian metric and measure is d -rectifiable. In this case, it is known that the condition (1.12) is true, see [8, Chapter I, Proposition 9.10], while the condition on the measure is obvious because the metric tensor is continuous. We refer to [11] for much more exotic examples of rectifiable spaces.

In the present paper we will prove the following theorem.

Theorem 1.1. *Let \mathcal{M} be a compact d -rectifiable metric space. Then for each N we have*

$$\rho_N(\mathcal{M}) \geq N^2 \langle \rho \rangle - d 2^{d-1} \text{Lip}(f) N^{1-\frac{1}{d}}. \quad (1.14)$$

If additionally, the measure ξ on the set of radii T satisfies the condition

$$\xi([a, b]) \leq c_0(\xi)|a - b|, \quad a \leq b, a, b \in T,$$

with a constant $c_0(\xi) > 0$, then for each N we also have

$$\lambda_N(\xi, \mathcal{M}) \leq d2^{d-2} \text{Lip}(f) c_0(\xi) N^{1-\frac{1}{d}}. \quad (1.15)$$

In (1.14) and (1.15) $\text{Lip}(f)$ is the Lipschitz constant of the map f in the definition of d -rectifiability of the space \mathcal{M} .

Under such general assumptions one can not expect that the bounds (1.14) and (1.15) are the best possible, and one can give examples of d -dimensional manifolds where the bounds (1.14) and (1.15) can be improved. At the same time, it is known that for the d -dimensional spheres $S^d \subset \mathbb{R}^{d+1}$ with the rotation invariant metric and measure the bounds (1.14) and (1.15) are the best possible.

For spheres S^d bounds of the type (1.14) and (1.15) were established by Stolarsky [14]. The opposite bounds

$$\rho_N(\mathcal{M}) \leq N^2 \langle \rho \rangle - c_4 N^{1-\frac{1}{d}} \quad (1.16)$$

and

$$\lambda_N(\mathcal{M}) \geq c_2 N^{1-\frac{1}{d}} \quad (1.17)$$

with positive constants c_1 and c_2 independent of N , in the case of $\mathcal{M} = S^d$, were proved by Beck [4]. We refer to [2, 5, 10] for a discussion of these results.

Spheres as homogeneous spaces $S^d = SO(d+1)/SO(d)$ are the simplest examples of compact Riemannian symmetric spaces of rank one. All such spaces are known, see, for example, [8]: besides the spheres they are the real, complex, and quaternionic projective spaces and the octonionic projective plane.

By Theorem 1.1 the bounds (1.14) and (1.15) hold for all these spaces. It turns out that the opposite bounds (1.16) and (1.17) are also true for all such spaces. This theorem will be proved in our forthcoming paper [13] with the help of methods of harmonic analysis on homogeneous spaces. Hence, relying on Theorem 1.1 in conjunction with the results of [13], we are able to establish the exact order of the extremal characteristics (1.10) and (1.11) as $N \rightarrow \infty$ for all compact Riemannian symmetric spaces of rank one.

In the present paper we use quite elementary methods. It should be recorded that these methods are related to the papers Alexander [1] and Stolarsky [14] dedicated to point distributions on spheres.

In Section 2 we give a generalization of Stolarsky's invariance principle to distance-invariant spaces (Theorem 2.1).

In Section 3 we give a probabilistic version of the invariance principle suitable for arbitrary compact metric spaces (Theorem 3.1). Using the probabilistic invariance principle, we obtain in Section 3 the basic bounds for the characteristics (1.10) and (1.11) in terms of equal measure partitions of a metric space (Theorem 3.2).

In Section 4 for d -rectifiable compact metric spaces we give an explicit construction of equal measure partitions with the optimum order of the average diameter of the subsets of the partition (Theorem 4.1). Using such partitions, we complete the proof of Theorem 1.1 in Section 4.

2. The invariance principle for distance-invariant spaces

On the space \mathcal{M} we introduce the following metrics associated with the original metric ρ and measure μ

$$\rho^*(\xi, y_1, y_2) = \int_T \rho_r^*(y_1, y_2) d\xi(r), \quad (2.1)$$

where

$$\rho_r^*(y_1, y_2) = \mu(B_r(y_1) \Delta B_r(y_2)) \quad (2.2)$$

is the "symmetric difference" metric for the balls

$$B_r(y_1) \Delta B_r(y_2) = B_r(y_1) \cup B_r(y_2) \setminus B_r(y_1) \cap B_r(y_2). \quad (2.3)$$

Therefore,

$$\begin{aligned}
\rho_r^*(y_1, y_2) &= \int_{\mathcal{M}} \chi(B_r(y_1) \Delta B_r(y_2), y) d\mu(y) \\
&= \int_{\mathcal{M}} [\chi(B_r(y_1), y) + \chi(B_r(y_2), y) - 2\chi(B_r(y_1), y)\chi(B_r(y_2), y)] d\mu(y) \\
&= \int_{\mathcal{M}} |\chi(B_r(y_1), y) - \chi(B_r(y_2), y)| d\mu(y). \tag{2.4}
\end{aligned}$$

For the average values of the metrics $\rho^*(\xi)$ and ρ_r^* we obtain

$$\langle \rho^*(\xi) \rangle = \int_T \langle \rho_r^* \rangle d\xi(r), \tag{2.5}$$

$$\langle \rho_r^* \rangle = \iint_{\mathcal{M} \times \mathcal{M}} \rho_r^*(y_1, y_2) d\mu(y_1) d\mu(y_2) = 2 \int_{\mathcal{M}} [\mu(B_r(y)) - \mu(B_r(y))^2] d\mu(y). \tag{2.6}$$

In view of symmetry of the metric ρ , we have

$$\chi(B_r(y), x) = \chi(B_r(x), y) = \chi(r - \rho(x, y)), \tag{2.7}$$

where $\chi(t)$, $t \in \mathbb{R}$ is the characteristic function of the half-axis $[0, \infty)$.

Lemma 2.1. (i) *We have the equality*

$$\rho^*(\xi, y_1, y_2) = \int_{\mathcal{M}} |\sigma(\rho(y_1, y)) - \sigma(\rho(y_2, y))| d\mu(y), \tag{2.8}$$

where

$$\sigma(r) = \xi([r, L]) = \int_r^L d\xi(t), \quad r \in T, \tag{2.9}$$

and $L = \sup\{r : r \in T\}$ is the diameter of \mathcal{M} .

(ii) *If the measure ξ satisfies the condition*

$$\xi([a, b]) \leq c_0(\xi)|a - b|, \quad a \leq b, a, b \in T, \tag{2.10}$$

with a constant $c_0(\xi) > 0$, then we also have the inequality

$$\rho^*(\xi, y_1, y_2) \leq c_0(\xi)\rho(y_1, y_2). \tag{2.11}$$

Proof. For short, we write $\rho(y_1, y) = \rho_1$, $\rho(y_2, y) = \rho_2$.

(i) Using the formulas (2.1), (2.4) and (2.7), we obtain

$$\begin{aligned} & \rho^*(\xi, y_1, y_2) \\ &= \int_{\mathcal{M}} d\mu(y) \int_T d\xi(r) [\chi(r - \rho_1) + \chi(r - \rho_2) - 2\chi(r - \rho_1)\chi(r - \rho_2)] \\ &= \int_{\mathcal{M}} d\mu(y) [\sigma(\rho_1) + \sigma(\rho_2) - 2\sigma(\max\{\rho_1, \rho_2\})] \end{aligned} \quad (2.12)$$

Since σ is a non-increasing function, we have

$$\begin{aligned} 2\sigma(\max\{\rho_1, \rho_2\}) &= 2\min\{\sigma(\rho_1), \sigma(\rho_2)\} \\ &= \sigma(\rho_1) + \sigma(\rho_2) - |\sigma(\rho_1) - \sigma(\rho_2)|. \end{aligned} \quad (2.13)$$

Substituting (2.13) to (2.12), we obtain (2.8).

(ii) Suppose that $\rho_1 \leq \rho_2$, then using (2.9), (2.10) and the triangle inequality for the metric ρ , we obtain

$$\begin{aligned} |\sigma(\rho_1) - \sigma(\rho_2)| &= \xi([\rho_1, L]) - \xi([\rho_2, L]) = \xi([\rho_1, \rho_2]) \\ &\leq c_0(\rho_2 - \rho_1) = c_0(\rho(y_2, y_1) - \rho(y_1, y)) \leq c_0(\xi)\rho(y_1, y_2). \end{aligned} \quad (2.14)$$

The similar inequality holds if $\rho_1 > \rho_2$. Substituting (2.14) to (2.8), we obtain (2.11).

The proof of Lemma 2.1 is complete. \square

Consider the kernel (1.7). Substituting (1.4) to (1.7), we obtain

$$\begin{aligned} \lambda_r(y_1, y_2) &= \int_{\mathcal{M}} d\mu(y) [\chi(B_r(y), y_1)\chi(B_r(y), y_2) \\ &\quad - \mu(B_r(y))\chi(B_r(y), y_1) - \mu(B_r(y))\chi(B_r(y), y_2) + \mu(B_r(y))^2]. \end{aligned} \quad (2.15)$$

Comparing the formulas (2.4) and (2.15), we see that

$$2\lambda_r(y_1, y_2) + \rho_r^*(y_1, y_2) = A_r^{(0)} + A_r^{(1)}(y_1) + A_r^{(1)}(y_2), \quad (2.16)$$

where

$$A_r^{(0)} = 2 \int_{\mathcal{M}} (B_r(y))^2 d\mu(y), \quad (2.17)$$

$$\begin{aligned}
A_r^{(1)}(x) &= \int_{\mathcal{M}} d\mu(y) [\chi(B_r(x), y) - 2\mu(B_r(y))\chi(B_r(y), x)] \\
&= \mu(B_r(x)) - 2 \int_{\mathcal{M}} \mu(B_r(y))\chi(B_r(y), x) d\mu(y) \\
&= \mu(B_r(x)) - 2 \int_{\mathcal{M}} \mu(B_r(y))\chi(B_r(x), y) d\mu(y), \tag{2.18}
\end{aligned}$$

here we used the formula (2.7).

Let us consider these formulas in the following special case. A metric space \mathcal{M} is called *distance-invariant*, if for each $r \in T$ the volume of ball $\mu(B_r(y))$ is independent of $y \in \mathcal{M}$, see [9].

In this case, the integrals in (2.17), (2.18) can be easily evaluated and we arrive at the following result.

Theorem 2.1. *Let \mathcal{M} be a compact distance-invariant metric space. Then we have the relations*

$$2\lambda_r(y_1, y_2) + \rho_r^*(y_1, y_2) = \langle \rho_r^* \rangle \tag{2.19}$$

$$2\lambda(\xi, y_1, y_2) + \rho^*(\xi, y_1, y_2) = \langle \rho^*(\xi) \rangle. \tag{2.20}$$

Particularly, for any N -point distribution $\mathcal{D}_N \subset \mathcal{M}$ we have the invariance principle

$$2\lambda[\xi, \mathcal{D}_N] + \rho^*[\xi, \mathcal{D}_N] = N^2 \langle \rho^*(\xi) \rangle. \tag{2.21}$$

Proof. For short, we write $v_r = \mu(B_r(y))$. By definition, v_r is a constant independent of $y \in \mathcal{M}$. and the formulas (2.17), (2.18) take the form

$$A_r^{(0)} = 2v_r^2, \quad A_r^{(1)}(x) = v_r - 2v_r^2, \quad x \in \mathcal{M}.$$

Therefore, the right side of the equality (2.16) is equal to $2(v_r - v_r^2)$. From the other hand, the average value (2.6) is also equal to $2(v_r - v_r^2)$. This proves the equality (2.19).

Integrating the equality (2.19) over $r \in T$ with the measure ξ , we obtain (2.20). Summing the equality (2.20) over $y_1, y_2 \in \mathcal{D}_N$, we obtain (2.21). \square

The typical examples of distance-invariant spaces are (finite or infinite) homogeneous spaces $\mathcal{M} = G/H$, where G is a compact group, $H \triangleleft G$ is a closed subgroup, while ρ and μ are some G -invariant metric and measure on \mathcal{M} .

Numerous examples of distance-invariant spaces are known in algebraic combinatorics as distance-regular graphs and metric association schemes (on finite or infinite sets). Such spaces are characterizing even a stronger condition: the volume of the intersection of any two balls $\mu(B_{r_1}(y_1) \cap B_{r_2}(y_2))$ depends only on $r_1, r_2, r_3 = \rho(y_1, y_2)$, see [3, 9].

For spheres S^d the identity (2.21) was established by Stolarsky [14] and called the invariance principle. The original proof in [14] was rather difficult, it was simplified in the recent papers Bilyk [6] and Brauchard and Dick [7].

Theorem 1.1 is a generalization of the invariance principle to arbitrary compact distance-invariant spaces. Probably, the above arguments provide the most adequate explanation of such relations.

Notice that the formula (2.8) enables us to calculate the metric ρ^* explicitly for some special spaces \mathcal{M} . For spheres $S^d \subset \mathbb{R}^{d+1}$ and a special measure ξ the metric ρ^* is equivalent to Euclidean metric in \mathbb{R}^{d+1} , this fact was established in [14], see also [6, 7]. In [13] we will show that for projective spaces and specific measure ξ the metric ρ^* is equivalent to the Fubini-Study metric.

3. Equal measure partitions and the probabilistic invariance principle

Whether it is possible to generalize the relations (2.19), (2.20)–(2.21) to arbitrary compact metric spaces? At first glance the answer should be negative. Nevertheless, a probabilistic generalization of these relations turns out to be possible.

First of all, we introduce some definitions and notations. Consider a partition $\mathcal{R}_N = \{V_i\}_1^N$ of a compact space \mathcal{M} into N measurable subsets $V_i \subset \mathcal{M}$,

$$\mu(\mathcal{M} \setminus \bigcup_{i=1}^N V_i) = 0, \quad \mu(V_i \cap V_j) = 0, \quad i \neq j \quad (3.1)$$

We write $\text{diam}(\rho, V) = \sup\{\rho(y_1, y_2), y_1, y_2 \in V\}$ for the diameter of a subset $V \subset \mathcal{M}$ with respect to the metric ρ . For a partition \mathcal{R}_N we introduce the *average diameter* $\|\mathcal{R}_N\|_1$ by

$$\|\mathcal{R}_N\|_1 = \frac{1}{N} \sum_{i=1}^N \text{diam}(\rho, V_i) \quad (3.2)$$

and the maximum diameter $\|\mathcal{R}_N\|_\infty$ by

$$\|\mathcal{R}_N\|_\infty = \max_{1 \leq i \leq N} \text{diam}(\rho, V_i). \quad (3.3)$$

A partition $\mathcal{R}_N = \{V_i\}_1^N$ is an *equal measure partition* if all subsets V_i have equal measure, $\mu(V_i) = N^{-1}$, $1 \leq i \leq N$.

Let an equal measure partition $\mathcal{R}_N = \{V_i\}_1^N$ of the space \mathcal{M} be given. We introduce a probability space Ω_N by

$$\Omega_N = \prod_{i=1}^N V_i = \{X_N = (x_1, \dots, x_N) : x_i \in V_i, 1 \leq i \leq N\} \quad (3.4)$$

with a probability measure $\omega_N = \prod_{i=1}^N \tilde{\mu}_i$, where $\tilde{\mu}_i = N\mu|_{V_i}$, and $\mu|_{V_i}$ denotes the restriction of the measure μ to a subset $V_i \subset \mathcal{M}$.

We write $\mathbb{E}_N F[\cdot]$ for the expectation of a random variable $F[X_N]$, $X_N \in \Omega_N$,

$$\mathbb{E}_N F[\cdot] = \int_{\Omega_N} F[X_N] d\omega_N = N^N \iint_{V_1 \times \dots \times V_N} F(x_1, \dots, x_N) d\mu(x_1) \dots d\mu(x_N). \quad (3.5)$$

Lemma 3.1. *Let $F^{(1)}[X_N]$ and $F^{(2)}[X_N]$, $X_N = (x_1, \dots, x_N) \in \Omega_N$, be the following random variables*

$$F^{(1)}[X_N] = \sum_i f(x_i), \quad F^{(2)}[X_N] = \sum_{i \neq j} f(x_i, x_j), \quad (3.6)$$

where $f(y)$ and $f(y_1, y_2)$ are integrable functions on \mathcal{M} and $\mathcal{M} \times \mathcal{M}$, correspondingly. Then

$$\mathbb{E}_N F^{(1)}[\cdot] = N \int_{\mathcal{M}} f(y) d\mu(y) \quad (3.7)$$

$$\begin{aligned} \mathbb{E}_N F^{(2)}[\cdot] &= N^2 \iint_{\mathcal{M} \times \mathcal{M}} f(y_1, y_2) d\mu(y_1) d\mu(y_2) \\ &\quad - N^2 \sum_{i=1}^N \iint_{V_i \times V_i} f(y_1, y_2) d\mu(y_1) d\mu(y_2). \end{aligned} \quad (3.8)$$

Proof. Substituting (3.6) to (3.5), we obtain

$$\mathbb{E}_N F^{(1)}[\cdot] = N \sum_i \int_{V_i} f(y) d\mu(y) = N \int_{\mathcal{M}} f(y) d\mu(y).$$

This proves (3.7).

Substituting (3.6) to (3.5), we obtain

$$\begin{aligned} \mathbb{E}_N F^{(2)}[\cdot] &= N^2 \sum_{i \neq j} \iint_{V_i \times V_j} f(y_1, y_2) d\mu(y_1) d\mu(y_2) \\ &= N^2 \sum_{i, j} \iint_{V_i \times V_j} f(y_1, y_2) d\mu(y_1) d\mu(y_2) - N^2 \sum_i \iint_{V_i \times V_i} f(y_1, y_2) d\mu(y_1) d\mu(y_2) \\ &= N^2 \iint_{\mathcal{M} \times \mathcal{M}} f(y_1, y_2) d\mu(y_1) d\mu(y_2) - N^2 \sum_i \iint_{V_i \times V_i} f(y_1, y_2) d\mu(y_1) d\mu(y_2). \end{aligned}$$

This proves (3.8). \square

Elements $X_N \in \Omega_N$ can be thought of as specific N -point distributions in the space \mathcal{M} and the corresponding sums of distances and discrepancies for $D_N = X_N = \{x_1, \dots, x_N\} \in \Omega_N$ as random variables on the probability space Ω_N . We put

$$\rho[X_N] = \sum_{i \neq j} \rho(x_i, x_j), \quad (3.9)$$

$$\rho_r^*[X_N] = \sum_{i \neq j} \rho^*(x_i, x_j), \quad (3.10)$$

$$\rho^*[\xi, X_N] = \sum_{i \neq j} \rho^*(\xi, x_i, x_j) \quad (3.11)$$

and

$$\lambda_r[X_N] = \sum_i \lambda_r(x_i, x_i) + \sum_{i \neq j} \lambda_r(x_i, x_j), \quad (3.12)$$

$$\lambda[\xi, X_N] = \sum_i \lambda(\xi, x_i, x_i) + \sum_{i \neq j} \lambda(\xi, x_i, x_j). \quad (3.13)$$

The probabilistic invariance principle can be stated as follows.

Theorem 3.1. *Let \mathcal{R}_N be an equal measure partition of a compact metric space \mathcal{M} . Then the expectations of the random variables (3.10), (3.11), (3.12) and (3.13) on the probability space Ω_N satisfy the following relations*

$$2\mathbb{E}_N \lambda_r[\cdot] + \mathbb{E}_N \rho_r^*[\cdot] = N^2 \langle \rho_r^* \rangle, \quad (3.14)$$

$$2\mathbb{E}_N \lambda[\xi, \cdot] + \mathbb{E}_N \rho^*[\xi, \cdot] = N^2 \langle \rho^*(\xi) \rangle. \quad (3.15)$$

Proof. Summing the equality (2.16) over $x_1, x_2 \in X_N$, we obtain

$$2\lambda_r[X_N] + \rho_r^*[X_N] = N^2 A_r^{(0)} + 2A_r^{(1)}[X_N], \quad (3.16)$$

where

$$A_r^{(1)}[X_N] = \sum_i A_r^{(1)}(x_i).$$

Now, we calculate the expectation \mathbb{E}_N of both sides in the equality (3.16). Using the equality (3.7) and the formulas (2.17), (2.18) (2.6), we find that

$$\begin{aligned} & 2\mathbb{E}_N \lambda_r[\cdot] + \mathbb{E}_N \rho_r^*[\cdot] \\ &= N^2 A_r^{(0)} + 2\mathbb{E}_N A_r^{(1)}[\cdot] = N^2 A_r^{(0)} + 2N^2 \int_{\mathcal{M}} A_r^{(1)}(y) d\mu(y) \\ &= 2N^2 \int_{\mathcal{M}} \mu(B_r(y)) d\mu(y) + 2N^2 \int_{\mathcal{M}} \mu(B_r(y)) d\mu(y) - 4N^2 \int_{\mathcal{M}} (B_r(y))^2 d\mu(y) \\ &= 2N^2 \int_{\mathcal{M}} [\mu(B_r(y)) - \mu(B_r(y))^2] d\mu(y) = \langle \rho_r^* \rangle. \end{aligned}$$

This proves the relation (3.14).

Integrating the relation (3.14) over $r \in T$ with the measure ξ , we obtain the relation (3.15).

The proof of Theorem 3.1 is complete. \square

Distributions $X_N \in \Omega_N$ form a subset in the set of all N -point distributions $D_N \subset \mathcal{M}$. Therefore,

$$\rho_N(\mathcal{M}) \geq \mathbb{E}_N \rho[\cdot], \quad (3.17)$$

$$\lambda_N(\xi, \mathcal{M}) \leq \mathbb{E}_N \lambda[\xi, \cdot]. \quad (3.18)$$

These inequalities in conjunction with Lemma 3.1 and Theorem 3.1 lead to the following basic bounds.

Theorem 3.2. *Let \mathcal{R} be an equal measure partition of a compact metric space \mathcal{M} . Then we have the following bound*

$$\rho_N(M) \geq N^2 \langle \rho \rangle - N \|\mathcal{R}_N\|_1. \quad (3.19)$$

If additionally, the measure ξ satisfies the condition (2.10), then we also have the following bound

$$\lambda_N(\xi, \mathcal{M}) \leq \frac{1}{2} c_0(\xi) N \|\mathcal{R}_N\|_1. \quad (3.20)$$

Proof. Applying the formula (3.8) to the random variable (3.9), we obtain

$$\mathbb{E}_N \rho[\cdot] = N^2 \langle \rho \rangle - N^2 Q_N(\rho),$$

where

$$Q_N(\rho) = \sum_i \iint_{V_i \times V_i} \rho(y_1, y_2) d\mu(y_1) d\mu(y_2) \leq N^{-2} \sum_i \text{diam}(\rho, V_i) = N^{-1} \|\mathcal{R}_N\|_1.$$

Therefore,

$$\mathbb{E}_N \rho[\cdot] \geq N^2 \langle \rho \rangle - N \|\mathcal{R}_N\|_1.$$

Comparing this bound with the inequality (3.17), we obtain the bound (3.19).

Let the measure ξ satisfy the condition (2.10). Applying the formula (3.8) to the random variable (3.11), we obtain

$$\mathbb{E}_N \rho^*[\xi, \cdot] = N^2 \langle \rho(\xi) \rangle - N^2 Q_N(\rho^*(\xi)),$$

where

$$\begin{aligned} Q_N(\rho^*(\xi)) &= \sum_i \iint_{V_i \times V_i} \rho^*(\xi, y_1, y_2) d\mu(y_1) d\mu(y_2) \\ &\leq N^{-2} \sum_i \text{diam}(\rho^*(\xi), V_i) \leq N^{-2} c_0(\xi) \sum_i \text{diam}(\rho, V_i) = N^{-1} c_0(\xi) \|\mathcal{R}\|_1. \end{aligned}$$

Therefore,

$$\mathbb{E}_N \rho^*[\xi, \cdot] \geq N^2 \langle \rho^*(\xi) \rangle - N c_0(\xi) \|\mathcal{R}_N\|_1.$$

Substituting this bound to the equality (3.15), we obtain

$$2\mathbb{E}_N \lambda[\xi, \cdot] \leq N c_0(\xi) \|\mathcal{R}_N\|_1$$

Comparing this bound with the inequality (3.18), we obtain the bound (3.20).

The proof of Theorem 3.2 is complete. \square

4. Construction of equal measure partitions. Proof of Theorem 1.1

In this section we will prove the following general theorem.

Theorem 4.1. *Let \mathcal{M} be a compact d -rectifiable metric space. Then, for each N there exists an equal measure partition \mathcal{R}_N of the space \mathcal{M} , such that*

$$\|\mathcal{R}_N\|_1 \leq d 2^{d-1} \text{Lip}(f) N^{-\frac{1}{d}}, \quad (4.1)$$

where $\text{Lip}(f)$ is the Lipschitz constant of the map f in the definition of d -rectifiability of the space \mathcal{M} .

Theorem 1.1 follows immediately from Theorems 3.2 and 4.1.

Proof of Theorem 1.1. It suffices to substitute the bound (4.1) to the bounds (3.19) and (3.20). \square

At the present time, for spheres S^d equal measure partitions are constructed to satisfy the bound

$$\|\mathcal{R}_N\|_\infty \leq c(d) N^{-\frac{1}{d}} \quad (4.2)$$

with a constant $c(d)$ independent of N . For subsequences $N = c_d m^d$, where $m > 0$ are integers, such partitions were described still in the paper [1] by Alexander. In the general case of all sufficiently large N such partitions for spheres S^d were constructed in the paper [12] by Rakhmanov, Saff and Zhou.

Certainly, the bound (4.2) is stronger than (4.1), because $\|\mathcal{R}_N\|_1 \leq \|\mathcal{R}_N\|_\infty$, see (3.2) and (3.3). However, the corresponding constructions in [1, 12] significantly depend on the geometry of spheres S^d as smooth submanifolds in \mathbb{R}^{d+1} , while the bound (4.1) can be established for arbitrary compact d -rectifiable metric spaces.

The proof of Theorem 4.1 is relying on three lemmas. Lemma 4.1 contains a very simple result which is needed at each step of our inductive construction. Our construction of partitions is described in Lemma 4.2 for a special case of a measure concentrated on the d -dimensional unite cube. The bound (4.1) for such equal measure partitions of the unit cube is given in Lemma 4.3. Once these partitions of the unite cube are constructed, the proof of Theorem 4.1 can be easily completed on the base of Definition 1.1.

Let ν_0 be a finite non-negative measure on the unite segment $I = [0, 1]$. Suppose that the measure ν_0 has not a discrete component. Then, its distribution function $\varphi(z) = \nu_0([0, z])$, $z \in I$, is continuous, non-decreasing, $\varphi(0) = 0$ and $\varphi(1) = \nu_0(I)$. Notice that there is a one-to-one correspondence between such functions and finite measures on I without discrete components.

Since the graph of φ can have horizontal parts, we define the inverse function φ^{-1} by

$$\varphi^{-1}(t) = \sup\{z : \varphi(z) = t\}, \quad t \in [0, \nu_0(I)]. \quad (4.3)$$

Let $1 \leq n \leq k$ be integers and

$$n = \sum_{i=1}^k n(i) \quad (4.4)$$

be an arbitrary representation of n as a sum of k non-negative summands $n(i) \geq 0$.

Define points $\lambda(0) = 0 < \lambda(1) \leq \dots \leq \lambda(k) = 1$, by

$$\lambda(j) = \varphi^{-1}\left(n^{-1} \sum_{i=1}^j n(i) \nu_0(I)\right), \quad 1 \leq j \leq k, \quad (4.5)$$

and consider the segments $\Delta(j) = [\lambda(j-1), \lambda(j)] \subset I$, of length $l(j) = \lambda(j) - \lambda(j-1)$, $1 \leq j \leq k$.

We have immediately the following result.

Lemma 4.1. *With the above assumptions, the segments $\{\Delta(j), 1 \leq j \leq k\}$ form a partition of the unite segment*

$$I = \bigcup_{j=1}^k \Delta(j), \quad \nu_0(\Delta(j_1) \cap \Delta(j_2)) = 0, \quad j_1 \neq j_2,$$

furthermore

$$\sum_{j=1}^k l(j) = 1 \quad (4.6)$$

and

$$\nu_0(\Delta(j)) = \frac{n(j)}{n} \nu_0(I). \quad (4.7)$$

Remark. If the measure $\nu_0 \equiv 0$ identically, then for any $n \geq 1$ the partition given in Lemma 4.1 takes the form

$$\Delta(1) = [0, 1], \quad \Delta(j) = [1, 1] = \{1\}, \quad 2 \leq j \leq k. \quad (4.8)$$

it is convenient to agree that for $\nu_0 \equiv 0$ the partition (4.8) takes also place for $n = 0$.

Now we wish to generalize Lemma 4.1 to the d -dimensional unite cube $I^d = [0, 1]^d$. Introduce some notation.

Let $N \geq 1$ be an integer, $k = \lceil N^{1/d} \rceil$, $N \leq k^d$, and

$$N = \sum_{i_1, \dots, i_d=1}^k N(i_1, \dots, i_d) \quad (4.9)$$

be an arbitrary representation of N as a sum of numbers $N(i_1, \dots, i_d)$ equal to 0 or 1.

Introduce the following non-negative integers

$$N(i_1, \dots, i_q) = \sum_{i_{q+1}, \dots, i_d=1}^k N(i_1, \dots, i_d), \quad i \leq q < d. \quad (4.10)$$

These integers satisfy the following relations

$$N(i_1, \dots, i_q) = \sum_{i_{q+1}=1}^k N(i_1, \dots, i_{q+1}), \quad i \leq q < d, \quad (4.11)$$

and

$$N = \sum_{i_1=1}^k N(i_1). \quad (4.12)$$

Lemma 4.2. *Let ν be a finite non-negative measure on I^d with a continuous distribution function*

$$\varphi(Z) = \nu([0, z_1] \times \dots \times [0, z_d]), \quad Z = (z_1, \dots, z_d) \in I^d. \quad (4.13)$$

Then, for any representation of an integer N as the sum (4.9) there exists a sequence of partitions

$$\mathcal{P}(q) = \{\Pi(i_1, \dots, i_q), \quad 1 \leq j \leq k, \quad 1 \leq j \leq q\}, \quad q = 1, \dots, d,$$

of the unite I^d into rectangular boxes of the form

$$\begin{aligned}\Pi(i_1, \dots, i_q) &= \prod_{j=1}^q \Delta(i_1, \dots, i_j) \times [0, 1]^{d-q} \\ &= \{Z = (z_1, \dots, z_d) \in I^d : z_j \in \Delta(i_1, \dots, i_j), \\ &1 \leq j \leq q, \quad y_j \in [0, 1], q+1 \leq j \leq d\},\end{aligned}\tag{4.14}$$

where $\Delta(i_1, \dots, i_j) \subset I$ are some segments.

For any fixed indexes i_1, \dots, i_{j-1} the segments $\{\Delta(i_1, \dots, i_{j-1}, i_j), i_j = 1, \dots, k\}$ form a partition of I ,

$$\sum_{i_j=1}^k l(i_1, \dots, i_{j-1}, i_j) = 1,\tag{4.15}$$

where $l(i_1, \dots, i_j)$ are lengths of the segments $\Delta(i_1, \dots, i_j)$.

The measures of the rectangular boxes (4.14) satisfy the relations

$$\nu(\Pi(i_1, \dots, i_q)) = \frac{N(i_1, \dots, i_q)}{N} \nu(I^d).\tag{4.16}$$

Proof. We construct the partitions $\mathcal{P}(q)$, $q = 1, \dots, d$, by induction on q . The partition $\mathcal{P}(1)$ is defined as follows.

Consider the following one-dimensional distribution function

$$\varphi(z_1) = \nu([0, z_1]) \times [0, 1]^{d-1} = \nu_0([0, z_1]), \quad z_1 \in I,$$

where ν_0 is the corresponding measure on I $\nu_0(I) = \nu(I^d)$. Applying Lemma 4.1 with $n = N$ $n(i_1) = N(i_1)$, see (4.4), to this function, we obtain a partition of I into segments $\Delta(i_1)$ of length $l(i_1)$, moreover,

$$\sum_{i_1=1}^k l(i_1) = 1.$$

see (4.6). We put

$$\Pi(i_1) = \Delta(i_1) \times [0, 1]^{d-1} \subset I^d,$$

then,

$$\nu(\Pi(i_1)) = \nu_0(\Delta(I_1)) = \frac{N(i_1)}{N} \nu(I^d).$$

see (4.7). Thus, the partition $\mathcal{P}(1)$ is constructed.

Suppose that the partition $\mathcal{P}(q)$ is already constructed for some q , $1 \leq q < d$. Then, the partition $\mathcal{P}(q+1)$ can be constructed as follows.

For each rectangular box (4.14) we consider the following one-dimensional distribution function

$$\begin{aligned}\varphi(i_1, \dots, i_q, z_{q+1}) &= \nu \left(\prod_{j=1}^q \Delta(i_1, \dots, i_j) \times [0, z_{q+1}] \times [0, 1]^{d-q-1} \right) \\ &= \nu_0^{(i_1, \dots, i_q)}([0, z_{q+1}]), \quad z_{q+1} \in I,\end{aligned}\tag{4.17}$$

where $\nu_0^{(i_1, \dots, i_q)}$ is the corresponding measure on I and

$$\nu_0^{(i_1, \dots, i_q)}(I) = \nu(\Pi(i_1, \dots, i_q)) = \frac{N(i_1, \dots, i_q)}{N} \nu(I^d),\tag{4.18}$$

see (4.16).

Applying Lemma 4.1 with $n = N(i_1, \dots, i_q)$, $n(i_{q+1}) = N(i_1, \dots, i_{q+1})$, to the function (4.17), we obtain a partition of I into segments $\Delta(i_1, \dots, i_1, i_{q+1})$ of length $l(i_1, \dots, i_q, i_{q+1})$, $1 \leq i_{q+1} \leq k$, moreover,

$$\sum_{i_{q+1}=1}^k l(i_1, \dots, i_q, i_{q+1}) = 1.$$

We put

$$\Pi(i_1, \dots, i_{q+1}) = \prod_{j=1}^{q+1} \Delta(i_1, \dots, i_j) \times [0, 1]^{d-q-1}$$

For these rectangular boxes, we have

$$\nu(\Pi(i_1, \dots, i_{q+1})) = \frac{N(i_1, \dots, i_{q+1})}{N} \nu(I^d).\tag{4.19}$$

Indeed, if $N(i_1, \dots, i_q) \geq 1$, then in view of (4.18), we obtain

$$\nu(\Pi(i_1, \dots, i_{q+1})) = \frac{N(i_1, \dots, i_{q+1})}{N(i_1, \dots, i_q)} \nu(\Pi(i_1, \dots, i_q)) = \frac{N(i_1, \dots, i_{q+1})}{N} \nu(I^d).$$

If $N(i_1, \dots, i_q) = 0$, then $\nu(\Pi(i_1, \dots, i_q)) = 0$ and the segments $\Delta(i_1, \dots, i_q, i_{q+1})$, $1 \leq i_{q+1} \leq k$, are defined by (4.8). Therefore, $\nu(\Pi(i_1, \dots, i_{q+1})) = 0$ and the equality (4.19) is also true.

Thus, the partition $\mathcal{P}(q+1)$ is constructed.

Induction on q completes the proof of Lemma 4.2. \square

Consider the partition $\mathcal{P}(d) = \{\Pi(i_1, \dots, i_d), i \leq i_j \leq k, i \leq j \leq d\}$ constructed in Lemma 4.2. For this partition the equality (4.16) takes the form

$$\nu(\Pi(i_1, \dots, i_d)) = \frac{N(i_1, \dots, i_d)}{N} \nu(I^d), \quad (4.20)$$

where, by definition, the numbers $N(i_1, \dots, i_d)$ are equal to 0 or 1, moreover, the number of $N(i_1, \dots, i_d) = 1$ is N , see (4.9).

We put $\mathcal{A} = \{\alpha = (i_1, \dots, i_d) : N(i_1, \dots, i_d) = 1\}$, $\#\mathcal{A} = N$, and introduce the following partition of the unite cube $\mathcal{P}_N = \{\Pi(\alpha), \alpha \in \mathcal{A}\}$. We write $\|\mathcal{P}_N\|_1$ for the average diameter of the rectangular boxes $\Pi(\alpha)$, $\alpha \in \mathcal{A}$, with respect to Euclidean metric $\|\cdot\|$ in \mathbb{R}^d , see also (4.2),

$$\|\mathcal{P}_N\|_1 = \frac{1}{N} \sum_{\alpha \in \mathcal{A}} \text{diam}(\|\cdot\|, \Pi_\alpha). \quad (4.21)$$

Lemma 4.3. $\mathcal{P}_N = \{\Pi(\alpha), \alpha \in \mathcal{A}\}$ is an equal measure partition,

$$\nu(\Pi(\alpha)) = N^{-1} \nu(I^d) \quad (4.22)$$

and we have the bound

$$\|\mathcal{P}_N\|_1 < d 2^{d-1} N^{-\frac{1}{d}} \quad (4.23)$$

Proof. The equality (4.22) follows from (4.20) and the definition of the partition \mathcal{P}_N .

The Euclidean diameter of a rectangular box $\Pi(i_1, \dots, i_d)$ does not exceed the sum of lengths of its sides:

$$\text{diam}(\|\cdot\|, \Pi(i_1, \dots, i_d)) \leq l(i_1) + l(i_1, i_2) + \dots + l(i_1, \dots, i_d), \quad (4.24)$$

where $l(i_1, \dots, i_j)$ are lengths of the segments $\Delta(i_1, \dots, i_j)$, $1 \leq j \leq d$, see (4.14).

Using (4.21), (4.24) and (4.15), we obtain

$$\begin{aligned} N \|\mathcal{P}_N\|_1 &\leq \sum_{i_1, \dots, i_d}^k \text{diam}(\|\cdot\|, \Pi(i_1, \dots, i_d)) \\ &\leq \sum_{j=1}^d \sum_{i_1, \dots, i_d=1}^k l(i_1, \dots, i_j) = \sum_{j=1}^d k^{d-j} \sum_{i_1, \dots, i_{j-1}=1}^k l(i_1, \dots, i_j) \\ &= \sum_{j=1}^d k^{d-1} = d k^{d-1}. \end{aligned}$$

Since $k = \lceil N^{1/d} \rceil$ and $k < N^{1/d} + 1$, we have

$$N \|\mathcal{P}_N\|_1 < d(N^{1/d} + 1)^{d-1} = d(1 + N^{-1/d})^{d-1} N^{1-\frac{1}{d}} \leq d 2^{d-1} N^{1-\frac{1}{d}},$$

that is equivalent to (4.23).

The proof of Lemma 4.3 is complete. \square

Proof of Theorem 4.1. Let \mathcal{M} d -rectifiable space. Without loss of generality, we can assume that in Definition 1.1 the measure ν is concentrated on the subset $\mathcal{O} \subset I^d$ and the measures μ ν are normalized by $\mu(\mathcal{M}) = \nu(\mathcal{O}) = 1$.

Since the measure ν is absolutely continuous with respect to Lebesgue measure on I^d , its distribution function (4.13) is continuous, and Lemmas 4.2 and 4.3 can be applied.

Let $\mathcal{P}_N = \{\Pi(\alpha), \alpha \in \mathcal{A}\}$ be an equal measure partition of the unit cube I^d given in Lemma 4.3 for the measure ν . Consider the following collection of subsets in the space \mathcal{M}

$$\mathcal{R}_N = \{V(\alpha), \alpha \in \mathcal{A}\}, \quad V(\alpha) = f(\Pi(\alpha) \cap \mathcal{O}). \quad (4.25)$$

Using the formula (1.13), we obtain

$$\mu(V(\alpha)) = \nu(\Pi(\alpha) \cap \mathcal{O}) = \nu(\Pi(\alpha)) = N^{-1},$$

since the measure ν is concentrated on \mathcal{O} .

By definition, the map $f : \mathcal{O} \rightarrow \mathcal{M}$ is an injection. Therefore

$$\mu(V(\alpha_1) \cap V(\alpha_2)) = \nu(\Pi(\alpha_1) \cap \Pi(\alpha_2) \cap \mathcal{O}) = 0, \quad \alpha_1 \neq \alpha_2.$$

Thus, the collection of subsets \mathcal{R}_N (4.25) is an equal measure partition of the \mathcal{M} .

By definition, the map $f : \mathcal{O} \rightarrow \mathcal{M}$ is also a Lipschitz map, see (1.12). Therefore

$$\text{diam}(\rho, V(\alpha)) \leq \text{Lip}(f) \text{diam}(\|\cdot\|, \Pi(\alpha) \cap \mathcal{O}) \leq \text{Lip}(f) \text{diam}(\|\cdot\|, \Pi(\alpha))$$

and

$$\|\mathcal{R}_N\|_1 \leq \text{Lip}(f) \|\mathcal{P}_N\|_1, \quad (4.26)$$

see (3.2) and (4.21).

Substituting the bound (4.23) to (4.26), we obtain the bound (4.1).

The proof of Theorem 4.1 is complete. \square

References

- [1] J. R. Alexander, *On the sum of distances between n points on a sphere.* — Acta Math. Hungar. **23** (3–4) (1972), 443–448.
- [2] J. R. Alexander, J. Beck, W. W. L. Chen, *Geometric discrepancy theory and uniform distributions.* — in Handbook of Discrete and Computational Geometry (J. E. Goodman and J. O’Rourke eds.), Chapter 10, pages 185–207. CRC Press LLC, Boca Raton, FL, 1997.
- [3] A. Barg, M. Skriganov, *Association schemes on general measure spaces and zero-dimensional Abelian groups.* — Advances in Math. **281** (2015), 142–247.
- [4] J. Beck, *Sums of distances between points on a sphere: An application of the theory of irregularities of distributions to distance geometry,* Mathematika **31** (1984), 33–41.
- [5] J. Beck, W. W. L. Chen, *Irregularities of Distribution.* — Cambridge Tracts in Math., vol. 89, Cambridge Univ. Press, 1987.
- [6] D. Bilyk, *Discrepancy problems as particle interactions,* Preprint, 2014.
- [7] J. S. Brauchart, J. Dick, *A simple proof of Stolarsky’s invariance principle.* — Proc. Amer. Math. Soc. **141** (2013), 2085–2096.
- [8] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces,* Academic Press Inc., London, 1978.
- [9] V. I. Levenshtein, *Universal bounds for codes and designs,* in Handbook of Coding Theory (V. S. Pless and W. C. Huffman eds.), Chapter 6, pages 499–648. Elsevier, Amsterdam, 1998.
- [10] J. Matoušek, *Geometric Discrepancy. An Illustrated Guide,* Springer-Verlag, Berlin, 1999.
- [11] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability,* Cambridge Univ. Press, Cambridge, 1995.
- [12] E. A. Rakhmanov, E. B. Saff, Y. M. Zhou, *Minimal discrete energy on the sphere.* — Mathematical Research Letters **1** (1994), 647–662.

- [13] M. M. Skriganov, *Point distributions in compact metric spaces*, II (in preparation).
- [14] K. B. Stolarsky, *Sums of distances between points on a sphere*, II. Proc. Amer. Math. Soc. **41** (1973), 575–582.