ПРЕПРИНТЫ ПОМИ РАН

Учредитель: Федеральное государственное бюджетное учреждение науки
Санкт-Петербургское отделение Математического института
им. В. А. Стеклова Российской академии наук

Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16 октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых коммуникаций

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

телефоны:(812)312-40-58; (812) 571-57-54

e-mail: admin@pdmi.ras.ru

http://www. pdmi.ras.ru /preprint/

Заведующая информационно-издательским сектором Симонова В.Н

POMI Preprint – 12/2014

Dyadic shift randomization in classical discrepancy theory

M. M. Skriganov

St. Petersburg Department of Steklov Institute of Mathematics
E-mail: maksim88138813@mail.ru

August 25, 2014

Abstract

Dyadic shifts $D \oplus T$ of point distributions D in the d-dimensional unit cube U^d are considered as a randomization. Explicit formulas for the L_q-discrepancies of such randomized distributions are given in the paper in terms of Rademacher functions. Relaying on the statistical independence of Rademacher functions, Khinchin's inequalities, and other related results, we obtain very sharp upper and lower bounds for the mean L_q-discrepancies. $0 < q \leq \infty$.

The upper bounds imply directly a generalization of the well known Chen's theorem on mean discrepancies to the case of dyadic shifts (Theorem 2.1).

From the lower bounds it follows that for an arbitrary N-point distribution D_N and any exponent $0 < q \leq 1$ there exist dyadic shifts $D_N \oplus T$ such that the L_q-discrepancy $\mathcal{L}_q[D_N \oplus T] > c_{d,q} (\log N)^{\frac{1}{2}(d-1)}$ (Theorem 2.2).

The lower bounds for the L_∞-discrepancy are also considered in the paper. It is shown that for an arbitrary N-point distribution D_N there exist dyadic shifts $D_N \oplus T$ such that $\mathcal{L}_\infty[D_N \oplus T] > c_d (\log N)^{\frac{1}{2}d}$ (Theorem 2.3).

Keywords: Uniform distributions, mean L_q-discrepancies, Rademacher functions, Khinchin's inequality

ПРЕПРИНТЫ
Санкт-Петербургского отделения
Математического института им. В. А. Стеклова
Российской академии наук

PREPRINTS
of the St. Petersburg Department of Steklov Institute of Mathematics

---

Contents

1. Dyadic shifts and the mean discrepancies

The classical discrepancy theory deals with the distribution of finite point sets in rectangular sub-boxes in the unit cube with sides parallel to the coodinate axes. A detailed discussion of numerous methods and results known in the field can be found in [1, 2, 11]. We recall only the main definitions and facts necessary for the purposes of our paper.

Let D be an arbitrary finite subset (distribution) in the unit cube $U^d = [0,1)^d$. The *local discrepancy* $\mathcal{L}[D,Y]$, $Y = (y_1, \dots, y_d) \in U^d$, is defined by

$$\mathcal{L}[D,Y] = |D \cap B_Y| - |D| \operatorname{vol} B_Y, \tag{1.1}$$

where $B_Y = [0,y_1) \times \cdots \times [0,y_d)$ is a recfngular box of volume $\operatorname{vol} B_Y = y_1, \dots, y_d$, and $|\cdot|$ denotes the cardinality of a set.

The L_q-*discrepancies* are defined by

$$\mathcal{L}_q[D] = \left(\int_{U_d} |\mathcal{L}[D,Y]|^q dY \right)^{1/q}, \quad 0 < q < \infty, \tag{1.2}$$

$$\mathcal{L}_\infty[D] = \sup_{Y \in U^d} |\mathcal{L}[D,Y]|. \tag{1.3}$$

We write $\mathbb{N}$ for the set of all positive integers, $\mathbb{N}_0$ for the set of all non-negative integers, $\mathbb{N}^d$ and $\mathbb{N}_0^d$ for the product of d copies of the corresponding sets. For $s \in \mathbb{N}_0$, we put

$$\mathbb{Q}(2^s) = \{x = m2^{-s} \in [0,1) : m = 0,1,\dots,2^s - 1\},$$
$$\mathbb{Q}^d(2^s) = \{X = (x_1,\dots,x_d) \in U_d : x_j \in \mathbb{Q}(2^s), j = 1,\dots,d\},$$
$$\mathbb{Q}(2^\infty) = \bigcup_{s\geq 0} \mathbb{Q}(2^s),\ \mathbb{Q}^d(2^\infty) = \bigcup_{s\geq 0} \mathbb{Q}^d(2^s).$$

The points of $\mathbb{Q}^d(2^\infty)$ are called *dyadic rational points.*

Any $y \in [0,1)$ can be represented in the form

$$y = \sum_{a\geq 1} \eta_a(y) 2^{-a}, \tag{1.4}$$

where $\eta_a(y) \in \{0,1\} \simeq \mathbb{F}_2$, $a \in \mathbb{N}$. Here $\mathbb{F}_2$ is the field of two elements identified with the set of residues $\{0,1\}$ mod 2.

The dyadic expansion (1.4) is unique if we agree that for each dyadic rational point the sum in (1.4) contains finitely many nonzero terms. With this agreement, $\eta_a(y) = 0$ for $a > s$ if $y \in \mathbb{Q}(2^s)$ or, in other words, for each point $y \in [0,1)$, the sequence $\{\eta_a(y), a \in \mathbb{N}\}$ contains infinitely many zeros.

In a natural way, the set of dyadic rational points can be endowed with the structure of a vector space over the finite field $\mathbb{F}_2$. For any two points x and y in $\mathbb{Q}(2^\infty)$, we define their sum $x \oplus y$ as follows

$$\eta_a(x \oplus y) = \eta_a(x) + \eta_a(y) \bmod 2,\ \ a \in \mathbb{N}, \tag{1.5}$$

and for any two points $X = (x,\dots,x_d)$ and $Y = (y_1,\dots,y_d)$ in $\mathbb{Q}^d(2^\infty)$ we define

$$X \oplus Y = (x_1 \oplus y_1,\dots,x_d \oplus y_d). \tag{1.6}$$

With respect to the addition $\oplus$ defined in this way, each set $\mathbb{Q}^d(2^s)$ is a vector space over the field $\mathbb{F}_2$, and $\dim \mathbb{Q}^d(2^s) = ds$.

Note that formulas (1.5), (1.6) consistently define the addition $\oplus$ for all pairs of points X and Y, whenever only one of the points, say Y, belongs to $\mathbb{Q}^d(2^\infty)$, while the other is an arbitrary point $X \in U^d$.

The said above shows that, for an arbitrary distribution D and any point $T \in \mathbb{Q}^d(2^\infty)$, we can define the dyadic shift $D \oplus T = \{X \oplus T : X \in D\}$ and view it as a new distribution. For each $s \in \mathbb{N}$, we can consider the family $\{D \oplus T : T \in \mathbb{Q}^d(2^s)\}$ as a randomization of D and the corresponding discrepancies $\mathcal{L}_q[D \oplus T]$ as random variables.

The aim of the present paper is to study the following *mean L_q-discrepancies*

$$\mathcal{M}_{s,q}[D] = \left(2^{-ds} \sum_{T \in \mathbb{Q}^d(2^s)} \mathcal{L}_q[D \oplus T]^q \right)^{1/q}, \quad 0 < q < \infty, \tag{1.7}$$

$$\mathcal{M}_{s,\infty}[D] = \max_{T \in \mathbb{Q}^d(2^s)} \mathcal{L}_\infty[D \oplus T]. \tag{1.8}$$

Our results are given in the next section in Theorems 2.1, 2.2 and 2.3. In Theorem 2.1 we will consider the upper bounds for $\mathcal{M}_{s,q}[D]$, $0 < q < \infty$, and specific distributions D, the so-called (δ, s, d)-nets. The lower bounds for $M_{s,q}[D]$ and arbitrary distributions D will be given in Theorems 2.2 and 2.3 for exponents $0 < q \leq 1$ and $q = \infty$, correspondingly.

Recall the definition of dyadic (δ, s, d)-nets. We refer to $[1, 2, 11]$ for details; notice that in [2] such (δ, s, d)-nets are called 2^δ-sets of class $s - t$.

Consider the *elementary intervals* $\Delta_s^m \subset [0, 1)$ of the form

$$\Delta_a^m = [m2^{-a}, (m+1)2^{-a}), a \in \mathbb{N}_0, m = 0, 1, \ldots, 2^a - 1, \tag{1.9}$$

and the *elementary boxes* $\Delta_A^M \subset U^d$, $A = (a_1, \ldots, a_d)$, $M = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$,

$$\Delta_A^M = \Delta_{a_1}^{m_1} \times \cdots \times \Delta_{a_d}^{m_d}, m_j = 0, 1, \ldots, 2^{a_j} - 1, j = 1, \ldots, d. \tag{1.10}$$

Every such box has volume $\operatorname{vol} \Delta_A^M = 2^{-a_1 - \cdots - a_d}$.

Let $0 \leq \delta \leq s$ be integers. A subset $D_{2^s} \subset U^d$ consisting of $N = 2^s$ points is called a *dyadic (δ, s, d)-net of deficienty δ* if each elementary box Δ_A^M of volume $2^{\delta - s}$ contains exactly 2^δ points of D_{2^s}.

It follows from the definition that any (δ, s, d)-net D_{2^s} has zero discrepancy in all elementary boxes of large volume. Precisely,

$$\left.\begin{aligned} |D_{2^s} \cap \Delta_A^M| &= 2^s \operatorname{vol} \Delta_A^M \quad \text{if} \quad \operatorname{vol}\ \Delta_A^M \geq 2^{\delta - s}, \\ |D_{2^s} \cap \Delta_A^M| &\leq 2^\delta \quad \text{if} \quad \operatorname{vol}\ \Delta_A^M < 2^\delta. \end{aligned}\right\} \tag{1.11}$$

Indeed, in the first case, each box Δ_A^M is a disjoint union of elementary boxes of volume $2^{\delta - s}$, and in the second, each box Δ_A^M is contained in an elementary box of volume $2^{\delta - s}$.

Notice also that for any (δ, s, d)-net D_{2^s} its shift $D_{2^s} \oplus T$, $T \in \mathbb{Q}^d(2^\infty)$, is a net with the same parameters.

Indeed, $|(D \oplus T) \cap \Delta_A^M| = |D \cap (\Delta_A^M \oplus T)|$, $T \in \mathbb{Q}^d(2^\infty)$, and $\Delta_A^M \oplus T = \Delta_A^{M(T)}$ with an index $M(T)$.

Replacing the base 2 in the definition and in (1.9), (1.10) by an arbitrary prime p, we arrive at (δ, s, d)-nets in the base p. In arbitrary dimensions d, first constructions of dyadic (δ, s, d)-nets with $\delta \leq d \log d$ were given by Sobol, and later, other constructions of nets in arbitrary base p were proposed by Faure, see [2].

It is significant that for each base p, the deficiency δ increases with the growth of the dimension d. Furthermore, $(0, s, d)$-nets in the base p and with arbitrary large s exist if and only if $d \leq p+1$; in particular, infinite sequences of dyadic nets with $\delta = 0$ exist only in dimensions $d = 1, 2$ and 3.

It is known that (δ, s, d)-nets D_{2^s} fill the unit cube very uniformly, and the L_∞-discrepancies admit the bounds

$$\mathcal{L}_\infty[D_{2^s}] < C_d 2^\delta s^{d-1}, \quad s \to \infty, \tag{1.12}$$

with a constant C_d depending only on dimension d. Furthermore, for arbitrary (δ, s, d)-nets the order of this bound as $s \to \infty$ can not be improved.

We recall that for an arbitrary N-point distribution $D_N \subset U^d$ the following bounds hold

$$\mathcal{L}_q[D_N] > c_{d,q} (\log N)^{\frac{1}{2}(d-1)}, \quad 1 < q < \infty, \tag{1.13}$$

with positive constants $c_{d,q}$ depending only on d and q.

These classical bounds are due to Roth for $2 \leq q \leq \infty$ and Schmidt for $1 < q < 2$. In two dimensions, it is known that bound (1.13) is also true for $q = 1$, this result is due to Halász.

The order of bound (1.13) is the best possible as $N \to \infty$. In the most general form, in all dimensions $d \geq 2$ and for all exponents $0 < q < \infty$ this fundamental fact was established by Chen. Previously, for $0 < q \leq 2$, this fact was established by Davenport, Roth and other authors.

It should be mentioned that Chen gave two different proofs of his theorem. In the first proof the averagings of the L_q-discrepancies were considered with respect to the usual Euclidean translations of point distributions. The original idea of the p-adic shifts was introduced and exploited in the second proof in the paper [7].

We refer to [1, 2, 11] for detailed discussion of all these questions.

2. Main results

Our first result concerns upper bounds for the mean L_q-discrepancies.

Theorem 2.1. *Let D_{2^s} be an arbitrary dyadic (δ, s, d)-net. Then, for each $0 < q < \infty$ the following bound holds*

$$\mathcal{M}_{s,q}[D_{2^s}] < 2^{-d+\delta+1} \left(\lceil \frac{1}{2} q \rceil (s+1) \right)^{\frac{1}{2}(d-1)} + d2^{\delta}. \tag{2.1}$$

In particular, there exist dyadic shifts $T \in \mathbb{Q}^d(2^s)$ such that

$$\mathcal{L}_q[D_{2^s} \oplus T] \leq 2^{-d+\delta+1} \left(\lceil \frac{1}{2} q \rceil (s+1) \right)^{\frac{1}{2}(d-1)} + d2^{\delta}. \tag{2.2}$$

Theorem 2.1 shows that in all dimensions there exist dyadic (δ, s, d)-nets which meet the lower bound (1.13).

For the first time, the results of such type were established by Chen for nets of definiciency $\delta = 0$ in an arbitrary prime base $p \geq 2$.

The original Chen's approach was relaying on an elaborated combinatorial analysis involving simultaneous induction on the parameters d, s, and even integers q. Under that approach, the assumption $\delta = 0$ turns out to be essential. As a result, for each fixed prime base p, Chen's theorem could be applied only to dimensions $d \leq p+1$, and for dyadic nets only in dimensions 1, 2 and 3.

In the author's paper [12] a new approach to the study of the mean L_q-discrepancies was proposed. Under this approach, the value of the deficiency δ turns out to be completely irrelevant. This approach is relaying on the theory of lacunary function series. In the case of dyadic nets, these are series of Rademacher functions, which form a lacunary subsystems for the Walsh functions, and in the case of nets in an arbitrary base p these series are lacunary subsystems for the corresponding Chrestenson–Levy functions. The detailed description of such functional systems can be found in [10].

A result similar to Theorem 2.1 was established previously in [13], see also [14], but with worth constants in the bounds. As functions of q the constants given above in bound (2.1) and (2.2) are optimal in the following sense. It can be shown that

$$\mathcal{L}_q[D_{2^s}] \leq \mathcal{L}_\infty[D_{2^s}] \leq 2^{d/\varepsilon} \left(\mathcal{L}_q[D_{2^s}] + d2^{\delta+1} \right), \tag{2.3}$$

where $q = \varepsilon s \to \infty$ and $\varepsilon > 0$ is an arbitrary constant, see Lemma 6.2.

Therefore, bounds (2.1) and (2.2) imply bound (1.12). Furthermore, if the order of the constants in (2.1) and (2.2) could be improved as $q \to \infty$, then the order of bound (1.12) could be also improved as $s \to \infty$ for a subsequence of (δ, s, d)-nets.

Now we consider lower bounds for the mean L_q-discrepancies. In what follows, log denotes the logarithm in base 2.

Theorem 2.2. *Let $D_N \subset U^d$, $d \geq 2$, be an arbitrary N-point distribution and an exponent $0 < q \leq 1$ be arbitrary fixed. Suppose that an integer s is choosen to satisfy*

$$s \geq \log N + \frac{2d+1}{q} + \frac{1}{2}(d-1)\log(d-1) + d + 1 + \log d. \tag{2.4}$$

Then, the following bound holds

$$\mathcal{M}_{s,q}[D_N] > \gamma_q(d)(\log N)^{\frac{1}{2}(d-1)}, \tag{2.5}$$

where

$$\gamma_q(d) = 2^{-(2d+1)/q-d-1}(d-1)^{-\frac{1}{2}(d-1)}. \tag{2.6}$$

In particular, there exist dyadic shifts $T \in \mathbb{Q}^d(2^s)$ such that

$$\mathcal{L}_q[D_N \oplus T] > \gamma_q(d) \quad (\log N)^{\frac{1}{2}(d-1)}. \tag{2.7}$$

Certainly, bounds (2.5) and (2.7) hold also for $1 < q < \infty$ but, in this case, these bounds follow at once from (1.13).

In dimensions $d \geq 3$, even the exact order of the L_1-discrepancy is unknown, and the L_q-discrepancies with $0 < q < 1$ were never considered at all.

Theorem 2.2 shows that, in contrast to the L_q-discrepancies of individual distributions, the mean L_q-discrepancies can be studied completely for all exponents $0 < q \leq 1$.

It is worth noting that Theorems 2.1 and 2.2 can be extended to the following *conditional mean L_q-discrepancies*

$$M_{s,q}[D, V] = \left(|V|^{-1} \sum_{T \in V} \mathcal{L}_q[D \oplus T]^q \right)^{1/q}, \quad 0 < q < \infty, \tag{2.8}$$

where V is a subset in $\mathbb{Q}^d(2^s)$.

It turns out that the conditional means (2.8) can meet the bounds of order (2.1) and (2.5) at very small averaging subsets V of cardinality $|V| = O(s^{\omega_q(d)})$ as $s \to \infty$; here $\omega_q(d)$ is a constant independent of s.

Certainly, such subsets V should be rather specific. Some results in this direction were obtained in [14], and further studies of these intriguing questions will be continued in the forthcoming papers.

Our result on the mean L_∞-discrepancy can be stated as follows.

Theorem 2.3. *Let $D_N \subset U^d$, $d \geq 3$, be an arbitrary N-point distribution. Suppose that an integer s is choosen to satisfy*

$$s \geq \log N + \frac{1}{2}(d-2)\log(d-2) + 2d + \log d. \tag{2.9}$$

Then, the following bound holds

$$\mathcal{M}_{s,\infty}[D_N] > \gamma_\infty(d) \quad (\log N)^{\frac{1}{2}d}, \tag{2.10}$$

where

$$\gamma_\infty(d) = 2^{-2d-1}(d-2)^{-\frac{1}{2}(d-2)}. \tag{2.11}$$

In particular, there exist dyadic shifts $T \in \mathbb{Q}^d(2^s)$ such that

$$\mathcal{L}_\infty[D_N \oplus T] > \gamma_\infty(d) \quad (\log N)^{\frac{1}{2}d}. \tag{2.12}$$

In dimensions $d \geq 3$ the exact order of the L_∞-discrepancy still remains an open question. In two dimensions the answer is known: the following Schmidt's lower bound is the best possible

$$\mathcal{L}_\infty[D_N] > c \ \log N, \quad D_N \subset U^2.$$

In higher dimensions, the following Beck's lower bound for the three-dimensional distributions remained the only known result over many years

$$\mathcal{L}_\infty[D_N] > c_\varepsilon \log N (\log\log N)^{\frac{1}{8}-\varepsilon}, D_N \subset U^3, \tag{2.13}$$

where $\varepsilon > 0$ is arbitrary small.

Rather recently, the following strong lower bounds were established in all dimensions $d \geq 3$

$$\mathcal{L}_\infty[D_N] > c_d(\log N)^{\frac{1}{2}(d-1)+\eta_d} \tag{2.14}$$

with small constants $\eta_d \gtrsim d^{-2}$ depending only on d.

These deep results are due to Bilyk and Lacey [3] in dimension $d = 3$ and Bilyk, Lacey and Vagharshakyan [4] in dimensions $d \geq 4$, see also the surveys [5, 6].

Traditiionally, a great number of specilists in the discrepancy theory believes that in all dimensions $d \geq 3$ the best possible lower bound is of the form

$$\mathcal{L}_\infty[D_N] > c_d(\log N)^{d-1}.$$

However, contrary to such a popular belief, it was conjectured that the best possible lower bound should have the form

$$\mathcal{L}_\infty[D_N] > c_d(\log N)^{\frac{1}{2}d}. \tag{2.15}$$

This conjecture is inspired by some very non-trivial parallels between the disrepancy theory and the theory of stochastic processed. The reader can consult the cited papers [3–6] for a more detailed discussion of these questions.

Theorem 2.3 shows that the hypothetical bound (2.15) is true for the mean L_∞-discrepancy.

We will see that the mean L_q-discrepancies can be represented in terms of the Rademacher series, see section 4. For such series, very sharp upper and lower L_q-bounds for any $0 < q < \infty$ can be given by Khinchin's inequality. In fact, Theorems 2.1 and 2.2 are corolaries of this inequality. At the same time, Theorem 2.3 is a corollary of a suitably modified Khinchin's inequality, adapted to the L_∞-norm, see Lemma 3.2.

Lower bounds (1.13), (2.13), (2.14) are obtained with the help of different variations of Roth's orthogonal function method, cf. [2, 5]. It is interesting to note that, in the proofs of Theorems 2.2 and 2.3, we will not use any auxiliary orthogonal functions. The corresponding lower bounds will be derived directly from the explicit formulas for discrepancies given in Lemma 4.3.

3. Rademacher functions and related inequalities

In this section all nosessary facts on Rademacher functions and related topics are collected.

In the one-dimensional case, the Rademacher functions $r_a(y)$, $y \in [0,1)$, $a \in \mathbb{N}$, can be defined by

$$r_a(y) = (-1)^{\eta_a(y)} = 1 - 2\eta_a(y), \tag{3.1}$$

where $\eta_a(y)$ are the coefficients in the dyadic expansion (1.4). It is convenient to put $r_0(y) \equiv 1$.

In these terms, expansion (1.4) takes the form

$$y = \frac{1}{2} - \frac{1}{2}\sum_{a\geq 1} 2^{-a} r_a(y). \tag{3.2}$$

The Rademacher functions $r_a(\cdot)$, $a \in \mathbb{N}$, form a sequence of independent random variables taking the values ± 1 with probability $1/2$. This fact can be expressed by the following relations

$$\operatorname{mes}\{y \in [0,1) : r_{a_1}(y) = \varepsilon_1, \dots, r_{a_l}(y) = \varepsilon_l\} = 2^{-l} \tag{3.3}$$

which hold for any $1 \leq a_1 < \cdots < a_l$, $l \in \mathbb{N}$, and any $\varepsilon_j = \pm 1$, $j = 1, \dots, l$, see, for example [9, 12].

Each function $r_a(y)$, $a \in \mathbb{N}$, is piecewise constant on elementary intervals $\Delta_a^m = [m2^{-a}, (m+1)2^{-a})$, $m = 0, 1, \dots, 2^a - 1$. Therefore, relations (3.3) are equivalent to their discret analogs

$$|\{y \in \mathbb{Q}(2^s) : r_{a_1}(y) = \varepsilon_1, \dots, r_{a_l}(y) = \varepsilon_l\}| = 2^{s-l}. \tag{3.4}$$

with any $1 \leq a_1 < \cdots < a_l \leq s$, $s \in \mathbb{N}$, and any $\varepsilon_j = \pm 1$, $j = 1, \dots, l$.

The k-dimensional Rademacher functions $r_A(Y)$, $Y = (y_1, \dots, y_k) \in U^k$, $A = (a_1, \dots, a_k) \in \mathbb{N}_0^k$, are defined by

$$r_A(Y) = \prod_{j=1}^{d} r_{a_j}(y_j). \tag{3.5}$$

In some formulas, we write k for dimension, because the formulas will be used in the subsequent text with $k = d$ and $k = d - 1$.

We introduce the linear space $\mathcal{R}_s^k$, $s \in \mathbb{N}_0$, consisting of all functions of the form

$$f(Y) = \sum_{A \in I_s^k} \lambda_A r_A(Y). \tag{3.6}$$

with real coefficients λ_A; here $I_s = \{0, 1, \dots, s\}$ and I_s^k denotes the product of k copies of I_s.

It follows from relations (3.4) that the set of functions $\{r_a(\cdot), a \in I_s\}$ is linear independent on $\mathbb{Q}(2^s)$, and therefore, the set $\{r_A(\cdot), A \in I_s^d\}$ is linear independent on $\mathbb{Q}^d(2^s)$. Thus, $\dim \mathcal{R}_s^k = (s+1)^k$, and $\mathcal{R}_s^k$ is a very small subspace in the large space $\mathcal{B}_s^k$ of dimension 2^{ks} consisting of all real-valued functions piecewise constant on elementary cubes

$$\begin{aligned}
&\Delta_s^M = [m_1 2^{-s}, (m_1+1)2^{-s}) \times \cdots \times [m_k 2^{-s}, (m_k+1)2^{-s}), \\
&m_j = 0, 1, \dots, 2^s - 1, j = 1, \dots, k.
\end{aligned}$$

Each function $f \in \mathcal{B}_s^k$ is determined by its values on dyadic rational points $\mathbb{Q}^k(2^s)$, and we have

$$\|f\|_q = \|f\|_{s,q}, \quad 0 < q \le \infty, \tag{3.7}$$

where

$$\begin{aligned}
\|f\|_q &= \left(\int_{U^k} |f(Y)|^q dY \right)^{1/q}, \quad 0 < q < \infty, \\
\|f\|_\infty &= \sup_{Y \in U^k} |f(Y)|, \\
\|f\|_{s,q} &= \left(2^{-ks} \sum_{Y \in \mathbb{Q}^k(2^s)} |f(Y)|^q \right)^{1/q}, \ 0 < q < \infty, \\
\|f\|_{s,\infty} &= \max_{Y \in \mathbb{Q}^k(2^s)} |f(Y)|.
\end{aligned}$$

The k-dimensional Khinchin's inequality: for each function $f \in \mathcal{R}_s^k$ and all $0 < q < \infty$, we have

$$\alpha_q^k Q_2[f] \le \|f\|_{s,q} \le \beta_q^k Q_2[f], \tag{3.8}$$

where

$$Q_2[f] = \left(\sum_{A \in I_s^k} \lambda_A^2 \right)^{1/2}, \tag{3.9}$$

The constants α_q^k and β_q^k are independent of f and s; they are the k-th power of the constants α_q and β_q, correspondingly, and

$$\alpha_q \geq \begin{cases} 2^{-(2-q)/q}, & \text{if} \quad 0 < q < 2 \\ 1, & \text{if} \quad 2 \leq q < \infty, \end{cases} \tag{3.10}$$

$$\beta_q \leq \lceil \frac{1}{2} q \rceil^{1/2}. \tag{3.11}$$

In the one-dimensional case inequality (3.8) is a corollary of the independence of Rademacher functions, see (3.3), (3.4), and its proof can be found in many texts on harmonic analysis and probability theory, see, for example, [9, Sec. 10.3, Thm. 1], [12], [16, Chap. 5, Thm. 8.4].

The extension of Khinchin's inequality to higher dimensions can be easily given by induction on k; we refer to [15, Appendix D] for details.

In the subsequent text we will use corolaries of Khinchin's inequality given below in Lemmas 3.1 and 3.2.

For $Y = (y_1, \dots, y_d) \in U^d$ and $A = (a_1, \dots, a_d) \in I_s^d$, $d \geq 2$, we put

$$\left. \begin{aligned} &Y = (\mathbf{Y}, y), \mathbf{Y} = (y_1, \dots, y_{d-1}) \in U^{d-1}, y = y_d \in [0, 1), \\ &A = (\mathbf{A}, a), \mathbf{A} = (a_1, \dots, a_{d-1}) \in I_s^{d-1}, a = a_d \in I_s. \end{aligned} \right\} \tag{3.12}$$

With notation (3.12), any function $f \in \mathcal{R}_s^d$ can be written in the following two forms

$$f(Y) = f(\mathbf{Y}, y) = \sum_{\mathbf{A} \in I_s^{d-1}} \Phi_{\mathbf{A}}(y) r_{\mathbf{A}}(\mathbf{Y}), \tag{3.13}$$

where

$$\Phi_{\mathbf{A}}(y) = \sum_{a \in I_s} \lambda_A r_a(y) \tag{3.14}$$

and

$$f(Y) = f(\mathbf{Y}, y) = \sum_{a \in I_s} \varphi_a(\mathbf{Y}) r_a(y), \tag{3.15}$$

where

$$\varphi_a(\mathbf{Y}) = \sum_{\mathbf{A} \in I_s^{d-1}} \lambda_A r_{\mathbf{A}}(\mathbf{Y}). \tag{3.16}$$

Lemma 3.1. *For each function $f \in \mathcal{R}_s^d$, we have the following bounds*

$$\|f\|_{s,q} \le \beta_q^{d-1} Q_{\infty,2}[f], \quad 0 < q < \infty, \tag{3.17}$$

where

$$Q_{\infty,2}[f] = \max_{y\in\mathbb{Q}(2^s)} \left(\sum_{\mathbf{A}\in I_s^{d-1}} \Phi_{\mathbf{A}}(y)^2 \right)^{1/2}, \tag{3.18}$$

and

$$\|f\|_{s,q} \ge \alpha_q^d Q_2[f], \tag{3.19}$$

where $Q_2[f]$ is defined in (3.9).

Proof. Applying the right inequality (3.8) with $k = d-1$ to function (3.13), we obtain (3.17). Bound (3.19) is just the left inequality (3.8) with $k = d$. $\square$

Lemma 3.1 will be used in the proof of Theorems 1.1 and 1.2. For the proof of Theorem 1.3 the following more specific result will be needed. This result can be thought of as a modification of Khinchin's inequality for the L_∞-norm.

Lemma 3.2. *For each function $f \in \mathcal{R}_s^d$, we have the following bound*

$$\|f\|_{s,\infty} \ge \alpha_1^{d-1} Q_{1,2}[f], \tag{3.20}$$

where

$$Q_{1,2}[f] = \sum_{a\in I_s} Q_2[\varphi_a], \tag{3.21}$$

$$Q_2[\varphi_a] = \left(\sum_{\mathbf{A}\in I_s^{d-1}} \lambda_A^2 \right)^{1/2}. \tag{3.22}$$

Proof. First of all, we observe that relations (3.4) imply the following equality for each one-dimensional function $\varphi \in \mathcal{R}_s$. Let

$$\varphi(y) = \sum_{a\in I_s} \varphi_a r_a(y), \quad y \in [0,1),$$

then, we have

$$\|\varphi\|_{s,\infty} = \sum_{a\in I_s} |\varphi_a|. \tag{3.23}$$

Indeed, we can assume always that $\varphi_0 \geq 0$, and in view of relations (3.4), there exists a point $y_0 \in \mathbb{Q}(2^s)$ such that $r_a(y_0) = \operatorname{sign} \varphi_a$ if $\varphi_a \neq 0$, $a \in I_s$. Therefore,

$$\|\varphi\|_{s,\infty} \geq |\varphi(y_0)| = \sum_{a \in I_s} |\varphi_a|.$$

The opposite inequality is obvious, and (3.23) follows.

Applying equality (3.23) to function (3.15), we obtain

$$\begin{aligned}
\|f\|_{s,\infty} &= \max_{\mathbf{Y} \in \mathbb{Q}^{d-1}(2^s)} \max_{y \in \mathbb{Q}(2^s)} |f(\mathbf{Y}, y)| \\
&= \max_{\mathbf{Y} \in \mathbb{Q}^{d-1}(2^s)} \sum_{a \in I_s} |\varphi_a(\mathbf{Y})| \\
&\geq 2^{-(d-1)s} \sum_{\mathbf{Y} \in \mathbb{Q}^{d-1}(2^s)} \sum_{a \in I_s} |\varphi_a(\mathbf{Y})| = \sum_{a \in I_s} \|\varphi_a(\cdot)\|_{s,1} \\
&\geq \alpha_1^{d-1} \sum_{a \in I_s} Q_2[\varphi_a] = \alpha_1^{d-1} Q_{1,2}[f],
\end{aligned}$$

where, on the last step, we used the left inequality (3.8) with $k = d-1$ and $q = 1$.

The proof of Lemma 3.2 is complete. $\square$

4. Rademacher functions and explicit formulas for discrepancies

For an arbitrary point $y \in [0,1)$ with dyadic expansion (1.4), we denote by $y^{(s)}$ its projection to $\mathbb{Q}(2^s)$:

$$y^{(s)} = \sum_{a=1}^{s} \eta_a(y) 2^{-a}, \quad s \in \mathbb{N}, \tag{4.1}$$

and for $s = 0$ we put $y^{(0)} = 0$, so that

$$y = y^{(s)} + \theta_s(y) 2^{-s}, \quad s \in \mathbb{N}_0, \tag{4.2}$$

where $\theta_s(y) \in [0,1)$ for all $y \in [0,1)$.

We put

$$\delta^{(s)}(x,y) = \begin{cases} 1, & \text{if} \quad x^{(s)} = y^{(s)} \\ 0, & \text{if} \quad x^{(s)} \neq y^{(s)} \end{cases} \tag{4.3}$$

It follows from (1.4) and (4.1) that elementary intervals Δ_s^m, $m = 0, 1, \ldots, 2^s-1$, see (1.9), can be written in the form

$$\Delta_s^m = [m2^{-s}, (m+1)2^s) = \{z \in [0,1) : z^{(s)} = m2^{-s}\}.$$

Therefore,

$$\delta^{(s)}(x,y) = \delta^{(s)}(x^{(s)} \oplus y^{(s)}) = \chi(\Delta_s^0, x^{(s)} \oplus y^{(s)}) \tag{4.4}$$

and

$$\delta^{(s)}(x^{(s)} \oplus y^{(s)}) = \sum_{m=0}^{2^s-1} \chi(\Delta_s^m, x)\chi(\Delta_s^m, y) \tag{4.5}$$

Hereinafter, we write $\chi(\mathcal{E}, \cdot)$ for the characteristic function of a set $\mathcal{E}$. Notice that

$$\chi(\Delta_s^m, x) = \chi(\Delta_s^m, x^{(s)}) = \chi(\Delta_s^m x^{(a)}) \tag{4.6}$$

for any $a \geq s$.

It follows from (4.4) and (4.5) that

$$\delta^{(s)}(x^{(s)} \oplus y^{(s)}) = \sum_{z \in \mathbb{Q}(2^s)} \delta^{(s)}(x^{(s)} \oplus z)\delta^{(s)}(z \oplus y^{(s)}).$$

Furthermore, $\delta^{(s)}(x^{(s)} \oplus y^{(s)})$ is the reproducing kerner for the space $\mathcal{B}_s$:

$$\begin{aligned} f(x) &= \sum_{y \in \mathbb{Q}(2^s)} \delta^{(s)}(x^{(s)}(x^{(s)} \oplus y^{(s)})f(y) \\ &= 2^s \int_0^1 \delta^{(s)}(x^{(s)} \oplus y^{(s)})f(y)dy, \quad f \in \mathcal{B}_s. \end{aligned} \tag{4.7}$$

Consider the following elementary intervals

$$\Pi_a = \Delta_a^1 = [2^{-a}, 2^{1-a}), \quad a \in \mathbb{N}. \tag{4.8}$$

It is convenient to put $\Pi_0 = [0,1)$.

In terms of dyadic expansion (1.4), intevals (4.8) can be described as follows

$$\Pi_a = \{z \in [0,1) : \eta_a(z) = 1, \eta_i(z) = 0 \quad \text{for} \quad i < a\}. \tag{4.9}$$

Notice that for each $s \in \mathbb{N}$ the set of intervals $\{\Pi_a, a > s\}$ form a partition of the open interval $(0, 2^{-s})$.

The following result is of crucial importance in the subsequent consideration.

Lemma 4.1. *For each $s \in \mathbb{N}$, the characteristic function $\chi([0,y),\cdot)$ of the interval $[0,y), y \in [0,1)$, has the following representation*

$$\chi([0,y),x) = \chi^{(s)}([0,y),x) + \varepsilon^{(s)}(x,y), \tag{4.10}$$

where

$$\chi^{(s)}([0,y),x) = \frac{1}{2} - \frac{1}{2}\sum_{a=1}^{s} \chi(\Pi_a, x^{(s)} \oplus y^{(s)}) r_a(y), \tag{4.11}$$

and for all x, $y \in [0,1)$, the following bounds hold

$$0 \leq \chi^{(s)}([0,y),x) \leq 1 \tag{4.12}$$

and

$$|\varepsilon^{(s)}(x,y)| \leq \frac{1}{2}\delta^{(s)}(x^{(s)} \oplus y^{(s)}). \tag{4.13}$$

Proof. We will check the statements of the lemma for all possible arrangements of points x and y.

If $x = y$, then $\chi([0,y),y) = 0$, $\chi^{(s)}(0,y),y) = 1/2$, $\varepsilon^{(s)}(y,y) = -1/2$, and bounds (4.12), (4.13) are true.

If $x \neq y$, we put

$$\nu = \nu(x,y) = \min\{a \in \mathbb{N} : \eta_a(x) \neq \eta_a(y)\}.$$

With (4.2), we obtain

$$y - x = (\eta_\nu(y) - \eta_\nu(x))2^{-\nu} + (\theta_\nu(y) - \theta_\nu(x))2^{-\nu}, \tag{4.14}$$

where $\eta_\nu(x) \neq \eta_\nu(y)$ and $0 \leq |\theta_\nu(y) - \theta_\nu(x)| < 1$.

From (4.14), we conclude the following:

(i) $x < y$, if and only if $\eta_\nu(y) = 1$ and $\eta_\nu(x) = 0$;

(ii) $x > y$, if and only if $\eta_\nu(y) = 0$ and $\eta_\nu(x) = 1$.

Furthemore, from (4.9) we conclude that

$$\chi(\Pi_a, x^{(a)} \oplus y^{(a)}) = \begin{cases} 1, & \text{if} \quad a = \nu, \\ 0, & \text{if} \quad a \neq \nu. \end{cases} \tag{4.15}$$

The said above can be expesed by the following explicit formulas

$$\chi([0,y),x) = \chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)})\eta(y) = \frac{1}{2}\chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)})(1 - r_\nu(y))$$
$$= \frac{1}{2} - \chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)}) r_\nu(y). \tag{4.16}$$

Now, taking formulas (4.16) and (4.15) into account, we consider the following two opportunities:

(i) $\nu \leq s$; in this case, equality (4.10) holds with $\varepsilon^{(s)}(x,y) = 0$, and bounds (4.12), (4.13) are obvious;

(ii) $\nu > s$; in this case, equality (4.10) holds with $\chi^{(s)}([0,y),x) = \frac{1}{2}$ and $\varepsilon^{(s)}(x,y) = -\frac{1}{2}\chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)}) r_\nu(y)$, and bound (4.12) is obvious, while bound (4.13) is true because $\Pi_\nu \subset \Delta_s^0$ and, therefore,

$$\chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)}) \leq \chi(\Delta_s^0, x^{(s)} \oplus y^{(s)}) = \delta^{(s)}(x^{(s)} \oplus y^{(s)}),$$

cf. (4.4), (4.6).

The proof of Lemma 4.1 is complete. □

We emphasize that relations (4.16) and (4.15) imply the following explicit formulas

$$\chi([0,y),x) = \sum_{a \in \mathbb{N}} \chi(\Pi_a, x^{(a)} \oplus y^{(a)}) \eta_a(y)$$
$$= \frac{1}{2} - \sum_{a \in \mathbb{N}} \chi(\Pi_a, x^{(a)} \oplus y^{(a)}) r_a(y) - \delta(x,y), \tag{4.17}$$

where $\delta(x,y) = 1$ if $x = y$ and is equal to 0 otherwise.

Furthermore, for any x and y the sums in (4.17) contain at most one nonzero term. In this sense, one can say that series (4.17) converge for all x and y, while the convergence is not uniform. Lemma 4.1 shows how to deal with such series: although the error terms $\varepsilon^{(s)}$ in (4.10) are not small, they are concentrated on small subsets.

Consider the multidemensional extention of the above result. For an arbitrary point $Y = (y_1, \dots, y_d) \in U^d$ we denote by $Y^{(s)} = (y_1^{(s)}, \dots, y_d^{(s)})$ its projection to $\mathbb{Q}^d(2^s)$, so that

$$Y = Y^{(s)} + \Theta_s(Y) 2^{-s}, \quad s \in \mathbb{N}_0,$$

where

$$\Theta_s(Y) = (\theta_s(y_1), \dots, \theta_s(y_d)) \in U^d. \tag{4.18}$$

Introduce elementary boxes of the form

$$\Pi_A = \Pi_{a_1} \times \dots \times \Pi_{a_d}, \quad A = (a_1, \dots, a_d) \in \mathbb{N}_0^d. \tag{4.19}$$

Each such box has volume $\operatorname{vol} \Pi_A = 2^{-a_1 - \dots - a_d}$.

We write $\varkappa(A)$ for the number of nonzero elements in $A = (a_1, \dots, a_d) \in \mathbb{N}_0^d$.

Multiplying formulas (4.10) with $x = x_j$, $y = y_j$, $j = 1, \dots, d$ (recall that $r_0(y) \equiv 1$ and $\Pi_0 = [0, 1)$), we obtian the following result

Lemma 4.2. *For each $s \in \mathbb{N}$, the characteristic function $\chi(B_Y, X)$ of the rectangular box $B_Y = [0, y_1) \times \dots \times [0, y_d)$, $Y \in U^d$, has the following representation*

$$\chi(B_Y, X) = \chi^{(s)}(B_Y, X) + \varepsilon^{(s)}(X, Y), \tag{4.20}$$

where

$$\chi^{(s)}(B_y, X) = 2^{-d} \sum_{A \in I_s^d} (-1)^{\varkappa(A)} \chi(\Pi_A, X^{(s)}) r_A(Y), \tag{4.21}$$

and for all $X = (x_1, \dots, x_d)$, $Y = (y_1, \dots, y_d) \in U^d$, the following bounds hold

$$0 \le \chi^{(s)}(B_Y, X) \le 1 \tag{4.22}$$

and

$$|\varepsilon^{(s)}(X, Y)| \le \frac{1}{2} \sum_{j=1}^{d} \delta^{(s)}(x_j^{(s)} \oplus y_j^{(s)}). \tag{4.23}$$

Proof. By definition

$$\chi^{(s)}(B_y, X) = \prod_{j=1}^{d} \chi^{(s)}([0, y_j), x_j,$$

and bound (4.22) follows from (4.12).

Using coordinates (3.12), we obtain

$$\begin{aligned}\chi(B_y, X) &= \chi(B_{\mathbf{Y}}, \mathbf{X})\chi([0,y), x) \\ &= \chi^{(s)}(B_{\mathbf{Y}}, \mathbf{X}) + \varepsilon^{(s)}(\mathbf{X}, \mathbf{Y}))(\chi^{(s)}([0,y), x) + \varepsilon^{(s)}(x, y)) \\ &= \chi^{(s)}(B_Y, X) + \varepsilon^{(s)}(X, Y),\end{aligned}$$

where

$$\varepsilon^{(s)}(X, Y) = \varepsilon^{(s)}(\mathbf{X}, \mathbf{Y})\chi^{(s)}([0,y), x) + \varepsilon^{(s)}(x, y)\chi(B_Y, X).$$

Therefore,

$$|\varepsilon^{(s)}(X, Y)| \leq |\varepsilon^{(s)}(\mathbf{X}, \mathbf{Y})| + |\varepsilon^{(s)}(x, y)|. \tag{4.24}$$

In the one-dimensional case bound (4.23) is given in (4.13). Using (4.24), we obtain bound (4.23) in all dimensions by induction on d. □

Multiplying formulas (3.2) with $y = y_j$, $j = 1, \ldots, d$, we obtain

$$y_1 \ldots y_d = 2^{-d} \sum_{A \in \mathbb{N}_0^d} (-1)^{\varkappa(A)} 2^{-a_1 - \cdots - a_d} r_A(Y)$$

Since $\operatorname{vol} B_Y = y_1 \ldots y_d$ and $\operatorname{vol} \Pi_A = 2^{-a_1 - \cdots - a_d}$, this formula can be writen in the form

$$\begin{aligned}\operatorname{vol} B_Y &= 2^{-d} \sum_{A \in \mathbb{N}_0^d} (-1)^{\varkappa(A)} \operatorname{vol} \Pi_A \, r_A(Y) \\ &= \operatorname{vol}^{(s)} B_Y + \varepsilon^{(s)}(Y), \quad s \in \mathbb{N}_0,\end{aligned} \tag{4.25}$$

where

$$\operatorname{vol}^{(s)} B_Y = 2^{-d} \sum_{A \in I_s^d} (-1)^{\varkappa(A)} \operatorname{vol} \Pi_A \, r_A(Y), \tag{4.26}$$

and $\varepsilon^{(s)}(Y)$ satisfies the bound

$$|\varepsilon^{(s)}(Y)| \leq d 2^{-s-1}, \quad Y \in U^d, \tag{4.27}$$

that can be easily proved by induction on d.

The local discrepancy (1.1) can be written in the form

$$\mathcal{L}[D, Y] = \sum_{X \in D} \mathcal{L}(X, Y), \quad \mathcal{L}(X, Y) = \chi(B_Y, X) - \operatorname{vol} B_Y. \tag{4.28}$$

Substituting formulas (4.20) and (4.25) to (4.28), we obain

$$\mathcal{L}(X,Y) = \mathcal{L}^{(s)}(X,Y) + \mathcal{E}^{(s)}(X,Y), \tag{4.29}$$

where

$$\mathcal{L}^{(s)}(X,Y) = 2^{-2} \sum_{A \in I_s^d} (-1)^{\varkappa(A)} \lambda_A(X^{(s)} \oplus Y^{(s)}) r_A(Y),$$
$$\lambda_A(X^{(s)}) \oplus Y^{(s)}) = \chi(\Pi_A, X^{(s)} \oplus) - \operatorname{vol} \Pi_A$$

and

$$\mathcal{E}^{(s)}(X,Y) = \varepsilon^{(s)}(X,Y) - \varepsilon^{(s)}(Y).$$

In view of bounds (4.23) and (4.27), we have

$$|\mathcal{E}^{(s)}(X,Y)| \le \frac{1}{2} \left(\sum_{j=1}^{d} \delta^{(s)}(x_j^{(s)} \oplus y_j^{(s)}) + d2^{-s} \right), \quad X,Y \in U^d.$$

For an arbitrary distribution $D \subset U^d$, we denote by $D^{(s)}$ its projection to $\mathbb{Q}^d(2^s)$:

$$D^{(s)} = \{X^{(s)} : X \in D\}, \quad s \in \mathbb{N}_0,$$

so that, $|D^{(s)}| = |D|$, while some points of $D^{(s)}$ may coincide.

We define the *microlocal discrepancies* by

$$\begin{aligned} \lambda_A[D^{(s)} \oplus Y^{(s)}] &= \sum_{X \in D} \lambda_A(X^{(s)} \oplus Y^{(s)}) = \sum_{X \in D} (\chi(\Pi_A, X^{(s)} \oplus Y^{(x)}) - \operatorname{vol} \Pi_A) \\ &= |(D^{(s)} \oplus Y^{(s)}) \cap \Pi_A| - |D| \operatorname{vol} \Pi_A, \end{aligned} \tag{4.30}$$

Substituting (4.29) to (4.28), we arrive at the following rusult summarizing the above discussion.

Lemma 4.3. *For each $s \in \mathbb{N}$, the local discrepancy $\mathcal{L}[D,Y]$ has the following representation*

$$\mathcal{L}[D,Y] = \mathcal{L}^{(s)}[D,Y] + \mathcal{E}^{(s)}[D,Y], \tag{4.31}$$

where

$$\mathcal{L}^{(s)}[D,Y] = 2^{-d} \sum_{A \in I_s^d} (-1)^{\varkappa(A)} \lambda_A[D^{(s)} \oplus Y^{(s)}]\, r_A(Y), \tag{4.32}$$

and the term $\mathcal{E}^{(s)}[D,Y]$ satisfies the bound

$$|\mathcal{E}^{(s)}[D,Y]| \leq \frac{1}{2}\left(\sum_{j=1}^{d} \delta^{(s)}[D^{(s)} \oplus Y^{(s)}] + d|D|2^{-s}\right), \tag{4.33}$$

where

$$\delta^{(s)}[D^{(s)} \oplus Y^{(s)}] = \sum_{X \in D} \delta_s(x_j^{(s)} \oplus y_j^{(s)}). \tag{34}$$

5. Explicit formulas and preliminary bounds for the mean discrepancies

Applying Lemma 4.3 to a shifted distribution $D \oplus T$, $T \in \mathbb{Q}^d(2^s)$, we obtain

$$\mathcal{L}[D \oplus T, Y] = \mathcal{L}^{(s)}[D \oplus T, Y] + \mathcal{E}^{(s)}[D \oplus T, Y], \tag{5.1}$$

where the term $\mathcal{L}^{(s)}[D \oplus T, Y]$ can be written in the form

$$\mathcal{L}^{(s)}[D \oplus T, Y] = \mathcal{F}^{(s)}[D, T \oplus Y^{(s)}, Y^{(s)}], \tag{5.2}$$

$$\mathcal{F}^{(s)}[D, Z, Y] = 2^{-d} \sum_{A \in I_s^d} (-1)^{\varkappa(A)} \lambda_A[D \oplus Z]\, r_A(Y), \tag{5.3}$$

$$\begin{aligned}\lambda_A[D \oplus Z] &= \sum_{X \in D} (\chi(\Pi_A, X^{(s)} \oplus Z) - \operatorname{vol}\Pi_A) \\ &= |(D \oplus Z)| \cap \Pi_A| - |D| \operatorname{vol}\Pi_A, \quad Z \in \mathbb{Q}^d(2^s).\end{aligned} \tag{5.4}$$

Let $L_q(\mathbb{Q}^d(2^s) \times U^d)$, $0 < q \leq \infty$, be the space consisting of all functions $f(T,Y)$, $T \in \mathbb{Q}^d(2^s)$, $Y \in U^d$, with $|||f|||_q < \infty$, where

$$|||f|||_q = \left(2^{-ds} \sum_{T \in \mathbb{Q}^d(2^s)} \int_{U^d} |f(T,Y)|^q dY\right)^{1/q}, \quad 0 < q < \infty,$$

$$|||f|||_\infty = \max_{T \in \mathbb{Q}^d(2^s)} \sup_{Y \in U^d} |f(T,Y)|.$$

For any two functions $f_1, f_2 \in L_q(\mathbb{Q}^d(2^s) \times U^d)$, we have

$$|||f_1 + f_2|||_q \leq |||f_1|||_q + |||f_2|||_q, \quad 1 \leq q \leq \infty, \tag{5.5}$$

$$|||f_1 + f_2|||_q^q \leq |||f_1|||_q^q + |||f_2|||_q^q, \quad 0 < q \leq 1. \tag{5.6}$$

For $1 \le q < \infty$, relation (5.5) is the standard Minkowski's inequality, while (5.6) is its modification for $0 < q < 1$, see [16, Chap. 1, Ineqs. (9.11), (9.13)].

With these notations, we put

$$\mathcal{M}_q^{(s)}[D] = |||\mathcal{L}^{(s)}[D \oplus .,.]\,|||_q, \quad 0 < q \le \infty, \tag{5.7}$$

and

$$\mathcal{E}_q^{(s)}[D] = |||\,\mathcal{E}^{(s)}[D \oplus .,.]\,|||_q, \quad 0 < q \le \infty. \tag{5.8}$$

Substituting (5.1) to definition (1.7) and using (5.7), we obtain the upper bound

$$\mathcal{M}_{s,q}[D] \le \mathcal{M}_q^{(s)}[D] + \mathcal{E}_q^{(s)}[D], \quad 1 \le q < \infty \tag{5.9}$$

and, for $0 < q \le 1$, we can merely put

$$\mathcal{M}_{s,q}[D] \le \mathcal{M}_{s,1}[D] \le \mathcal{M}_1^{(s)}[D] + \mathcal{E}_1^{(s)}[D], \quad 0 < q \le 1. \tag{5.10}$$

Similarly, using (5.6), we obtain the lower bound

$$\mathcal{M}_{s,q}[D]^q \ge \mathcal{M}_q^{(s)}[D]^q - \mathcal{E}_q^{(s)}[D]^q, \quad 0 < q \le 1. \tag{5.11}$$

Bounds (5.9), (5.10) and (5.11) will be used in the proofs of Theorems 1.1 and 1.2, correspondingly.

It follows from formulas (5.2) and (5.3) that $\mathcal{L}^{(s)}[D \oplus T, Y]$ as a function of $Y \in U^d$ belongs to the space $\mathcal{B}_s^d$. Hence, we can use equality (3.7), and write (5.7) in the form

$$\begin{aligned}\mathcal{M}_q^{(s)}[D] &= \left(2^{-ds} \sum_{T \in \mathbb{Q}^d(2^s)} |\mathcal{L}^{(s)}[D \oplus T, .]\,|_{s,q}^q\right)^{1/q} \\ &= \left(2^{-2ds} \sum_{T,Y \in \mathbb{Q}^d(2^s)} |\mathcal{L}^{(s)}[D \oplus T, Y]\,|^q\right)^{1/q}, \quad 0 < q < \infty \end{aligned}\tag{5.12}$$

The following simple observation explains why the mean L_q-discrepancies can be expressed in terms of Rademacher series.

In the vector space of pairs $(T, Y) \in \mathbb{Q}^d(2^s) \times \mathbb{Q}^d(2^s) \simeq \mathbb{F}_2^{2ds}$, we consider the following linear mapping

$$\tau : (T, Y) \to (T \oplus Y, Y) \tag{5.13}$$

Obviously, $\tau^2 = \mathbb{1}$, $\tau^{-1} = r$. Hence, τ is a one-to-one mapping, and in the double sum in (5.12), the variables $Z = T \oplus Y$ and Y can be viewed as independent. As a result, we have

$$\mathcal{M}_q^{(s)}[D] = \left(2^{-ds} \sum_{Z \in \mathbb{Q}^d(2^s)} \mathcal{F}_q^{(s)}[D, Z]^q \right)^{1/q}, \quad 0 < q < \infty \tag{5.14}$$

where

$$\mathcal{F}_q^{(s)}[D, Z] = \left(2^{-ds} \sum_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]|^q \right)^{1/q}. \tag{5.15}$$

Bounds (5.9), (5.10), (5.11) and formulas (5.14), (5.15) will be used in the proof of Theorems 1.1 and 2.2.

In the case of the mean L_∞-discrepancy the above arguments should be slightly modified. First of all, using definitions (1.8) and (1.13), we can write

$$\mathcal{M}_{s,\infty}[D] = \max_{T \in \mathbb{Q}^d(2^s)} \sup_{Y \in U^d} |\mathcal{L}[D \oplus T, Y]\,| \geq \max_{T, Y \in \mathbb{Q}^d(2^s)} |\mathcal{L}[D \oplus, Y]\,|. \tag{5.16}$$

For $Z, Y \in \mathbb{Q}^d(2^s)$, we put $T = Z \oplus Y$ and

$$\mathcal{F}[D, Z, Y] = \mathcal{L}[D \oplus Z \oplus Y, Y]. \tag{5.17}$$

With this notation, formula (5.1) takes the form

$$\mathcal{F}[D, Z, Y] = \mathcal{F}^{(s)}[D, Z, Y] + \mathcal{E}^{(s)}[D, Z, Y], \tag{5.18}$$

where $\mathcal{F}^{(s)}[D, Z, Y]$ is defined in (5.3) and

$$\mathcal{E}^{(s)}[D, Z, Y] = \mathcal{E}^{(s)}[D \oplus Z \oplus Y, Y]. \tag{5.19}$$

Since τ defined in (5.13) is a one-to one mapping, we have the equality

$$\max_{T, Y \in \mathbb{Q}^d(2^s)} |\mathcal{L}[D \oplus T, Y]\,| = \max_{Z, Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]\,| \tag{5.20}$$

This relation can be continued as follows

$$\begin{aligned} &\max_{Z, Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]\,| = \max_{Z \in \mathbb{Q}^d(2^s)} \max_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]\,| \\ &\geq 2^{-ds} \sum_{Z \in \mathbb{Q}^d(2^s)} \max_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]\,| \geq \mathcal{F}_{1,\infty}^{(s)}[D] - \mathcal{E}_{1,\infty}^{(s)}[D]. \end{aligned} \tag{5.21}$$

where

$$\mathcal{F}_{1,\infty}^{(s)}[D] = 2^{-ds} \sum_{Z \in \mathbb{Q}^d(2^s)} \mathcal{F}_{\infty}^{(s)}[D, Z], \tag{5.22}$$

$$\mathcal{F}_{\infty}^{(s)}[D, Z] = \max_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}^{(s)}[D, Z, Y]\,| \tag{5.23}$$

and

$$\mathcal{E}_{1,\infty}^{(s)}[D] = 2^{-2s} \sum_{Z \in \mathbb{Q}^d(2^s)} \max_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{E}^{(s)}[D, Z, Y]\,|. \tag{5.24}$$

Comparing (5.16), (5.20) and (5.21), we obtain the lower bound

$$\mathcal{M}_{s,\infty}[D] \geq \mathcal{F}_{1,\infty}^{(s)}[D] - \mathcal{E}_{1,\infty}^{(s)}[D]. \tag{5.25}$$

This bound will be used in the proof of Theorem 2.3.

We will call the quantities $\mathcal{M}_q^{(s)}[D]$ and $\mathcal{F}_{1,\infty}^{(s)}[D]$ as the *principal terms* while the $\mathcal{E}_q^{(s)}[D]$ and $\mathcal{E}_{1,\infty}^{(s)}[D]$ as the *eror terms.*

6. Bounds for the error terms and some auxiliary bounds

Lemma 6.1. (i) *Let D_{2^s} be an arbitrary dyadic (δ, s, d)-net. Then, the following bound holds*

$$\mathcal{E}_q^{(s)}[D_{2^s}] \leq d2^{\delta}, \quad 0 < q \leq \infty \tag{6.1}$$

(ii) *Let $D_N \subset U^d$ be an arbitrary N-point distribution. Then, the following bounds hold*

$$\mathcal{E}_q^{(s)}[D_N] \leq dN2^{-s}, \quad 0 < q \leq 1, \tag{6.2}$$

and

$$\mathcal{E}_{1,\infty}^{s}[D_N] \leq dN2^{-s}. \tag{6.3}$$

Proof. The functions $\delta_j^{(s)}[D^{(s)} \oplus Y^{(s)}]$, $j = 1, \ldots, d$, defined in (4.34), belong to the space $\mathcal{B}_s^d$ and satisfy equality (3.7). We put

$$\delta_{j,q}^{(s)}[D] = \|\delta_j^{(s)}[D^{(s)} \oplus .]\,\|_q = \|\delta_j^{(s)}[D^{(s)} \oplus .]\,\|_{s,q}, \quad 0 < 1 \leq \infty. \tag{6.4}$$

Obviously,

$$\delta_{j,q}^{(s)}[D \oplus Z] = \delta_{j,q}^{(s)}[D], \quad Z \in \mathbb{Q}^d(2^s). \tag{6.5}$$

Applying formula (4.5) to definition (4.34), we obtain

$$\delta_j^{(s)}[D^s \oplus Z] = \sum_{m=0}^{2^s-1} N_{j,m} \quad \chi(\Delta_s^m, z_j), \tag{6.7}$$

where

$$N_{j,m} = \sum_{X \in D} \chi(\Delta_s^m, x_j^{(s)}) = |D \cap \Delta_{s,j}^m|,$$

and $\Delta_{s,j}^m$ denotes the following elementary box

$$\Delta_{s,j}^m = \{X = (x_1, \ldots, x_d) \in U^d : x_j \in \Delta_s^m,\, x_i \in [0,1),\, i \neq j\}$$

Notice that $\operatorname{vol} \Delta_{s,j}^m = 2^{-s}$ and for each $j = 1, \ldots, d$ the boxes $\Delta_{s,j}^m$ $m = 0, 1, \ldots, 2^s - 1$, form a partition of the unit cube U^d. Therefore,

$$\sum_{m=0}^{2^s-1} N_{j,m} = N = |D|. \tag{6.8}$$

(i) From (6.7), we obtain the bound

$$\delta_{j,q}^{(s)}[D] \leq \delta_{j,\infty}^{(s)} \leq \max_m N_{j,m}, \quad 0 \leq \infty. \tag{6.9}$$

Using definition (5.8), bound (4.33), and equality (6.5), we obtain

$$\mathcal{E}_q^{(s)}[D \oplus T] \leq \frac{1}{2} \left(\sum_{j=1}^{d} \delta_{j,\infty}^{(s)}[D] + d|D|2^{-s} \right), \quad 0 < q \leq \infty. \tag{6.10}$$

If $D_{2^s}$is an arbitrary (δ, s, d)-net, then $N = 2^s$ and $N_{j,m} \leq 2^\delta$ for all j and m, see (1.11). Comparing bounds (6.9) and (6.10) for such a net, we obtain bound (6.1).

(ii) From (6.7) and (6.8), we obtain the bound

$$\delta_{j,q}^{(s)}[D] \leq \delta_{j,1}^{(s)}[D] = \sum_{j=1}^{2^s-1} N_{j,m} 2^{-s} = N 2^{-s}, \quad 0 < q \leq 1 \tag{6.11}$$

Using definition (5.8), bound (4.33) and equality (6.5), we obtain

$$\mathcal{E}_q^{(s)}[D \oplus T] \leq \mathcal{E}_1^{(s)}[D \oplus T] \leq \frac{1}{2} \left(\sum_{j=1}^{d} \delta_{j,1}^{(s)}[D] + d|D|2^{-s} \right), \quad 0 < q \leq 1. \tag{6.12}$$

If D_N is an arbitrary N-point distribution, then bounds (6.11) and (6.12) imply bound (6.2).

For function (5.19) bound (4.33) takes the form

$$|\mathcal{E}^{(s)}[D,Z,Y]\,| = \,|\mathcal{E}^{(s)}[D\oplus Z\oplus Y,Y]| \le \frac{1}{2}\left(\sum_{j=1}^{d}\delta_j^{(s)}[D^{(s)}\oplus Z]+d|D|2^{-s}\right), \tag{6.13}$$

where the right hand side is independent of Y.

Substituting (6.13) to definition (5.24), we obtain

$$\mathcal{E}_{1,\infty}^{(s)}[D] \le \frac{1}{2}\left(\sum_{j=1}^{d}\delta_{j,1}^{(s)}[D]+d|D|2^{-s}\right). \tag{6.14}$$

For an arbitrary N-point distribution D_N, bound (6.14) implies bound (6.3).

The proof of Lemma 6.1 is complete. □

In the comments to Theorem 2.1 we have mentioned bound (2.3); its proof is given in the following

Lemma 6.2. *For an arbitrary distribution $D \subset U^d$, the following bound holds*

$$\mathcal{L}_q[D] \le \mathcal{L}_\infty[D] \le 2^{ds/q}(\mathcal{L}_q[D]+2\mathcal{E}_\infty^{(s)}[D]), \quad 1\le q<\infty, \tag{6.15}$$

where the term $\mathcal{E}_\infty^{(s)}[D]$ is defined in (5.8).

In particular, for an arbitrary (δ,s,d)-net D_{2^s} and $q=\varepsilon s$, $\varepsilon>0$, bound (6.15) *takes the form*

$$\mathcal{L}_q[D_{2^s}] \le \mathcal{L}_\infty[D_{2^s}] \le 2^{d/\varepsilon}(\mathcal{L}_q[D_{2^s}]+d2^{\delta+1}). \tag{6.16}$$

Proof. It follows from (4.7) that the function

$$\delta^{(s)}(X^{(s)}\oplus Y^{(s)}) = \prod_{j=1}^{d}\delta^{(s)}(x_j^{(x)}\oplus y_j^{(s)})$$

is the reproducing kerner for the space $\mathcal{B}_s^d$:

$$f(X) = \sum_{Y\in\mathbb{Q}^d(2^s)}\delta^{(s)}(X^{(s)}\oplus Y^{(s)})f(Y), \quad f\in\mathcal{B}_s^d. \tag{6.17}$$

Applying Hölder's inequality to the sum in (6.17) and taking (3.7) into account, we obtain

$$\|f\|_\infty = \|f\|_{s,\infty} \le \left(\sum_{Y \in \mathbb{Q}^d(2^s)} |f(Y)|^q \right)^{1/q}$$
$$= 2^{ds/q} \|f\|_{s,q} = 2^{ds/q} \|f\|_q, \quad 1 \le q < \infty.$$

In particular,

$$\|\mathcal{L}^{(s)}[D,.]\|_\infty \le 2^{ds/q} \|\mathcal{L}^{(s)}[D,.]\|_q, \tag{6.18}$$

where the $\mathcal{L}^{(s)}[D,.]$ is defined in (4.32).

On the other hand, we derive from (4.31) and (5.8) that

$$\|\mathcal{L}^{(s)}[D,.]\|_\infty \ge \|\mathcal{L}[D,.]\,\|_\infty - \|\mathcal{E}[D,.]\,\|_\infty \ge \mathcal{L}_\infty[D] - \mathcal{E}^{(s)}_\infty[D]$$

and

$$\|\mathcal{L}^{(s)}[D,.]\|_q \le \|\mathcal{L}[D,.]\,\|_q + \|\mathcal{E}[D,.]\,\|_\infty \le \mathcal{L}_q[D] + \mathcal{E}^{(s)}_\infty[D]$$

Comparing these inequalities with (6.18), we obtain

$$\mathcal{L}_\infty[D] \le 2^{ds/q}(\mathcal{L}_q[D] + \mathcal{E}^{(s)}_\infty[D]) + \mathcal{E}^{(s)}_\infty[D]$$
$$\le 2^{ds/q}(\mathcal{L}_q[D] + 2\mathcal{E}^{(s)}_\infty[D]),$$

that proves the right bound (6.15), while the left bound is obvious.

If D_{2^s} is a (δ, s, d)-net and $q = \varepsilon s$, $\varepsilon > 0$, then substituting bound (6.1) from Lemma 6.1, we obtain (6.16).

The proof of Lemma 6.2 is complete. □

In conclusion of this section, we give one further auxiliary result that will be used in the proofs of Theorems 2.2 and 2.3.

Consider the following subset of the k-dimensional elementary boxes $\Pi_A \subset U^k$, $k \ge 2$, see (4.19),

$$J^k_\sigma(s) = \{\Pi_A : A \in I^k_s,\, \mathrm{vol}\,\Pi_A = 2^{-\sigma}\}, \quad \sigma \in \mathbb{N} \tag{6.19}$$

Lemma 6.3. *If $s \ge \sigma$, then the subset $J^k_\sigma(s) = J^k_\sigma$ is independent of s, and the following bound holds*

$$|J^k_\sigma| \ge \left(\frac{\sigma}{k-1} \right)^{k-1}. \tag{6.20}$$

Proof. Since $\operatorname{vol} \Pi_A = 2^{-a_1-\cdots-a_k}$, subset (6.19) consists of boxes Π_A with

$$A = (a_1, \ldots, a_k) \in I_s^k : a_1 + \cdots + a_k = \sigma \tag{6.21}$$

Each solution of equation (6.21) satisfies $0 \le a_j \le \min\{\sigma, s\}$, $j = 1, \ldots, k$, and for $s \ge \sigma$ the set of all solutions is independent of s.

If $s \ge \sigma$, then for any $(a_1, \ldots, a_{k-1}) \in \mathbb{N}_0^{k-1}$ with $0 \le a_j \le \lfloor \sigma/(k-1) \rfloor$, $j = 1, \ldots, k-1$, the integer $a_k = \sigma - a_1 - \cdots - a_{k-1}$ satisfies $0 \le a_k \le \sigma$. Therefore, $A = (a_1, \ldots, a_k)$ is a solution of (6.21), and

$$|J_\sigma^k| \ge (1 + \lfloor \sigma/(k-1) \rfloor)^{k-1} \ge \left(\frac{\sigma}{k-1} \right)^{k-1}.$$

$\square$

7. Proofs of Theorems 2.1, 2.2 and 2.3

The proof of each of Theorems 2.1, 2.2 and 2.3 consists of two steps. At first, relaying on the bounds for sums of Rademacher functions given in Lemmas 2.1 and 2.2, we establish very good bounds for the principal terms $\mathcal{M}_q^{(s)}[D]$ and $\mathcal{F}_{1,\infty}^{(s)}[D]$. Next, relaying on the upper bounds for the error terms $\mathcal{E}_q^{(s)}[D]$ and $\mathcal{E}_{1,\infty}^{(s)}$ given in Lemma 6.1, we compare the principal terms with the corresponding mean discrepancies $\mathcal{M}_{s,q}[D]$.

Proof of Theorem 2.1. Let D_{2^s} be a (δ, s, d)-net. Applying bound (3.18) from Lemma 3.1 to function (5.3), we obtain the following bound for quantity (5.15)

$$\mathcal{F}_q^{(s)}[D_{2^s}, Z] \le \beta_q^{d-1}\, Q_{\infty,2}[\mathcal{F}^{(s)}], \tag{7.1}$$

where

$$Q_{\infty,2}[\mathcal{F}^{(s)}] = 2^{-d} \max_{y \in \mathbb{Q}(2^s)} \left(\sum_{\mathbf{A} \in I_s^{d-1}} \Phi_{\mathbf{A}}(Z, y)^2 \right)^{1/2}, \tag{7.2}$$

$$\Phi_{\mathbf{A}}(Z, y) = \sum_{a \in I_s} \lambda_A[D_{2^s} \oplus Z] r_a(y), \tag{7.3}$$

and the coefficients $\lambda_A\ [D_{2^s} \oplus Z]$ are defined in (5.4).

For each $Z \in \mathbb{Q}^d(2^s)$ the shift $D_{2^s} \oplus Z$ is a (δ, s, d)-net, and it follows from (1.11) that

$$\lambda_A[D_{2^s} \oplus Z] = 0 \quad \text{if} \quad \operatorname{vol} \Pi_A \geq 2^{\delta - s}.$$

The condition on volumes can be written as

$$\operatorname{vol} \Pi_A = \operatorname{vol} \Pi_{\mathbf{A}} \operatorname{vol} \Pi_a = 2^{-a_1 - \cdots - a_{d-1} - a} \geq 2^{\delta - s}$$

or $a \leq s - \delta - a_1 - \cdots - a_{d-1}$. Therefore, the summation in (7.3) is extended to

$$s \geq a \geq l, \quad l = \max\{0, s - \delta - a_1 - \cdots - a_{d-1} + 1\}.$$

Elementary boxes Π_A are mutually disjoint, and, for a given $\mathbf{A}$, all boxes $\Pi_A = \Pi_{\mathbf{A}} \times \Pi_a$, $s \geq a \geq l$, are embedded to the elementary box $\Pi_{\mathbf{A}} \times \Delta$, where $\Delta = \Delta^0_{l-1}$ if $l \geq 1$ and $\Delta = [0, 1)$ if $l = 0$. In both cases, $\operatorname{vol} \Pi_{\mathbf{A}} \times \Pi_a \leq 2^{\delta - s}$. Hence, $|(D_{2^s} \oplus Z) \cap (\Pi_{\mathbf{A}} \times \Delta)| \leq 2^\delta$ by the definition of (δ, s, d)-nets, see (1.11).

With these bounds, function (7.3) can be estimated as follows

$$\begin{aligned}
|\Phi_{\mathbf{A}}(Z, y)| &\leq \sum_{a=l}^{s} |\lambda_A[D_{2^s} \oplus Z]| \\
&\leq \sum_{a=l}^{s} |(D_{2^s} \oplus Z) \cap \Pi_A| + |D_{2^s}| \sum_{a=l}^{s} \operatorname{vol} \Pi_A \\
&\leq |(D_{2^s} \oplus Z) \cap (\Pi_{\mathbf{A}} \times \Delta)| + 2^s \operatorname{vol}(\Pi_{\mathbf{A}} \times \Delta) \leq 2^{\delta+1}.
\end{aligned}$$

Substituting this bound to (7.2), we obtain

$$Q_{\infty,2}[\mathcal{F}^{(s)}] \leq 2^{-d+\delta+1} |I_s^{d-1}|^{1/2} = 2^{-d+\delta+1} (s+1)^{\frac{1}{2}(d-1)},$$

and, therefore,

$$\mathcal{F}_q^{(s)}[D_{2^s, Z}] \leq \beta_q^{d-1} 2^{-d+\delta+1} (s+1)^{\frac{1}{2}(d-1)}.$$

With this bound, the principal term (5.14) can be estimated as follows

$$\mathcal{M}_q^{(s)}[D_{2^s}] \leq 2^{-d+\delta+1} \lceil \frac{1}{2} q \rceil^{\frac{1}{2}(d-1)} (s+1)^{\frac{1}{2}(d-1)}, \tag{7.4}$$

where bound (3.11) for the constant β_q has been also used.

Substituting bound (7.4) and bound (6.1) from Lemma 6.1 to inequalities (5.9) and (5.10), we obtain

$$\mathcal{M}_{s,q}[D_{2^s}] < 2^{-d+\delta+1}\left(\lceil\frac{1}{2}q\rceil(s+1)\right)^{\frac{1}{2}(d-1)} + d2^{\delta}, \quad 0 < q < \infty.$$

The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. Let $D_N \subset U^d$, $d \geq 2$, be an N-point distribution. Applying bound (3.19) from Lemma 3.1 to function (5.3), we obtain the following bound for quantity (5.15)

$$\mathcal{F}_q^{(s)}[D_N, Z] \geq \alpha_q^d Q_2[\mathcal{F}^{(s)}], \tag{7.5}$$

where

$$Q_2[\mathcal{F}^{(s)}] = 2^{-d}\left(\sum_{A\in I_s^d} \lambda_A[D_N \oplus Z]^2\right)^{1/2}. \tag{7.6}$$

The coefficients $\lambda_A[D_N \oplus Z]$ are defined in (5.4), and it is clear that

$$|\lambda_A[D_N \oplus Z]| \geq \quad \ll N \operatorname{vol}\Pi_A \gg, \tag{7.7}$$

where $\ll t \gg = \min\{|t-n| : n \in \mathbb{Z}\}$ is the distance of a number $t \in \mathbb{R}$ from the set of all integers $\mathbb{Z}$.

With bound (7.7), we have

$$Q_2[\mathcal{F}^{(s)}] \geq 2^{-d}\left(\sum_{A\in I_s^d} \ll N \operatorname{vol}\Pi_A \gg^2\right)^{1/2}. \tag{7.8}$$

Let $\sigma \in \mathbb{N}$ be chosen to satisfy

$$2^{-2} < N2^{-\sigma} \leq 2^{-1},$$

then $\ll N \operatorname{vol}\Pi_A \gg > 2^{-2}$ for all boxes Π_A with $\operatorname{vol}\Pi_A = 2^{-\sigma}$.

Let $s \in \mathbb{N}$ be chosen to satisfy

$$s \geq \sigma = \lceil \log N \rceil + 1 \geq \log N + 1, \tag{7.9}$$

then, using Lemma 6.3 with $k = d$, we can estimate the sum in (7.8) as follows

$$\sum_{A \in I_s^d} \ll N \operatorname{vol} \Pi_A \gg^2 \geq \sum_{A \in J_\sigma^d} \ll N \operatorname{vol} \Pi_A \gg^2$$
$$> 2^{-4} |J_\sigma^d| \geq 2^{-4} \left(\frac{\log N + 1}{d-1} \right)^{d-1} .$$

Sibstituting this bound to (7.8), we obtain

$$Q_2[\mathcal{F}^{(s)}] > 2^{-d-2}(d-1)^{-\frac{1}{2}(d-1)}(\log N + 1)^{\frac{1}{2}(d-1)},$$

and, therefore,

$$\mathcal{F}^{(s)}[D_N, Z] > \alpha_q^d 2^{-d-2}(d-1)^{-\frac{1}{2}(d-1)}(\log N + 1)^{\frac{1}{2}(d-1)}.$$

With this bound, the principal term (5.14) can be estimated as follows

$$\mathcal{M}_q^{(s)}[D_N] > c_q(d)(\log N + 1)^{\frac{1}{2}(d-1)}, \quad 0 < q \leq 1,$$
$$c_q(d) = 2^{-2d/q-d-1}(d-1)^{-\frac{1}{2}(d-1)}, \tag{7.10}$$

where bound (3.10) for the constant α_q has been also used.

Substituting bounded (7.10) and bound (6.2) from Lemma 6.1 to inequality (5.11), we obtain

$$\mathcal{M}_{s,q}[D_N]^q > c_q(d)^q(\log N + 1)^{\frac{1}{2}(d-1)q} - (dN2^{-s})^q$$
$$\geq c_q(d)^q(\log N + 1)^{\frac{1}{2}(d-1)q}(1 - \xi_q(s)), \quad 0 < q \leq 1,$$

where

$$\xi_q(s) = c_q(d)^{-q}(dN2^{-s})^q.$$

Let s be chosen sufficiently large to satisfy $\xi_q(s) \leq 1/2$. To do this, we put

$$s \geq \log N + \frac{2d+1}{q} + \frac{1}{2}(d-1)\log(d-1) + d + 1 + \log d,$$

and in this case the above condition (7.9) will be also satisfied.

As a result, we have

$$\mathcal{M}_{s,q}[D_N] > \gamma_q(d)(\log N + 1)^{\frac{1}{2}(d-1)}, \quad 0 < q \leq 1,$$

where

$$\gamma_q(d) = 2^{-1/q}c_q(d) = 2^{-(2d+1)/q-d-1}(d-1)^{-\frac{1}{2}(d-1)}$$

The proof of Theorem 2.2 is complete. □

Proof of Theorem 2.3. Let $D_N \subset U^d$, $d \geq 3$, be an N-point distribution. Applying bound (3.20) from Lemma 3.2 to function (5.3), we obtain the following bound for quantity (5.23)

$$\mathcal{F}_{\infty}^{(s)}[D_N, Z] \geq \alpha_1^{d-1} Q_{1,2}[\mathcal{F}^{(s)}], \tag{7.11}$$

where

$$Q_{1,2}[\mathcal{F}^{(s)}] = 2^{-d} \sum_{a \in I_s} Q_2[\varphi_a], \tag{7.12}$$

$$Q_2[\varphi_a] = \left(\sum_{\mathbf{A} \in I_s^{d-1}} \lambda_A [D_N \oplus Z]^2 \right)^{1/2}. \tag{7.13}$$

With bound (7.7), we have

$$Q_2[\varphi_a] \geq \left(\sum_{\mathbf{A} \in I_s^{d-1}} \ll N \operatorname{vol} \Pi_A \gg^2 \right)^{1/2}. \tag{7.14}$$

Notice that $\operatorname{vol} \Pi_A = \operatorname{vol} \Pi_A \operatorname{vol} \Pi_a = \operatorname{vol} \Pi_A 2^{-a}$, and define $\sigma_a \in \mathbb{N}$ by

$$2^{-2} < N 2^{-\sigma_a - a} \leq 2^{-1},$$

then $\ll N \operatorname{vol} \Pi_A \gg > 2^{-2}$ for all boxes $\Pi_{\mathbf{A}}$ with $\operatorname{vol} \Pi_{\mathbf{A}} = 2^{-\sigma_a}$.

It is clear that $\sigma_a = \sigma - a$, $0 \leq a \leq \sigma$, where

$$\sigma = \lceil \log N \rceil + 1 \geq \log N + 1.$$

In what follows, we assume that

$$0 \leq a \leq \frac{1}{2}\sigma \quad \text{and} \quad \sigma \geq \sigma_a \geq \frac{1}{2}\sigma.$$

Let $s \in \mathbb{N}$ be chosen to satisfy $s \geq \sigma$, then

$$s \geq \sigma = \sigma_0 > \sigma_1 > \sigma_2 > \dots, \tag{7.15}$$

and, using Lemma 6.3 with $k = d-1$, we can estimate the sum in (7.14) as follows

$$\sum_{\mathbf{A} \in I_s^{d-1}} \ll N \operatorname{vol} \Pi_A \gg^2 \geq \sum_{\mathbf{A} \in J_{\sigma_a}^{d-1}} \ll N \operatorname{vol} \Pi_A \gg^2$$
$$> 2^{-4} |J_{\sigma_a}^{d-1}| \geq 2^{-4} \left(\frac{\sigma_a}{d-2} \right)^{d-2} \geq 2^{-4} \left(\frac{\sigma/2}{d-2} \right)^{d-2}.$$

Hence, for quantities (7.13), we have the bound

$$Q_2[\varphi_a] > 2^{-2}(d-2)^{-\frac{1}{2}(d-2)}(\sigma/2)^{\frac{1}{2}d-1}, 0 \leq a \leq \sigma/2.$$

Substituting this bound to (7.12), we obtain

$$\begin{aligned} Q_{1,2}[\mathcal{F}^{(s)}] &\geq 2^{-d} \sum_{0 \leq a \leq \sigma/2} Q_2[\varphi_a] > 2^{-d-2}(d-2)^{-\frac{1}{2}(d-2)}(\sigma/2)^{\frac{1}{2}d} \\ &\geq 2^{-\frac{3}{2}d-2}(d-2)^{-\frac{1}{2}(d-2)}(\log N + 1)^{\frac{1}{2}d}, \end{aligned}$$

and, therefore,

$$\mathcal{F}^{(s)}_{\infty}[D_N, Z] > \alpha_1^{d-1} 2^{-\frac{3}{2}d-2}(d-2)^{-\frac{1}{2}(d-2)}(\log N + 1)^{\frac{1}{2}d}.$$

With this bound, the principal term (5.22) can be estimated as follows

$$\mathcal{F}^{(s)}_{1,\infty}[D_N] > c_\infty(d)(\log N + 1)^{\frac{1}{2}d}, \quad c_\infty(d) = 2^{-2d}(d-2)^{-\frac{1}{2}(d-2)} \tag{7.16}$$

where bound (3.10) for the constant α_1 has been also used.

Substituting bound (7.16) and bound (6.3) from Lemma 6.1 to inequality (5.25), we obtain

$$\begin{aligned} \mathcal{M}_{s,\infty}[D_N] &> c_\infty(d)(\log N + 1)^{\frac{1}{2}d} - \frac{1}{2}(d+1)N2^{-s} \\ &\geq c_\infty(d)(\log N + 1)^{\frac{1}{2}d}(1 - \xi_\infty(s)), \end{aligned}$$

where

$$\xi_\infty(d) = c_\infty(s)^{-1}(dN2^{-s}).$$

Let s be chosen sufficiently large to satisfy $\xi_\infty(s) \leq \frac{1}{2}$. To do this, we put

$$s \geq \log N + \frac{1}{2}(d-2)\log(d-2) + 2d + \log d,$$

and in this case the above condition (7.15) will be also satisfied.

As a result, we have

$$\mathcal{M}_{s,\infty}[D_N] > \gamma_\infty(d)(\log N + 1)^{\frac{1}{2}d},$$

where

$$\gamma_\infty(d) = \frac{1}{2}c_\infty(d) = 2^{-2d-1}(d-2)^{-\frac{1}{2}(d-2)}.$$

The proof of Theorem 1.3 is complete. $\square$

References

[1] *A panorama of discrepancy theory* (W. W. L. Chen, A. Srivastav, G. Travaglini Eds.) Springer-Verlag, to apppear.

[2] J. Beck , W. W. L. Chen, *Irregularities of distribution*, Cambridge Tracts in Math., vol. 89, Cambridge Unive Press, Cambridge, 1987.

[3] D. Bilyk, M. T. Lacey, *On the small ball inequality in three dimensions*, Duke Math. J., **143**, №1 (2008), 81–115.

[4] D. Bilyk, M. T. Lacey, A. Vagharshakyan, *On the small ball inequality in all dimensions*, J. Funct. Anal., **254** (2008), 2470–2502.

[5] D. Bilyk, *Roth's orthogonal function method in discrepancy theory and some new connection*, in [1] pp. 65–149.

[6] D. Bilyk, M. T. Lacey, *The supremum norm of the discrepancy function: recent results and connections*, Preprint.

[7] W. W. L. Chen, *On irregularities of distrigution.* II, Quart. J. Math. Oxford, Ser. 2, **34** (1983), 257–279.

[8] W. W. L. Chen, M. M. Skriganov, *Upper bounds in classical discrepancy theory*, in [1] pp. 3–64.

[9] Y. S. Chow, H. Teicher, *Probability theory. Independence, interchangeability, martingales*, 3rd edn., Springer-verlag, N. Y., 1997.

[10] B. I. Golubov, A. V. Efimov, V. A. Skvortsov, *Walsh series and transformations. Theory and applications*, Nauka, Moscow, 1987; English transl., Math. Appl. (Soviet Ser.) vol. 64, Kluwer Acad. Publ. Group, Dordrecht, 1991.

[11] J. Matoušek, *Geometric discrepancy. An illustrated guide*, 2nd edn., Springer-Verlag, Berlin, 2010.

[12] G. Peshkir, A. N. Shiryaev, *Khinchin inequality and a martingale extention of the sphere of their action*, Uspekhi Mat. Nauk **50**, no. 5 (1995), 3–62; English transl., Russian Math. Surveys, **50**, no. 5 (1995), 849–904.

[13] M. M. Skriganov, *Khinchin's inequality and Chen's theorem*, Algebra i Analiz, **23**, no. 4 (2011), 179–204; English transl., St. Petersburg Math. J., **23**, no. 4 (2012), 761–778.

[14] M. M. Skriganov, *On mean values of the L_q-discrepancies of point distributions*, Algebra i Analiz, **24**, no. 6 (2012), 196–225; English transl., St. Petersburg Math. J., **24**, no. 6 (2013), 991–1012.

[15] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Ser., No. 30, Princeton Univ. Press, Princeton, NJ, 1970.

[16] A. Zygmund, *Trigonometric series*, vols. 1 and 2, 3rd edn., Cambridge Univ. Press, Cambridge, 2002.