ПРЕПРИНТЫ ПОМИ РАН

Учредитель: Федеральное государственное бюджетное учреждение науки
Санкт-Петербургское отделение Математического института
им. В. А. Стеклова Российской академии наук

Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16 октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых коммуникаций

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

телефоны:(812)312-40-58; (812) 571-57-54

e-mail: admin@pdmi.ras.ru

http://www. pdmi.ras.ru /preprint/

Заведующая информационно-издательским сектором Симонова В.Н

PDMI Preprint 03/2014

The Riemann Hypothesis and eigenvalues of related Hankel matrices. I

Yu. V. Matiyasevich[1,2]

St.Petersburg Department of V. A. Steklov Institute of Mathematics of Russian Academy of Sciences

yumat@pdmi.ras.ru

March 7, 2014

Abstract: The Riemann Hypothesis is reformulated as statements about the eigenvalues of certain Hankel matrices, entries of which are defined via the Taylor series coefficients of the zeta function. These eigenvalues demonstrate very interesting visual patterns allowing one to state a number of new conjectures related to the Riemann Hypothesis.

Key words: Riemann Hypothesis, Hankel matrix, eigenvalue.

[1]The research was supported in the framework of the Program of Fundamental Research of the Division of Mathematical Sciences of the Russian Academy of Sciences "Modern problems of theoretical mathematics".

[2]The author is very grateful to Peter Zvengrowski (University of Calgary) for his help with the English.

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Санкт-Петербургского отделения
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PREPRINTS
of the St.Petersburg Department
of Steklov Institute of Mathematics

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1 Riemann's zeta function and the Riemann Hypothesis

One of the most interesting and important objects in mathematics is Riemann's zeta function $\zeta(z)$. It can be defined for $\Re(z) > 1$ by the Dirichlet series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{1.1}$$

The function can be analytically extended to the entire complex z-plane with the exception of the point $z = 1$ which is the only pole of $\zeta(z)$.

Leonhard Euler studied this function for real values of z, in particular he determined the values of $\zeta(z)$ for all negative integer values of z (without having the notion of analytical continuation!). His celebrated formula

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-z} &= \prod_{p \text{ prime}} \frac{1}{1 - p^{-z}} && (1.2) \\ &= \prod_{p \text{ prime}} \left(1 + p^{-z} + p^{-2z} + \dots\right) && (1.3) \end{aligned}$$

is known as *Euler product*. It can be viewed as an analytical form of the *Fundamental Theorem of Arithmetic* stating that every natural number has a unique factorization into product of powers of primes—just expand (1.3) and get the left hand side of (1.2)!

The identity (1.2) "explains" why Riemann's zeta function $\zeta(z)$ plays such an important role in the study of prime numbers. In particular, Euler gave a new proof of the infinitude of primes: *if the number of primes were finite, then the divergent harmonic series, that is, the left hand side of* (1.2) *for* $z = 1$, *would have a finite value, the right hand side of* (1.2).

A closer relationship with the distribution of prime numbers was discovered by Bernhard Riemann. Already Euler knew that $\zeta(-2m) = 0$ for $m = 1, 2, \dots$ and today points $\mathrm{z}_1 = -2,\ \mathrm{z}_2 = -4,\ \dots,\ \mathrm{z}_n = -2n, \dots$ are called the *trivial zeroes* of the zeta-function. They are the only real zeroes of this function. Riemann showed the role of the complex zeroes of the zeta-function for the study of the distribution of primes. In a more transparent way this relationship can be seen from the following formula established by Hans Carl Friedrich von

MANGOLD [3]: *for non-integer x greater than* 1

$$\psi(x) \;=\; x - \sum_{m=1} \frac{x^{-2m}}{-2m} - \sum_{\substack{\zeta(\rho)=0 \\ \Im(\rho)\neq 0}} \frac{x^{\rho}}{\rho} - \ln(2\pi) \tag{1.4}$$

where

$$\psi(x) \;=\; \ln(\mathrm{LCM}(1, 2, ..., \lfloor x \rfloor)). \tag{1.5}$$

The first summation in (1.4) is performed over the trivial zeroes $-2, -4, \ldots, -2m, \ldots$ and similar the second summation is over all the others, the *non-trivial* complex zeroes of the zeta function. The function $\psi(x)$, introduced by PAFNUTIJ CHEBYSHEV, has a jump of size $\ln(p)$ at every prime p and at its powers.

According to (1.4), the growth of the difference $\psi(x) - x$ is related to the real parts of the non-trivial zeroes. We have the celebrated

Riemann Hypothesis (version 1). *All non-trivial zeroes of the function $\zeta(z)$ lie on the* critical line $\Re(z) = \frac{1}{2}$.

In terms of the function $\psi(x)$ the hypothesis, RH for short, can be reformulated as

Riemann Hypothesis (version 2). *For $x \to +\infty$*

$$\psi(x) = x + O(x^{\frac{1}{2}} \ln^2(x)). \tag{1.6}$$

Formula (1.6) allows one to give a good approximation to the function $\pi(x)$, the prime counting function equal to the number of primes not exceeding x. Similar to the function $\psi(x)$, the function $\pi(x)$ has a jump at every prime but only of size 1, and, in contrast to $\psi(x)$, function $\pi(x)$ has no jumps at other powers of primes. These prime powers can be ignored because their number below an x is of order $O(x^{\frac{1}{2}})$, so in terms of function $\pi(x)$ we have

Riemann Hypothesis (version 3).

$$\pi(x) = \int^{x} \frac{\mathrm{d}t}{\ln(t)} + O(x^{\frac{1}{2}} \ln(x)). \tag{1.7}$$

2 Riemann's xi function and subhypotheses of the Riemann Hypothesis

There is a tradition (taking its origin from RIEMANN's only paper [12] about this subject) to get rid of the trivial zeroes by dealing with the entire function

$$\xi(z) = \pi^{-\frac{z}{2}}(z-1)\Gamma(1+\tfrac{z}{2})\zeta(z) \tag{2.1}$$

rather than with the function $\zeta(z)$ itself (we use modern notation for this function, RIEMANN used $\xi(t)$ to denote the function which today is usually denoted $\Xi(t)$). The poles of the factor $\Gamma(1+\frac{z}{2})$ in (2.1) cancel the trivial zeroes of $\zeta(z)$ and similarly the factor $z-1$ cancels the pole of $\zeta(z)$. The factor $\pi^{-\frac{z}{2}}$ influences neither zeroes nor poles but it allows us to state the *functional equation* in a pretty form:

$$\xi(z) = \xi(1-z). \tag{2.2}$$

In this paper we won't deprive the zeta function of its trivial zeroes but try to take advantage of our knowledge of the precise positions of these zeroes. To this end we will work with the entire function

$$\zeta^*(z) = 2(z-1)\zeta(z). \tag{2.3}$$

For our purpose we could also omit the factor $z-1$ and/or use the factor $\pi^{-\frac{z}{2}}$; this would change the picture(s) so probably separate paper(s) could be devoted to these variations. The factor 2 in (2.3) results in the equality

$$\zeta^*(0) = 1 \tag{2.4}$$

which slightly simplifies some forthcoming formulas.

If the zeta function had a zero z_0 with the real part greater than $\frac{1}{2}$, then according to (2.2) $1-z_0$ would be a zero with the real part less than $\frac{1}{2}$. Thus, we have

Riemann Hypothesis (version 4). *The trivial zeroes* $\mathrm{z}_1 = -2,\ \mathrm{z}_2 = -4,\ \ldots,\ \mathrm{z}_n = -2n, \ldots$ *are the only zeroes of the function* $\zeta^*(z)$ *lying in the half-plane* $\Re(z) < \frac{1}{2}$.

A half-plane is a natural object when one deals with Dirichlet series. However, we are going to deal with Taylor series, and for them disks are more natural regions. So we make a change of variable:

$$z = \frac{w}{w+1}, \qquad w = \frac{z}{1-z}. \tag{2.5}$$

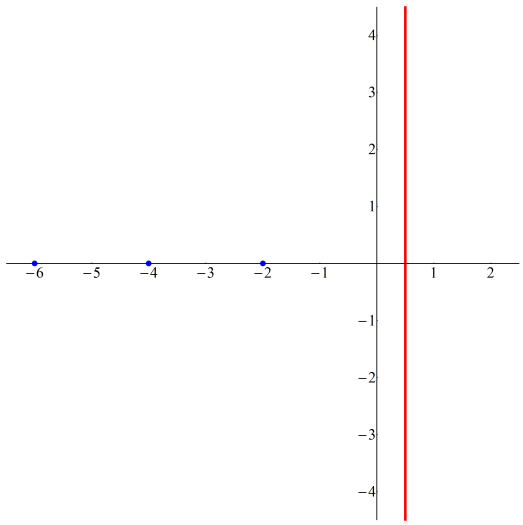

Figure 1: z-plane and the critical line

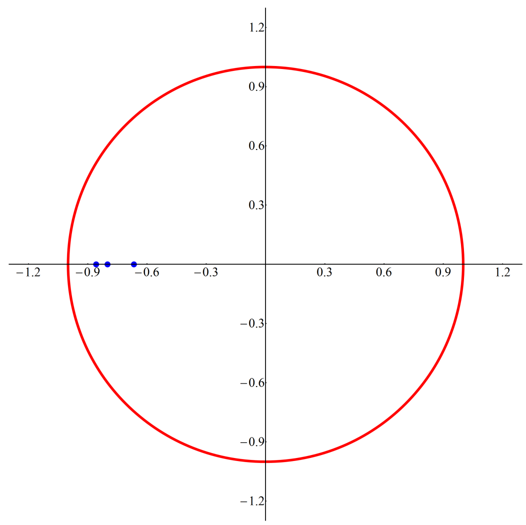

Figure 2: w-plane and the critical circle

Under this transformation the critical line becomes the *critical circle* $|w| = 1$, the half-plane $\Re(z) < \frac{1}{2}$ becomes the interior of this circle, and points

$$\mathrm{w}_1 = \frac{\mathrm{z}_1}{1 - \mathrm{z}_1} = -\frac{2}{3}, \ \ldots, \ \mathrm{w}_n = \frac{\mathrm{z}_n}{1 - \mathrm{z}_n} = -\frac{2n}{2n+1}, \ldots \tag{2.6}$$

become the *trivial zeroes of the function*

$$\tilde{\zeta}(w) = \zeta^*(\tfrac{w}{w+1}). \tag{2.7}$$

With this new notation we have

Riemann Hypothesis (version 5). *The trivial zeroes* $\mathrm{w}_1 = -\frac{2}{3}, \ \ldots, \ \mathrm{w}_n = -\frac{2n}{2n+1}, \ldots$ *are the only zeroes of the function* $\tilde{\zeta}(w)$ *lying in the open disk* $\{w : |w| < 1\}$.

It isn't convenient to work near the critical circle (full of zeroes) so we split RH into an infinite series of weaker subhypotheses.

Subhypothesis $\mathbf{RH_n}$. *The trivial zeroes* $\mathrm{w}_1 = -\frac{2}{3}, \ \ldots, \ \mathrm{w}_n = -\frac{2n}{2n+1}$ *are the only zeroes of the function* $\tilde{\zeta}(w)$ *lying in the closed disk* $\{w : |w| \leq \frac{2n+1}{2n+2}\}$.

While each of these subhypotheses is weaker than RH, taken together, they, evidently, are equivalent to it:

Riemann Hypothesis (version 6). *For every n the subhypothesis* RH_n *is true.*

3 Padé approximations and a theorem of de Montessus de Ballore

In order "to see" where the smallest (in absolute value) zeroes of $\tilde{\zeta}(w)$ lie, we can approximate this function by rational functions: let $P_{n,m}(w)$ and $Q_{n,m}(w)$ be polynomials such that

$$\tilde{\zeta}(w) \approx \frac{P_{n,m}(w)}{Q_{n,m}(w)} = \frac{1 + p_{n,m,1}w + \cdots + p_{n,m,n}w^n}{1 + q_{n,m,1}w + \cdots + q_{n,m,m}w^m} \tag{3.1}$$

$$= \tilde{\zeta}(w) + O(w^k) \tag{3.2}$$

where k has the maximal possible value; since in (3.3) we have at our disposal $n+m$ coefficients

$$p_{n,m,1}, \ldots, p_{n,m,n}, q_{n,m,1}, \ldots, q_{n,m,m}, \tag{3.3}$$

k should be equal to $n+m+1$.

A theorem of Robert de Montessus de Ballore [10, 11] (see also [1]) tells us about the behaviour of the numerators in (3.1) for special choices of n. Namely, let us say that a number n is good if there is a positive number R such that the closed disk

$$\{w : |w| \leq R\} \tag{3.4}$$

contains exactly n zeroes of the function $\tilde{\zeta}(w)$. Then, for any fixed good n, with the growth of m the n zeroes of $P_{n,m}$ approach the zeroes from disk (3.4).

Our subhypothesis RH_n implies that n is good and furthermore, for $m \to \infty$,

$$P_{n,m}(w) \to \prod_{k=1}^{n} \left(1 - \frac{w}{\mathrm{w}_n}\right). \tag{3.5}$$

We are going to deal only with the absolute value of the leading coefficient of $P_{n,m}(w)$ for which (3.5) implies the weaker

Subhypothesis $\mathbf{RH}_{\boldsymbol{n}}^{\mathbf{w}}$. *For $m \to \infty$*

$$|p_{n,m,n}| \to \mathrm{W}_n \tag{3.6}$$

where

$$\mathrm{W}_n = \prod_{k=1}^{n} \frac{1}{|\mathrm{w}_n|} = \prod_{k=1}^{n} \frac{2k+1}{2k}. \tag{3.7}$$

Each subhypothesis $\mathrm{RH}_n^{\mathrm{w}}$ is, formally, weaker than the corresponding subhypothesis RH_n, nevertheless, taken together the subhypotheses $\mathrm{RH}_n^{\mathrm{w}}$ are also equivalent to RH. In order to see why it is so, suppose that RH isn't valid, and let $\check{\mathrm{w}}$ be a non-trivial zero of $\tilde{\zeta}(w)$ violating it in version 5 above. We will assume that $\check{\mathrm{w}}$ has the least possible absolute value denoted R. Disk (3.4) contains only finitely many, say, n, zeroes of $\tilde{\zeta}(w)$ because there are only finitely many zeroes of $\zeta^*(z)$ in the preimage of the disk on z-plane. Let these zeroes be denoted $\check{\mathrm{w}}_1$, $\dots$, $\check{\mathrm{w}}_n$ and enumerated in such a way that $|\check{\mathrm{w}}_1| \le \dots \le |\check{\mathrm{w}}_n|$. Clearly, n is a good number and hence by the theorem of de Montessus for $m \to \infty$

$$P_{n,m}(w) \to \prod_{k=1}^{n} \left(1 - \frac{w}{\check{\mathrm{w}}_n}\right) \tag{3.8}$$

and respectively

$$|p_{n,m,n}| \to \prod_{k=1}^{n} \frac{1}{|\check{\mathrm{w}}_n|}. \tag{3.9}$$

It is easy to see that $|\check{\mathrm{w}}_1| \le |\mathrm{w}_1|$, $\dots$, $|\check{\mathrm{w}}_{n-1}| \le |\mathrm{w}_{n-1}|$ and $|\check{\mathrm{w}}_n| < |\mathrm{w}_n|$. Thus,

$$\prod_{k=1}^{n} \frac{1}{|\check{\mathrm{w}}_n|} > \prod_{k=1}^{n} \frac{1}{|\mathrm{w}_n|} = \mathrm{W}_n, \tag{3.10}$$

which gives the required contradiction with (3.6) and we have

Riemann Hypothesis (version 7). *For every n the subhypothesis $\mathrm{RH}_n^{\mathrm{w}}$ is true.*

It follows from the above consideration that in this version of RH we can restrict n to good numbers.

4 Determinants and eigenvalues

It is easy to understand that coefficients (3.3) can be expressed via the coefficients in the Taylor expansion

$$\tilde{\zeta}(w) = 1 + \theta_1 w + \dots + \theta_k w^k + \dots \tag{4.1}$$

In order to simplify further notation we put $\theta_0 = 1$ and $\theta_k = 0$ for $k < 0$. Explicit expressions for $p_{n,m,n}$ can be given (Carl Jacobi [2], see also [1]) in terms of Toeplitz matrices

$$L_{n,m} = \begin{pmatrix} \theta_n & \theta_{n-1} & \dots & \theta_{n-m+1} \\ \theta_{n+1} & \theta_n & \dots & \theta_{n-m+2} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{n+m-1} & \theta_{n+m-2} & \dots & \theta_n \end{pmatrix} \tag{4.2}$$

or in terms of the dual Hankel matrices

$$M_{n,m} =(-1)^{n+m} \begin{pmatrix} \theta_{n+m-1} & \theta_{n+m-2} & \dots & \theta_n \\ \theta_{n+m-2} & \theta_{n+m-3} & \dots & \theta_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_n & \theta_{n-1} & \dots & \theta_{n-m+1} \end{pmatrix}, \tag{4.3}$$

namely,

$$p_{n,m,n} = \frac{\det(L_{n,m+1})}{\det(L_{n,m})} \tag{4.4}$$

$$= (-1)^{n+m+1}\frac{\det(M_{n,m+1})}{\det(M_{n,m})}. \tag{4.5}$$

Representation (4.4) was investigated in [5, 7, 8, 9], in this paper we will study representation (4.5) which was considered in [6, 7, 9]; ongoing study of both representations can be followed on [4].

If for some constant C for $m \to \infty$

$$|p_{n,m,n}| \to C \tag{4.6}$$

then (4.5) implies that

$$|\det(M_{n,m})|^{\frac{1}{m}} \to C. \tag{4.7}$$

Thus subhypothesis $\mathrm{RH}_n^{\mathrm{w}}$ has the following (formally) weaker corollary:

Subhypothesis $\mathbf{RH}_n^{\mathbf{ww}}$. *For $m \to \infty$*

$$|\det(M_{n,m})|^{\frac{1}{m}} \to \mathrm{W}_n. \tag{4.8}$$

Again, taken together, subhypotheses $\mathrm{RH}_n^{\mathrm{ww}}$ imply RH. Indeed, we see from (3.9), (3.10), (4.6), and (4.7) that if RH were not valid, then for some good n

the values of $|p_{n,m,n}|$ and, respectively, of $|\det(M_{n,m})|^{\frac{1}{m}}$ would tend to a quantity greater than W_n. Thus we have

Riemann Hypothesis (version 8). *For every n the subhypotheses $\mathrm{RH}_n^{\mathrm{ww}}$ is true.*

One could try to prove (4.8) by induction on n. We know that -2 is the zero of the zeta function nearest to the origin, hence $\mathrm{RH}_1^{\mathrm{ww}}$ is true and it remains to show the validity of

Subhypothesis $\mathbf{RH}_n^{\mathbf{r}}$. *For $m \to \infty$*

$$\left|\frac{\det(M_{n+1,m})}{\det(M_{n,m})}\right|^{\frac{1}{m}} \to \frac{\mathrm{W}_{n+1}}{\mathrm{W}_n} = \frac{1}{|\mathrm{w}_{n+1}|} = \frac{2n+3}{2n+2}. \tag{4.9}$$

Let $\mu_{n,m,1},\ \mu_{n,m,2},\ \ldots,\ \mu_{n,m,m}$ be the eigenvalues of the matrix $M_{n,m}$ (they are real thanks to matrix being Hankel). In terms of these eigenvalues we can restate

Subhypothesis $\mathbf{RH}_n^{\mathbf{ww}}$ (equivalent form). *For $m \to \infty$*

$$\frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n,m,k}| \to \ln(\mathrm{W}_n) \tag{4.10}$$

and

Subhypothesis $\mathbf{RH}_n^{\mathbf{r}}$ (equivalent form). For $m \to \infty$

$$\frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n+1,m,k}| - \frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n,m,k}| \to -\ln(|\mathrm{w}_{n+1}|) = \ln\left(\frac{2n+3}{2n+2}\right). \tag{4.11}$$

5 Visual patterns of eigenvalues

According to (4.10), RH can be viewed as a statement about the behavior of the eigenvalues of the matrices (4.3) as a whole, but doesn't tells us anything, at least directly, about the distribution of the eigenvalues. The author was curious to perform calculations for small m in order to see whether one could find interesting patterns in values of individual eigenvalues. Indeed, being properly exhibited, the eigenvalues show definite patterns allowing one to state a number of conjectures.

The (multi)set $\{\mu_{n,m,1},\ \mu_{n,m,2}, \dots \mu_{n,m,m}\}$ of eigenvalues of matrix (4.3) will be called the *μ-spectrum* of the function $\tilde{\zeta}$ and will be denoted $\mathrm{Spec}^{\mu}_{n,m}(\tilde{\zeta})$ or, for simplicity, just $\mathrm{Spec}^{\mu}_{n,m}$ unless we need to consider similar spectra for other functions. We shall always suppose that the eigenvalues are numbered in such a way that

$$|\mu_{n,m,1}| \leq \cdots \leq |\mu_{n,m,k}| \leq |\mu_{n,m,k+1}| \leq \cdots \leq |\mu_{n,m,m}|. \tag{5.1}$$

It turned out that spectra $\mathrm{Spec}^{\mu}_{n,m}$ contain both very large and very small (in absolute value) numbers, so it is more convenient to use logarithmic scaling. We define the *logarithmic μ-spectrum* as

$$\mathrm{Spec}^{\ln\mu}_{n,m} \;=\; \{\ln(|\mu|) : \mu \in \mathrm{Spec}^{\mu}_{n,m}\}, \tag{5.2}$$

its elements will be called *logarithmic eigenvalues*.

It is very useful to keep track of the signs of the eigenvalues missing in (5.2), so we split $\mathrm{Spec}^{\ln\mu}_{n,m}$ into the *positive logarithmic μ-spectrum* and the *negative logarithmic μ-spectrum*, defined as follows

$$\mathrm{Spec}^{\ln^+\mu}_{n,m} \;=\; \{\ln(\mu) : \mu \in \mathrm{Spec}^{\mu}_{n,m} \& \mu > 0\}, \tag{5.3}$$

$$\mathrm{Spec}^{\ln^-\mu}_{n,m} \;=\; \{\ln(-\mu) : \mu \in \mathrm{Spec}^{\mu}_{n,m} \& \mu < 0\}. \tag{5.4}$$

When exhibiting several logarithmic μ-spectra on a single picture, we will shift them vertically, that is, an eigenvalue μ from $\mathrm{Spec}^{\ln^\pm\mu}_{n,m}$ will produce a point with coordinates $(x, y) = (\ln|\mu|, m)$ or at point $(x, y) = (\ln|\mu|, m + \frac{1}{2})$ in the case when we need to show two spectra on the same picture.

Figures 3–8 (more pictures can be downloaded from [4]) show spectra $\mathrm{Spec}^{\ln\mu}_{n,m}$ for $n = 1, \dots, 6$. First of all, we notice that spectra $\mathrm{Spec}^{\ln^+\mu}_{n,m}$ and $\mathrm{Spec}^{\ln^+\mu}_{n,m+1}$ are close one to another, and so are spectra $\mathrm{Spec}^{\ln^-\mu}_{n,m}$ and $\mathrm{Spec}^{\ln^-\mu}_{n,m+1}$ as well. As a consequence, we can see "trajectories" and, treating m as (discrete) time, we can imagine "particles" moving along these trajectories.

In order to introduce this notion formally, we will enumerate the logarithmic eigenvalues in two ways, from left to right and vice versa. Let $N^+_{n,m}$ and $N^-_{n,m}$ be the number of elements in sets (5.3) and (5.4) respectively and let

$$\mathrm{Spec}^{\ln^+\mu}_{n,m} \;=\; \left\{\mu^{+<}_{n,m,1}, \dots, \mu^{+<}_{n,m,N^+_{n,m}}\right\} \tag{5.5}$$

$$\;=\; \{\mu^{+>}_{n,m,1}, \dots, \mu^{+>}_{n,m,N^+_{n,m}}\} \tag{5.6}$$

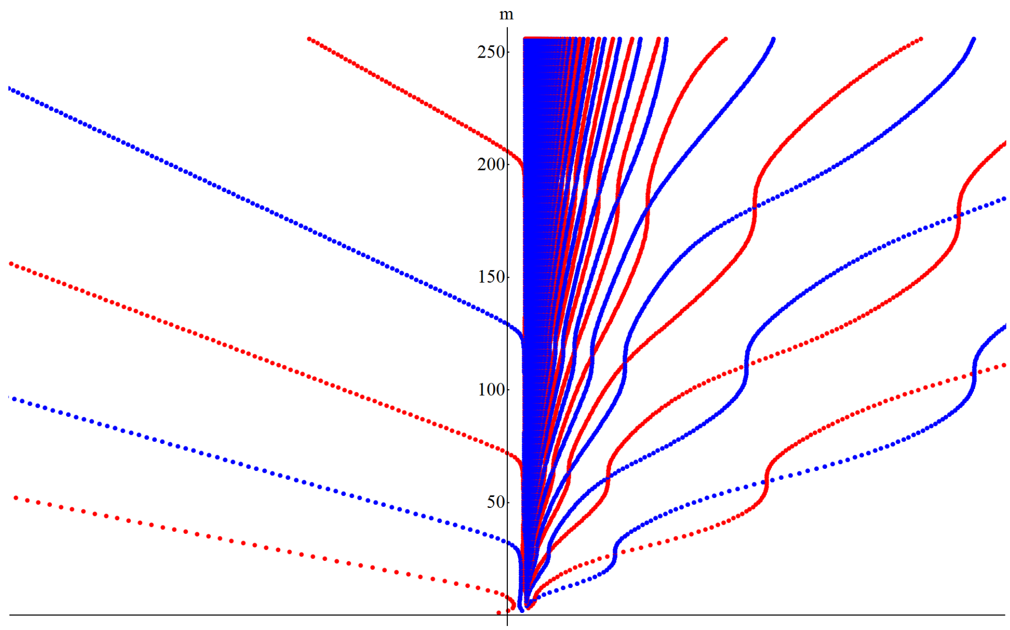

Figure 3: Positive and negative logarithmic μ-sprectra $\mathrm{Spec}_{1,m}^{\ln^{\pm}\mu}$, $m = 1 \ldots 256$

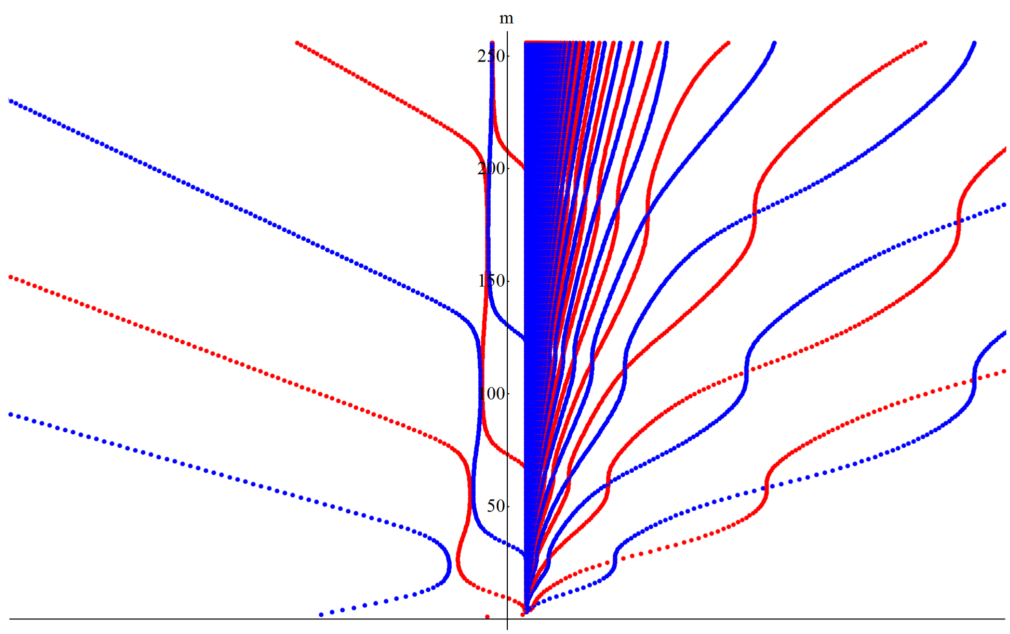

Figure 4: Positive and negative logarithmic μ-sprectra $\mathrm{Spec}_{2,m}^{\ln^{\pm}\mu}$, $m = 1 \ldots 256$

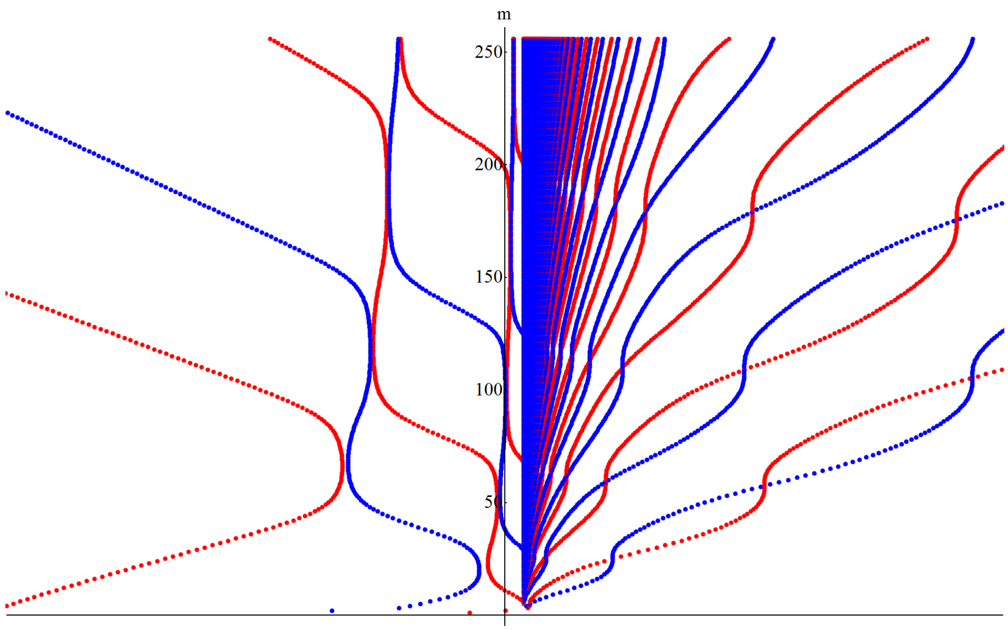

Figure 5: Positive and negative logarithmic μ-sprectra $\mathrm{Spec}_{3,m}^{\ln^{\pm}\mu}$, $m = 1 \ldots 256$

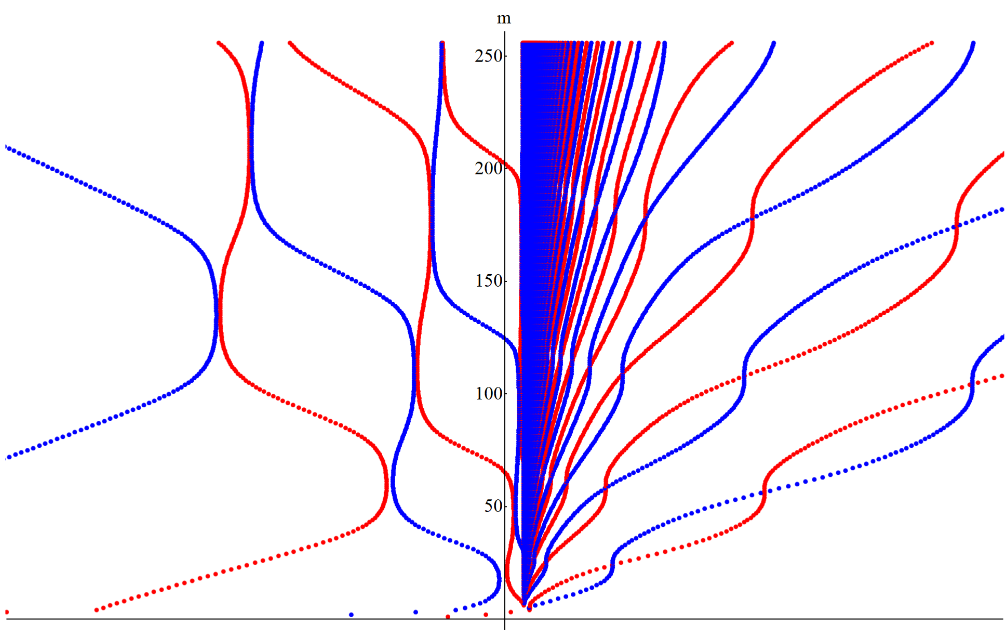

Figure 6: Positive and negative logarithmic μ-sprectra $\mathrm{Spec}_{4,m}^{\ln^{\pm}\mu}$, $m = 1 \ldots 256$

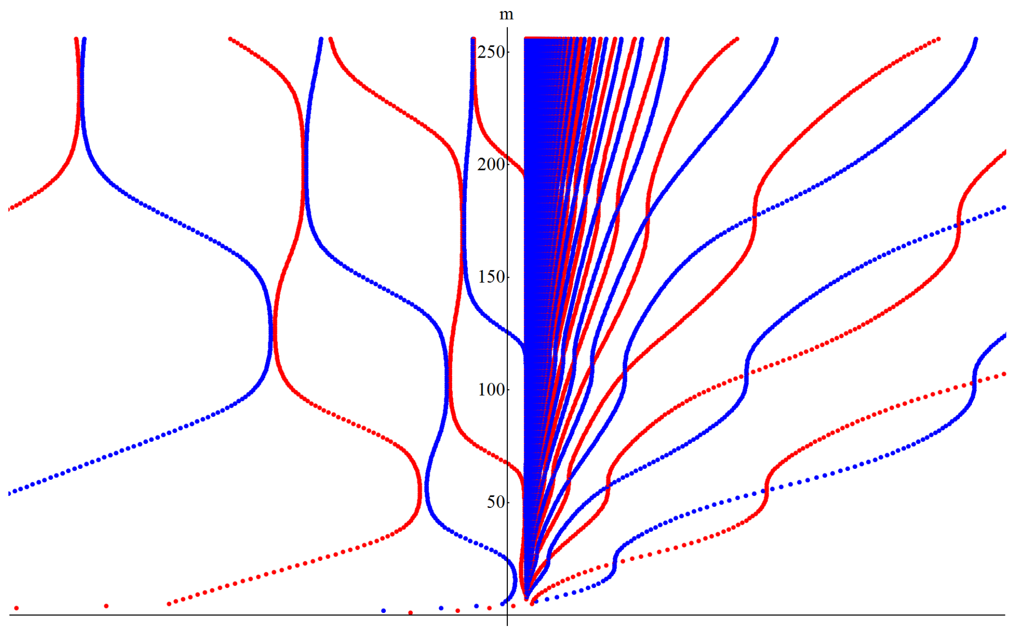

Figure 7: Positive and negative logarithmic μ-sprectra $\mathrm{Spec}_{5,m}^{\ln^{\pm}\mu}$, $m = 1 \ldots 256$

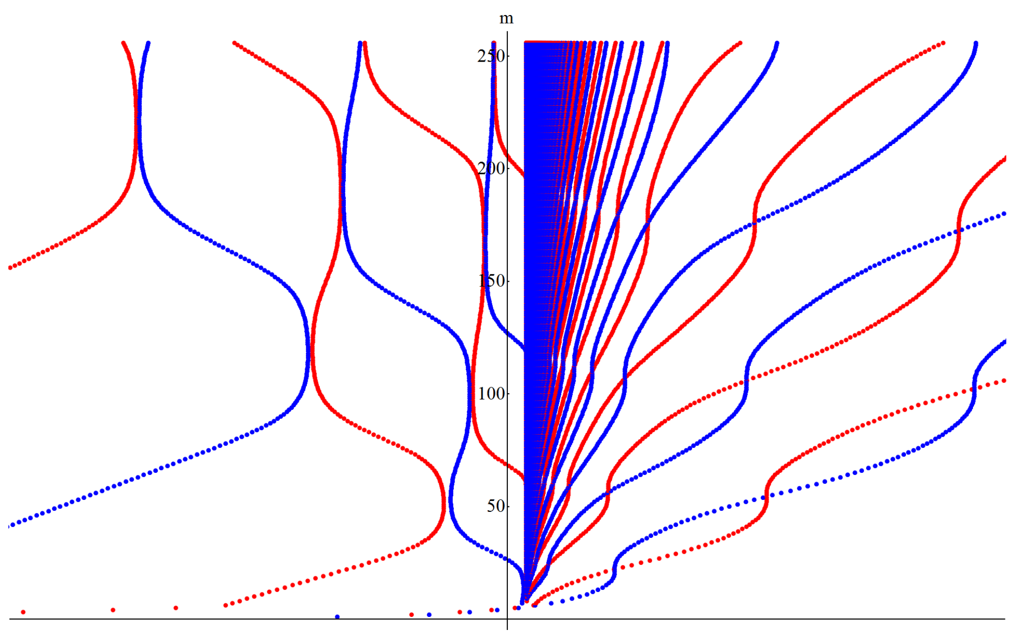

Figure 8: Positive and negative logarithmic μ-sprectra $\mathrm{Spec}_{6,m}^{\ln^{\pm}\mu}$, $m = 1 \ldots 256$

with

$$\mu^{+<}_{n,m,1} \le \cdots \le \mu^{+<}_{n,m,k} \le \mu^{+<}_{n,m,k+1} \le \cdots \le \mu^{+<}_{n,m,N^+_{n,m}}, \tag{5.7}$$
$$\mu^{+>}_{n,m,1} \ge \cdots \ge \mu^{+>}_{n,m,k} \ge \mu^{+>}_{n,m,k+1} \ge \cdots \ge \mu^{+>}_{n,m,N^+_{n,m}} \tag{5.8}$$

and

$$\begin{aligned} \mathrm{Spec}^{\ln^-\mu}_{n,m} &= \{\mu^{-<}_{n,m,1}, \dots, \mu^{-<}_{n,m,N^-_{n,m}}\} & (5.9)\\ &= \{\mu^{->}_{n,m,1}, \dots, \mu^{->}_{n,m,N^-_{n,m}}\} & (5.10)\end{aligned}$$

with

$$\mu^{-<}_{n,m,1} \le \cdots \le \mu^{-<}_{n,m,k} \le \mu^{-<}_{n,m,k+1} \le \cdots \le \mu^{-<}_{n,m,N^-_{n,m}}, \tag{5.11}$$
$$\mu^{->}_{n,m,1} \ge \cdots \ge \mu^{->}_{n,m,k} \ge \mu^{->}_{n,m,k+1} \ge \cdots \ge \mu^{->}_{n,m,N^+_{n,m}}. \tag{5.12}$$

According to the upper indices we can distinguish four kinds of particles, $\pi^{+<}_{n,k}$, $\pi^{-<}_{n,k}$, $\pi^{+>}_{n,k}$, and $\pi^{->}_{n,k}$. At time moment m these four particles have coordinates $\mu^{+<}_{n,m,k}$, $\mu^{+>}_{n,m,k}$, $\mu^{-<}_{n,m,k}$, and $\mu^{->}_{n,m,k}$ respectively. Each position is always occupied by two particles, either by $\pi^{+<}_{n,k_1}$ $\pi^{+>}_{n,k_2}$ or by $\pi^{-<}_{n,k_1}$ $\pi^{->}_{n,k_2}$ with $k_1 + k_2 = N^+_{n,m} + 1$ or $k_1 + k_2 = N^-_{n,m} + 1$ respectively. From each pair one particle will finally go to $-\infty$, and the other to $+\infty$. In this paper we deal with the former kind of particles, and a forthcoming paper will be devoted to the latter kind.

Particles $\pi^{+<}_{n,k}$, $\pi^{-<}_{n,k}$ moving towards $-\infty$ were named “electrons” in [6]. This name is due to the following observation which can be made on the basis of Figures 3–8 (but the best is to watch animations of the spectra which can be downloaded from [4]). We see there that the trajectories of electrons corresponding to positive and negative eigenvalues interleave, the particles oscillate but never touch one another, that is, they behave as particles having similar charges – repelling one another and thus bouncing.

It seems that after changing the direction a few times, electrons start moving towards $-\infty$ with stabilizing speed, and this speed is related in a remarkable way to the trivial zeroes $\mathrm{z}_1, \mathrm{z}_2, \dots$ of the zeta function, or, equivalently, to the trivial zeroes $\mathrm{w}_1, \mathrm{w}_2, \dots$ of function $\tilde{\zeta}(w)$. Namely, the numerical data suggest

Conjecture A. *For every n, k there is a positive number $C_{n,k}$ such that for $m \to \infty$*

$$\mu_{n,m,k} = (-1)^{n+k}(C_{n,k} + o(1))|\mathrm{w}_{n+k}|^{-m}, \tag{5.13}$$

moreover, for every n there is a number C_n such that for $k \to \infty$

$$C_{n,k} \to C_n. \tag{5.14}$$

Geometrically (see Figure (9)) Conjecture A implies that electrons corresponding to eigenvalues $\mu_{n',m,k'}$ and $\mu_{n'',m,k''}$ eventually start to "follow parallel courses" provided that $n' + k' = n'' + k''$.

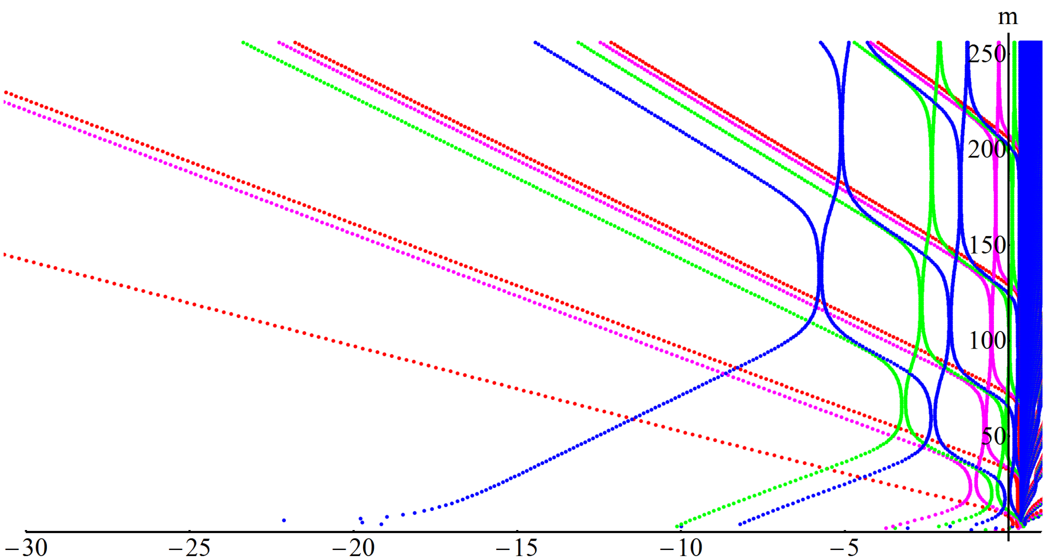

Figure 9: Logarithmic μ-sprectra $\mathrm{Spec}_{n,m}^{\ln \mu}$ for $n = 1, 2, 3, 4$

It seems that Conjecture A is independent from RH in the sense that there is no straightforward way to deduce this conjecture from RH. On the other hand, the conjecture allows one to give one more reformulation of RH and to put forth a conjecture stronger than RH.

Namely, according to Conjecture A for every $k > 1$ the electron corresponding to $\mu_{n,m,k}$ has its "elder brother" $\mu_{n+1,m,k-1}$ moving with the same speed, but $\mu_{n,m,1}$ has only "younger brothers". This suggests rewriting (4.11) as

$$\frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n+1,m,k}| - \frac{1}{m}\sum_{k=1}^{m+1} \ln|\mu_{n,m+1,k}| \to -\ln(|w_{n+1}|) \tag{5.15}$$

(replacing m by $m+1$ is justified by the inductive hypothesis according to which there exists a finite limiting value in (4.10)). Now we have:

$$\frac{1}{m}\sum_{k=1}^{m}\ln|\mu_{n+1,m,k}| - \frac{1}{m}\sum_{k=1}^{m+1}\ln|\mu_{n,m+1,k}| = \tag{5.16}$$

$$-\frac{\ln|\mu_{n,m+1,1}|}{m} + \frac{1}{m}\sum_{k=1}^{m}(\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \tag{5.17}$$

and Conjecture A tells us that already the first summand in (5.17) accounts for the limiting values in (5.15). In other words, Conjecture A implies that RH is equivalent to

$$\sum_{k=1}^{m}(\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) = o(m). \tag{5.18}$$

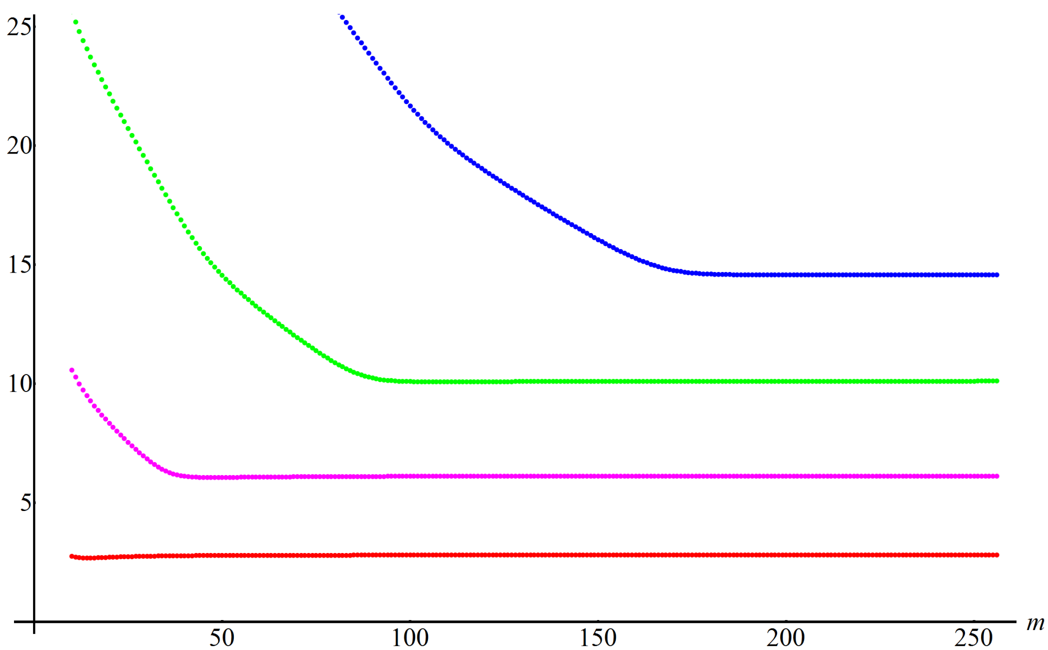

Figure 10: The left-hand side of (5.18) for $n = 1, 2, 3, 4$

Visually (see Figure 9) values of $\ln|\mu_{n+1,m,k}|$ and $\ln|\mu_{n,m+1,k+1}|$) are indeed very close, Figure 10 exhibits the left-hand side of (5.18) for $n = 1, \ldots, 4$ and $m = 10, \ldots, 256$, and suggests

Conjecture B. *For every n there exists a number D_n such that for $m \to \infty$*

$$\sum_{k=1}^{m} (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \to D_n. \tag{5.19}$$

Clearly, Conjectures A and B, taken together, imply RH.

References

[1] Baker G. A., Jr. *Essntials of Padé Approximations.* Academic Press, New York, SanFrancisco, London, 1975.

[2] Jacobi, C. G. J. Über die Darstellung einer Reihe Gegebner Werthe durch eine Gebrochne Rationale Function. *J. Reine Angew. Math.*, 30, 127–156, 1846.

[3] Mangoldt, H. C. F. Zu Riemanns Abhandlung 'Über die Anzhal der Primzahlen unter einer gegebenen Grösse'. *J. Reine Agew.Math.*, 114:255–305, 1985.

[4] Matiyasevich, Yu. *Hidden Life of Riemann's Zeta Function.* `http://logic.pdmi.ras.ru/~yumat/personaljournal/zetahiddenlife`

[5] Matiyasevich, Yu. *Hidden Life of Riemann's Zeta Function 1. Arrow, Bow, and Targets.* `http://arxiv.org/abs/0707.1983`, 2007.

[6] Matiyasevich, Yu. *Hidden Life of Riemann's Zeta Function 2. Electrons and Trains.* `http://arxiv.org/abs/0709.0028`, 2007.

[7] Matiyasevich, Yu. Riemann's zeta function: Some computations and conjectures. In Proceedings of Conference on Algorithmic Number Theory (Turku, 2007), A.-M. Ernvall-Hytönen, M. Jutila, J Karhumäki and A. Lepsitö, Eds., TUCS General Publication Series 46, pp. 87–112, Turku, 2007, 978-952-12-2014-2. Available on line `http://logic.pdmi.ras.ru/~yumat/personaljournal/zetahiddenlife/talks/turku2007/Matiyasevich_2007.pdf` and `http://tucs.fi/publications/attachment.php?fname=G46.pdf` (entire volume).

[8] Matiyasevich, Yu. Riemann's Zeta Function: More Computations and Conjectures. In *Voronoï's Impact on Modern Science*, Book 4, Volume 1. Proceedings of the 4th International Conference on Analytical

Number Theory and Spacial Tesselations (Kyiv, 2008). A. Laurinčikas and J. Steuding, Eds., pp. 2-11, Kyiv, 2008, ISBN 978-966-02-4891-5. Available online http://logic.pdmi.ras.ru/~yumat/personaljournal/zetahiddenlife/talks/kiev_2008/Matiyasevich_2008.pdf.

[9] Matiyasevich, Yu. Hidden life of Riemann's zeta function. Video of a talk given on December 18, 2008 at Steklov Mathematical Institute Seminar, available at http://www.mathnet.ru/php/seminars.phtml?&presentid=260&option_lang=eng

[10] de Montessus de Ballore, R. Sur le fraction continues algébriques. *Bull. Soc. Math. France*, 30, 28–36, 1902.

[11] de Montessus de Ballore, R. Sur le fraction continues algébriques. *Rend. Circ. Math. Palermo*, 19, 1–73, 1905.

[12] Riemann, B. Über die Anzhal der Primzahlen unter einer gegebenen Grösse. *Monatsberichter der Berliner Akademie*, November, 1859. Included into: Riemann, B. *Gesammelte Werke*. Teubner, Leipzig, 1892; reprinted by Dover Books, New York, 1953, http://www.claymath.org/millennium/Riemann_Hypothesis/1859_manuscript/zeta.pdf. English translation: http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/EZeta.pdf.