ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

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Time-dependent correlation functions for a bimodal Bose-Hubbard model *

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Abstract

A collection of finite number of interacting bosonic atoms in a double-well trap is considered and studied by using the two-site Bose-Hubbard Hamiltonian. The application of the Quantum Inverse Method allows to obtain the exact expressions for the time-dependent correlation functions.

Keywords: quantum correlations; ultracold atoms; double-well potential; correlation functions.

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The recent interest to the bimodal Bose-Hubbard model is caused by the possibility of its applications in the studies of weakly interacting ultracold atoms in a double-well traps [1, 2, 3, 4, 5], quantum dynamics in the bosonic Josephson junctions [6, 7], quantum metrology [8] and quantum information processing [9].

The Quantum Inverse Method (QIM) [10, 11] allowed to solve the model exactly [12, 13]. In this paper we shall use the QIM approach to the calculation of the time-dependent correlation functions of the model.

Consider a collection of N bosonic atoms at zero temperature in a double-well optical trap sufficiently deep that only the lowest state in each well is populated. In this so-called two mode approximation, the atoms' dynamics is described by the Bose-Hubbard Hamiltonian

$$\hat{\mathcal{H}} = \epsilon(n_b - n_a) - J(a^\dagger b + ab^\dagger) + \frac{U}{2}\left(n_a(n_a-1) + n_b(n_b-1)\right) , \tag{1}$$

where $a, a^\dagger$ and $b, b^\dagger$ are bosonic annihilation and creation operators in the ath and bth wells respectively $[a, a^\dagger] = [b, b^\dagger] = 1$, and operators in different wells mutually commute. The number operators of particles in the wells $n_a = a^\dagger a, n_b = b^\dagger b$. The bias potential, coupling, and interaction energies are denoted as ε, J, and U, respectively. The bias ε may be positive or negative, depending on the energy detuning between the two modes, $U > 0$ corresponding to repulsive inter- actions and vice versa. The total number operator of particles

$$\hat{N} = a^\dagger a + b^\dagger b \tag{2}$$

is conserved: $[\hat{H}, \hat{N}] = 0$.

Eliminating insignificant c-number terms, we can rewrite (1) as a spin Hamiltonian,

$$\hat{H}_L = \varepsilon \hat{L}_z - J\hat{L}_x + U\hat{L}_z^2 , \tag{3}$$

where the $SU(2)$ generators in the Schwinger representation

$$\hat{L}_x = \frac{1}{2}(a^\dagger b + ab^\dagger) , \hat{L}_y = \frac{1}{2i}(a^\dagger b - ab^\dagger) , \hat{L}_z = \frac{1}{2}(a^\dagger a - b^\dagger b) \tag{4}$$

are defined in terms of the boson annihilation and creation operators $a, a^\dagger, b, b^\dagger$. In the spin form (3) the bimodal Bose-Hubbard model is equivalent to the Lipkin-Meshkov-Glick model [14] of the nuclear physics.

The conservation of the total number operator allows us to set

$$\hat{H} = -J^{-1}\left\{\hat{\mathcal{H}} - \frac{U}{2}\hat{N}(\hat{N}-1) - \epsilon\hat{N}\right\} ,$$

and $[\hat{H}, \hat{\mathcal{H}}] = 0$. We can, therefore, write

$$\hat{H} = \Delta b^\dagger b + (a^\dagger b + ab^\dagger) + c^2 a^\dagger a b^\dagger b , \tag{5}$$

where $\Delta = 2J^{-1}\epsilon,\ c^2 = J^{-1}U$. Henceforth, we shall consider $\hat{H}$ and give its exact solution, but the same results can immediately be extended to the model with Hamiltonian (1) through the mapping given above.

To apply QIM to the solution of the model we consider the two 2×2 matrix operators $\mathbf{L}_a(\lambda)$ and $\mathbf{L}_b(\lambda)$:

$$\mathbf{L}_a(\lambda) = \begin{pmatrix} \lambda - c^{-1}\Delta - ca^\dagger a & a^\dagger \\ a & -c^{-1} \end{pmatrix}, \tag{6}$$

$$\mathbf{L}_b(\lambda) = \begin{pmatrix} \lambda - cb^\dagger b & b^\dagger \\ b & -c^{-1} \end{pmatrix}, \tag{7}$$

in which λ is a complex number. Note that the elements of $\mathbf{L}_a(\lambda)$ each commute with each of the elements of $\mathbf{L}_b(\lambda)$. These L-operators were introduced in [15, 16] and may be obtained as a special limit of a general boson L-operator [17]. For the monodromy matrix $T(\lambda)$ we can now set

$$\mathbf{T}(\lambda) = \mathbf{L}_a(\lambda)\mathbf{L}_b(\lambda) = \begin{pmatrix} \mathbf{A}(\lambda) & \mathbf{B}(\lambda) \\ \mathbf{C}(\lambda) & \mathbf{D}(\lambda) \end{pmatrix}, \tag{8}$$

so that

$$\begin{aligned} \mathbf{B}(\lambda) &= \lambda b^\dagger - \mathbf{X}; \\ \mathbf{X} &= c^{-1}\Delta b^\dagger + ca^\dagger a b^\dagger + c^{-1}a^\dagger; \\ [b^\dagger, \mathbf{X}] &= 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \mathbf{C}(\lambda) &= \lambda a - \mathbf{Y}; \\ \mathbf{Y} &= c^{-1}b + cab^\dagger b\,; \\ [a, \mathbf{Y}] &= 0\,. \end{aligned} \tag{10}$$

We also have

$$\begin{aligned} \mathbf{A}(\lambda) &= \lambda^2 - \lambda(ca^\dagger a + cb^\dagger b + \Delta c^{-1}) + \Delta b^\dagger b + a^\dagger b + c^2 a^\dagger a b^\dagger b\,; \\ \mathbf{D}(\lambda) &= ab^\dagger + c^{-2}\,. \end{aligned} \tag{11}$$

The trace of the monodromy matrix (8)

$$\tau(\lambda) = Tr\mathbf{T}(\lambda) = \mathbf{A}(\lambda) + \mathbf{D}(\lambda) \tag{12}$$

in the explicite form

$$\tau(\lambda) = \lambda^2 - \lambda(c\hat{N} + \Delta c^{-1}) + \hat{H} + c^{-2}, \tag{13}$$

where $\hat{N}$ is the number operator (2) and $\hat{H}$ is the Hamiltonian (5). It can be checked that

$$\begin{aligned} \hat{H} &= \tau(0) - c^{-2}, \\ \hat{N} &= -\,c^{-1}\frac{\partial\tau(\lambda)}{\partial\lambda}\bigg|_{\lambda=0} - c^{-1}\Delta \end{aligned} \tag{14}$$

We can define the monodromy matrix of the model multiplying L-operators in the opposite order

$$\widetilde{\mathbf{T}}(\lambda) = \mathbf{L}_b(\lambda)\mathbf{L}_a(\lambda) = \begin{pmatrix} \widetilde{\mathbf{A}}(\lambda) & \widetilde{\mathbf{B}}(\lambda) \\ \widetilde{\mathbf{C}}(\lambda) & \widetilde{\mathbf{D}}(\lambda) \end{pmatrix}. \tag{15}$$

Note that $Tr\mathbf{T}(\lambda) = Tr\widetilde{\mathbf{T}}(\lambda) = \tau(\lambda)$, and

$$\widetilde{\mathbf{B}}(\lambda) = \mathbf{C}^{+}(\lambda),\ \widetilde{\mathbf{C}}(\lambda) = \mathbf{B}^{+}(\lambda). \tag{16}$$

The introduced L-operators satisfy the intertwining relations

$$\mathbf{R}(\lambda,\mu)\mathbf{L}_{a,b}(\lambda)\otimes\mathbf{L}_{a,b}(\mu) = \mathbf{L}_{a,b}(\mu)\otimes\mathbf{L}_{a,b}(\lambda)\mathbf{R}(\lambda,\mu) \tag{17}$$

with the rational R-matrix

$$\mathbf{R}(\lambda,\mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}, \tag{18}$$

with the enries

$$f(\mu,\lambda) = 1 - \frac{c}{\mu-\lambda}, \qquad g(\mu,\lambda) = -\frac{c}{\mu-\lambda}.$$

The mutual commutativity of the entries of the operator valued matrices $\mathbf{L}_B$ and $\mathbf{L}_S$ leads to the relation

$$\mathbf{R}(\lambda,\mu)\mathbf{T}(\lambda)\otimes\mathbf{T}(\mu) = \mathbf{T}(\mu)\otimes\mathbf{T}(\lambda)\mathbf{R}(\lambda,\mu) \tag{19}$$

which defines the algebra of the entries $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$. The consequence of this relation is the commutativity of the transfer matrices (12) for arbitrary complex numbers λ,μ:

$$[\tau(\lambda),\tau(\mu)] = 0. \tag{20}$$

It can be shown that $\hat{N}\mathbf{B}(\lambda) = \mathbf{B}(\lambda)(\hat{N}+1)$. So as it is already evident $\mathbf{B}(\lambda)$ acts as a creation operator for the quasi-particles of the theory while $\mathbf{C}(\lambda)$ is an annihilation operator.

The N-particle state vectors are constructed in the usual fashion for the QIM method

$$\mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle = \prod_{j=1}^{N}\mathbf{B}(\lambda_j)\mid\Omega\rangle = \prod_{j=1}^{N}(\lambda_j b^{\dagger} - \mathbf{X})\mid\Omega\rangle, \tag{21}$$

where the vacuum state $\mid\Omega\rangle = \mid 0\rangle_a \otimes \mid 0\rangle_b$ ($a\mid 0\rangle_a = 0; b\mid 0\rangle_b = 0$) is the eigenstate of the operators $\mathbf{A},\mathbf{D}$ (11):

$$\begin{aligned} \mathbf{A}(\lambda)\mid\Omega\rangle = a(\lambda)\mid\Omega\rangle = \lambda(\lambda-\Delta c^{-1}), \\ \mathbf{D}(\lambda)\mid\Omega\rangle = d(\lambda)\mid\Omega\rangle = -c^{-1}, \end{aligned} \tag{22}$$

while

$$\mathbf{C}(\lambda)\mid\Omega\rangle = 0.$$

Throughout the paper we shall use as short-hand notations for appropriate N-tuples of parameters $\{\mathbf{x}\}\equiv(x_1,x_2,\ldots,x_N)$.

It follows from (21) that the state vector (21) may be written as

$$\mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle = \sum_{m=0}^{N}(-1)^{m+1}e_m(b^{\dagger})^m\mathbf{X}^{N-m}\mid\Omega\rangle, \tag{23}$$

where

$$e_m = \sum_{i_1<i_2<\ldots<i_m} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_m}$$

is the elementary symmetric function [18].

The conjugated N-particle state vectors is equal to

$$\langle \mathbf{\Psi}_N(\{\boldsymbol{\mu}\}) \mid = \langle \Omega \mid \prod_{j=1}^{N} \mathbf{C}(\mu_j) = \langle \Omega \mid \sum_{m=0}^{N} (-1)^{m+1} e_m a^m \mathbf{Y}^{N-m} \tag{24}$$

and

$$\langle \Omega \mid \mathbf{B}(\mu) = 0.$$

It can be shown that the state vector (21) is the eigenvector of the transfer matrix (12)

$$\tau(\mu) \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle = \theta_N(\mu) \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle \tag{25}$$

and thus of the Hamiltonian if $\{\boldsymbol{\lambda}\}$ are, in fact, the roots of the Bethe equations which here take the form, for $n = 1, 2, \ldots, N$,

$$c\lambda_n(c\lambda_n - \Delta) = \prod_{j=1, j\neq n}^{N} \frac{\lambda_n - \lambda_j + c}{\lambda_n - \lambda_j - c}. \tag{26}$$

Taking the complex conjugation of these equations we find that the complex conjugated $\{\boldsymbol{\lambda}^*\}$ satisfy the same equations. It means that there are real and complex-valued solutions of the equations. The complex valued roots are pairwise conjugate. Thus $\{\boldsymbol{\lambda}^*\} = \{\boldsymbol{\lambda}\}$.

This way we can find the eigenvalues of the transfer matrix (12)

$$\begin{aligned} \theta_N(\mu) &= a(\mu)\prod_{j=1}^{N} f(\mu, \lambda_j) + d(\mu)\prod_{j=1}^{N} f(\lambda_j, \mu) \\ &= \mu(\mu - \Delta c^{-1})\prod_{j=1}^{N}\left(1 - \frac{c}{\mu - \lambda_j}\right) + c^{-2}\prod_{j=1}^{N}\left(1 + \frac{c}{\mu - \lambda_j}\right). \end{aligned} \tag{27}$$

The conjugated N-particle state vector (24) is the eigenvector of the transfer matrix (12)

$$\langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\}) \mid \tau(\mu) = \theta_N(\mu)\langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\}) \mid \tag{28}$$

with the same eigenvalue (27) if $\{\boldsymbol{\lambda}\}$ are the roots of Bethe equations (26).

From the equation (14) we find that the N-particle eigenenergies of the Hamiltonian (5) are equal to:

$$\begin{aligned} \hat{H} \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle &= E_N \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle \\ E_N &= -c^{-2} + c^{-2}\prod_{j=1}^{N}\left(1 - \frac{c}{\lambda_j}\right). \end{aligned} \tag{29}$$

The ground state of the Hamiltonian (5) corresponds to the minimal value of egenenergy. The set of solutions of Bethe equations that defines this state will be labeled by σ_g: $\{\boldsymbol{\lambda}^{\sigma_g}\}$.

Consider the state vectors constructed by operators (16)

$$| \widetilde{\mathbf{\Psi}}_N(\{\boldsymbol{\lambda}\})\rangle = \prod_{j=1}^{N} \widetilde{\mathbf{B}}(\lambda_j) \mid \Omega\rangle, \ \langle\widetilde{\mathbf{\Psi}}_N(\{\boldsymbol{\mu}\}) \mid = \langle\Omega \mid \prod_{j=1}^{N} \widetilde{\mathbf{C}}(\mu_j)\,. \tag{30}$$

It was proved in [17] that on the solutions of Bethe equations

$$| \widetilde{\mathbf{\Psi}}_N(\{\boldsymbol{\lambda}\})\rangle = \nu_N \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle, \ \langle\widetilde{\mathbf{\Psi}}_N(\{\boldsymbol{\mu}\}) \mid = \nu_N^{-1}\langle\mathbf{\Psi}_N(\{\boldsymbol{\mu}\}) \mid, \tag{31}$$

where for the model under consideration

$$\nu_N = \prod_{n=1}^{N}(\Delta - c\lambda_n) = (-1)^N \prod_{n=1}^{N} \frac{1}{c\lambda_n}\,. \tag{32}$$

To calculate the correlators we shall use the well known formula [19, 20] for the scalar product of the state vectors of integrable models adopted for the model under consideration:

$$\begin{aligned} S_N(\{\boldsymbol{\mu}\},\{\boldsymbol{\lambda}\}) &= \langle\mathbf{\Psi}_N(\{\boldsymbol{\mu}\}) \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle \\ &= c^N \frac{\prod_{j=1}^{N}\prod_{\alpha=1}^{N}(\mu_j - \lambda_\alpha)}{\prod_{j>k}(\mu_k - \mu_j)\prod_{\alpha<\beta}(\lambda_\beta - \lambda_\alpha)} \det T(\{\boldsymbol{\mu}\},\{\boldsymbol{\lambda}\}), \end{aligned} \tag{33}$$

where the entries of the $N \times N$ matrix T are

$$\begin{aligned} T_{ab} &= c^{-1}\frac{\partial}{\partial\lambda_a}\tau(\mu_b,\{\boldsymbol{\lambda}\}) \\ &= \frac{1}{(\mu_b - \lambda_a)^2}\left\{-a(\mu_b)\prod_{j=1,j\neq a}^{N}(1 - \frac{c}{\mu_b - \lambda_j}) + d(\mu_b)\prod_{j=1,j\neq a}^{N}(1 + \frac{c}{\mu_b - \lambda_j})\right\}. \end{aligned} \tag{34}$$

It is supposed here that $\{\boldsymbol{\lambda}\}$ are the solutions of Bethe equations, while $\{\boldsymbol{\mu}\}$ is the set of arbitrary parameters. The square of the norm of the eigenvectors is calculated by the Gaudin formula [21]:

$$\mathcal{N}^2 = S_N(\{\boldsymbol{\lambda}\},\{\boldsymbol{\lambda}\}) = c^N \prod_{j=1}^{N} d^2(\lambda_j) \prod_{\alpha\neq\beta} \frac{\lambda_\alpha - \lambda_\beta + c}{\lambda_\alpha - \lambda_\beta} \det\Phi(\{\boldsymbol{\lambda}\}), \tag{35}$$

where the entries of the $N \times N$ matrix Φ are

$$\Phi_{ab} = -\frac{\partial}{\partial\lambda_b}\ln\frac{a(\lambda_a)}{d(\lambda_a)}\prod_{k=1,k\neq a}^{M}\frac{\lambda_a - \lambda_k - c}{\lambda_k - \lambda_a - c}\,. \tag{36}$$

The Bethe state vectors form a complete orthogonal set

$$\langle\mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma_1}\}) \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma_2}\})\rangle = \delta_{\sigma_1,\sigma_2}\,,$$

$$\sum_{\sigma}\frac{\mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma}\})\rangle\langle\mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma}\}) \mid}{\mathcal{N}_\sigma^2} = 1\,. \tag{37}$$

Index σ denotes the sets of solutions of Bethe equations (26), and the summation is over all K sets of independent solutions.

The determinant representation (33) may be used in calculation of the transition element of the photon annihilation operator

$$\langle \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}'\}) \mid a^n \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle\,, \tag{38}$$

where $\{\boldsymbol{\lambda}\}$ and $\{\boldsymbol{\lambda}'\}$ are the solutions of Bethe equations (26) for N and $N-n$ particles respectively. Really, notice that from the definition (10) it follows that $\lim_{\lambda\to\infty}\lambda^{-1}C(\lambda) = a$ and hence

$$\langle \mathbf{\Psi}_{N-n}(\{\boldsymbol{\mu}\}) \mid a^n \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle = \frac{1}{\mu^n}\lim_{\mu_1,\mu_2,\ldots,\mu_n\to\infty} S_N(\{\boldsymbol{\mu}\},\{\boldsymbol{\lambda}\}). \tag{39}$$

Replacing the arbitrary parameters $\{\boldsymbol{\mu}\}$ by the solutions of Bethe equations $\{\boldsymbol{\lambda}'\}$ we obtain the answer for (38). The limit in (39) may be found with the help of the formula

$$\lim_{v_1,v_2,\ldots,v_n\to\infty}\frac{\det\{G(v_j,u_k)\}}{\prod_{M\geq j>k\geq 1}(v_k-v_j)} = \frac{\det\left\{\frac{1}{(j-1)!}\frac{\partial^{j-1}}{\partial v^{j-1}}G(v,u_k)\mid_{v\to\infty}\right\}\Big|_{j=1}^{n}}{\prod_{M\geq j>k>n}(v_k-v_j)}\,, \tag{40}$$

where $G(u,v)$ is an arbitrary differentiable for at least M times function of two variables. Denoting $V_{jk}\equiv\frac{1}{(j-1)!}\mu_j^{n(n-1)}\frac{\partial^{j-1}}{\partial\mu_j^{j-1}}T(\mu_j,u_k)\mid_{\mu_j\to\infty}$ we obtain the answer

$$\begin{aligned} A_{N,n}(\{\boldsymbol{\mu}\},\{\boldsymbol{\lambda}\}) &\equiv \langle \mathbf{\Psi}_{N-n}(\{\boldsymbol{\mu}\}) \mid a^n \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle \\ &= c^N\frac{\prod_{j=n+1}^{N}\prod_{\alpha=1}^{N}(\mu_j-\lambda_\alpha)}{\prod_{j>k>n}(\mu_k-\mu_j)\prod_{\alpha<\beta}(\lambda_\beta-\lambda_\alpha)}\det T_{(n)}(\{\boldsymbol{\mu}\},\{\boldsymbol{\lambda}\})\,. \end{aligned} \tag{41}$$

The entries of the $N\times N$ matrix $T_{(n)}$ are equal to V_{ab} for $a\leq n$ and are T_{ab} (34) for $a>n$, $1\leq b\leq N$. To obtain the answer for the transition element (38) we have to change parameters $\{\boldsymbol{\mu}\}$ on the solutions of Bethe equations for $N-n$ particles.

For instance, for $n=1$ the answer for the transition element is

$$\begin{aligned} &\langle \mathbf{\Psi}_{N-1}(\{\boldsymbol{\lambda}'\}) \mid a \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle \\ &\qquad = c^N\frac{\prod_{j=3}^{N}\prod_{\alpha=1}^{N}(\lambda'_j-\lambda_\alpha)}{\prod_{j>k>2}(\lambda'_k-\lambda'_j)\prod_{\alpha<\beta}(\lambda_\beta-\lambda_\alpha)}\det T_{(1)}(\{\boldsymbol{\lambda}'\},\{\boldsymbol{\lambda}\})\,, \end{aligned} \tag{42}$$

where matrix

$$T_{(1)}(\{\boldsymbol{\lambda}'\},\{\boldsymbol{\lambda}\}) = \begin{pmatrix} -1 & T_{12} & \ldots & T_{1N} \\ -1 & T_{22} & \ldots & T_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & T_{N2} & \ldots & T_{NN} \end{pmatrix}.$$

To find the transition element of the creation operator one can take the complex

conjugation of the transition element of the annihilation operator:

$$\begin{aligned}\langle \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}'\}) \mid a^n \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle^* &= \langle \Omega \mid \prod_{j=1}^{N-n} \mathbf{C}(\lambda'_j) a^n \prod_{j=1}^{N} \mathbf{B}(\lambda_j) \mid \Omega\rangle^* \\ &= \langle \Omega \mid \prod_{j=1}^{N} \mathbf{B}^+(\lambda_j)(a^\dagger)^n \prod_{j=1}^{N-n} \mathbf{C}^+(\lambda'_j) \mid \Omega\rangle \\ &= \langle \Omega \mid \prod_{j=1}^{N} \widetilde{\mathbf{C}}(\lambda_j)(a^\dagger)^n \prod_{j=1}^{N-n} \widetilde{\mathbf{B}}(\lambda'_j) \mid \Omega\rangle \\ &= \frac{\nu'_{N-n}}{\nu_N} \langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\}) \mid (a^\dagger)^n \mid \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}'\})\rangle\,, \quad (43)\end{aligned}$$

where the definition (30) and the property (31) have been used, and $\boldsymbol{\lambda}'$ are the solutions of Bethe equations (26) for $N-n$ particles. The obtained formula allows to express the transition element of the creation operator in the determinantal form:

$$\begin{aligned}A^+_{N,n}(\{\boldsymbol{\lambda}\},\{\boldsymbol{\lambda}'\}) &\equiv \langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\}) \mid (a^\dagger)^n \mid \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}'\})\rangle \\ &= c^N \frac{\nu_N}{\nu'_{N-n}} \frac{\prod_{j=n+1}^{N}\prod_{\alpha=1}^{N}(\lambda'_j-\lambda_\alpha)}{\prod_{j>k>n}(\lambda'_k-\lambda'_j)\prod_{\alpha<\beta}(\lambda_\beta-\lambda_\alpha)} \det T^+_{(n)}(\{\boldsymbol{\lambda}'\},\{\boldsymbol{\lambda}\})\,, \quad (44)\end{aligned}$$

where $T^+_{(n)}(\{\boldsymbol{\lambda}'\},\{\boldsymbol{\lambda}\})$ is the Hermitian conjugated matrix (41) on the solutions of Bethe equations.

The obtained representations for the transition elements (41) and (44) allow us to calculate different n-photon time-dependent correlation functions.

We define the n-photon Green function $\langle (a^\dagger)^n a^n(t)\rangle_N$ as the average taken over the N-particle ground state of the model $\mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma_g}\})\rangle$:

$$\begin{aligned}\langle (a^\dagger)^n a^n(t)\rangle_N &= \frac{1}{\mathcal{N}^2_{\sigma_g}} \langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma_g}\}) \mid (a^\dagger)^n e^{-i\mathbf{H}t} a^n e^{i\mathbf{H}t} \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma_g}\})\rangle \\ &= \sum_\sigma \frac{e^{it(E^{\sigma_g}_N - E^{\sigma}_{N-n})}}{\mathcal{N}^2_{\sigma_g}\mathcal{N}^2_\sigma} \langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma}\}) \mid (a^\dagger)^n \mid \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}^{\sigma_g}\})\rangle \\ &\qquad \times \langle \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}^{\sigma_g}\}) \mid a^n \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma}\})\rangle \\ &= \sum_\sigma \frac{e^{it(E^{\sigma_g}_N - E^{\sigma}_{N-n})}}{\mathcal{N}^2_{\sigma_g}\mathcal{N}^2_\sigma} \left|\langle \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}^{\sigma_g}\}) \mid a^n \mid \mathbf{\Psi}_{S,N}(\{\boldsymbol{\lambda}^{\sigma}\})\rangle\right|^2 \frac{\nu^{\sigma}_{N-n}}{\nu^{\sigma_g}_N}\,, \quad (45)\end{aligned}$$

where the sum is taken over the sets of solutions of Bethe equations (26) for $M+n$ particles $\boldsymbol{\lambda}^\sigma$ labeled by σ, and the complete orthogonal set (37) of eigenstates of the Hamiltonian (5) was used. Substituting formulas (29), (32), (35) and (41) into (45) we obtain the final answer.

From (9) it follows that $\lim_{\lambda\to\infty} \lambda^{-1}B(\lambda) = b^\dagger$. Repeating the derivation of the transition element of the operator a (38) we find that

$$\begin{aligned}B^+_{N,n}(\{\boldsymbol{\lambda}\},\{\boldsymbol{\lambda}'\}) &\equiv \langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\}) \mid (b^\dagger)^n \mid \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}'\})\rangle \\ &= c^N \frac{\prod_{j=n+1}^{N}\prod_{\alpha=1}^{N}(\lambda'_j-\lambda_\alpha)}{\prod_{j>k>n}(\lambda'_k-\lambda'_j)\prod_{\alpha<\beta}(\lambda_\beta-\lambda_\alpha)} \det T^+_{(n)}(\{\boldsymbol{\lambda}'\},\{\boldsymbol{\lambda}\})\,, \quad (46)\end{aligned}$$

where matrix T is defined in (41), (44), and λ, λ' are the solutions of corresponding Bethe equations. The expression (46) differs from (44) by the factor ν_N/ν'_{N-n} and thus

$$\langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\}) \mid (b^\dagger)^n \mid \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}'\})\rangle = \langle \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}'\}) \mid a^n \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}\})\rangle \,.$$

The Green function

$$\begin{aligned}\langle (b^\dagger)^n a^n(t)\rangle_N &= \frac{1}{\mathcal{N}^2_{\sigma_g}} \langle \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma_g}\}) \mid (b^\dagger)^n e^{-i\mathbf{H}t} a^n e^{i\mathbf{H}t} \mid \mathbf{\Psi}_N(\{\boldsymbol{\lambda}^{\sigma_g}\})\rangle \\ &= \sum_\sigma \frac{e^{it(E_N^{\sigma_g} - E^\sigma_{N-n})}}{\mathcal{N}^2_{\sigma_g}\mathcal{N}^2_\sigma} \left| \langle \mathbf{\Psi}_{N-n}(\{\boldsymbol{\lambda}^{\sigma_g}\}) \mid a^n \mid \mathbf{\Psi}_{S,N}(\{\boldsymbol{\lambda}^\sigma\})\rangle \right|^2 , \quad (47)\end{aligned}$$

An important resource in quantum metrology is the Einstein-Podolsky-Rosen entanglement which is characterized by the sign of the observable

$$\mathcal{E} = \langle a^\dagger b\rangle_N \langle b^\dagger a\rangle_N - \langle a^\dagger a b^\dagger b\rangle_N \,.$$

The wells are said to be spatially entangled whenever $\mathcal{E} > 0$.

The other important quantity which measures the visibility of the interference fringes is given by

$$\alpha = \frac{2|\langle a^\dagger b\rangle_N|}{N} \quad (48)$$

where the expectation value is taken with respect to the ground state. This quantity characterizes the degree of coherence between the two wells, related to the left-right (and back) tunneling.

The obtained results for the time-dependent correlation functions will allow to study the dynamics of entanglement and visibility in details.

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