

ПРЕПРИНТЫ ПОМИ РАН

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**On a relation between the basic representation
of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$
and a Schur–Weyl representation of the infinite symmetric group**

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ABSTRACT:

We prove that there is a natural grading-preserving isomorphism of \mathfrak{sl}_2 -modules between the basic module of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ (with the homogeneous grading) and a Schur–Weyl module of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ with a grading defined through the combinatorial notion of the major index of a Young tableau, and study the properties of this isomorphism. The results reveal new and deep interrelations between the representation theory of $\widehat{\mathfrak{sl}}_2$ and the Virasoro algebra on the one hand, and the representation theory of $\mathfrak{S}_{\mathbb{N}}$ and the related combinatorics on the other hand.

Key words: infinite symmetric group, Schur–Weyl representation, affine Lie algebra, Virasoro algebra

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1 Introduction

In this paper we reveal some new and deep interrelations between two well developed branches of representation theory: the representation theory of the infinite symmetric group and that of the affine Lie and Virasoro algebras. Our starting point was the analogy observed in [10] between the decomposition (3) of a so-called Schur–Weyl representation of the infinite symmetric group $\mathfrak{S}_\mathbb{N}$ into irreducibles,

$$\mathcal{X} = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k,$$

and the decomposition (17) of the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ into irreducible representations of the Virasoro algebra Vir ,

$$\mathcal{H}_0 = \bigoplus_{k=0}^{\infty} M_{2k+1} \otimes L(1, k^2);$$

in these formulas Π_k is an irreducible representation of $\mathfrak{S}_\mathbb{N}$, $L(1, k^2)$ is an irreducible representation of Vir , and M_{2k+1} is the $(2k + 1)$ -dimensional irreducible representation of \mathfrak{sl}_2 ; in both cases, the operator algebras generated by the actions of $\mathfrak{S}_\mathbb{N}$ or Vir and \mathfrak{sl}_2 are mutual commutants. This analogy suggested that there should be a natural action of Vir in the $\mathfrak{S}_\mathbb{N}$ -module \mathcal{X} , or, equivalently, a natural action of $\mathfrak{S}_\mathbb{N}$ in the $\widehat{\mathfrak{sl}}_2$ -module \mathcal{H}_0 . The aim of this paper is to describe and study the underlying natural isomorphism of \mathfrak{sl}_2 -modules.

For this, we use the result of B. Feigin and E. Feigin [2] that the level 1 irreducible highest weight representations of $\widehat{\mathfrak{sl}}_2$ can be realized as certain inductive limits of tensor powers $(\mathbb{C}^2)^{\otimes N}$ of the two-dimensional irreducible representation of \mathfrak{sl}_2 . The construction of [2] is based on the notion of the fusion product of representations, whose main ingredient is, in turn, a special grading in the space $(\mathbb{C}^2)^{\otimes N}$. A key observation underlying the results of this paper is that the fusion product under consideration can be realized in an $\mathfrak{S}_\mathbb{N}$ -module so that this special grading essentially coincides with a well-known combinatorial characteristic of Young tableaux called the major index (see Sec. 4 and Theorem 1). Thus our results provide, in particular, a kind of combinatorial description of the fusion product and show that the combinatorial notion of the major index of a Young tableau has new and rich representation-theoretic meaning. For instance, Corollary 3 in Sec. 7 shows that the so-called stable major indices of infinite Young tableaux are the eigenvalues of the Virasoro L_0 operator, the Gelfand–Tsetlin basis of the Schur–Weyl module being its eigenbasis.

The paper is organized as follows. In Secs. 2 and 3 we briefly reproduce the necessary background on the notion of Schur–Weyl duality and the fusion product

of representations, respectively. Section 4 contains our finite-dimensional Theorem 1 with combinatorial interpretation of the fusion product grading via the major index of Young tableaux. In Sec. 5, we prove its infinite-dimensional version, our main Theorem 2, which states that the grading-preserving isomorphism of \mathfrak{sl}_2 -modules constructed in Theorem 1 extends, through the corresponding inductive limits, to a grading-preserving isomorphism of \mathfrak{sl}_2 -modules between the basic $\widehat{\mathfrak{sl}_2}$ -module $L_{0,1}$ and the Schur–Weyl module \mathcal{X} . The remaining part of the paper is devoted to studying the key isomorphism in more detail. With this aim, in Sec. 6 we describe the Fock space realizations of the involved representations of $\widehat{\mathfrak{sl}_2}$ and Vir, and then, in Sec. 7, prove some properties of our isomorphism (Theorem 3).

For definiteness, in what follows we consider only the even case $N = 2n$. The odd case can be treated in exactly the same way; instead of the basic representation $L_{0,1}$, it leads to the other level 1 highest weight representation $L_{1,1}$ of $\widehat{\mathfrak{sl}_2}$.

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2 Infinite-dimensional Schur–Weyl duality

In [10], the notion of infinite-dimensional Schur–Weyl duality was introduced. Namely, starting from the classical Schur–Weyl duality

$$(\mathbb{C}^2)^{\otimes N} = \sum_{k=0}^n M_{2k+1} \otimes \pi_k, \quad (1)$$

where π_k is the irreducible representation of the symmetric group \mathfrak{S}_N corresponding to the two-row Young diagram $\lambda^{(k)} = (n+k, n-k)$ and M_{2k+1} is the $(2k+1)$ -dimensional irreducible representation of the special linear group $SL(2, \mathbb{C})$, we consider so-called Schur–Weyl embeddings $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2)}$ that preserve this Schur–Weyl structure, i.e., respect both the actions of $SL(2, \mathbb{C})$ and \mathfrak{S}_N , and the inductive limits of chains

$$(\mathbb{C}^2)^{\otimes 0} \hookrightarrow (\mathbb{C}^2)^{\otimes 2} \hookrightarrow (\mathbb{C}^2)^{\otimes 4} \hookrightarrow \dots \quad (2)$$

Such an inductive limit has the form

$$\mathcal{X} = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k, \quad (3)$$

where Π_k is an irreducible representation of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ (an inductive limit of the sequence of irreducible representations π_k of \mathfrak{S}_N); the operator algebras generated by the actions of $\mathfrak{S}_{\mathbb{N}}$ and $SL(2, \mathbb{C})$ are mutual commutants.

3 Fusion product

The notion of the fusion product of finite-dimensional representations of \mathfrak{sl}_2 was introduced in [3]. Given an \mathfrak{sl}_2 -representation ρ and $z \in \mathbb{C}$, let $\rho(z)$ be the evaluation representation of the polynomial current algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$, defined as $(x \otimes t^i)v = z^i \cdot xv$ for $x \in \mathfrak{sl}_2$, $v \in \rho$. Now, given a collection ρ_1, \dots, ρ_N of irreducible representations of \mathfrak{sl}_2 with lowest weight vectors v_1, \dots, v_N , and a collection z_1, \dots, z_N of pairwise distinct complex numbers, we consider the tensor product of the corresponding evaluation representations: $V_N = \rho_1(z_1) \otimes \dots \otimes \rho_N(z_N)$. The crucial step is introducing a special grading in V_N by setting

$$V_N^{(m)} = U^{(m)}(e \otimes \mathbb{C}[t])(v_1 \otimes \dots \otimes v_N) \subset V_N,$$

where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the raising operator in \mathfrak{sl}_2 and $U^{(m)}$ is spanned by homogeneous elements of degree m in t . In other words, $V_N^{(m)}$ is spanned by the monomials of the form

$$e_{i_1} \dots e_{i_k}, \quad i_1 + \dots + i_k = m,$$

where $e_j = e \otimes t^j$. Then we consider the corresponding filtration on V_N :

$$V_N^{(\leq m)} = \bigoplus_{k \leq m} V_N^{(k)}.$$

The fusion product of ρ_1, \dots, ρ_N is the graded representation with respect to the above filtration:

$$V_N^* = \text{gr } V_N = V_N^{(\leq 0)} \oplus V_N^{(\leq 1)} / V_N^{(\leq 0)} \oplus V_N^{(\leq 2)} / V_N^{(\leq 1)} \oplus \dots \quad (4)$$

The space $V_N^*[k] = V_N^{(\leq k)} / V_N^{(\leq k-1)}$ is the subspace of elements of degree k , and elements of the form $x \otimes t^l \in \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ send $V_N^*[k]$ to $V_N^*[\widetilde{k+l}]$. The degree of an element with respect to this grading will be denoted by deg .

It is proved in [3] that V_N^* is an $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^n)$ -module that does not depend on z_1, \dots, z_N provided that they are pairwise distinct. Moreover, V_N^* is isomorphic to $\rho_1 \otimes \dots \otimes \rho_N$ as an \mathfrak{sl}_2 -module.

We apply this construction to the case where $\rho_1 = \dots = \rho_N = M_2$ with $M_2 = \mathbb{C}^2$ being the two-dimensional irreducible representation of \mathfrak{sl}_2 with the lowest weight vector v_0 . In this case,

$$V_N^* \simeq (\mathbb{C}^2)^{\otimes N} \quad \text{as an } \mathfrak{sl}_2\text{-module.}$$

We equip V_N^* with the inner product such that the corresponding representation of \mathfrak{sl}_2 is unitary.

Consider the decomposition of V_N^* into irreducible \mathfrak{sl}_2 -modules:

$$V_N^* = \bigoplus_{k=0}^n M_{2k+1} \otimes \mathcal{M}_k.$$

By the classical Schur–Weyl duality (1), we know that the multiplicity space \mathcal{M}_k is the space of the irreducible representation π_k of \mathfrak{S}_N . On the other hand, it inherits the grading from V_N^* :

$$\mathcal{M}_k = \bigoplus_{i \geq 0} \mathcal{M}_k[i], \quad (5)$$

where $\mathcal{M}_k[i] = \mathcal{M}_k \cap V^*[i]$. Consider the corresponding q -character

$$\text{ch}_q \mathcal{M}_k = \sum_{i \geq 0} q^i \dim \mathcal{M}_k[i].$$

It was proved in [5] that

$$\text{ch}_q \mathcal{M}_k = q^{\frac{N(N-1)}{2}} \cdot K_{\lambda_k, 1^N}(1/q), \quad (6)$$

where $K_{\lambda, \mu}$ is the Kostka–Foulkes polynomial (see [7, Sec. III.6]).

4 Major index and the tableaux realization of the fusion product

Let T_N be the set of all standard Young tableaux of length N with at most two rows.

As was proved in [6],

$$K_{\lambda, 1^N}(q) = \sum_{\tau \in [\lambda]} q^{c(\tau)}, \quad (7)$$

where $[\lambda]$ is the set of standard Young tableaux of shape λ and $c(\tau)$ is the so-called charge of a tableau $\tau \in T_N$, defined as the sum of $i \leq N - 1$ such that in τ the element $i + 1$ lies to the right of i (see [7]).

It is more convenient for our purposes to use another statistic on Young tableaux, namely, the major index, defined as follows (see [9, Sec. 7.19]):

$$\text{maj}(\tau) = \sum_{i \in \text{des}(\tau)} i,$$

where, for $\tau \in T_N$,

$$\text{des}(\tau) = \{i \leq N - 1 : \text{the element } i + 1 \text{ in } \tau \text{ lies lower than } i\}$$

is the descent set of τ . Obviously, for $\tau \in T_N$ we have $\text{maj}(\tau) = \frac{N(N-1)}{2} - c(\tau)$. Then it follows from (6) and (7) that

$$\dim \mathcal{M}_k[i] = \#\{\tau \in [(n+k, n-k)] : \text{maj}(\tau) = i\}. \quad (8)$$

Denote by \mathcal{X}_N the space $(\mathbb{C}^2)^{\otimes N} = \sum_{k=0}^n M_{2k+1} \otimes \pi_k$ (see (1)) in which the irreducible representation π_k of \mathfrak{S}_N is realized in the space spanned by the standard Young tableaux of shape $(n+k, n-k)$ equipped with the standard inner product under which the representation is unitary. Note that this is an \mathfrak{sl}_2 -module endowed additionally with the grading maj .

Theorem 1. *There is a grading-preserving unitary isomorphism of the fusion product V_N^* (with the grading deg) and the space \mathcal{X}_N (with the grading maj) as \mathfrak{sl}_2 -modules such that the multiplicity space \mathcal{M}_k is spanned by the standard Young tableaux τ of shape $(n+k, n-k)$ (and hence $\mathcal{M}_k[i]$ is spanned by τ with $\text{maj}(\tau) = i$).*

Proof. Follows from the fact that the fusion product V_N^* is isomorphic to $(\mathbb{C}^2)^{\otimes N}$ as an \mathfrak{sl}_2 -module and equation (8). \square

Remark 1. Observe that the isomorphism from Theorem 1 is not unique.

Remark 2. The isomorphism from Theorem 1 determines an action of the symmetric group \mathfrak{S}_N on the space V_N^* . It does not coincide with the original action of \mathfrak{S}_N on $\mathbb{C}^{\otimes N}$.

Given $\tau \in T_N$, let $k(\tau)$ be half the difference of the lengths of the first and the second row of τ . Then, in view of the Schur–Weyl duality, we can write

$$V_N^* = \bigoplus_{\tau \in T_N} M_{2k(\tau)+1}(\tau),$$

where $M_{2k(\tau)+1}(\tau)$ is the $(2k(\tau) + 1)$ -dimensional \mathfrak{sl}_2 -module parametrized by τ as an element of the multiplicity space.

5 Embeddings and the limit

It is proved in [2] that there is an embedding

$$j_N : V_N^* \rightarrow V_{N+2}^*$$

equivariant with respect to the action of $\mathfrak{sl}_2 \otimes (\mathbb{C}[t^{-1}]/t^{-n})$, and the corresponding inductive limit

$$\mathcal{V} = \lim(V_N, j_N)$$

is isomorphic to the basic representation $L_{0,1}$ of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. This embedding satisfies

$$\widetilde{\deg}(j_N x) = \widetilde{\deg}(x) - (N + 1). \quad (9)$$

Now consider the following natural embedding $i_N : T_N \rightarrow T_{N+2}$: given a standard Young tableau τ of length N , its image $i_N(\tau)$ is the standard Young tableau of length $N + 2$ obtained from τ by adding the element $N + 1$ to the first row and the element $N + 2$ to the second row.

Note that i_N is, obviously, a Schur–Weyl embedding in the sense of [10] (see Sec. 2). Let \mathcal{X} be the corresponding inductive limit (3). Then Π_k is the discrete representation of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ associated with the tableau

$$\tau_k = \begin{array}{cccccccc} 1 & 2 & \dots & 2k & 2k+1 & 2k+3 & \dots & \\ 2k+2 & 2k+4 & \dots & & & & & \end{array}, \quad (10)$$

which can be realized in the space (which, by abuse of notation, will also be denoted by Π_k) spanned by the infinite two-row Young tableaux tail-equivalent to τ_k (we denote the set of such tableaux by \mathcal{T}_k). In what follows, the tableaux τ_k will be called *principal*.

Obviously,

$$\text{maj}(i_N(\tau)) = \text{maj}(\tau) + (N + 1). \quad (11)$$

Given $N = 2n$ and $\tau \in T_N$, denote $r_N(\tau) = n^2 - \text{maj}(\tau)$. Then $r_{N+2}(i_N(\tau)) = r_N(\tau)$, so that we have a well-defined grading on the space $\Pi = \bigoplus_{k=0}^{\infty} \Pi_k$:

$$r(\tau) = \lim_{n \rightarrow \infty} r_{2n}([\tau]_{2n}) = \lim_{n \rightarrow \infty} (n^2 - \text{maj}([\tau]_{2n})), \quad (12)$$

where $[\tau]_l$ is the initial part of length l of the infinite tableau τ . We will call $r(\tau)$ the *stable major index* of τ . Obviously, $r(\tau_k) = k^2$.

Our main theorem is the following.

Theorem 2. *The grading-preserving unitary isomorphism of \mathfrak{sl}_2 -modules described in Theorem 1 extends to a grading-preserving unitary isomorphism of \mathfrak{sl}_2 -modules between the spaces \mathcal{V} and \mathcal{X} :*

$$\mathcal{V} \simeq \mathcal{X} = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k. \quad (13)$$

Thus in the Schur–Weyl module \mathcal{X} , which is an \mathfrak{sl}_2 -module and an \mathfrak{S}_N -module, there is also a structure of the basic $\widehat{\mathfrak{sl}}_2$ -module $L_{0,1}$. The corresponding grading is given by the stable major index (12), that is, for $w = x \otimes \tau \in M_{2k+1} \otimes \Pi_k$, we have $\deg w = r(\tau)$.

Remark. As mentioned in the introduction, we consider in detail only the even case just for simplicity of notation. Considering instead of (2) the chain $(\mathbb{C}^2)^{\otimes 1} \hookrightarrow (\mathbb{C}^2)^{\otimes 3} \hookrightarrow (\mathbb{C}^2)^{\otimes 5} \hookrightarrow \dots$ and reproducing exactly the same arguments, we will obtain a grading-preserving isomorphism of the corresponding Schur–Weyl representation with the other level 1 highest weight representation $L_{1,1}$ of $\widehat{\mathfrak{sl}}_2$.

Proof. Since we are now considering $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}]$ instead of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$, we should slightly modify the previous constructions to take the minus sign into account. Namely, instead of (5) we now have $\mathcal{M}_k = \bigoplus_{i \geq 0} \mathcal{M}_k[-i]$, and the isomorphism of Theorem 1 identifies $\mathcal{M}_k[-i]$ with the space spanned by the tableaux τ of shape $(n+k, n-k)$ such that $\text{maj}(\tau) = i$. Denote this isomorphism between V_N^* and \mathcal{X}_N by ρ_N . Observe that the only conditions we impose on ρ_N are as follows: (a) ρ_N is a unitary isomorphism of \mathfrak{sl}_2 -modules and (b) $\rho_N \circ \deg = -\text{maj}$.

Now, to prove Theorem 2, we need to show that we can choose a sequence of isomorphisms ρ_N such that the diagram

$$\begin{array}{ccc} V_N^* & \xrightarrow{\rho_N} & X_N \\ \downarrow j_N & & \downarrow i_N \\ V_{N+2}^* & \xrightarrow{\rho_{N+2}} & X_{N+2} \end{array}$$

is commutative for all N . We use induction on N . The base being obvious, assume that we have already constructed ρ_N , and let us construct ρ_{N+2} .

We have $V_{N+2}^* = j_N(V_N^*) \oplus (j_N(V_N^*))^\perp$. On the first subspace, we set $\rho_{N+2}(x) := i_N(\rho_N(j_{N+2}^{-1}(x)))$. On the second one, we define it in an arbitrary way to satisfy the desired conditions (a) and (b). The fact that this definition is correct and provides us with a desired isomorphism between V_{N+2}^* and \mathcal{X}_{N+2} follows from (9) and (11). \square

Corollary 1. *The embedding $j_N : V_N^* \rightarrow V_{N+2}^*$ is equivariant with respect to the action of the symmetric group \mathfrak{S}_N (see Remark 2 after Theorem 1). Thus the limit space \mathcal{V} , isomorphic to $L_{0,1}$, has the structure of a representation of the infinite symmetric group $\mathfrak{S}_\mathbb{N}$.*

Let ω_{-2k} be the lowest vector in M_{2k+1} . Then a natural basis of \mathcal{V} is $\{e_0^m \omega_{-2k} \otimes \tau : m = 0, 1, \dots, 2k, \tau \in \mathcal{T}_k\}$. Denoting $\mathcal{V}_k = M_{2k+1} \otimes \Pi_k$ and $\mathcal{V}_k[0] = \{v \in \mathcal{V}_k : h_0 v = 0\}$, we have $\mathcal{V}_k[0] = e_0^k \omega_{-2k} \otimes \Pi_k$, so that we may identify $\mathcal{V}_k[0]$ with Π_k by the correspondence

$$c(t) \cdot e_0^k \omega_{-2k} \otimes t \leftrightarrow t, \quad t \in \Pi_k,$$

where $c(t)$ is a normalizing constant. Thus we have

$$\mathcal{V}[0] := \{v \in \mathcal{V} : h_0 v = 0\} \longleftrightarrow \Pi = \bigoplus_{k=0}^{\infty} \Pi_k, \quad (14)$$

where Π is the space spanned by all infinite two-row Young tableaux with ‘‘correct’’ tail behavior, i.e., tail-equivalent to τ_k (see (10)) for some k .

Our aim in the remaining part of the paper is to study the isomorphism from Theorem 2 in more detail. For this, we first describe the Fock space realization of the basic $\widehat{\mathfrak{sl}}_2$ -module and the fusion product.

6 The Fock space

6.1 The Fock space and the level 1 highest weight representations of $\widehat{\mathfrak{sl}}_2$

Let \mathcal{F} be the fermionic Fock space constructed as the infinite wedge space over the linear space with basis $\{u_k\}_{k \in \mathbb{Z}} \cup \{v_k\}_{k \in \mathbb{Z}}$. That is, \mathcal{F} is spanned by the semi-infinite forms

$$u_{i_1} \wedge \dots \wedge u_{i_k} \wedge v_{j_1} \wedge \dots \wedge v_{j_l} \wedge u_N \wedge v_N \wedge u_{N-1} \wedge v_{N-1} \wedge \dots, \\ N \in \mathbb{Z}, i_1 > \dots > i_k > N, j_1 > \dots > j_l > N,$$

and is equipped with the inner product in which such monomials are orthonormal. Let ϕ_k be the exterior multiplication by u_k and ψ_k be the exterior multiplication by v_k , and denote by ϕ_k^* , ψ_k^* the corresponding adjoint operators. Then this family of operators satisfies the canonical anticommutation relations (CAR):

$$\phi_k \phi_k^* + \phi_k^* \phi_k = 1, \quad \psi_k \psi_k^* + \psi_k^* \psi_k = 1,$$

all the other anticommutators being zero.

Consider the generating functions

$$\phi(z) = \sum_{i \in \mathbb{Z}} \phi_i z^{-(i+1)}, \quad \psi(z) = \sum_{i \in \mathbb{Z}} \psi_i z^{-(i+1)}, \quad \phi^*(z) = \sum_{i \in \mathbb{Z}} \phi_i^* z^i, \quad \psi^*(z) = \sum_{i \in \mathbb{Z}} \psi_i^* z^i.$$

Let a_n^ϕ and a_n^ψ be the systems of bosons constructed from the fermions $\{\phi_k\}$ and $\{\psi_k\}$, respectively:

$$a_0^\phi = \sum_{n=1}^{\infty} \phi_n \phi_n^* - \sum_{n=0}^{\infty} \phi_{-n}^* \phi_{-n}, \quad a_n^\phi = \sum_{k \in \mathbb{Z}} \phi_k \phi_{k+n}^*, \quad n \neq 0,$$

and similarly for a^ψ . They satisfy the canonical commutation relations (CCR)

$$[a_n^\phi, a_m^\phi] = n\delta_{n,-m}, \quad [a_n^\psi, a_m^\psi] = n\delta_{n,-m}, \quad (15)$$

i.e., form a representation of the Heisenberg algebra \mathfrak{A} . Denote

$$a^\phi(z) = \sum_{n \in \mathbb{Z}} a_n^\phi z^{-(n+1)}, \quad a^\psi(z) = \sum_{n \in \mathbb{Z}} a_n^\psi z^{-(n+1)}.$$

Let V be the operator in \mathcal{F} that shifts the indices by 1:

$$V(w_{i_1} \wedge w_{i_2} \wedge \dots) = V_0(w_{i_1}) \wedge V_0(w_{i_2}) \wedge \dots, \quad V_0(u_i) = u_{i+1}, \quad V_0(v_i) = v_{i-1}.$$

The vacuum vector in \mathcal{F} is $\Omega = u_{-1} \wedge v_{-1} \wedge u_{-2} \wedge v_{-2} \wedge \dots$. We also consider the family of vectors

$$\Omega_0 = \Omega, \quad \Omega_{2n} = V^{-n}\Omega_0, \quad n \in \mathbb{Z}.$$

In the space \mathcal{F} we have a canonical representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, which is given by the following formulas. Given $x \in \mathfrak{sl}_2$, denote $X(z) = \sum_{n \in \mathbb{Z}} x_n z^{-(n+1)}$. Then

$$E(z) = \psi(z)\phi^*(z), \quad F(z) = \phi(z)\psi^*(z),$$

$$h_n = a_{-n}^\psi - a_{-n}^\phi, \quad d = \frac{h_0^2}{2} + \sum_{n=1}^{\infty} h_{-n}h_n, \quad c = 1.$$

We have

$$\mathcal{F} = \mathcal{H}_0 \otimes \mathcal{K}_0 + \mathcal{H}_1 \otimes \mathcal{K}_1,$$

where $\mathcal{H}_0 \simeq L_{0,1}$ and $\mathcal{H}_1 \simeq L_{1,1}$ are the irreducible level 1 highest weight representations of $\widehat{\mathfrak{sl}}_2$ and \mathcal{K}_0 and \mathcal{K}_1 are the multiplicity spaces. Observe also that

$$e_{-(N+1)}\Omega_{-N} = \Omega_{-(N+2)}.$$

Note that the operators $a_n = \frac{1}{\sqrt{2}}h_n$ satisfy the CCR (15), i.e., form a system of free bosons, or generate the Heisenberg algebra \mathfrak{A}_h . The vectors $\{\Omega_{2n}\}_{n \in \mathbb{Z}}$ introduced above are exactly singular vectors for this Heisenberg algebra: $h_k \Omega_m = 0$ for $m < 0$, $h_0 \Omega_m = m \Omega_m$. The representation of \mathfrak{A}_h in \mathcal{H}_0 breaks into a direct sum of irreducible representations:

$$\mathcal{H}_0 = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_0[2k], \quad (16)$$

where $\mathcal{H}_0[2k]$ is the charge $2k$ subspace, i.e., the eigenspace of h_0 with eigenvalue $2k$:

$$\mathcal{H}_0[2k] = \{v \in \mathcal{H}_0 : h_0 v = 2kv\} = \mathbb{C}[h_0, h_1, \dots] \Omega_{2k}.$$

6.2 The representation of the Virasoro algebra associated with the basic representation of $\widehat{\mathfrak{sl}}_2$

Given a representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$, we can use the Sugawara construction to obtain the corresponding representation of the Virasoro algebra Vir. It can also be described in the following way. As noted above, the operators $a_n = \frac{1}{\sqrt{2}}h_n$ form a system of free bosons. Given such a system, a representation of Vir can be constructed as follows ([4, Ex. 9.17]):

$$L_n = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j} a_{j+n}, \quad n \neq 0; \quad L_0 = \sum_{j=1}^{\infty} a_{-j} a_j.$$

Thus we obtain a representation of Vir in \mathcal{F} and, in particular, in \mathcal{H}_0 . In this representation, the algebras generated by the operators of Vir and $\mathfrak{sl}_2 \subset \widehat{\mathfrak{sl}}_2$ are mutual commutants, and we have the decomposition

$$\mathcal{H}_0 = \bigoplus_{k=0}^{\infty} M_{2k+1} \otimes L(1, k^2), \quad (17)$$

where M_{2k+1} is the $(2k+1)$ -dimensional irreducible representation of \mathfrak{sl}_2 and $L(1, k^2)$ is the irreducible representation of Vir with central charge 1 and conformal dimension k^2 .

The charge k subspace $\mathcal{H}_0[k]$ contains a series of singular vectors $\xi_{k,m}$ of Vir with energy $(k+m)^2$:

$$L_n \xi_{k,m} = 0 \text{ for } n = 1, 2, \dots, \quad L_0 \xi_{k,m} = (k+m)^2.$$

Let us use the so-called homogeneous vertex operator construction of the basic representation of $\widehat{\mathfrak{sl}}_2$ (see [4, Sec. 14.8]). In this realization,

$$E(z) = \Gamma_-(z)\Gamma_+(z)z^{-h_0}V^{-1}, \quad F(z) = \Gamma_+(z)\Gamma_-(z)z^{h_0}V, \quad (18)$$

where

$$\Gamma_{\pm}(z) = \exp\left(\mp \sum_{j=1}^{\infty} \frac{z^{\pm j}}{j} h_{\pm j}\right)$$

and the operators $\Gamma_{\pm}(z)$ satisfy the commutation relation

$$\Gamma_+(z)\Gamma_-(w) = \Gamma_-(w)\Gamma_+(z) \left(1 - \frac{z}{w}\right)^2. \quad (19)$$

Using the boson–fermion correspondence (see [4, Ch. 14]), we can identify \mathcal{H}_0 with the space $\Lambda \otimes \mathbb{C}[q, q^{-1}]$, where Λ is the algebra of symmetric functions (see [7]). In particular, consider the charge 0 subspace $\mathcal{H}[0] = \mathcal{H}_0[0]$, which is identified with Λ . We can use the following representation of the Heisenberg algebra generated by $\{h_n\}_{n \in \mathbb{Z}}$:

$$h_n \leftrightarrow 2n \frac{\partial}{\partial p_n}, \quad h_{-n} = p_n, \quad n > 0, \quad (20)$$

where p_j are Newton’s power sums. Then the corresponding Virasoro operators are

$$\begin{aligned} L_n &= \sum_{r=n+1}^{\infty} p_{n-r} \cdot r \frac{\partial}{\partial p_r} + \sum_{r=1}^{n-1} r(n-r) \cdot \frac{\partial}{\partial p_r} \frac{\partial}{\partial p_{n-r}}, \\ L_{-n} &= \sum_{r=1}^{\infty} p_{n+r} \cdot r \frac{\partial}{\partial p_r} + \frac{1}{4} \sum_{r=1}^{n-1} p_r p_{n-r}, \quad n > 0. \end{aligned} \quad (21)$$

Note that the representation (20) of the Heisenberg algebra, and hence the representation (21) of the Virasoro algebra, are not unitary with respect to the standard inner product in Λ . To make it unitary, we should consider the inner product in Λ defined by

$$\langle p_{\lambda}, p_{\mu} \rangle = \delta_{\lambda\mu} \cdot z_{\lambda} \cdot 2^{l(\lambda)}, \quad (22)$$

where p_{λ} are the power sum symmetric functions, $z_{\lambda} = \prod_i i^{m_i} m_i!$ for a Young diagram λ with m_i parts of length i , and $l(\lambda)$ is the length (number of nonzero rows) of λ .

Denote the singular vectors of Vir in $\mathcal{H}[0]$ by $\xi_m := \xi_{0,m}$. According to a result by Segal [8], in the symmetric function realization (20),

$$\xi_n \leftrightarrow c \cdot s_{(n^n)}, \quad (23)$$

where $s_{(n^n)}$ is the Schur function indexed by the $n \times n$ square Young diagram and c is a numerical coefficient.

6.3 Fusion product and the Fock space

It is shown in [2] that

$$V_{2n}^* \simeq \mathbb{C}[e_0, \dots, e_{-(2n-1)}]\Omega_{-2n} \subset \mathcal{F}$$

as an $\mathfrak{sl}_2 \otimes (\mathbb{C}[t^{-1}]/t^{-2n})$ -module, the embedding j_{2n} under this isomorphism coincides with the natural inclusion

$$\mathbb{C}[e_0, \dots, e_{-(2n-1)}]\Omega_{-2n} \subset \mathbb{C}[e_0, \dots, e_{-(2n+1)}]\Omega_{-2(n+1)},$$

and the limit space \mathcal{V} coincides with \mathcal{H}_0 .

Using results of [2], one can easily prove the following lemma.

Lemma 1. *A basis in $F_{2n} = \mathbb{C}[e_0, \dots, e_{-(2n-1)}]\Omega_{-2n}$ is*

$$\{e_0^{i_0} e_{-1}^{i_1} \dots e_{-(2n-1)}^{i_{2n-1}} : 0 \leq k \leq 2n - (i_0 + \dots + i_{2n-1})\}\Omega_{-2n}.$$

Observe that under this “fusion–Fock” correspondence, the charge 0 subspace $\mathcal{H}[0]$ is identified with $\mathcal{V}[0]$. It follows from Lemma 1 that a basis of $F_{2n}[0] = F_{2n} \cap \mathcal{H}[0]$ is

$$\{\prod e_0^{i_0} e_{-1}^{i_1} \dots e_{-n}^{i_n} : i_0 + i_1 + \dots + i_n = n\}\Omega_{-2n}. \quad (24)$$

7 The key isomorphism in more detail

Comparing (13) and (17), we obtain the following result.

Corollary 2. *The space Π_k of the discrete representation of the infinite symmetric group corresponding to the tableau τ_k has a natural structure of the Virasoro module $L(1, k^2)$.*

Our aim is to study this Virasoro representation in Π_k (or, which is equivalent, the corresponding representation of the infinite symmetric group in the Fock space). In particular, from the known theory of the basic module $L_{0,1}$, we immediately obtain the following result.

Corollary 3. *In the above realization of the Virasoro module $L(1, k^2)$, the Gelfand–Tsetlin basis in Π_k (which consists of the infinite two-row Young tableaux tail-equivalent to τ_k) is the eigenbasis of L_0 , and the eigenvalues are given by the stable major index r :*

$$L_0 \tau = r(\tau) \tau.$$

Note that, in view of (14) and the remark after Lemma 1, the charge 0 subspace $\mathcal{H}[0]$ is identified with the space Π spanned by all infinite two-row Young tableaux with “correct” tail behavior. Thus we obtain the following corollary.

Corollary 4. *The space Π , which is the countable sum of discrete representations of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$, has a structure of an irreducible representation of the Heisenberg algebra \mathfrak{A} .*

On the other hand, as mentioned above, $\mathcal{H}[0]$ can be identified with the algebra of symmetric functions Λ via (20). Denote by Φ the obtained isomorphism between Π and Λ , which thus associates with every tableau $\tau \in \Pi$ a symmetric function $\Phi(\tau) \in \Lambda$ such that $r(\tau) = \deg \Phi(\tau)$.

Denote by $T^{(N)}$ the (finite) set of two-row tableaux that coincide with some τ_n , $n = 0, 1, \dots$, from the N th level. Let $\Pi^{(N)}$ be the subspace in Π spanned by all $\tau \in T^{(N)}$. It follows from all the above identifications that $\Pi^{(2k)} \leftrightarrow F_{2k}[0]$.

Theorem 3. *Under the isomorphism Φ ,*

- 1) *the principal tableaux (10) correspond to the Schur functions with square Young diagrams:*

$$\Phi(\tau_k) = \text{const} \cdot s_{(k^k)};$$

- 2) *the subspace $\Pi^{(2k)}$ correspond to the subspace $\Lambda_{k \times k}$ of Λ spanned by the Schur functions indexed by Young diagrams lying in the $k \times k$ square; the correspondence between the Schur function basis in $\Lambda_{k \times k}$ and the basis (24) in $\Pi^{(2k)} \simeq F_{2k}[0]$ is given by formula (29) below.*

Proof. We follow Wasserman’s [11] proof of Segal’s result (23).

Let $0 \leq i_1, \dots, i_k \leq k$. Then, obviously,

$$e_{-i_1} \dots e_{-i_k} \Omega_{-2k} = \left[\prod_{j=1}^k z_j^{i_j-1} \right] E(z_k) \dots E(z_1) \Omega_{-2k},$$

where by $[\text{monomial}]F(z_1, \dots, z_m)$ we denote the coefficient of this monomial in $F(z_1, \dots, z_m)$. Now, using the representation (18), the commutation relation (19), and the obvious facts that $V^{-k} \Omega_{-2k} = \Omega_0$ and $\Gamma_+(z) \Omega_0 = \Omega_0$, we obtain

$$E(z_k) \dots E(z_1) \Omega_{-2k} = \prod_{j=1}^k z_j^{2(k-j)} \prod_{1 \leq j < i \leq k} \left(1 - \frac{z_i}{z_j} \right)^2 \Gamma_-(z_k) \dots \Gamma_-(z_1) \Omega_0.$$

Observe that, in view of (20) and the well-known fact from the theory of symmetric functions, $\Gamma_-(z)$ is exactly the generating function of the complete symmetric

functions. Hence, expanding the product $\Gamma_-(z_k) \dots \Gamma_-(z_1) \Omega_0$ by the Cauchy identity ([7, I.4.3]) and making simple transformations, we obtain

$$E(z_k) \dots E(z_1) \Omega_{-2k} = (-1)^{k(k-1)/2} \prod_{j=1}^k z_j^{k-1} a_\delta(z) a_\delta(z^{-1}) \sum_{\lambda: l(\lambda) \leq k} s_\lambda(z^{-1}) s_\lambda,$$

where

$$a_\delta(z) = \prod_{1 \leq i < j \leq k} (z_i - z_j) = \det[z_i^{k-j}]_{1 \leq i, j \leq k}$$

is the Vandermonde determinant, $a_\delta(z^{-1})$ is the similar determinant for the variables $z^{-1} = (z_1^{-1}, \dots, z_k^{-1})$, $l(\lambda)$ is the length of the diagram λ (the number of nonzero rows), $s_\lambda(z^{-1})$ is the Schur function calculated at the variables z^{-1} , and s_λ is the Schur function as an element of Λ identified with $\mathcal{H}[0]$. Thus we have

$$e_{-i_1} \dots e_{-i_k} \Omega_{-2k} = (-1)^{k(k-1)/2} \cdot [1] \left(\prod_{j=1}^k z_j^{k-i_j} a_\delta(z) a_\delta(z^{-1}) \sum_{\lambda} s_\lambda(z^{-1}) s_\lambda \right).$$

First consider the case where $i_1 = \dots = i_k = m$. Then, by the definition of the Schur functions [7, I.3.1],

$$\prod_{j=1}^k z_j^{k-i_j} a_\delta(z) = \det[z_i^{2k-m-j}]_{1 \leq i, j \leq k} = a_\delta(z) s_{((k-m)^k)}(z),$$

where $((k-m)^k)$ is the rectangular Young diagram with k rows of length $k-m$, and the standard orthogonality relations imply that

$$e_{-m}^k \Omega_{-2k} = (-1)^{k(k-1)/2} k! \cdot s_{((k-m)^k)}. \quad (25)$$

Since $\xi_k = e_0^k \Omega_{-2k}$, for $m = 0$ this is Segal's result (23), which we have now extended to the case of rectangular diagrams. It is easy to see that the singular vector of Vir in $\mathcal{V}_k[0]$ is just $e_0^k \omega_{-2k} \otimes \tau_k$, so that the first claim of the theorem follows.

We now turn to the case of i_1, \dots, i_k that are not necessarily equal. For convenience, set $\tilde{e}_p := e_{-(k-p)}$, $0 \leq p \leq k$. Given $0 \leq \alpha_1, \dots, \alpha_k \leq k$, we have

$$\tilde{e}_{\alpha_1} \dots \tilde{e}_{\alpha_k} \Omega_{-2k} = [1] \left(\prod_{j=1}^k z_j^{\alpha_j} a_\delta(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1}) s_\lambda \right), \quad (26)$$

where $a_{\lambda+\delta}(x) = \det[x_i^{\lambda_j+k-j}]_{1 \leq i, j \leq k} = s_\lambda(x) a_\delta(x)$. Consider a Young diagram $\mu = (\mu_1, \dots, \mu_k) = (0^{r_0} 1^{r_1} 2^{r_2} \dots)$. Let us sum (26) over all different permutations

$\alpha = (\alpha_1, \dots, \alpha_k)$ of the sequence (μ_1, \dots, μ_k) . Note that the operators e_j commute with each other, so that the left-hand side does not depend on the order of the factors. In the right-hand side, $\sum_{\alpha} \prod z_j^{\alpha_j} = m_{\mu}(z)$, a monomial symmetric function. Thus we have

$$\frac{k!}{\prod_{j=0}^k r_j!} \tilde{e}_{\mu_1} \dots \tilde{e}_{\mu_k} = [1] \left(m_{\mu}(z) a_{\delta}(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1}) s_{\lambda} \right). \quad (27)$$

Let ν be a Young diagram with at most k rows and at most k columns, i.e., $\nu \subset (k^k)$. We have

$$s_{\nu}(z) = \sum_{\mu} K_{\nu\mu} m_{\mu}(z), \quad (28)$$

where $K_{\nu\mu}$ are Kostka numbers. It is well known that $K_{\nu\mu} = 0$ unless $\mu \leq \nu$, where \leq is the standard ordering on partitions: $\mu \leq \nu \iff \mu_1 + \dots + \mu_i \leq \nu_1 + \dots + \nu_i$ for every $i \geq 1$. In particular, $\mu_1 \leq \nu_1 \leq k$. Besides, since we consider only k nonzero variables z_1, \dots, z_k , it also follows that $m_{\mu}(z) = 0$ unless $l(\mu) \leq k$. Thus the sum in (28) can be taken only over diagrams $\mu \subset (k^k)$, for which equation (27) holds. Multiplying this equation by $K_{\nu\mu}$ and summing over μ yields

$$\sum_{\mu=(0^{r_0} 1^{r_1} 2^{r_2} \dots) \subset (k^k)} \frac{k!}{\prod_{j=0}^k r_j!} K_{\nu\mu} \tilde{e}_{\mu_1} \dots \tilde{e}_{\mu_k} = [1] \left(s_{\nu}(z) a_{\delta}(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1}) s_{\lambda} \right).$$

By the orthogonality relations, the right-hand side is equal to $k! s_{\nu}$. Thus we obtain the following formula:

$$s_{\nu} = \sum_{\mu=(0^{r_0} 1^{r_1} 2^{r_2} \dots) \subset (k^k)} \frac{K_{\nu\mu}}{\prod_{j=0}^k r_j!} e_{-(k-\mu_1)} \dots e_{-(k-\mu_k)} \Omega_{-2k}. \quad (29)$$

Observe that for rectangular diagrams this formula reduces to (25). Indeed, for $\nu = ((k-m)^k)$, all diagrams μ with $\mu < \nu$ have $l(\mu) > k$, hence the only nonzero term in the right-hand side of (29) corresponds to $\mu = \nu$, with $K_{\nu\nu} = 1$ and $r_j = k! \delta_{j,k-m}$.

It follows from (29) that $\Lambda_{k \times k} \subset \Pi^{2k}$. On the other hand, the generating functions for the tableaux from $T^{(2k)}$ and for the Young diagrams lying in the $k \times k$ square coincide:

$$\sum_{\tau \in T^{(2k)}} q^{r(\tau)} = \sum_{\lambda \subset (k^k)} q^{|\lambda|} = \left[\begin{matrix} 2k \\ k \end{matrix} \right]_q,$$

where $|\lambda|$ is the number of cells in a Young diagram λ and $\begin{bmatrix} 2k \\ k \end{bmatrix}_q$ is the q -binomial coefficient (the equation for Young diagrams can be found in [1, Theorem 3.1]; for tableaux, it can be deduced from the known results on the major index given, e.g., in [9]). This implies, in particular, that $\dim \Lambda_{k \times k} = \dim \Pi^{2k}$ and completes the proof. \square

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