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New exact solutions of the Born-Infeld model [1]

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ABSTRACT

The Lagrangian and Hamiltonian of the Born-Infeld model in the cartesian as well as in the light cone variables are given. Using the auto-Backlund transformation the new solutions of the corresponding nonlinear equation are constructed. In particular, the "dressed" Barbashov-Chernikov's solution is obtained.

Key words: Born-Infeld model, Backlund transformation, solitonic waves.

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The Born-Infeld model (BI) [1], suggested at the beginning of 1930-th, is a nonlinear generalization of the Maxwell electrodynamics (a detailed report on its advantages as well as shortcomings "hot on the trail" was given in [2]). Let us notice in this connection, that a growth of interest to this model is caused by some of its remarkable features such as relativistic invariance, Hamiltonian structure, finiteness of energy, some additional symmetries and by rather recently established property of complete integrability - an existence of a Lax pair [3], which gives a possibility to construct families of solitonic solutions. All this stimulated an extension of the area of its applications, and, the present time, the BI model itself as its generalizations are actively exploited, in particular, in the theory of bosonic strings, superstrings [4] and in the cosmology [5].

The present work is devoted to important aspects of the BI model: Hamiltonian structure (including variant of the equation in the light cone variables) and to construction its new solutions using of the Backlund transformation.

1. The main object of the theory is a real scalar massless field $\phi = \phi(x,t)$, related to the function of two invariants of the Maxwell electrodynamics: $I_1 = (1/2)F_{ik}F^{ik} = (\mathbf{B}^2 - \mathbf{E}^2)/2$ and $I_2 = (1/4)e^{iklm}F_{ik}F_{lm} = \mathbf{EB}$, where F_{ik} is electromagnetic field tensor, e_{iklm} is the Levi-Civita symbol. In the case of (1+1)-dimensional pseudoeuclidean space with the metric $\epsilon_{\mu\nu} = diag(1,-1)$ this connection is given by the following relation: $b^2(I_1 - b^2 I_2^2) = \epsilon_{\mu\nu}\phi_\mu\phi_\nu = \phi_x^2 - \phi_t^2$, where b is the Born constant, which has a sense of a quantity inverse to some "maximal" field $\sim E_0^2$.

A nonlinear equation for the field ϕ is generated either under the condition of vanishing of divergence the vector $G_\mu = \epsilon_{\mu\nu}\phi_\nu/\sqrt{1 + b^2(I_1 - b^2 I_2^2)}$, or, equivalently, by the Born-Infeld action with the Lagrangian density $\mathcal{L} = (1/b^2)\{1 - \sqrt{1 + \phi_x^2 - \phi_t^2}\}$ (for simplicity the constant b will be put below equal to unity):

$$S(\phi) = \int\int \mathcal{L}\, dxdt = \int\int \{1 - \sqrt{1 + \phi_x^2 - \phi_t^2}\}\, dxdt. \tag{1}$$

The equation is written in the form

$$(1 + \phi_x^2)\phi_{tt} - 2\phi_x\phi_t\phi_{xt} - (1 - \phi_t^2)\phi_{xx} = 0. \tag{2}$$

This equation belongs to the hyperbolic class of nonlinear differential equations ([4]), provided that the following condition is valid

$$1 + \phi_x^2 - \phi_t^2 > 0. \tag{3}$$

Additionally we require of rather strong decrease of the function $\phi(x) = \phi(x,0)$ and its derivatives in the sense of Shwartz space. Let us also notice that Eq. (2) has an obvious geometric sense: it represents a minimal graph in the pseudoeuclidean space of variables $\{x,\ t,\ \phi(x,t)\}$ and has a natural analog in the euclidean space as a minimal surface graph in $\mathbb{R}^3$.

Let us put $\pi(x,t) = \partial\mathcal{L}/\partial\phi_t = \phi_t/\sqrt{1 + \phi_x^2 - \phi_t^2}$, then the set of functions $(\phi(x), \pi(x))$, where $\phi(x) = \phi(x,0)$, $\pi(x) = \pi(x,0)$, form a phase space Γ of the dynamical system (2), and it has the Hamiltonian ($\mathcal{H} = \pi\phi_t + \sqrt{1 + \phi_x^2 - \phi_t^2} - 1$ is the Hamiltonian density):

$$H = \int_{-\infty}^{\infty} \mathcal{H} dx = \int_{-\infty}^{\infty} (\pi\phi_t + \sqrt{1 + \phi_x^2 - \phi_t^2} - 1) dx. \tag{4}$$

[4] Also it is necessary to notice, that the BI-equation up to an exchange $t \to y$, where y is the second plane coordinate, coincides with the equation for the velocity potential of two-dimensional presonic gas flow [6].

Introducing the Poisson structure on Γ by the brackets

$$\{\phi(x), \phi(y)\} = \{\pi(x), \pi(y)\} = 0, \quad \{\phi(x), \pi(y)\} = \delta(x-y), \tag{5}$$

one may readily make sure, that the equation of motion (2), will take the Hamiltonian form

$$\pi_t = \{\pi, H\} = -\frac{\delta H}{\delta \phi}. \tag{6}$$

Besides the global momentum of the model is equal to

$$P = -\int_{-\infty}^{\infty} \pi \phi_x dx, \tag{7}$$

while the Lorentz boost generator (a subgroup of the Poincare group) is

$$K = \int_{-\infty}^{\infty} x(\pi\phi_t + \sqrt{1+\phi_x^2 - \phi_t^2} - 1)dx. \tag{8}$$

The Poisson brackets related to the corresponding Lie algebra are the following

$$\{H, P\} = 0, \quad \{H, K\} = P, \quad \{K, P\} = -H, \tag{9}$$

what is identical to relations for these generators in field models including, for example, the models which are described by the sin - Gordon equation [7].

Let us introduce the light cone variables which will be used below $\xi = (t-x)/2, \eta = (t+x)/2$. Equation (2) has the form

$$2\hat{\phi}_{\xi\eta}(\hat{\phi}_\xi\hat{\phi}_\eta - 2) - (\hat{\phi}_\xi^2\hat{\phi}_{\eta\eta} + \hat{\phi}_\eta^2\hat{\phi}_{\xi\xi}) = 0, \tag{10}$$

and

$$\hat{\phi}(\xi, \eta) = \phi(\eta - \xi, \eta + \xi), \ \phi(x, t) = \hat{\phi}(\frac{t-x}{2}, \frac{t+x}{2}). \tag{11}$$

In terms of these variables the action (1) takes the form (with condition of hyperbolicity $\hat{\phi}_\xi\hat{\phi}_\eta < 1$)

$$\hat{S}(\hat{\phi}) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{1 - \sqrt{1 - \hat{\phi}_\xi\hat{\phi}_\eta}\,\} \, d\xi d\eta. \tag{12}$$

The phase space $\hat{\Gamma}$ of the dynamical system (10) is formed now by the canonical variables $\hat{\phi}(\xi)$ and $\hat{\pi}(\xi)$ (under an assumption of its strong decreasing at $\xi \to \pm\infty$), where $\hat{\phi}(\xi) = \hat{\phi}(\xi, \eta = 0)$, $\hat{\pi}(\xi) = (\partial\hat{\mathcal{L}}/\partial\hat{\phi}_\eta)_{|\eta=0}$, $\hat{\mathcal{L}}$ is the Lagrangian density (the integrand of Eq. (12)), and the Hamiltonian turns into the following one:

$$\hat{H} = \int_{-\infty}^{\infty} [\hat{\pi}\hat{\phi}_\eta + 2\sqrt{1 - \hat{\phi}_\xi\hat{\phi}_\eta} - 2] d\xi; \tag{13}$$

with that the Poisson structure on $\hat{\Gamma}$ will has the form

$$\{\hat{\phi}(\xi), \hat{\phi}(\xi')\} = 0, \ \{\hat{\pi}(\xi), \hat{\pi}(\xi')\} = 0, \ \{\hat{\phi}(\xi), \hat{\pi}(\xi')\} = \delta(\xi - \xi'). \tag{14}$$

Following (13) and (14) Eq. (10) can be represented in the Hamiltonian form:

$$\hat{\pi}_\eta = \{\hat{H}, \hat{\pi}\} = -\frac{\delta \hat{H}}{\delta \hat{\phi}}. \tag{15}$$

As the momentum of the model one should take the functional

$$\hat{P} = -\int_{-\infty}^{\infty} \hat{\pi}\hat{\phi}_\eta d\xi, \tag{16}$$

and the boost generator is

$$\hat{K} = \int_{-\infty}^{\infty} \xi(\hat{\pi}\hat{\phi}_\eta + 2\sqrt{1 - \hat{\phi}_\xi\hat{\phi}_\eta} - 2)d\xi. \tag{17}$$

2. Equation (2) possess a Backlund autotransformation, at first time introduced, apparently, in [8]. In order to show this we notice that the differential forms

$$\omega_1 = \frac{\phi_t}{1-\mathcal{L}}dx + \frac{\phi_x}{1-\mathcal{L}}dt \quad \text{and} \quad \omega_2 = \phi_x dx + \phi_t dt \tag{18}$$

are exact and, hence, the relations

$$\Phi_x = \frac{\phi_t}{1-\mathcal{L}}, \qquad \Phi_t = \frac{\phi_x}{1-\mathcal{L}} \tag{19}$$

form the sought transformation. Hence, a new solution of Eq. (2) may be written as a curvilinear integral, independent on the integration curve:

$$\Phi(x,t) = \int_{(x_0,t_0)}^{(x,t)} \frac{\phi_t}{\sqrt{1+\phi_x^2-\phi_t^2}}\,dx + \frac{\phi_x}{\sqrt{1+\phi_x^2-\phi_t^2}}\,dt. \tag{20}$$

In the cone variables the autotransformation (19) will take the form:

$$\hat{\Phi}_\xi = -\frac{\hat{\phi}_\xi}{\sqrt{1-\hat{\phi}_\xi\hat{\phi}_\eta}}, \qquad \hat{\Phi}_\eta = \frac{\hat{\phi}_\eta}{\sqrt{1-\hat{\phi}_\xi\hat{\phi}_\eta}}, \tag{21}$$

and, hence, the new solution will be also represented in the form independent on the integration curve

$$\hat{\Phi}(\xi,\eta) = \int_{(\xi_0,\eta_0)}^{(\xi,\eta)} -\frac{\hat{\phi}_\xi}{\sqrt{1-\hat{\phi}_\xi\hat{\phi}_\eta}}\,d\xi + \frac{\hat{\phi}_\eta}{\sqrt{1-\hat{\phi}_\xi\hat{\phi}_\eta}}\,d\eta \tag{22}$$

(we imply that the integrals in (20) and (22) exist; it is necessary to notice also that the transformations (19) and (21) do not contain any parameter).

3. In the following we shall construct some simplest solutions of the Eq. (2) ((10)), and also will illustrate an application of the formulas (20) and (22).

a). It is obvious that the functions

$$\hat{\phi}(\xi,\eta) = \hat{\phi}_1(\xi) \text{ and } \hat{\phi}(\xi,\eta) = \hat{\phi}_2(\eta), \tag{23}$$

where $\hat{\phi}_1$, $\hat{\phi}_2$ are arbitrary functions, are solutions of Eq. (10). They describe one-dimensional solitary waves, propagating along the characteristics (correspondingly $\xi = const$ and $\eta = const$).

Substituting correspond each of these solutions into (22), one may readily see, that in the result we shall obtain (in view of an invariance of Eq. (10) under a shifts of solution on an arbitrary constant we omit it here and below): $\hat{\Phi}(\xi,\eta) = -\hat{\phi}_1(\xi)$ and $\hat{\Phi}(\xi,\eta) = \hat{\phi}_2(\eta)$ correspondingly. The first case reflect the fact, that Eq. (10) is invariant under a substitution $\phi(\xi,\eta) \to -\phi(\xi,\eta)$, while in the second one - the Backlund transformation leaves the solution unchanged. It is clear, that, for example, a repeated application of this transformation will result in the initial picture.

b). Let us consider a solution of the form: $\hat{\phi}(\xi,\eta) = \hat{\phi}_1(\xi) + \hat{\phi}_2(\eta)$. After its substitution into (10) and integration we shall obtain:

$$\hat{\phi}(\xi,\eta) = \pm\frac{1}{c_0}\ln|\frac{c_0\xi + c_1}{c_0\eta + c_2}|, \tag{24}$$

where c_0, c_1, c_2 are arbitrary constants. From the physical point of view the solutions (24) have the meaning of the nonlinear superposition states of two solitonic waves, propagating along the corresponding characteristics. Substituting, for example, the first of them into (22), and performing integration, we shall have:

$$\hat{\Phi}(\xi,\eta) = \frac{1}{c_0}\ln|\frac{2b_0(c_0\eta + c_2) - 2b_0 - 1 + \sqrt{[2b_0(c_0\eta + c_2) - 1]^2 - 1}}{2a_0(c_0\xi + c_1) - 2a_0 - 1 + \sqrt{[2a_0(c_0\xi + c_1) - 1]^2 - 1}}|, \tag{25}$$

where $a_0 = c_0\eta_0 + c_2$, $b_0 = c_0\xi_0 + c_1$. As well as (24), relation (25) describes (a more complicated) nonlinear superposition of waves on characteristics.

c). Let us consider a solution in the form $\phi(x,t) = \Phi(z)$, $z = x^2 - t^2$. After its substitution into (2) we shall obtain an equation on the function Φ

$$z\Phi_{zz} + \Phi_z + 2z\Phi_z^3 = 0. \tag{26}$$

As a result of its integration we find:

$$\phi_{1,2}(x,t) = \frac{1}{\sqrt{c_1}}\ln|c_1(x^2 - t^2) - 2 \pm \sqrt{c_1^2(x^2 - t^2)^2 - 4c_1(x^2 - t^2)}|, \tag{27}$$

where $c_1 > 0$ is an arbitrary constant. These solutions correct in the region $|x^2 - t^2| > 2/c_1$, describe also (however distinct with respect to (24) and (25)) nonlinear superposition of two opposite directions solitonic waves. However, in this case an application of the transformation (19) gives a rather complicated relation and is not presented here.

d). Let us consider the well know Barbashov-Chernikov solution of the Eq. (2) [9] (see also the monograph [10], where this solution was obtained with the use of a hodograph transformation). It corresponds to linear interaction of two solitonic waves with opposite directions (solitons) and it has the form:

$$\phi(x,t) = \phi_{10}(x - t + \varkappa_1) + \phi_{20}(x + t + \varkappa_2), \tag{28}$$

where ϕ_{10}, ϕ_{20} are arbitrary localized and smooth functions, $\varkappa_1$, $\varkappa_2$ are constants equal

$$\int_{-\infty}^{\infty}(\phi_{20q})^2\,dq \quad \text{and} \quad \int_{-\infty}^{\infty}(\phi_{10s})^2\,ds$$

at $t \to \infty$ and having a sense of initial wave phases.

For construction of a new ("dressed") solution it is convenient to pass to another related to ξ, η, light cone variables of the form $\xi_1 = (x - t + \varkappa_1)/2$, $\eta_1 = (x + t + \varkappa_2)/2$, so, that

$$\phi(x,t)=\hat{\hat{\phi}}\Big(\frac{x-t+\varkappa_1}{2},\frac{x+t+\varkappa_2}{2}\Big),\ \ \hat{\hat{\phi}}(\eta_1,\xi_1)=\phi\Big(\eta_1+\xi_1-\frac{\varkappa_2+\varkappa_1}{2},\eta_1-\xi_1-\frac{\varkappa_2-\varkappa_1}{2}\Big). \quad (29)$$

Using these variables Eq. (10) turns into the next (under the hyperbolicity condition $1+\hat{\hat{\phi}}_{\xi_1}\hat{\hat{\phi}}_{\eta_1}>0$):

$$2\hat{\hat{\phi}}_{\xi_1\eta_1}(2+\hat{\hat{\phi}}_{\xi_1}\hat{\hat{\phi}}_{\eta_1})-(\hat{\hat{\phi}}^2_{\eta_1}\hat{\hat{\phi}}_{\xi_1\xi_1}+\hat{\hat{\phi}}^2_{\xi_1}\hat{\hat{\phi}}_{\eta_1\eta_1})=0, \quad (30)$$

and from (19) will have the corresponding Backlund autotransformation:

$$\hat{\hat{\Phi}}_{\xi_1}=-\frac{\hat{\hat{\phi}}_{\xi_1}}{\sqrt{1+\hat{\hat{\phi}}_{\xi_1}\hat{\hat{\phi}}_{\eta_1}}},\quad \hat{\hat{\Phi}}_{\eta_1}=\frac{\hat{\hat{\phi}}_{\eta_1}}{\sqrt{1+\hat{\hat{\phi}}_{\xi_1}\hat{\hat{\phi}}_{\eta_1}}}. \quad (31)$$

Hence we find a "dressed" Barbashov-Chernikov solution (($\xi_0,\ \eta_0$) is the initial point):

$$\hat{\hat{\Phi}}(\xi_1,\eta_1)=-\int_{(\xi_0,\eta_0)}^{(\xi_1,\eta_1)}\frac{\hat{\hat{\phi}}_{10\xi_1}}{\sqrt{1+\hat{\hat{\phi}}_{10\xi_1}\hat{\hat{\phi}}_{20\eta_1}}}\,d\xi_1+\frac{\hat{\hat{\phi}}_{20\eta_1}}{\sqrt{1+\hat{\hat{\phi}}_{10\xi_1}\hat{\hat{\phi}}_{20\eta_1}}}\,d\eta_1. \quad (32)$$

Let us choose as an example of an application of this relation the functions $\hat{\hat{\phi}}_{10}$ и $\hat{\hat{\phi}}_{20}$ in the form: $\hat{\hat{\phi}}_{10}(\xi_1)=A\sin\xi_1$ и $\hat{\hat{\phi}}_{20}(\eta_1)=B\sin\eta_1$, where A, B are the wave amplitudes (we suppose that they are given positive values). Then the formula (32) will be rewritten as

$$\hat{\hat{\Phi}}(\xi_1,\eta_1)=-\int_{(\xi_0,\eta_0)}^{(\xi_1,\eta_1)}\frac{A\cos\xi_1}{\sqrt{1+C_1\cos\xi_1}}\,d\xi_1+\frac{B\cos\eta_1}{\sqrt{1+C_2\cos\eta_1}}\,d\eta_1, \quad (33)$$

where $C_1=C_1(\eta_0)=AB\cos\eta_0$, $C_2=C_2(\xi_0)=AB\cos\xi_0$. Supposing for definiteness, that $C_1,\ C_2>0$, i.e. $\eta_0,\ \xi_0\in(-\pi/2,\ \pi/2)$ (another combinations of signs are considered in a similar way [5]) and performing the integration, we shall obtain (see, for example, [11]):

$$\hat{\hat{\Phi}}(\xi_1,\eta_1)=-\frac{2A}{C_1\sqrt{1+C_1}}\Big[(1+C_1)E\Big(\frac{\xi_1}{2},k_1\Big)-F\Big(\frac{\xi_1}{2},k_1\Big)\Big]$$

$$(34)$$

$$+\frac{2B}{C_2\sqrt{1+C_2}}\Big[(1+C_2)E\Big(\frac{\eta_1}{2},k_2\Big)-F\Big(\frac{\eta_1}{2},k_2\Big)\Big],\ \ \xi_1\in[max(\xi_0,0),\pi],\ \ \eta_1\in[max(\eta_0,0),\pi],$$

where

$$F(\Psi,k)=\int_0^{\Psi}\frac{d\Psi}{\sqrt{1-k^2\sin^2\Psi}}\ \ \text{и}\ \ E(\Psi,k)=\int_0^{\Psi}\sqrt{1-k^2\sin^2\Psi}\,d\Psi \quad (35)$$

are correspondingly elliptic integrals of the first and second kind, $k_1^2=2C_1/(1+C_1)$, $k_2^2=2C_2/(1+C_2)$, where in this case $0<k_1,\ k_2\le 1$, and, hence, $C_1,\ C_2\le 1$. Taking into account, that at $k\ll 1$ the integrals (35) have asymptotics of the form $F(\Psi,k)=\Psi+(1/4)(\Psi-\sin 2\Psi/2)k^2+O(k^4)$, $E(\Psi,k)=\Psi-(1/4)(\Psi-\sin 2\Psi/2)k^2+O(k^4)$, from (34) we come to the

[5]For example, the solution at $C_1<0$ will be expressed from elliptic integrals of the first and third kind.

conclusion that an asymptotic representation of the "dressed" Barbashov-Chernikov solution at $k_1,\ k_2 \ll 1,\ k_1/k_2 = O(1)$ has the form:

$$\hat{\hat{\Phi}}(\xi_1, \eta_1) = -\frac{A}{\sqrt{1+C_1}}[\xi_1 - \frac{2+C_1}{2(1+C_1)}(\xi_1 - \sin\xi_1)]+$$
$$\frac{B}{\sqrt{1+C_2}}[\eta_1 - \frac{2+C_2}{2(1+C_2)}(\eta_1 - \sin\eta_1)] + O(k_1^2) + O(k_2^2). \tag{36}$$

Let us, also, notice that according to periodicity of the integrand (33), the solution (34) (36) will not be changed under the substitutions $\xi_0 \to \xi_0 + 2\pi m_1$, $\xi_1 \to \xi_1 + 2\pi m_1$, $\eta_0 \to \eta_0 + 2\pi n_1$, $\eta_1 \to \eta_1 + 2\pi n_1$, $m_1,\ n_1 = 0,\ \pm 1,\ \pm 2,\ \ldots$, and, hence, it extends on the exterior of the region, pointed in (34)

From the physical point of view the formula (34) describes a propagation of nonlinear waves along the characteristics on a background of linear ones, while the "dressing" procedure - as it follows from (36) - at small k_1 and k_2 turns to "renormalization" of bare solution amplitudes and linear additional terms.

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