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L_p-THEORY OF FREE BOUNDARY PROBLEMS OF MAGNETOHYDRODYNAMICS IN MULTI-CONNECTED DOMAINS

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ABSTRACT:

We develop the L_p-theory of solvability of free boundary problems of magnetohydrodynamics of viscous incompressible fluids in multi-connected domains constructed in the paper [1] for $p = 2$. The case of simply connected domains is studied in [2,3].

Key words: free boundaries, Sobolev spaces, magnetohydrodynamics, divergence free vector fields

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1 Introduction.

As in [1], we consider the problem of finding a bounded variable domain $\Omega_{1t} \subset \mathbb{R}^3$ with the boundary Γ_t, $t > 0$, filled with the fluid, together with the vector fields of velocity $\boldsymbol{v}(x,t)$, magnetic and electric fields $\boldsymbol{H}(x,t)$, $\boldsymbol{E}(x,t)$ and the pressure function $p(x,t)$. We assume that the fluid is subject to the mass forces, capillary force at the free boundary Γ_t and forces due to the presence of the magnetic and electric fields that are generated by the electric current $\boldsymbol{j}(x,t)$ given in a fixed domain Ω_3 that is bounded away from Ω_{1t}. The function $\boldsymbol{j}(x,t)$ satisfies the condition

$$\boldsymbol{j}(x,t)\cdot\boldsymbol{n}(x) = 0, \quad x \in \partial\Omega_3 \equiv S_3 \tag{1.1}$$

and vanishes outside Ω_3. Both Ω_{1t} and Ω_3 are surrounded by a vacuum region Ω_{2t}. The domain $\Omega = \bar{\Omega}_{1t} \cup \bar{\Omega}_3 \cup \Omega_{2t}$ is bounded by a perfectly conducting surface S. The governing equations are the Navier-Stokes equations with the magnetic field $\boldsymbol{H}$ and the Maxwell equations without displacement current (i.e., without the derivative $\boldsymbol{E}_t$) - see [4.5]:

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla\cdot T(\boldsymbol{v},p) - \nabla\cdot T_M(\boldsymbol{H}) = \boldsymbol{f}(x,t), \\ \nabla\cdot\boldsymbol{v}(x,t) = 0, \quad x \in \Omega_{1t}, \ t > 0, \end{cases} \tag{1.2}$$

$$\begin{cases} \mu\boldsymbol{H}_t = -\mathrm{rot}\boldsymbol{E}, \quad \nabla\cdot\boldsymbol{H} = 0, \qquad x \in \Omega_{1t}\cup\Omega_{2t}\cup\Omega_3, \\ \mathrm{rot}\boldsymbol{H} = \alpha_1(\boldsymbol{E} + \mu_1(\boldsymbol{v}\times\boldsymbol{H})), \quad x \in \Omega_{1t}, \ t > 0, \\ \mathrm{rot}\boldsymbol{H} = \alpha_3\boldsymbol{E} + \boldsymbol{j}(x,t), \quad x \in \Omega_3, \\ \mathrm{rot}\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{E} = 0, \quad x \in \Omega_{2t}. \end{cases} \tag{1.3}$$

These equations are completed by the following boundary and initial conditions:

$$\begin{cases} \big(T(\boldsymbol{v},p) + [T_M(\boldsymbol{H})]\big)\boldsymbol{n} = \sigma\boldsymbol{n}H, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x \in \Gamma_t, \\ \mathfrak{n}_t[\mu\boldsymbol{H}] + [\mathfrak{n}_x \times \boldsymbol{E}] = 0, \quad x \in \Gamma_t, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad x \in \Gamma_t \cup S_3, \quad [\boldsymbol{E}_\tau] = 0, \quad x \in S_3, \\ \boldsymbol{H}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{E}_\tau = 0, \quad x \in S, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_{10}, \quad \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x), \quad x \in \Omega_{10}\cup\Omega_{20}\cup\Omega_3. \end{cases} \tag{1.4}$$

.

Here $T(\boldsymbol{v},p)$ is the viscous stress tensor: $T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$, $S(\boldsymbol{v}) = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T$ is the doubled rate-of-strain tensor, $T_M(\boldsymbol{H}) = \mu(\boldsymbol{H}\otimes\boldsymbol{H} - \frac{1}{2}|\boldsymbol{H}|^2 I)$, is the magnetic stress tensor, μ and α are piece-wise constant functions equal to μ_i and α_i in Ω_{it}, $\mu = \mu_3$ in Ω_3, $\alpha = 0$ in Ω_{2t}, $\boldsymbol{n}$ is the normal to Γ_t, S_3, S, exterior with respect to Ω_{1t}, Ω_3, Ω, V_n is the velocity of evolution of Γ_t in the direction $\boldsymbol{n}$, $\boldsymbol{H}_\tau = \boldsymbol{H} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{H})$ is the tangential component of $\boldsymbol{H}$, H is the doubled mean curvature of Γ_t negative for convex domains. The parameters ν, μ_i, α_1, α_3, σ are positive constants and $[u]$ is the jump of the function $u(x)$, $x \in \Omega_i$, $i = 1,2,3$, on Γ_t or S_3. We set $[u]|_{\Gamma_t} = u^{(1)} - u^{(2)}$, $[u]|_{S_3} = u^{(3)} - u^{(2)}$, $u^{(i)} = u(x,t)|_{x\in\bar{\Omega}_i}$.

By $\mathfrak{n}_x = (\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3)$ and $\mathfrak{n}_t$ we mean the components of the normal vector $\mathfrak{n}$ to the surface $\mathfrak{G} = \{x \in \Gamma_t, t > 0\}$ in $\mathbb{R}^4$.

We assume that Γ_0 is located in the neighborhood of a smooth connected surface $\mathcal{G}$ of arbitrary shape and can be regarded as a normal perturbation of $\mathcal{G}$:

$$\Gamma_0 = \{x = y + \boldsymbol{N}(y)\rho_0(y), \quad y \in \mathcal{G}\},$$

where ρ_0 is a given small function and $\boldsymbol{N}(y)$ is the normal to $\mathcal{G}$ exterior with respect to Ω_0. Moreover, we assume that also for $t > 0$

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho(y,t), \quad y \in \mathcal{G}\},$$

with an unknown function $\rho(y,t)$ such that $\rho(y,0) = \rho_0(y)$. We extend $\boldsymbol{N}(y)$ and $\rho(y,t)$ from $\mathcal{G}$ into Ω in such a way that the extension $\boldsymbol{N}^*$ of $\boldsymbol{N}$ is a smooth non-zero regular function in Ω and ρ^* vanishes near $S \cup S_3$ and satisfies the inequalities (1.23) (hence ρ^* is small for small ρ).

Let $\mathcal{F}_1$ be the domain bounded by $\mathcal{G}$, $\mathcal{F}_3 = \Omega_3$, $\mathcal{F}_2 = \Omega \setminus (\bar{\mathcal{F}}_1 \cup \bar{\mathcal{F}}_3)$. The transformation

$$x \equiv e_\rho(y,t) = y + \boldsymbol{N}^*(y)\rho^*(y,t) \tag{1.5}$$

maps $\mathcal{F}_1$ on Ω_{1t}, $\mathcal{F}_2$ on Ω_{2t}, $\mathcal{F}_3$ on itself and, as shown in [2], it converts (1.2)-(1.4) in

$$\begin{cases} \boldsymbol{u}_t - \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u} + (\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \\ \quad -\widetilde{\nabla}\cdot\widetilde{T}(\boldsymbol{u},q) - \widetilde{\nabla}\cdot T_M(\dfrac{\mathcal{L}}{L}\boldsymbol{h}) = \boldsymbol{f}(e_\rho,t), \\ \nabla\cdot\widehat{\mathcal{L}}\boldsymbol{u} = 0, \quad y\in\mathcal{F}_1, \ \ t>0, \\ \widetilde{T}(\boldsymbol{u},q)\boldsymbol{n}(e_\rho,t) + [T(\dfrac{\mathcal{L}}{L}\boldsymbol{h})\boldsymbol{n}(e_\rho,t)] = \sigma H\boldsymbol{n}, \quad \rho_t = \dfrac{\boldsymbol{u}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}{\Lambda(y,\rho)}, \quad y\in\mathcal{G}, \\ \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y) = \boldsymbol{v}_0(e_{\rho_0}), \quad y\in\mathcal{F}_1, \quad \rho(y,0) = \rho_0(y), \quad y\in\mathcal{G}, \end{cases} \tag{1.6}$$

$$\begin{cases} \mu(\boldsymbol{h}_t - \dfrac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h} - \rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\dfrac{1}{L}\mathcal{L}\boldsymbol{h}) = -\mathrm{rot}\mathcal{P}\boldsymbol{e}, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \\ \mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h} = \alpha(\mathcal{P}\boldsymbol{e} + \mu(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y\in\mathcal{F}_1, \\ \mathrm{rot}\mathcal{P}\boldsymbol{h} = 0, \quad \nabla\cdot\boldsymbol{h} = 0, \quad \nabla\cdot\boldsymbol{e} = 0, \quad x\in\mathcal{F}_2, \\ \mathrm{rot}\boldsymbol{h} = \alpha\boldsymbol{e} + \boldsymbol{j}(y,t), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y\in\mathcal{F}_3, \end{cases} \tag{1.7}$$

$$\begin{cases} \boldsymbol{h}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{e}_\tau = 0, \quad y\in S, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad [\boldsymbol{e}_\tau] = 0, \quad y\in S_3, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h} - \dfrac{\widetilde{\mathcal{L}}\widetilde{\mathcal{L}}^T\boldsymbol{N}}{|\widetilde{\mathcal{L}}^T\boldsymbol{N}|^2}(\boldsymbol{h}\cdot\boldsymbol{N})] = 0, \\ -\Lambda(y,\rho)\rho_t[\mu\boldsymbol{h}] + L[\boldsymbol{N}\times\mathcal{P}\boldsymbol{e}] = 0, \quad y\in\mathcal{G}, \\ \boldsymbol{h}_0(y,0) = \boldsymbol{h}_0(y) = \widehat{\mathcal{L}}(y,\rho_0^*)\boldsymbol{H}(e_{\rho_0}), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \end{cases} \tag{1.8}$$

where

$$\boldsymbol{u}(y,t) = \boldsymbol{v}(e_\rho,t), \quad q(y,t) = p(e_\rho,t), \quad \boldsymbol{h} = \widehat{\mathcal{L}}\boldsymbol{H}(e_\rho,t), \quad \boldsymbol{e} = \widehat{\mathcal{L}}\boldsymbol{E}(e_\rho,t), \tag{1.9}$$

$\mathcal{L} = \mathcal{L}(y,\rho^*) = (l_{ij})_{i,j=1,2,3}$ is the Jacobi matrix of the transformation (1.5), $L = det\mathcal{L}$,
$\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$ is the co-factors matrix of $\mathcal{L}$;
$\mathcal{P}(y,\rho^*) = \frac{\mathcal{L}^T}{L}\mathcal{L}$,
$\widetilde{\nabla} = \mathcal{L}^{-T}\nabla_y$ is the transformed gradient ∇_x ("T" means transposition, $\mathcal{L}^{-T} = (\mathcal{L}^{-1})^T$);
$\widetilde{S}(\boldsymbol{u}) = \widetilde{\nabla}\boldsymbol{u} + (\widetilde{\nabla}\boldsymbol{u})^T$ is the transformed rate-of-strain tensor,
$\widetilde{T}(\boldsymbol{u},q) = -qI + \nu\widetilde{S}(\boldsymbol{u})$ is the transformed stress tensor,
$\Lambda(y,\rho) = \boldsymbol{N}(y)\cdot\widehat{\mathcal{L}}(y,\rho)\boldsymbol{N}(y) = 1 - \rho\mathcal{H}(y) + \rho^2\mathcal{K}(y)$
and $\mathcal{H}$, $\mathcal{K}$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively.

We study the problem (1.6)-(1.8) in the spaces

$$W_p^{2,1}(D_T)=W_p^{2,0}(D_T)\cap W_p^{0,1}(D_T),\quad p>3,\quad D_T=D\times(0,T),\quad D\subset\mathbb{R}^n,$$

where $W_p^{2,0}(D_T)=L_p(0,T;W_p^2(D))$, $W_2^{0,1}(D_T)=W_p^1(0,T;L_p(D))$. The norm in $W_p^{2,1}(D_T)$ is defined by

$$\|u\|^p_{W_p^{2,1}(D_T)}=\int_0^T\|u(\cdot,t)\|^p_{W_p^2(D)}dt+\int_0^T(\|u_t(\cdot,t)\|^p_{L_p(D)}+\|u(\cdot,t)\|^p_{L_p(D)})dt, \tag{1.10}$$

and the integrals in the right-hand side represent the norms in $W_p^{2,0}(D_T)$ and $W_p^{0,1}(D_T)$, respectively.

By $W_p^l(D)$ we mean the space of functions $u(x)$, $x\in D$, with finite norm

$$\|u\|_{W_p^l(\Omega)}=\Big(\sum_{|j|\le l}\|D^ju\|^p_{L_p(D)}\Big)^{1/p},$$

if l is an integer, and

$$\|u\|_{W_p^l(D)}=\Big(\|u\|^p_{W_p^{[l]}(D)}+\sum_{|j|=[l]}\int_D\int_D\frac{|D^ju(x)-D^ju(y)|^p}{|x-y|^{n+p\lambda}}dxdy\Big)^{1/p},$$

if $l=[l]+\lambda$, $0<\lambda<1$. The anisotropic space $W_p^{l,l/2}(D_T)$ is defined as $W_p^{l,0}(D_T)\cap W_p^{0,1/2}(D_T)=L_p(0,T;W_p^l(D))\cap W_p^{l/2}(0,T;L_p(D))$.

The spaces W_p^l, $W_p^{l,l/2}$ on smooth manifolds are defined, as usual, with the help of local maps and partition of unity.

We recall the trace and extension theorems for the space $W_p^{2,1}(D_T)$ (see [6,7]): if $u\in W_p^{2,1}(D_T)$, then $u|_{t=t_0}\in W_p^{2-2/p}(D)$, $u|_{x\in\partial D}\in W_p^{2-1/p,1-1/(2p)}(\partial D\times(0,T))$, and

$$\|u(\cdot,t_0)\|_{W_p^{2-2/p}(D)}\le c\|u\|_{W_p^{2,1}(D_T)},$$

$$\|u\|_{W_p^{2-1/p,1-1/(2p)}(\partial D\times(0,T))}\le c\|u\|_{W_p^{2,1}(D_T)}.$$

For arbitrary $\varphi\in W_p^{2-2/p}(D)$ there exists such $u\in W_p^{2,1}(D_T)$ that $u(x,0)=\varphi(x)$ and

$$\|u\|_{W_p^{2,1}(D_T)}\le c\|\varphi\|_{W_p^{2-2/p}(D)};$$

for arbitrary $\psi\in W_p^{2-1/p,1-1/(2p)}(\partial D\times(0,T))$ there exists such $u\in W_p^{2,1}(D_T)$ that $u|_{x\in\partial D}=\psi(x,t)$ and

$$\|u\|_{W_p^{2,1}(D_T)}\le c\|\psi\|_{W_p^{2-1/p,1-1/(2p)}(\partial D\times(0,T))}.$$

Moreover, if $\rho\in W_p^{3-1/p,0}(\partial D\times(0,T))\cap W_p^1(0,T;W_p^{2-1/p}(\partial D))$, then $\rho(\cdot,t_0)\in W_p^{3-2/p}(\partial D)$ and

$$\|\rho(\cdot,t_0)\|_{W_p^{3-2/p}(\partial D)}\le c(\|\rho\|_{W_p^{3-1/p,0}(\partial D\times(0,T))}+\|\rho_t\|_{W_p^{2-1/p,0}(\partial D\times(0,T))}). \tag{1.11}$$

For arbitrary $\rho_0\in W_p^{3-2/p}(\partial D)$ there exists $\rho\in W_p^{3-1/p,0}(\partial D\times(0,T))$ with $\rho_t\in W_p^{2-1/p,1-1/2p}(\partial D\times(0,T))$ such that $\rho(x,0)=\rho_0(x)$ and

$$\|\rho\|_{W_p^{3-1/p,0}(\partial D\times(0,T))}+\|\rho_t\|_{W_p^{2-1/p,1-1/2p}(\partial D\times(0,T))}\le c\|\rho_0\|_{W_p^{3-2/p}(\partial D)}.$$

Following [8,9], we introduce some spaces of divergence free vector fields. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary. We define finite-dimensional spaces

$$U_n(\mathcal{D}) = \{\boldsymbol{u} \in W^1_p(\mathcal{D}), \quad \nabla \cdot \boldsymbol{u} = 0, \quad \mathrm{rot}\boldsymbol{u} = 0, \quad \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\mathcal{D}} = 0\},$$
$$U_d(\mathcal{D}) = \{\boldsymbol{u} \in W^1_p(\mathcal{D}), \quad \nabla \cdot \boldsymbol{u} = 0, \quad \mathrm{rot}\boldsymbol{u} = 0, \quad \boldsymbol{u}_\tau|_{\partial\mathcal{D}} = 0\}$$

of the Neumann and Dirichlet vector fields. The dimensions of these spaces are equal to the first and the second Betti number of the domain $\mathcal{D}$, $b_1(\mathcal{D})$ and $b_2(\mathcal{D})$, respectively.

If $b_1(\mathcal{D}) > 0$, then there exist $b_1(\mathcal{D})$ smooth closed contours in $\mathcal{D}^c = \mathbb{R}^3 \setminus \overline{\mathcal{D}}$, generating the first homology group of $\mathcal{D}^c$, and every such contour Λ generates the Neumann vector field of the form

$$\boldsymbol{u}(x) = \boldsymbol{u}_1(x) + \boldsymbol{u}_2(x), \quad x \in \mathcal{D}, \tag{1.12}$$

where $\boldsymbol{u}_1$ is defined through the Bio-Savard law:

$$\boldsymbol{u}_1(x) = \int\limits_\Lambda \frac{x-y}{|x-y|^3} \times d\boldsymbol{l}_y,$$

and $\boldsymbol{u}_2 = \nabla\varphi$,

$$\nabla^2\varphi(x) = 0, \quad x \in \mathcal{D}, \quad \frac{\partial\varphi}{\partial n}|_{\partial\mathcal{D}} = -\boldsymbol{u}_1 \cdot \boldsymbol{n}|_{\partial\mathcal{D}}.$$

As for $b_2(\mathcal{D})$, this number is equal to the number of the connected components of the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ minus 1, and if $\partial\mathcal{D} = \bigcup\limits_{k=0}^{b_2(\mathcal{D})} \Sigma_k$, then the basis in $U_d(\mathcal{D})$ is formed by the vector fields $\boldsymbol{v}_k = \nabla\Phi_k$, where Φ_k are the solutions to the problems

$$\nabla^2\Phi_k = 0, \quad \Phi_k|_{\Sigma_j} = \delta_{jk}, \quad k, j = 1, \dots, b_2(\mathcal{D}), \quad \Phi_k|_{\Sigma_0} = 0. \tag{1.13}$$

If $\Sigma_k \in C^m$, then $\boldsymbol{v}_k$, $\boldsymbol{u}_j \in W^{m-1}_p(\mathcal{D})$ for arbitrary $p > 1$.

For the domain $\Omega = \bar{\mathcal{F}}_1 \cup \mathcal{F}_2 \cup \bar{\mathcal{F}}_3$ introduced above we have

$$b_1(\mathcal{F}_2) = b_1(\Omega) + b_1(\mathcal{F}_1 \cup \mathcal{F}_3).$$

The elements $\boldsymbol{u}_j \in U_n(\mathcal{F}_2)$ can be generated by the contours $\Lambda_j \subset \mathbb{R}^3 \setminus \bar{\Omega}$ (in this case $j = 1, ..., b_1(\Omega)$) or $\Lambda_j \subset \mathcal{F}_1$, $\Lambda_j \subset \mathcal{F}_3$ ($j = b_1(\Omega) + 1, ..., b_1(\Omega) + b_1(\mathcal{F}_1 \cup \mathcal{F}_3)$). It is easily seen that in the first case $\boldsymbol{u}^{\mathcal{F}}_j \in U_n(\mathcal{F}_2)$ are connected with $\boldsymbol{u}^{\Omega}_j \in U_n(\Omega)$ by

$$\boldsymbol{u}^{\mathcal{F}}_j(x) = \boldsymbol{u}^{\Omega}_j(x) + \nabla\omega_j(x), \quad x \in \mathcal{F}_2,$$
$$\nabla^2\omega_j(x) = 0, \quad x \in \mathcal{F}_2, \quad \frac{\partial\omega_j(x)}{\partial n} = 0, \quad x \in S,$$
$$\frac{\partial\omega_j(x)}{\partial N} = -\boldsymbol{u}^{\Omega}_j \cdot \boldsymbol{N}, \quad x \in \mathcal{G}, \quad \frac{\partial\omega_j(x)}{\partial n} = -\boldsymbol{u}^{\Omega}_j \cdot \boldsymbol{n}, \quad x \in S_3.$$

Finally, we introduce in $\Omega = \bar{\mathcal{F}}_1 \cup \mathcal{F}_2 \cup \bar{\mathcal{F}}_3$ the space $\widetilde{U}_n(\Omega)$ of the modified Neumann vector fields $\widetilde{\boldsymbol{u}}_q$, $q = 1, ..., b_1(\Omega)$, satisfying the conditions

$$\begin{cases} \mathrm{rot}\widetilde{\boldsymbol{u}}_q(x) = 0, \quad \nabla \cdot \widetilde{\boldsymbol{u}}_q(x) = 0, \quad x \in \mathcal{F}_i, \quad i = 1, 2, 3, \\ [\mu\widetilde{\boldsymbol{u}}_q \cdot \boldsymbol{N}] = 0, \quad [\widetilde{\boldsymbol{u}}_{q,\tau}] = 0, \quad x \in \mathcal{G}, \quad [\mu\widetilde{\boldsymbol{u}}_q \cdot \boldsymbol{n}] = 0, \quad [\widetilde{\boldsymbol{u}}_{q,\tau}] = 0, \quad x \in S_3, \\ \widetilde{\boldsymbol{u}}_q \cdot \boldsymbol{n} = 0, \quad x \in S. \end{cases} \tag{1.14}$$

It is clear that

$$\widetilde{\boldsymbol{u}}_q = \boldsymbol{u}_q + \nabla\phi_q(x), \quad \boldsymbol{u}_q \in U_n(\Omega),$$

$$\nabla^2\phi_q(x) = 0, \quad x \in \mathcal{F}_i. \quad i = 1,2,3, \quad \frac{\partial\phi_q}{\partial n} = 0, \quad x \in S,$$

$$[\phi_q] = 0, \quad [\mu\frac{\partial\phi_q}{\partial N}] = -[\mu]\boldsymbol{u}_q \cdot \boldsymbol{N}, \quad x \in \mathcal{G}, \quad [\phi_q] = 0, \quad [\mu\frac{\partial\phi_q}{\partial n}] = -[\mu]\boldsymbol{u}_q \cdot \boldsymbol{n}, \quad x \in S_3.$$

Let us go back to the problem (1.6)-(1.8). We separate the determination of $(\boldsymbol{u}, q, \rho, \boldsymbol{h})$ from that of $\boldsymbol{e}$. It is easily seen that (1.7), (1.8) imply

$$\begin{cases} \mu(\boldsymbol{h}_t - \boldsymbol{\Phi}) + \alpha^{-1}\mathrm{rot}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h} = \mathrm{rot}\boldsymbol{J}, \quad \nabla\cdot\boldsymbol{h}(y,t) = 0, \quad y \in \mathcal{F}_1 \cup \mathcal{F}_3, \\ \mathrm{rot}\mathcal{P}\boldsymbol{h}(y,t) = 0, \quad \nabla\cdot\boldsymbol{h} = 0, \quad y \in \mathcal{F}_2, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \Big(\dfrac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2} - \boldsymbol{N}\Big)[\boldsymbol{h}\cdot\boldsymbol{N}], \quad y \in \mathcal{G}, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad y \in S_3, \quad \boldsymbol{h}\cdot\boldsymbol{n} = 0, \quad y \in S, \\ \boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3, \end{cases} \tag{1.15}$$

where $\boldsymbol{J} = \mu_1\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}$ in $\mathcal{F}_1$, $\boldsymbol{J} = \alpha^{-1}\boldsymbol{j}(y,t)$ in $\mathcal{F}_3$, and

$$\boldsymbol{\Phi} = \frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h} + \rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h}.$$

As shown in [2], the vector field $\boldsymbol{\Phi}$ is divergence free.

In the case $b_1(\mathcal{F}_2) > 0$ equations (1.15) should be supplemented by some orthogonality conditions. Multiplying the first equation in (1.7) by $\boldsymbol{u}_q^{\mathcal{F}} \in U_n(\mathcal{F}_2)$ and integrating over $\mathcal{F}_2$ we obtain

$$\begin{aligned} &\int_{\mathcal{F}_2}\mu(\boldsymbol{h}_t - \boldsymbol{\Phi})\cdot\boldsymbol{u}_q^{\mathcal{F}}dy = -\int_{\mathcal{F}_2}\mathrm{rot}\mathcal{P}\boldsymbol{e}^{(2)}\cdot\boldsymbol{u}_q^{\mathcal{F}}dy = \int_{\mathcal{G}}(\boldsymbol{N}\times\mathcal{P}\boldsymbol{e}^{(2)})\cdot\boldsymbol{u}_q^{\mathcal{F}}dS \\ &+\int_{S_3}(\boldsymbol{n}\times\boldsymbol{e}^{(2)})\cdot\boldsymbol{u}_q^{\mathcal{F}}dS = \int_{\mathcal{G}}(\boldsymbol{N}\times\mathcal{P}\boldsymbol{e}^{(1)})\cdot\boldsymbol{u}_q^{\mathcal{F}}dS + \int_{S_3}(\boldsymbol{n}\times\boldsymbol{e}^{(3)})\cdot\boldsymbol{u}_q^{\mathcal{F}}dS - \int_{\mathcal{G}}\boldsymbol{\Psi}\cdot\boldsymbol{u}_q^{\mathcal{F}}dS \\ &= \int_{\mathcal{G}}(\boldsymbol{N}\times(\alpha^{-1}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h}^{(1)} - \boldsymbol{J}))\cdot\boldsymbol{u}_q^{\mathcal{F}}dS + \int_{S_3}(\boldsymbol{n}\times\alpha^{-1}(\mathrm{rot}\boldsymbol{h}^{(3)} - \boldsymbol{j}(y,t)))\cdot\boldsymbol{u}_q^{\mathcal{F}}(y,t)dS \\ &-\int_{\mathcal{G}}\boldsymbol{\Psi}\cdot\boldsymbol{u}_q^{\mathcal{F}}dS, \quad q = 1,...b_1(\mathcal{F}_2), \end{aligned} \tag{1.16}$$

where $\boldsymbol{h}^{(i)} = \boldsymbol{h}|_{y\in\overline{\mathcal{F}}_i}$, $\boldsymbol{e}^{(i)} = \boldsymbol{e}|_{y\in\overline{\mathcal{F}}_i}$,

$$\boldsymbol{\Psi} = \frac{\rho_t\Lambda(y,\rho)}{L(y,\rho)}[\mu\boldsymbol{h}], \quad y \in \mathcal{G}.$$

The same equations are satisfied for $\boldsymbol{u}_q^{\Omega}(x)$, $q = 1,...,b_1(\Omega)$:

$$\begin{aligned} &\int_{\mathcal{F}_2}\mu(\boldsymbol{h}_t - \boldsymbol{\Phi})\cdot\boldsymbol{u}_q^{\Omega}dy = \int_{\mathcal{G}}\boldsymbol{N}\times(\alpha^{-1}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h}^{(1)} - \boldsymbol{J})\cdot\boldsymbol{u}_q^{\Omega}dS \\ &+\int_{S_3}\boldsymbol{n}\times\alpha^{-1}(\mathrm{rot}\boldsymbol{h}^{(3)} - \boldsymbol{j}(y,t))\cdot\boldsymbol{u}_q^{\Omega}(y,t)dS - \int_{\mathcal{G}}\boldsymbol{\Psi}\cdot\boldsymbol{u}_q^{\Omega}dS \end{aligned} \tag{1.17}$$

and, since

$$\int_{\mathcal{F}_1\cup\mathcal{F}_3}\mu(\boldsymbol{h}_t - \boldsymbol{\Phi})\cdot\boldsymbol{u}_q^{\Omega}dy = \int_{\mathcal{F}_1\cup\mathcal{F}_3}(-\alpha^{-1}\mathrm{rot}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h} + \mathrm{rot}\boldsymbol{J})\cdot\boldsymbol{u}_q^{\Omega}dy$$

$$= -\int_{\mathcal{G}} \boldsymbol{N} \times (\alpha^{-1}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h}^{(1)} - \boldsymbol{J}) \cdot \boldsymbol{u}_q^{\Omega} dS - \int_{S_3} \boldsymbol{n} \times \alpha^{-1}(\mathrm{rot}\boldsymbol{h}^{(3)} - \boldsymbol{j}) \cdot \boldsymbol{u}_q^{\Omega} dS,$$

it holds

$$\int_{\Omega} \mu(\boldsymbol{h}_t - \boldsymbol{\Phi}) \cdot \boldsymbol{u}_q^{\Omega} dy = -\int_{\mathcal{G}} \boldsymbol{\Psi} \cdot \boldsymbol{u}_q^{\Omega} dS, \quad q = 1, ..., b_1(\Omega). \tag{1.18}$$

Hence, in addition to (1.15), we have (1.16), (1.18).

Equations (1.6), (1.15), (1.16) constitute the main problem for $\boldsymbol{u}, q, \rho, \boldsymbol{h}$ that is solved in Sec. 2 and 3. Making use of (1.18), we construct in Sec.4 $\boldsymbol{e}(y,t)$ satisfying (1.7), (1.8).

Theorem 1. *Let $\boldsymbol{u}_0 \in W_p^{2-2/p}(\mathcal{F}_1)$, $\rho_0 \in W_p^{3-2/p}(\mathcal{G})$, $\boldsymbol{h}_0 \in W_r^{2-2/r}(\mathcal{F}_i)$, $i = 1,2,3$, with*

$$3 < r < p, \quad 1/r - 1/p \le 1/5(1 - 2/p), \tag{1.19}$$

and let the compatibility conditions

$$\begin{aligned}
&\widetilde{\nabla} \cdot \boldsymbol{u}_0 = 0, \quad y \in \mathcal{F}_1, \quad \widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n}_0(e_{\rho_0}) - \boldsymbol{n}_0(\boldsymbol{n}_0 \cdot \widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n}_0) = 0, \quad y \in \mathcal{G},\\
&\nabla \cdot \boldsymbol{h}_0 = 0, \quad y \in \mathcal{F}_i, \quad i = 1,2,3,\\
&\mathrm{rot}\mathcal{P}(y, \rho_0^*)\boldsymbol{h}_0 = 0, \quad y \in \mathcal{F}_2,\\
&[\mu \boldsymbol{h}_0 \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{h}_{0\tau}] = \big(\frac{\widehat{\mathcal{L}}(y,\rho_0)\widehat{\mathcal{L}}^T \boldsymbol{N}}{|\widehat{\mathcal{L}}^T \boldsymbol{N}|^2} - \boldsymbol{N}\big)[\boldsymbol{h}_0 \cdot \boldsymbol{N}], \quad y \in \mathcal{G},\\
&[\mu \boldsymbol{h}_0 \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{h}_{0\tau}] = 0, \quad y \in S_3, \quad \boldsymbol{h}_0 \cdot \boldsymbol{n} = 0, \quad y \in S,
\end{aligned} \tag{1.20}$$

where $\boldsymbol{n}_0$ is the normal to Γ_0, as well as the smallness condition

$$\|\rho_0\|_{W_p^{2-1/p}(\mathcal{G})} \le \epsilon \ll 1 \tag{1.21}$$

be satisfied. Assume also that

$$\boldsymbol{f} \in W_p^1(\mathbb{R}^3), \quad \forall t \in (0, T_0), \quad \boldsymbol{j} \in W_r^{1,0}(\mathcal{F}_3 \times (0, T_0)), \tag{1.22}$$

(1.1) holds and the extension ρ^* of ρ in (1.5) satisfies the conditions

$$\begin{aligned}
&\frac{\partial \rho^*(y,t)}{\partial N}\Big|_{\mathcal{G}} = 0, \quad \rho^*(y,t) = 0 \ \text{near} \ S \cup S_3 \ \text{and in} \ \mathcal{F}_3,\\
&\|\rho^*(\cdot,t)\|_{W_p^l(\Omega)} \le c\|\rho\|_{W_p^{l-1/p}(\mathcal{G})}, \quad l \in (1/p, 3],\\
&\|\rho_t^*(\cdot,t)\|_{W_p^l(\Omega)} \le c\|\rho_t\|_{W_p^{l-1/p}(\mathcal{G})}, \quad r \in (1/p, 2].
\end{aligned} \tag{1.23}$$

Then the problem (1.6), (1.15), (1.16) has a unique solution defined in a certain (small) time interval $(0,T)$, $T \le T_0$, with the following regularity properties:

$$\begin{aligned}
&\boldsymbol{u} \in W_p^{2.1}(Q_T^1), \quad \nabla q \in L_p(Q_T^1), \quad q \in W_p^{l-1/p,0}(G_T), \quad \rho \in W_p^{3-3/p,0}(G_T),\\
&\rho_t \in W_p^{2-1/p,1-1/2p}(G_T), \quad \boldsymbol{h} \in W_r^{2,1}(Q_T^i), \quad i = 1,2,3,
\end{aligned}$$

where $Q_T^i=\mathcal{F}_i\times(0,T)$, $G_T=\mathcal{G}\times(0,T)$. *The solution satisfies the inequality*

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)}+\|\nabla q\|_{L_p(Q_T^1)}+\|q\|_{W_p^{1-1/p,0}(G_T)}+\sup_{t<T}\|\boldsymbol{u}(\cdot,t)\|_{W_p^{2-2/p}(\mathcal{F}_1)}+\sup_{t<T}\|\rho\|_{W_p^{3-2/p}(\mathcal{G})}\\
&+\|\rho\|_{W_p^{3-1/p,0}(G_T)}+\|\rho_t\|_{W_p^{2-1/p,1-1/2p}(G_T))}+\sum_{i=1}^{3}\|\boldsymbol{h}^{(i)}\|_{W_r^{2,1}(Q_T^i)}\\
&+\sum_{i=1}^{3}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^{2-2/r}(\mathcal{F}_i)}\Big)\le c\Big(\|\boldsymbol{f}\|_{L_p(Q_T^1)}+\|\mathcal{H}\|_{W_p^{1-1/p}(\mathcal{G})}+\|\boldsymbol{u}_0\|_{W_p^{2-2/p}(\mathcal{F}_1)}\\
&+\|\rho_0\|_{W_p^{3-2/p}(\mathcal{G})}+\sum_{i=1}^{3}\|\boldsymbol{h}_0\|_{W_r^{2-2/r}(\Omega_{i0})}+\|\boldsymbol{j}\|_{W_r^{1,0}(Q_T^3)}\Big).
\end{aligned}
\tag{1.24}
$$

Once the solution with the above-mentioned properties is obtained, it can be shown that

$$
\boldsymbol{h}^{(i)}\in W_p^{2,1}(Q_T^i),\quad i=1,2,3,
$$

provided $\boldsymbol{h}_0\in W_p^{2-2/p}(\Omega_{i0})$, $i=1,2,3$, $\boldsymbol{j}\in W_p^{1,0}(Q_T^3)$ (cf. [3, Sec. 5]).

2 Linear problems

The proof of Theorem 1 is based on the analysis of the following non-homogeneous linear problems:

1. Find $(\boldsymbol{v},p,\rho)$ such that

$$
\begin{cases}
\boldsymbol{v}_t-\nu\nabla^2\boldsymbol{v}+\nabla p=\boldsymbol{f}(y,t),\\
\nabla\cdot\boldsymbol{v}=f(y,t)=\nabla\cdot\boldsymbol{F}(y,t),\quad y\in\mathcal{F}_1,\ \ t>0,\\
T(\boldsymbol{v},p)\boldsymbol{N}(y)+\sigma\boldsymbol{N}(y)\mathfrak{B}\rho=\boldsymbol{d}(y,t),\\
\rho_t+\boldsymbol{V}(x)\cdot\nabla_\tau\rho-\boldsymbol{v}\cdot\boldsymbol{N}(y)=g(y,t),\quad y\in\mathcal{G},\\
\boldsymbol{v}(y,0)=\boldsymbol{v}_0(y),\quad y\in\mathcal{F}_1,\quad \rho(y,0)=\rho_0(y),\quad y\in\mathcal{G},
\end{cases}
\tag{2.1}
$$

where $\mathfrak{B}\rho=-\Delta_{\mathcal{G}}\rho-b(y)\rho$, $b=(\mathcal{H}^2-2\mathcal{K})$, $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$, $\boldsymbol{V}$ is a given vector field from $W_p^{2-1/p}(\mathcal{G})$.

2. Find the vector field $\boldsymbol{H}(y,t)$, satisfying the equations

$$
\begin{cases}
\mu\boldsymbol{H}_t(y,t)+\alpha^{-1}\mathrm{rotrot}\boldsymbol{H}(y,t)=\boldsymbol{G}(y,t),\quad \nabla\cdot\boldsymbol{H}(y,t)=0,\quad y\in\mathcal{F}_1\cup\mathcal{F}_3,\\
\mathrm{rot}\boldsymbol{H}(y,t)=\mathrm{rot}\boldsymbol{\ell}(y,t),\quad \nabla\cdot\boldsymbol{H}(y,t)=0,\quad y\in\mathcal{F}_2\\
[\mu\boldsymbol{H}\cdot\boldsymbol{N}]=0,\quad [\boldsymbol{H}_\tau]=\boldsymbol{a}(y,t),\quad y\in\mathcal{G},\\
[\mu\boldsymbol{H}\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{H}_\tau]=0,\qquad y\in S_3,\quad \boldsymbol{H}\cdot\boldsymbol{n}=0,\quad y\in S,\\
\displaystyle\int_{\mathcal{F}_2}\mu\boldsymbol{H}_t\cdot\boldsymbol{u}_q^{\mathcal{F}}dy=\int_{\mathcal{G}}(\boldsymbol{N}\times\alpha^{-1}\mathrm{rot}\boldsymbol{H}^{(1)})\cdot\boldsymbol{u}_q^{\mathcal{F}}(y)dS\\
\displaystyle+\int_{S_3}(\boldsymbol{n}\times\alpha^{-1}\mathrm{rot}\boldsymbol{H}^{(3)})\cdot\boldsymbol{u}_q^{\mathcal{F}}(y)dS+\boldsymbol{M}_q(t),\quad q=1,...,b_1(\mathcal{F}_2),\\
\boldsymbol{H}(y,0)=\boldsymbol{H}_0(y),\quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3.
\end{cases}
\tag{2.2}
$$

In addition, we need to consider the auxiliary problem

3.
$$\begin{cases} \mathrm{rot}\boldsymbol{h}(y) = \boldsymbol{k}(y), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \boldsymbol{a}, \quad y\in\mathcal{G}, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad y\in S_3, \\ \boldsymbol{h}\cdot\boldsymbol{n}(y) = 0, \quad y\in S. \end{cases} \tag{2.3}$$

Theorem 2. *Assume that*

$$\boldsymbol{f}\in L_p(Q_T^1), \quad f\in W_p^{1,0}(Q_T^1), \quad f = \nabla\boldsymbol{F}, \quad \boldsymbol{F}\in W_p^{0,1}(Q_T^1),$$

$$\boldsymbol{d}\cdot\boldsymbol{N}\in W_2^{1-1/p,0}(G_T), \quad \boldsymbol{d}-\boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})\equiv\boldsymbol{d}_\tau\in W_p^{1-1/p,1/2-1/(2p)}(G_T),$$

$$\boldsymbol{g}\in W_p^{2-1/p,1-1/2p}(G_T), \quad \boldsymbol{v}_0\in W_p^{2-2/p}(\mathcal{F}_1),$$

$$\rho_0\in W_p^{3-2/p}(\mathcal{G}), \quad \boldsymbol{V}\in W_p^{2-1/p}(\mathcal{G})$$

with $p>3$ and let the compatibility conditions

$$\nabla\cdot\boldsymbol{v}_0(x) = f(x,0), \quad x\in\mathcal{F}_1, \quad \nu(S(\boldsymbol{v}_0)\boldsymbol{N})_\tau = \boldsymbol{d}_\tau(x,0), \quad x\in\mathcal{G}, \tag{2.4}$$

be satisfied. Then the problem (2.1) has a unique solution $\boldsymbol{v},p,\rho$ such that $\boldsymbol{v}\in W_p^{2,1}(Q_T^1)$, $\nabla p\in L_p(Q_T^1)$, $p\in W_p^{1-1/p,0}(G_T)$, $\rho\in W_p^{3-1/p,0}(G_T)$, $\rho_t\in W_p^{2-1/p,1-1/2p}(G_T)$, $\rho(\cdot,t)\in W_p^{3-2/p}(\mathcal{G})$, $\forall t\in(0,T)$, and the solution satisfies the inequality

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_p^{2,1}(Q_T^1)} + \|\nabla p\|_{L_p(Q_T^1)} + \|p\|_{W_p^{1-1/p,0}(G_T)} + \sup_{t<T}\|\boldsymbol{v}(\cdot,t)\|_{W_p^{2-2/p}(\mathcal{F}_1)} \\ &+ \|\rho\|_{W_p^{3-1/p,0}(G_T)} + \|\rho_t\|_{W_2^{2-1/p,1-1/2p}(G_T)} + \sup_{t<T}\|\rho\|_{W_p^{3-2/p}(\mathcal{G})} \\ &\leq c(T)\Big(\|\boldsymbol{f}\|_{L_p(Q_T^1)} + \|f\|_{W_p^{1,0}(Q_T^1)} + \|\boldsymbol{F}\|_{W_p^{0,1}(Q_T^1)} \\ &+ \|\boldsymbol{d}_\tau\|_{W_2^{1-1/p,l/2-1/(2p)}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1-1/p,0}(G_T)} \\ &+ \|g\|_{W_p^{2-1/p,1-1/2p}(G_T)} + \|\boldsymbol{v}_0\|_{W_p^{2-2/p}(\mathcal{F}_1)} + \|\rho_0\|_{W_p^{3-2/p}(\mathcal{G})}\Big). \end{aligned} \tag{2.5}$$

The constant $c(T)$ in (2.5) is an increasing functions of T.

The condition $\nabla\cdot\boldsymbol{v}_0(x) = f(x,0), \quad x\in\mathcal{F}_1$ can be understood in a weak sense as $\int_{\mathcal{F}_1}(\boldsymbol{v}_0(y)-\boldsymbol{F}(y,0))\cdot\nabla\eta(y)dy = 0$ for arbitrary smooth η such that $\eta|_{\mathcal{G}} = 0$.

The theorem is proved in [10].

Theorem 3. *Assume that $\boldsymbol{k} = \mathrm{rot}\boldsymbol{K}(y,t)$, $\boldsymbol{a} = [\boldsymbol{A}]$, $\boldsymbol{K},\boldsymbol{A}\in W_p^{2,1}(Q_T^i)$, $i=1,2,3$,*

$$\begin{aligned} &\boldsymbol{K} = \boldsymbol{A} = 0, \quad y\in S, \quad \boldsymbol{A} = 0, \quad y\in\mathcal{F}_3, \quad [\boldsymbol{K}_\tau]-[\boldsymbol{A}_\tau] = 0. \quad y\in S_3, \\ &[\boldsymbol{K}_\tau] = \boldsymbol{a}, \quad \boldsymbol{A}^{(1)}\cdot\boldsymbol{N}(y) = \boldsymbol{A}^{(2)}\cdot\boldsymbol{N} = 0, \quad y\in\mathcal{G}. \end{aligned} \tag{2.6}$$

Then the problem (2.3) has a unique solution $\boldsymbol{h}\in W_p^{2,1}(\mathcal{F}_i)$, $i=1,2,3$, orthogonal to the space $\widetilde{U}_n(\Omega)$:

$$\int_\Omega \mu\boldsymbol{h}\cdot\widetilde{\boldsymbol{u}}_q dx = 0, \quad q = 1,...,b_1(\Omega). \tag{2.7}$$

The solution satisfies the inequality

$$\sum_{i=1}^3\|\boldsymbol{h}\|_{W_p^{2,1}(\mathcal{F}_i)} \leq c\sum_{i=1}^3(\|\boldsymbol{K}\|_{W_p^{2,1}(Q_T^i)} + \|\boldsymbol{A}\|_{W_p^{2,1}(Q_T^i)}). \tag{2.8}$$

Proof. We restrict ourselves with the formula for the solution of the problem (2.3):

$$\boldsymbol{h} = \boldsymbol{A} + \nabla\phi + \boldsymbol{X} + \sum_{j=1}^{b_1(\Omega)} c_j \widetilde{\boldsymbol{u}}_j.$$

It differs from a similar formula in [3], Theorem 3, only by the last term. The functions ϕ, $\boldsymbol{X}$ are the same as in [3]; they solve the problems

$$\begin{cases} \nabla^2\phi(x) = -\nabla\cdot\boldsymbol{A}(x), \quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad \dfrac{\partial\phi}{\partial N} = 0, \quad x\in S, \\ [\phi] = 0, \quad [\mu\dfrac{\partial\phi}{\partial N}] = 0, \quad x\in\mathcal{G}, \quad [\phi] = 0, \quad [\mu\dfrac{\partial\phi}{\partial n}] = 0, \quad x\in S_3, \end{cases} \tag{2.9}$$

$$\begin{cases} \mathrm{rot}\boldsymbol{X} = \mathrm{rot}(\boldsymbol{K}(x,t) - \boldsymbol{A}(x,t)), \quad \nabla\cdot\boldsymbol{X} = 0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad \boldsymbol{X}\cdot\boldsymbol{N} = 0, \ x\in S, \\ [\mu\boldsymbol{X}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{X}_\tau] = 0, \quad x\in\mathcal{G}, \quad [\mu\boldsymbol{X}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{X}_\tau] = 0, \quad x\in S_3, \end{cases}$$

$$\begin{aligned} &\boldsymbol{X}(x) = \boldsymbol{X}_1(x) + \nabla U(x), \\ &\boldsymbol{X}_1(x) = \frac{1}{4\pi}\mathrm{rot}\int_\Omega \frac{\mathrm{rot}(\boldsymbol{K}(y) - \boldsymbol{A}(y))}{|x-y|}dy, \end{aligned}$$

$$\begin{cases} \nabla^2 U(x) = 0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad [U(x)] = 0, \quad [\mu\dfrac{\partial U}{\partial N}] = -[\mu]\boldsymbol{X}_1\cdot\boldsymbol{N}, \quad x\in\mathcal{G}, \\ [U(x)] = 0, \quad [\mu\dfrac{\partial U}{\partial n}] = -[\mu]\boldsymbol{X}_1\cdot\boldsymbol{n}, \quad x\in S_3, \quad \dfrac{\partial U}{\partial n} = -\boldsymbol{X}_1\cdot\boldsymbol{n}, \quad x\in S, \end{cases}$$

and they are estimated as in [3]. The orthogonality condition for $\boldsymbol{h}$ yields the formulas for $c_j(t)$: if $\int_\Omega \mu\widetilde{\boldsymbol{u}}_j\cdot\widetilde{\boldsymbol{u}}_q dy = \delta_{jq}, \quad j,q = 1,...,b_1(\mathcal{F}_2)$, then

$$0 = \int_\Omega \mu\boldsymbol{h}\cdot\widetilde{\boldsymbol{u}}_q dy = \int_\Omega \mu(\boldsymbol{A} + \nabla\phi + \boldsymbol{X})\cdot\widetilde{\boldsymbol{u}}_q dy + c_q(t),$$

This formula and the estimates of ϕ, $\boldsymbol{X}$ (see [3]) imply (2.8).

The uniqueness of the solution follows from the fact that in the case $\boldsymbol{K} = 0, \boldsymbol{A} = 0$ the solution belongs to $\widetilde{U}_n(\Omega)$. The theorem is proved.

Theorem 4. *Assume that the data of the problem (2.2) possess the following properties:* $\boldsymbol{G}\in L_p(Q_T^1)$, $\boldsymbol{H}_0\in W_p^{2-2/p}(\mathcal{F}_j)$, $j = 1,2,3$, $\boldsymbol{\ell}\in W_p^{2,1}(Q_T^2)$, $\boldsymbol{\ell}|_{x\in S} = 0$, $\boldsymbol{a}\in W_p^{2-1/p,1-1/2p}(G_T)$, *moreover,* $\boldsymbol{a} = [\boldsymbol{A}]$ *with* $\boldsymbol{A}^{(i)}\in W_p^{2,1}(Q_T^i)$, $i = 1,2$, *satisfying (2.6), finally, let the compatibility conditions*

$$\begin{aligned} &\nabla\cdot\boldsymbol{G}(x,t) = 0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_3, \\ &\nabla\cdot\boldsymbol{H}_0(x) = 0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad \mathrm{rot}\boldsymbol{H}_0(x) = \mathrm{rot}\boldsymbol{\ell}(x,0), \quad x\in\mathcal{F}_2, \\ &[\mu\boldsymbol{H}_0\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{H}_{0\tau}] = 0, \quad x\in S_3, \\ &[\mu\boldsymbol{H}_0\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{H}_{0\tau}] = \boldsymbol{a}(x,0) = [\boldsymbol{A}(x,0)], \quad x\in\mathcal{G}, \quad \boldsymbol{H}_0\cdot\boldsymbol{n} = 0, \quad x\in S \end{aligned} \tag{2.10}$$

hold. Then the problem (2.2) has a unique solution $\boldsymbol{H}\in W_p^{2,1}(Q_T^i), \quad i = 1,2,3,$ *and*

$$\begin{aligned} \sum_{i=1}^3 \|\boldsymbol{H}\|_{W_p^{2,1}(Q_T^i)} \leq c\Big(&\sum_{i=1,3}\|\boldsymbol{G}\|_{L_p(Q_T^1)} + \sum_{i=1}^3\|\boldsymbol{H}_0\|_{W_p^{2-2/p}(\mathcal{F}_i)} \\ &+ \|\boldsymbol{\ell}\|_{W_p^{2,1}(Q_T^2)} + \sum_{i=1}^2\|\boldsymbol{A}\|_{W_p^{2,1}(Q_T^i)} + \sum_{q=1}^{b_1(\mathcal{F}_2)}\|\boldsymbol{M}_q\|_{L_p(0,T)}\Big) \end{aligned} \tag{2.11}$$

Proof. We reduce (2.2) to a similar problem with $\boldsymbol{\ell} = 0$, $\boldsymbol{a} = 0$. We extend $\boldsymbol{\ell}$ into Ω with the preservation of class, i.e., so that the extended function $\boldsymbol{\ell}^* \in W_p^{2,1}(Q_T)$ satisfies

$$\|\boldsymbol{\ell}^*\|_{W_p^{2,1}(Q_T)} \le c\|\boldsymbol{\ell}\|_{W_p^{2,1}(Q_T^2)}, \tag{2.12}$$

and we define

$$\boldsymbol{a}^*(x,t) = \boldsymbol{A}^{(1)}(x,t) - \boldsymbol{A}^{(2)*}(x,t), \quad x \in \mathcal{F}_1,$$
$$\boldsymbol{a}^*(x,t) = 0, \quad x \in \mathcal{F}_2 \cup \mathcal{F}_3,$$

where $\boldsymbol{A}^{(2)*}$ is the extension of $\boldsymbol{A}^{(2)}$ such that

$$\|\boldsymbol{A}^{(2)*}\|_{W_p^{2,1}(Q_T^1 \cup Q_T^2)} \le c\|\boldsymbol{A}^{(2)}\|_{W_p^{2,1}(Q_T^2)}.$$

We set $\boldsymbol{A}^{(3)} = 0$. It is easily verified that

$$[\boldsymbol{a}^*] = [\boldsymbol{A}] = \boldsymbol{a}, \quad [\boldsymbol{\ell}^* + \boldsymbol{a}^*] = \boldsymbol{a}, \quad x \in \mathcal{G}.$$

Now we define $\boldsymbol{h}_1(x,t)$ as a solution of the problem (2.3) with $\boldsymbol{k}(x,t) = \mathrm{rot}(\boldsymbol{\ell}^* + \boldsymbol{a}^*)$. By (2.8),

$$\sum_{i=1}^{3} \|\boldsymbol{h}_1\|_{W_p^{2,1}(Q_T^i)} \le c(\|\boldsymbol{\ell}\|_{W_p^{2,1}(Q_T^2)} + \sum_{i=1}^{2} \|\boldsymbol{A}\|_{W_p^{2,1}(Q_T^i)}). \tag{2.13}$$

For $\boldsymbol{h} = \boldsymbol{H} - \boldsymbol{h}_1$ we obtain the problem

$$\begin{cases} \mu \boldsymbol{h}_t(y,t) + \alpha^{-1}\mathrm{rot}\mathrm{rot}\boldsymbol{h}(y,t) = \boldsymbol{G}'(y,t), \quad \nabla \cdot \boldsymbol{h}(y,t) = 0, \quad y \in \mathcal{F}_1 \cup \mathcal{F}_3, \\ \mathrm{rot}\boldsymbol{h}(y,t) = 0, \quad \nabla \cdot \boldsymbol{h}(y,t) = 0, \quad y \in \mathcal{F}_2 \\ [\mu \boldsymbol{h} \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad y \in \mathcal{G}, \\ [\mu \boldsymbol{h} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \qquad y \in S_3, \quad \boldsymbol{h} \cdot \boldsymbol{n} = 0, \quad y \in S, \\ \boldsymbol{h}(y,0) = \boldsymbol{h}_0(y) = \boldsymbol{H}_0(y) - \boldsymbol{h}_1(y,0), \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3, \end{cases} \tag{2.14}$$

$$\begin{cases} \displaystyle\int_{\mathcal{F}_2} \mu \boldsymbol{h}_t \cdot \boldsymbol{u}_q^{\mathcal{F}} dy = \int_{\mathcal{G}} (\boldsymbol{N} \times \alpha^{-1}\mathrm{rot}\boldsymbol{h}^{(1)}) \cdot \boldsymbol{u}_q^{\mathcal{F}}(y) dS \\ \displaystyle\qquad + \int_{S_3} (\boldsymbol{n} \times \alpha^{-1}\mathrm{rot}\boldsymbol{h}^{(3)}) \cdot \boldsymbol{u}_q^{\mathcal{F}}(y) dS + \boldsymbol{M}'_q(t), \quad q = 1, ..., b_1(\mathcal{F}_2), \end{cases} \tag{2.15}$$

where

$$\boldsymbol{G}' = \boldsymbol{G} - \mu \boldsymbol{h}_{1t} - \alpha^{-1}\mathrm{rot}\mathrm{rot}\boldsymbol{h}_1,$$
$$\boldsymbol{M}'_q = \boldsymbol{M}_q + \int_{\mathcal{G}} (\boldsymbol{N} \times \mathrm{rot}\boldsymbol{h}_1^{(1)}) \cdot \boldsymbol{u}_q^{\mathcal{F}}(y) dS$$
$$+ \int_{S_3} (\boldsymbol{n} \times \mathrm{rot}\boldsymbol{h}_1^{(3)}) \cdot \boldsymbol{u}_q^{\mathcal{F}}(y) dS - \int_{\mathcal{F}_2} \mu \boldsymbol{h}_{1t} \cdot \boldsymbol{u}_q^{\mathcal{F}} dy, \quad q = 1, ..., b_1(\mathcal{F}_2).$$

The equations $\mathrm{rot}\boldsymbol{h}(y,t) = 0, \quad \nabla \cdot \boldsymbol{h} = 0, \quad y \in \mathcal{F}_2$ imply

$$\boldsymbol{h}^{(2)} = \nabla\varphi(y,t) + \sum_{j=1}^{b_1(\mathcal{F}_2)} k_j(t) \boldsymbol{u}_j^{\mathcal{F}}(y), \quad y \in \mathcal{F}_2, \tag{2.16}$$

where φ is a solution to the problem

$$\begin{cases} \nabla^2\varphi(y,t)=0, \quad y\in\mathcal{F}_2, \quad \dfrac{\partial\varphi}{\partial n}=0, \quad y\in S, \\ \mu_2\dfrac{\partial\varphi}{\partial N}=\mu_1\boldsymbol{h}^{(1)}\cdot\boldsymbol{N}, \quad y\in\mathcal{G}, \quad \mu_2\dfrac{\partial\varphi}{\partial n}=\mu_3\boldsymbol{h}^{(3)}\cdot\boldsymbol{n}, \quad y\in S_3. \end{cases} \tag{2.17}$$

It follows that $\boldsymbol{h}^{(1)}$, $\boldsymbol{h}^{(3)}$ and φ satisfy

$$\begin{cases} \mu\boldsymbol{h}_t(y,t)+\alpha^{-1}\mathrm{rotrot}\boldsymbol{h}(y,t)=\boldsymbol{G}'(y,t), \quad \nabla\cdot\boldsymbol{h}(y,t)=0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_3, \\ \nabla^2\varphi(y,t)=0, \quad y\in\mathcal{F}_2, \quad \dfrac{\partial\varphi}{\partial n}=0, \quad y\in S, \\ \mu_2\dfrac{\partial\varphi}{\partial N}=\mu_1\boldsymbol{h}^{(1)}\cdot\boldsymbol{N}, \quad y\in\mathcal{G}, \quad \mu_2\dfrac{\partial\varphi}{\partial n}=\mu_3\boldsymbol{h}^{(3)}\cdot\boldsymbol{n}, \quad y\in S_3, \\ \boldsymbol{h}_\tau^{(1)}-\nabla_\tau\varphi=\displaystyle\sum_{j=1}^{b_1(\mathcal{F}_2)}k_j(t)\boldsymbol{u}_j^{\mathcal{F}}(y), \quad y\in\mathcal{G}, \\ \boldsymbol{h}_\tau^{(3)}-\nabla_\tau\varphi=\displaystyle\sum_{j=1}^{b_1(\mathcal{F}_2)}k_j(t)\boldsymbol{u}_j^{\mathcal{F}}(y), \quad y\in S_3, \\ \boldsymbol{h}(y,0)=\boldsymbol{h}_0(y)=\boldsymbol{H}_0(y)-\boldsymbol{h}_1(y,0), \quad y\in\mathcal{F}_1\cup\mathcal{F}_3. \end{cases} \tag{2.18}$$

This problem is studied in [11] under the assumptions $k_j=0$, $S_3=\emptyset$, but the result remains valid in the case considered here, in particular, there holds the a-priori estimate

$$\begin{aligned} &\sum_{i=1,3}\|\boldsymbol{h}^{(i)}\|_{W_p^{2,1}(Q_T^i)}+\|\nabla\varphi\|_{W_p^{2,1}(Q_T^2)} \\ &\le c\Big(\sum_{i=1,3}(\|\boldsymbol{G}'\|_{L_p(Q_T^i)}+\|\boldsymbol{h}_0\|_{W_p^{2-1/p}(\mathcal{F}_i)})+\|\boldsymbol{k}\|_{W_p^{1-1/2p}(0,T)}\Big). \end{aligned} \tag{2.19}$$

The functions $k_j(t)$ can be expressed in terms of $\boldsymbol{h}$ with the help of equations (2.15), (2.16): if $\int_{\mathcal{F}_2}\mu\boldsymbol{u}_m^{\mathcal{F}}\cdot\boldsymbol{u}_q^{\mathcal{F}}dy=\delta_{mq}$, $q,m=1,...,b_1(\mathcal{F}_2)$, then (2.15) imply

$$\begin{aligned} k_j(t)=k_j(0)+\int_0^t\boldsymbol{M}'_j(\tau)d\tau+\int_0^t d\tau\Big(\int_{\mathcal{G}}(\boldsymbol{N}\times\mathrm{rot}\boldsymbol{h}^{(1)})\cdot\boldsymbol{u}_j^{\mathcal{F}}dS \\ +\int_{S_3}(\boldsymbol{n}\times\mathrm{rot}\boldsymbol{h}^{(3)})\cdot\boldsymbol{u}_j^{\mathcal{F}}dS\Big), \quad j=1,...,b_1(\mathcal{F}_2), \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &\|\boldsymbol{k}\|_{W_p^1(0,T)}\le c(T)(|\boldsymbol{k}(0)|+\sum_q\|\boldsymbol{M}'_q\|_{L_p(0,T)} \\ &+\|\mathrm{rot}\boldsymbol{h}^{(1)}\|_{L_p(G_T)}+\|\mathrm{rot}\boldsymbol{h}^{(3)}\|_{L_p(S_3\times(0,T))}+\|\boldsymbol{k}\|_{W_p^{1-1/2p}(0,T)}). \end{aligned} \tag{2.21}$$

The norms of $\boldsymbol{h}$ in (2.21) can be estimated by (2.19) and the interpolation inequalities, which yields

$$\begin{aligned} &\sum_{i=1,3}\|\boldsymbol{h}^{(i)}\|_{W_p^{2,1}(Q_T^i)}+\|\nabla\varphi\|_{W_p^{2,1}(Q_T^2)}+\|\boldsymbol{k}\|_{W_p^1(0,T)} \\ &\le c(T)(\sum_{i=1,3}\|\boldsymbol{G}'\|_{L_p(Q_T^i)}+\sum_{i=1}^3\|\boldsymbol{h}_0\|_{W_p^{2-1/p}(\mathcal{F}_i)}+\sum_q\|\boldsymbol{M}'_q\|_{L_p(0,T)}+\|\boldsymbol{h}^{(i)}\|_{L_p(Q_T)} \\ &+\|\boldsymbol{k}\|_{L_p(0,T)}). \end{aligned} \tag{2.22}$$

This inequality holds for arbitrary $T' < T$, hence, by the Gronwall lemma,

$$\sum_{i=1,3} \|\boldsymbol{h}^{(i)}\|_{W_p^{2,1}(Q_T^i)} + \|\nabla\varphi\|_{W_p^{2,1}(Q_T^2)} + \|\boldsymbol{k}\|_{W_p^1(0,T)}$$
$$\leq c(T)\Big(\sum_{i=1,3} \|\boldsymbol{G}'\|_{L_p(Q_T^i)} + \sum_{i=1}^{3} \|\boldsymbol{h}_0\|_{W_p^{2-1/p}(\mathcal{F}_i)} + \sum_q \|\boldsymbol{M}'_q\|_{L_p(0,T)}\Big) \tag{2.23}$$

Finally, the estimate of $\boldsymbol{h}^{(2)}$ is obtained from (2.16). Taking (2.13) into account, we arrive at (2.11).

Our arguments show that the term $\sum_{j=1}^{b_1(\mathcal{F}_2} k_j(t)\boldsymbol{u}_j(y)$ in (2.18) generates a compact operator, which makes it possible to deduce the solvability of the problem (2.18), (2.15) from the solvability of the same problem with $k_j = 0$ (established in [11]) by standard methods. Theorem 4 is proved.

3 Nonlinear problem

In this section we outline main ideas of the proof of Theorem 1. We consider the problem (1.6), (1.15), (1.16) that can be written in the form

$$\begin{cases} \boldsymbol{u}_t(y,t) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q, \boldsymbol{h}, \rho) + \boldsymbol{f}(y,t), \\ \nabla\cdot\boldsymbol{u} = l_2(\boldsymbol{u},\rho), \quad y \in \mathcal{F}_1,\ t > 0, \\ \Pi_{\mathcal{G}} S(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u},\rho), \\ -q + \nu\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}(y) + \sigma\mathfrak{B}(\rho) = l_4(\boldsymbol{u},\boldsymbol{h},\rho) + l_5(\rho) + \sigma\mathcal{H}(y), \\ \rho_t + \boldsymbol{V}(x)\cdot\nabla_\tau\rho - \boldsymbol{u}\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u},\rho), \quad y \in \mathcal{G}, \\ \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y \in \mathcal{F}_1, \quad \rho(y,0) = \rho_0(y), \quad y \in \mathcal{G}, \end{cases} \tag{3.1}$$

$$\begin{cases} \mu\boldsymbol{h}_t + \alpha^{-1}\mathrm{rotrot}\boldsymbol{h} = \boldsymbol{l}_7(\boldsymbol{h},\boldsymbol{u},\rho), \\ \nabla\cdot\boldsymbol{h} = 0, \quad y \in \mathcal{F}_1, \\ \mu\boldsymbol{h}_t + \alpha^{-1}\mathrm{rotrot}\boldsymbol{h} = \alpha^{-1}\mathrm{rot}\boldsymbol{j}(y,t), \\ \nabla\cdot\boldsymbol{h} = 0, \quad y \in \mathcal{F}_3, \\ \mathrm{rot}\boldsymbol{h} = \mathrm{rot}\boldsymbol{l}_8(\boldsymbol{h},\rho), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y \in \mathcal{F}_2, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \boldsymbol{l}_9(\boldsymbol{h},\rho), \quad y \in \mathcal{G}, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad y \in S_3, \\ \boldsymbol{h}\cdot\boldsymbol{n} = 0, \quad y \in S, \\ \boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y \in \mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \end{cases} \tag{3.2}$$

$$\begin{cases} \int_{\mathcal{F}_2} \mu\boldsymbol{h}_t\cdot\boldsymbol{u}_q^{\mathcal{F}} dy - \int_{\mathcal{G}} \boldsymbol{N}\times\alpha^{-1}\mathrm{rot}\boldsymbol{h}^{(1)}\cdot\boldsymbol{u}_q^{\mathcal{F}} dS - \int_{S_3} \boldsymbol{n}\times\alpha^{-1}\mathrm{rot}\boldsymbol{h}^{(3)}\cdot\boldsymbol{u}_q^{\mathcal{F}} dS \\ = l_{10,q}(\boldsymbol{u},\rho,\boldsymbol{h}) - \int_{S_3} \alpha^{-1}\boldsymbol{j}\cdot\boldsymbol{u}_q^{\mathcal{F}} dS, \quad q = 1,...,b_1(\mathcal{F}_2), \end{cases} \tag{3.3}$$

where

$$\begin{cases} \boldsymbol{l}_1(\boldsymbol{u},q,\rho) = \nu(\widetilde{\nabla}^2-\nabla^2)\boldsymbol{u} + (\nabla-\widetilde{\nabla})q + \rho_t^*(\mathcal{L}^{-1}(y,\rho^*)\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u} \\ -(\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \widetilde{\nabla}\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h}) + \int_0^1 \rho^*(y,t)(\boldsymbol{N}^*(y)\cdot\nabla_x)\boldsymbol{f}(x,t)|_{x=e_{s\rho}}ds, \\ l_2(\boldsymbol{u},\rho) = (I-\widehat{\mathcal{L}}^T)\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{L}(\boldsymbol{u},\rho), \quad \boldsymbol{L} = (I-\widehat{\mathcal{L}})\boldsymbol{u}, \quad y\in\mathcal{F}_1, \\ \boldsymbol{l}_3(\boldsymbol{u},\rho) = \Pi_{\mathcal{G}}(\Pi_{\mathcal{G}}S(\boldsymbol{u})\boldsymbol{N})(y) - \Pi\widetilde{S}(\boldsymbol{u})\boldsymbol{n}(e_\rho(y)), \\ l_4(\boldsymbol{u},\boldsymbol{h},\rho) = \nu(\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{n}\cdot\widetilde{S}(\boldsymbol{u})\boldsymbol{n}) - [T_M(\frac{\mathcal{L}}{L}\boldsymbol{h})]\boldsymbol{n}, \\ l_5(\rho) = -\int_0^1(1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla\cdot\frac{\mathcal{L}^T(y,s\rho)\boldsymbol{N}}{|\mathcal{L}^T(y,s\rho)\boldsymbol{N}|}ds, \\ l_6(\boldsymbol{u},\boldsymbol{h},\rho) = (\frac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{\boldsymbol{N}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}|} + \nabla_\tau\rho - \boldsymbol{N})\cdot\boldsymbol{u} + (\boldsymbol{V}-\boldsymbol{u})\cdot\nabla_\tau\rho, \quad y\in\mathcal{G}, \\ \boldsymbol{l}_7(\boldsymbol{h},\rho) = \alpha^{-1}\mathrm{rot}(\mathrm{rot}\boldsymbol{h} - \mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h}) + \frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h} \\ \qquad + \rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h} + \mu_1\mathrm{rot}(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}), \quad y\in\mathcal{F}_1, \\ \boldsymbol{l}_8(\boldsymbol{h},\rho) = (I-\mathcal{P})\boldsymbol{h}, \quad y\in\mathcal{F}_2, \\ l_9(\boldsymbol{h},\rho) = (\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2} - \boldsymbol{N})[\boldsymbol{h}\cdot\boldsymbol{N}] = [\boldsymbol{A}(\boldsymbol{h},\rho)], \quad y\in\mathcal{G}, \\ \boldsymbol{A}^{(i)} = (\frac{\widehat{\mathcal{L}}(y,\rho^*)\widehat{\mathcal{L}}^T\boldsymbol{N}^*(y)}{|\widehat{\mathcal{L}}^T\boldsymbol{N}^*|^2} - \frac{\boldsymbol{N}^*(y)}{|\boldsymbol{N}^*|^2})(\boldsymbol{N}^*\cdot\boldsymbol{h}^{(i)}), \quad i=1,2, \quad \boldsymbol{A}^{(3)}=0, \\ l_{10,q}(\boldsymbol{u},\rho,\boldsymbol{h}) = \int_{\mathcal{F}_2}\mu\boldsymbol{\Phi}\cdot\boldsymbol{u}_q^{\mathcal{F}}dy + \int_{\mathcal{G}}\boldsymbol{N}\times\alpha^{-1}(\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h}^{(1)} - \mathrm{rot}\boldsymbol{h}^{(1)})\cdot\boldsymbol{u}_q^{\mathcal{F}}dS \\ -\int_{\mathcal{G}}(\boldsymbol{N}\times\mu(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}^{(1)}) - \boldsymbol{\Psi})\cdot\boldsymbol{u}_q^{\mathcal{F}}dS, \quad q=1,...,b_1(\mathcal{F}_2). \end{cases} \tag{3.4}$$

We have used the formula for the variation of the mean curvature under the normal perturbation of the surface:

$$H(e_\rho) - \mathcal{H}(y) = -\mathfrak{B}\rho - \int_0^1(1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla\cdot\frac{\mathcal{L}^T(y,s\rho)\boldsymbol{N}}{|\mathcal{L}^T(y,s\rho)\boldsymbol{N}|}ds.$$

and we have set

$$\Pi_{\mathcal{G}}\boldsymbol{a} = \boldsymbol{a} - \boldsymbol{N}(\boldsymbol{a}\cdot\boldsymbol{N}), \quad \Pi\boldsymbol{a} = \boldsymbol{a} - \boldsymbol{n}(\boldsymbol{a}\cdot\boldsymbol{n}).$$

The vector field $\boldsymbol{V}\in W_p^{2-1/p}(\mathcal{G})$ is introduced to optimize the estimate of the expression l_6 in (3.4).

The solvability of the problem (3.1)-(3.4) is proved by successive approximations, according to a standard scheme:

$$\begin{cases} \boldsymbol{u}_{m+1,t}(y,t) - \nu\nabla^2\boldsymbol{u}_{m+1} + \nabla q_{m+1} = \boldsymbol{l}_1(\boldsymbol{u}_m,q_m,\boldsymbol{h}_m,\rho_m) + \boldsymbol{f}(y,t), \\ \nabla\cdot\boldsymbol{u}_{m+1} = l_2(\boldsymbol{u}_m,\rho_m), \quad y\in\mathcal{F}_1, \ t>0, \\ \Pi_{\mathcal{G}}S(\boldsymbol{u}_{m+1})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}_m,\rho_m), \\ -q_{m+1} + \nu\boldsymbol{N}\cdot S(\boldsymbol{u}_{m+1})\boldsymbol{N}(y) + \sigma\mathfrak{B}(\rho_{m+1}) = l_4(\boldsymbol{u}_m,\boldsymbol{h}_m,\rho_m) + l_5(\rho_m) + \sigma\mathcal{H}(y), \\ \rho_{m+1,t} + \boldsymbol{V}(x)\cdot\nabla_\tau\rho_{m+1} - \boldsymbol{u}_{m+1}\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u}_m,\rho_m), \quad y\in\mathcal{G}, \\ \boldsymbol{u}_{m+1}(y,0) = \boldsymbol{u}_0(y), \quad y\in\mathcal{F}_1, \quad \rho_{m+1}(y,0) = \rho_0(y), \quad y\in\mathcal{G}, \end{cases} \tag{3.5}$$

$$
\begin{cases}
\mu\boldsymbol{h}_{m+1,t}+\alpha^{-1}\mathrm{rotrot}\boldsymbol{h}_{m+1}=\boldsymbol{l}_7(\boldsymbol{h}_m,\boldsymbol{u}_m,\rho_m),\\
\nabla\cdot\boldsymbol{h}_{m+1}=0,\quad y\in\mathcal{F}_1,\\
\mu\boldsymbol{h}_{m+1,t}+\alpha^{-1}\mathrm{rotrot}\boldsymbol{h}_{m+1}=\alpha^{-1}\mathrm{rot}\boldsymbol{j}(y,t),\\
\nabla\cdot\boldsymbol{h}_{m+1}=0,\quad y\in\mathcal{F}_3,\\
\mathrm{rot}\boldsymbol{h}_{m+1}=\mathrm{rot}\boldsymbol{l}_8(\boldsymbol{h}_m,\rho_m),\quad \nabla\cdot\boldsymbol{h}_{m+1}=0,\quad y\in\mathcal{F}_2,\\
[\mu\boldsymbol{h}_{m+1}\cdot\boldsymbol{N}]=0,\quad [\boldsymbol{h}_{m+1,\tau}]=\boldsymbol{l}_9(\boldsymbol{h}_m,\rho_m),\quad y\in\mathcal{G},\\
[\mu\boldsymbol{h}_{m+1}\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{h}_{m+1,\tau}]=0,\quad y\in S_3,\\
\boldsymbol{h}_{m+1}\cdot\boldsymbol{n}=0,\quad y\in S,\\
\displaystyle\int_{\mathcal{F}_2}\mu\boldsymbol{h}_{m+1,t}\cdot\boldsymbol{u}_q^{\mathcal{F}}dy-\int_{\mathcal{G}}\boldsymbol{N}\times\alpha^{-1}\mathrm{rot}\boldsymbol{h}_{m+1}\cdot\boldsymbol{u}_q^{\mathcal{F}}dS-\int_{S_3}\boldsymbol{n}\times\alpha^{-1}\mathrm{rot}\boldsymbol{h}_{m+1}\cdot\boldsymbol{u}_q^{\mathcal{F}}dS\\
\displaystyle=l_{10,q}(\boldsymbol{u}_m,\rho_m,\boldsymbol{h}_m)-\int_{S_3}\boldsymbol{j}\cdot\boldsymbol{u}_q^{\mathcal{F}}dS,\quad q=1,...,b_1(\mathcal{F}_2),\\
\boldsymbol{h}_{m+1}(y,0)=\boldsymbol{h}_0(y),\quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3.
\end{cases}
\tag{3.6}
$$

The first approximation, $(\boldsymbol{u}_1,q_1,\rho_1,\boldsymbol{h}_1)$, is defined as follows: $q_1=0$, $\boldsymbol{u}_1$, ρ_1 $\boldsymbol{h}_1$ satisfy

$$
\boldsymbol{u}_1(y,0)=\boldsymbol{u}_0(y),\ \ y\in\mathcal{F}_1,\ \ \rho_1(y,0)=\rho_0(y),\ \ y\in\mathcal{G},\ \ \boldsymbol{h}_1(y,0)=\boldsymbol{h}_0(y),\ \ y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3
$$

and the inequalities

$$
\begin{aligned}
&\|\boldsymbol{u}_1\|_{W_p^{2,1}(Q_T^1)}\le c\|\boldsymbol{u}_1\|_{W_P^{2,1}(Q_\infty^1)}\le c\|\boldsymbol{u}_0\|_{W_p^{2-2/p}(\mathcal{F}_1)},\\
&\|\rho_1\|_{W_p^{3-1/p,0}(G_\infty)}+\|\rho_{1,t}\|_{W_p^{2-1/p,1-1/2p}(G_\infty)}\ \le c\|\rho_0\|_{W_p^{3-2/p}(\mathcal{G})},\\
&\sum_{i=1}^2\|\boldsymbol{h}_1^{(i)}\|_{W_r^{2,1}(Q_T^i)}\le c\|\boldsymbol{h}_0^{(1)}\|_{W_r^{2-2/r}(\mathcal{F}_1)};
\end{aligned}
\tag{3.7}
$$

in addition, $\boldsymbol{h}_1$ is divergence free and satisfies (2.16), (2.17) (for the construction of such $\boldsymbol{h}_1$, see Theorem 4 in [2]).

The proof of the solvability of (3.4) is based on Theorems 2, 4 (with the exponent r instead of p) and on the estimates of nonlinear terms (3.2) obtained in [3], Sec. 5.

Theorem 5. *Assume that $\boldsymbol{u}_m\in W_p^{2,1}(Q_T^1)$, $\nabla q_m\in L_p(Q_T^1)$, $\rho_m\in W_p^{3-1/p,0}(G_T)$, $\rho_{m,t}\in W_p^{2-1/p,1-1/2p}(G_T)$, $\boldsymbol{h}_m\in W_r^{2,1}(Q_T^i)$, $i=1,2,3$, $\boldsymbol{V}\in W_p^{2-1/p}(\mathcal{G})$, and the conditions (1.19), (1.22), as well as*

$$
\begin{aligned}
&\|\rho_m(\cdot,t)\|_{W_p^{1-1/p}(\mathcal{G})}\le\delta\ll 1,\\
&\|\boldsymbol{V}-\boldsymbol{u}_0\|_{W_p^{1-1/p}(\mathcal{G})}\le\delta,\\
&T^{1/2-1/p}\|\boldsymbol{V}\|_{W_p^{2-1/p}(\mathcal{G})}\le\delta
\end{aligned}
\tag{3.8}
$$

are satisfied. Then

$$
Z_m(T)\le\vartheta(\delta,T)(Y_m(T)+Y_m^2(T)),
\tag{3.9}
$$

where

$$Z_m(T) = \|\boldsymbol{l}_1(\boldsymbol{u}_m, q_m, \boldsymbol{h}_m, \rho)\|_{L_p(Q_T^1)} + \|l_2(\boldsymbol{u}_m, \rho_m)\|_{W_p^{1,0}(Q_T^1)} + \|\boldsymbol{L}(\boldsymbol{u}_m, \rho_m)\|_{W_p^{0,l}(Q_T^1)}$$
$$+ \|\boldsymbol{l}_3(\boldsymbol{u}_m, \rho_m)\|_{W_p^{1-1/p,1/2-1/2p}(G_T)} + \|l_4(\boldsymbol{u}_m, \boldsymbol{h}_m, \rho_m)\|_{W_p^{1-1/p,0}(G_T)}$$
$$+ \|l_5(\rho_m)\|_{W_p^{1-1/p,0}(G_T)} + \|l_6(\boldsymbol{u}_m, \rho_m)\|_{W_p^{2-1/p,1-1/2p}(G_T)} + \sum_{i=1}^{3} \|\boldsymbol{l}_7(\boldsymbol{h}, \rho)\|_{L_r(Q_T^1)}$$
$$+ \|\boldsymbol{l}_8(\boldsymbol{h}_m, \rho_m)\|_{W_r^{2,0}(Q_T^2)} + \sum_{i=1}^{2} \|\boldsymbol{A}^{(i)}\|_{W_r^{2,1}(Q_T^i)}\Big),$$

and $\vartheta(\delta, T)$ *is a constant that can be made arbitrarily small by the choice of small* δ *and* T.

Inequality (3.9) is established in [3], Sec. 5, except for the estimate of $\rho^* \boldsymbol{N}^* \cdot \nabla \boldsymbol{f}|_{x=e_s\rho}$ that is obtained as follows:

$$\begin{aligned} \|\rho^* \boldsymbol{N}^* \cdot \nabla \boldsymbol{f}|_{x=e_s\rho}\|_{L_p(Q_T)} &\leq cT^{1/p} \sup_{Q_T^1} |\rho^*(x,t)| \sup_{t<T} \|\nabla \boldsymbol{f}(\cdot,t)|_{L_p(\mathbb{R}^3)} \\ &\leq cT^{1/p} \sup_{t<T} \|\rho\|_{W_p^{1-1/p}(\mathcal{G})}. \end{aligned} \tag{3.10}$$

The nonlinear expressions in $l_{10,q}$ are also estimated in [3].

If $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$ are found, then, by Theorems 2, 4,5 the problem (3.5), (3.6) is uniquely solvable and

$$Y_{m+1}(T) \leq c\theta(\delta, T)(Y_m(T) + Y_m^2(T)) + c_1 N, \tag{3.11}$$

where

$$N = \|\boldsymbol{u}_0\|_{W_p^{2-2/p}(\mathcal{F}_1)} + \|\rho_0\|_{W_p^{3-2/p}(\mathcal{G})} + \|\mathcal{H}\|_{W_p^{1-1/p}(\mathcal{G})} + \sum_{i=1}^{3} \|\boldsymbol{h}_0\|_{W_r^{2-2/r}(\mathcal{F}_i)}, \tag{3.12}$$

$$\begin{aligned} Y_m(T) &= \|\boldsymbol{u}_m\|_{W_p^{2,1}(Q_T^1)} + \sup_{t<T} \|\boldsymbol{u}_m\|_{W_p^{2-1/p}(\mathcal{F}_1)} + \|\nabla q_m\|_{L_p(Q_T^1)} \\ &+ \|q_m\|_{W_p^{1-1/p}(G_T)} + \|\rho_m\|_{W_p^{3-1/p,0}(G_T)} + \|\rho_{m,t}\|_{W_p^{2-2/p,1-1/2p}(G_T)} \\ &+ \sum_{i=1}^{2} (\|\boldsymbol{h}_m\|_{W_r^{2,1}(Q_T^i)} + \sup_{t<T} \|\boldsymbol{h}_m\|_{W_r^{2-2/r}(\mathcal{F}_i)}) \equiv Y(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m). \end{aligned} \tag{3.13}$$

It follows that

$$Y_{m+1}(T) \leq 2c_1 N, \tag{3.14}$$

if $Y_m(T) \leq 2c_1 N$ and ϑ is so small that

$$\vartheta(\delta, T)(2c_1 N + 4(c_1 N)^2) \leq c_1 N$$

i.e.,

$$\vartheta(\delta, T) \leq \frac{1}{2(1 + 2c_1 N)}; \tag{3.15}$$

in addition, for small ϵ and T,

$$\begin{aligned} &\|\rho_{m+1}(\cdot,t)\|_{W_p^{2-2/p}(\mathcal{G})} \leq \|\rho_0\|_{W_p^{2-1/p}(\mathcal{G})} + \int_0^t \|\rho_{m+1,\tau}(\cdot,\tau)\|_{W_p^{2-1/p}(\mathcal{G})} d\tau \\ &\leq \epsilon + 2c_1 T^{1-1/p} N \leq \delta. \end{aligned} \tag{3.16}$$

In the case $m = 0$ this inequality follows from (3.7), (1.11). Hence if c_1 is chosen in such a way that (3.14) holds for $m = 0$, then (3.14) and (3.16) are satisfied for all m.

In order to prove the convergence of the sequence $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$, we estimate the differences

$$\boldsymbol{w}_{m+1} = \boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \ \ s_{m+1} = q_{m+1} - q_m, \ \ r_{m+1} = \rho_{m+1} - \rho_m, \ \ \boldsymbol{k}_{m+1} = \boldsymbol{h}_{m+1} - \boldsymbol{h}_m.$$

They satisfy the relations

$$\begin{cases} \boldsymbol{w}_{m+1,t}(y,t) - \nu\nabla^2 \boldsymbol{w}_{m+1} + \nabla s_{m+1} = \boldsymbol{l}_1(\boldsymbol{u}_m, q_m, \boldsymbol{h}_m, \rho_m) - \boldsymbol{l}_1(\boldsymbol{u}_{m-1}, q_{m-1}, \boldsymbol{h}_{m-1}, \rho_{m-1}), \\ \nabla \cdot \boldsymbol{w}_{m+1} = l_2(\boldsymbol{u}_m, \rho_m) - l_2(\boldsymbol{u}_{m-1}, \rho_{m-1}), \quad y \in \mathcal{F}_1, \ t > 0, \\ \Pi_{\mathcal{G}} S(\boldsymbol{w}_{m+1})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}_m, \rho_m) - \boldsymbol{l}_3(\boldsymbol{u}_{m-1}, \rho_{m-1}), \\ -s_{m+1} + \nu \boldsymbol{N} \cdot S(\boldsymbol{w}_{m+1})\boldsymbol{N}(y) + \sigma \mathfrak{B} r_{m+1}) \\ \ = l_4(\boldsymbol{u}_m, \boldsymbol{h}_m, \rho_m) - l_4(\boldsymbol{u}_{m-1}, \boldsymbol{h}_{m-1}, \rho_{m-1}) + l_5(\rho_m) - l_5(\rho_{m-1}), \\ r_{m+1,t} + \boldsymbol{V}(x) \cdot \nabla_\tau r_{m+1} - \boldsymbol{w}_{m+1} \cdot \boldsymbol{N}(y) = l_6(\boldsymbol{u}_m, \rho_m) - l_6(\boldsymbol{u}_{m-1}, \rho_{m-1}), \quad y \in \mathcal{G}, \\ \boldsymbol{w}_{m+1}(y, 0) = 0, \quad y \in \mathcal{F}_1, \quad r_{m+1}(y, 0) = 0, \quad y \in \mathcal{G}, \end{cases} \tag{3.17}$$

$$\begin{cases} \mu \boldsymbol{k}_{m+1,t} + \alpha^{-1} \mathrm{rot}\mathrm{rot} \boldsymbol{k}_{m+1} = \boldsymbol{l}_7(\boldsymbol{h}_m, \boldsymbol{u}_m, \rho_m) - \boldsymbol{l}_7(\boldsymbol{h}_{m-1}, \boldsymbol{u}_{m-1}, \rho_{m-1}), \\ \nabla \cdot \boldsymbol{k}_{m+1} = 0, \quad y \in \mathcal{F}_1, \\ \mu \boldsymbol{k}_{m+1,t} + \alpha^{-1} \mathrm{rot}\mathrm{rot} \boldsymbol{k}_{m+1} = 0, \quad \nabla \cdot \boldsymbol{k}_{m+1} = 0, \quad y \in \mathcal{F}_3, \\ \mathrm{rot} \boldsymbol{k}_{m+1} = \mathrm{rot}(\boldsymbol{l}_8(\boldsymbol{h}_m, \rho_m) - \boldsymbol{l}_8(\boldsymbol{h}_{m-1}, \rho_{m-1})), \quad \nabla \cdot \boldsymbol{k}_{m+1} = 0, \quad y \in \mathcal{F}_2, \\ [\mu \boldsymbol{k}_{m+1} \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{k}_{m+1,\tau}] = \boldsymbol{l}_9(\boldsymbol{h}_m, \rho_m) - \boldsymbol{l}_9(\boldsymbol{h}_{m-1}, \rho_{m-1}), \quad y \in \mathcal{G}, \\ [\mu \boldsymbol{k}_{m+1} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{k}_{m+1,\tau}] = 0, \quad y \in S_3, \\ \boldsymbol{k}_{m+1} \cdot \boldsymbol{n} = 0, \quad y \in S, \\ \displaystyle\int_{\mathcal{F}_2} \mu \boldsymbol{k}_{m+1,t} \cdot \boldsymbol{u}_q^{\mathcal{F}} dy - \int_{\mathcal{G}} \boldsymbol{N} \times \alpha^{-1} \mathrm{rot} \boldsymbol{k}_{m+1} \cdot \boldsymbol{u}_q^{\mathcal{F}} dS - \int_{S_3} \boldsymbol{n} \times \alpha^{-1} \mathrm{rot} \boldsymbol{k}_{m+1} \cdot \boldsymbol{u}_q^{\mathcal{F}} dS \\ \ = l_{10,q}(\boldsymbol{u}_m, \rho_m, \boldsymbol{h}_m) - l_{10,q}(\boldsymbol{u}_{m-1}, \rho_{m-1}, \boldsymbol{h}_{m-1}), \quad q = 1, ..., b_1(\mathcal{F}_2), \\ \boldsymbol{k}_{m+1}(y, 0) = 0, \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3. \end{cases} \tag{3.18}$$

Using Theorems 2, 4 and the estimates of the differences of nonlinear terms in (3.17), (3.18), we show that

$$Y(\boldsymbol{w}_{m+1)}, s_{m+1}, r_{m+1}, \boldsymbol{k}_{m+1}) \leq \vartheta_1 Y(\boldsymbol{w}_m, s_m, r_m, \boldsymbol{k}_m),$$

$m > 1$, with a small ϑ_1, which guarantees the boundedness of

$$\sum_{m=2}^{\infty} Y(\boldsymbol{u}_{m+1)} - \boldsymbol{u}_m, q_{m+1} - q_m, \rho_{m+1} - \rho_m, \boldsymbol{h}_{m+1} - \boldsymbol{h}_m)$$

and the convergence of $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$. The idea of the proof is the same as in [2,7] in the case $p, r = 2$, and we omit the details.

It is clear that $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$ tend to the solution of the problem (5.1), as $m \to \infty$, and the solution satisfies (3.14):

$$Y(T) \leq 2c_1 N. \tag{3.19}$$

4 Construction of the electric field

We have proved the existence of $(\boldsymbol{u}, q, \rho, \boldsymbol{h})$ satisfying (1.6), (1.15), (1.16) and the additional condition (1.18). Now we find $\boldsymbol{e}$ satisfying (1.7), (1.8). We introduce the vector field $\boldsymbol{e}_1$ such that

$$\begin{aligned}&\nabla \cdot \boldsymbol{e}_1(y,t) = 0, \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3, \quad [\boldsymbol{N} \times \mathcal{P}\boldsymbol{e}_1] = \boldsymbol{\Psi}, \quad [\boldsymbol{N} \cdot \boldsymbol{e}_1] = 0, \quad y \in \mathcal{G}, \\ &[\boldsymbol{n} \cdot \boldsymbol{e}_1] = 0, \quad y \in S_3, \quad \boldsymbol{e}_1(y,t) = 0. \quad y \in S,\end{aligned} \tag{4.1}$$

$$\sum_{i=1}^{3} \|\boldsymbol{e}_1\|_{W_r^{1,0}(Q_T^i)} \leq c\|\boldsymbol{\Psi}\|_{W_r^{1-1/r}(G_T)}. \tag{4.2}$$

Since the matrix $\mathcal{P}$ is close to I, such $\boldsymbol{e}_1$ exists. As shown in [1, Proposition 1], see also [3], the conditions

$$\nabla \cdot \boldsymbol{h}(y,t) = 0, \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad [\mu \boldsymbol{h} \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{N} \times \mathcal{P}\boldsymbol{e}_1] = \boldsymbol{\Psi}, \quad y \in \mathcal{G}, \tag{4.3}$$

imply $[\boldsymbol{N} \cdot \mathrm{rot}\mathcal{P}\boldsymbol{e}_1] = -[\mu(\boldsymbol{h}_t - \boldsymbol{\Phi}) \cdot \boldsymbol{N}] = [\mu\boldsymbol{\Phi} \cdot \boldsymbol{N}], \quad y \in \mathcal{G}$. In addition, we have

$$\boldsymbol{n} \cdot ((\mu \boldsymbol{h}_t - \boldsymbol{\Phi}) + \mathrm{rot}\mathcal{P}\boldsymbol{e}_1)|_{y \in S} = 0. \tag{4.4}$$

Now, we consider the problem

$$\mathrm{rot}\boldsymbol{\mathcal{E}} = -\mu(\boldsymbol{h}_t - \boldsymbol{\Phi}) - \mathrm{rot}\mathcal{P}\boldsymbol{e}_1, \quad \nabla \cdot \mathcal{P}^{-1}\boldsymbol{\mathcal{E}} = 0, \quad y \in \Omega, \quad \boldsymbol{\mathcal{E}}_\tau = 0, \quad y \in S. \tag{4.5}$$

Theorem 6. *The problem (4.5) has a unique solution $\boldsymbol{\mathcal{E}} \in W_r^{1,0}(Q_T)$ orthogonal to the $b_2(\Omega)$-dimensional space $\widetilde{U}_d(\Omega)$ of vector fields $\boldsymbol{v}_j(y) = \nabla W_j(y)$ such that*

$$\nabla \cdot \mathcal{P}^{-1}\nabla W_j(y) = 0, \quad y \in \Omega, \quad W_j|_{S'_k} = \delta_{jk}, \quad W_j|_{S'_0} = 0, \tag{4.6}$$

where S'_k, $k = 0, ..., b_2(\Omega)$, are all the components of the boundary S of Ω. The solution satisfies the inequality

$$\|\boldsymbol{\mathcal{E}}\|_{W_r^{1,0}(Q_T)} \leq c(\|\boldsymbol{h}_t\|_{L_r(Q_T)} + \|\boldsymbol{\Phi}\|_{L_r(Q_T)} + \|\boldsymbol{\Psi}\|_{W_r^{1-1/r}(G_T)}). \tag{4.7}$$

Proof. As in [3], the solution may be expressed by the explicit formula

$$\begin{aligned}&\boldsymbol{\mathcal{E}}(y,t) = \boldsymbol{\mathcal{E}}_1(y,t) + \nabla Z(y,t) + \sum_{j=1}^{b_2(\Omega)} A_j(t)\nabla V_j(y), \\ &\boldsymbol{\mathcal{E}}_1(y,t) = -\frac{1}{4\pi}\mathrm{rot}\int_\Omega \frac{\mu(\boldsymbol{h}_t(z,t) - \boldsymbol{\Phi}(z,t)) + \mathrm{rot}\mathcal{P}\boldsymbol{e}_1(z,t)}{|y-z|}dz, \\ &\nabla \cdot \mathcal{P}^{-1}\nabla Z(y,t) = -\nabla \cdot \mathcal{P}^{-1}\boldsymbol{\mathcal{E}}_1, \quad y \in \Omega, \quad Z(y,t) = -g(y,t), \quad y \in S,\end{aligned}$$

The function g is defined as follows. In view of (4.4), $\int_\Sigma \mathrm{rot}\boldsymbol{\mathcal{E}}_1 \cdot \boldsymbol{n} dS = 0$ for arbitrary $\Sigma \subset S$ and, moreover, for the surfaces $\Sigma_k \subset \Omega$ such that the domain $\Omega \setminus \cup_k \Sigma_k$ is simply connected. This follows from the condition (1.18), i.e.,

$$\int_\Omega (\mu(\boldsymbol{h}_t - \boldsymbol{\Phi}) + \mathrm{rot}\mathcal{P}\boldsymbol{e}_1) \cdot \boldsymbol{u}_q^\Omega dy = 0, \quad q = 1, ..., b_1(\Omega),$$

see [8,9] for details. Hence $\int_L \boldsymbol{\mathcal{E}}_1 \cdot d\boldsymbol{l} = 0$ for arbitrary closed contour $L \subset S$, which means that $\boldsymbol{\mathcal{E}}_1 = \nabla_\tau g$ and $\boldsymbol{\mathcal{E}}_{1,\tau} + \nabla_\tau Z|_S = 0$.

From the Calderon-Zygmund theorem and elliptic estimates it follows that

$$\|\boldsymbol{\mathcal{E}}_1 + \nabla Z\|_{W_r^{1,0}(Q_T)} \le c(\|\boldsymbol{h}_t\|_{L_r(Q_T)} + \|\boldsymbol{\Phi}\|_{L_r(Q_T)} + \|\boldsymbol{\Psi}\|_{W_r^{1-1/r,0}(G_T)}).$$

Moreover, the orthogonality condition

$$0 = \int_\Omega (\boldsymbol{\mathcal{E}}_1 + \nabla Z + \sum_{j=1}^{b_2(\Omega)} A_k(t)\nabla V_j) \cdot \nabla V_q dy$$

yields the estimate of A_k by $\boldsymbol{\mathcal{E}}_1 + \nabla Z$, which completes the proof of (4.7) and of Theorem 6.

Now we pass to the definition of $\boldsymbol{e}$. We set

$$\begin{aligned} &\mathcal{P}\boldsymbol{e}^{(1)} = \alpha_1^{-1}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h} - \mu(\mathcal{L}^{-1}\boldsymbol{u} \times \boldsymbol{h}), \quad y \in \mathcal{F}_1, \\ &\boldsymbol{e}^{(3)} = \alpha^{-1}(\mathrm{rot}\boldsymbol{h} - \boldsymbol{j}), \quad y \in \mathcal{F}_3. \end{aligned} \tag{4.8}$$

Then

$$\begin{aligned} &\mathrm{rot}\mathcal{P}\boldsymbol{e} = -\mu\boldsymbol{h}_t + \mu\boldsymbol{\Phi} = \mathrm{rot}\boldsymbol{\mathcal{E}} + \mathrm{rot}\mathcal{P}\boldsymbol{e}_1, \quad y \in \mathcal{F}_1, \\ &\mathrm{rot}\boldsymbol{e} = \mathrm{rot}\boldsymbol{\mathcal{E}} + \mathrm{rot}\boldsymbol{e}_1, \quad y \in \mathcal{F}_3, \end{aligned} \tag{4.9}$$

which implies

$$\begin{aligned} &\mathcal{P}\boldsymbol{e} = \boldsymbol{\mathcal{E}} + \mathcal{P}\boldsymbol{e}_1 + \nabla\chi_1, \quad y \in \mathcal{F}_1, \\ &\boldsymbol{e} = \boldsymbol{\mathcal{E}} + \boldsymbol{e}_1 + \nabla\chi_3, \quad y \in \mathcal{F}_3. \end{aligned} \tag{4.10}$$

Since $\mathcal{F}_1$ and $\mathcal{F}_3$ may be multi-connected, this conclusion should be justified. We show that

$$\int_{\mathcal{F}_1} (\mathcal{P}\boldsymbol{e} - \boldsymbol{\mathcal{E}} - \mathcal{P}\boldsymbol{e}_1) \cdot \mathrm{rot}\boldsymbol{\psi}(y) dy + \int_{\mathcal{F}_3} (\boldsymbol{e} - \boldsymbol{\mathcal{E}} - \boldsymbol{e}_1) \cdot \mathrm{rot}\boldsymbol{\psi}(y) dy = 0, \quad \forall \boldsymbol{\psi} \in \mathcal{H}^{(1)}(\Omega), \tag{4.11}$$

where $\mathcal{H}^{(1)}(\Omega)$ is the space of solenoidal vector fields from $W_2^1(\mathcal{F}_1) \cap W_2^1(\mathcal{F}_2) \cap W_2^1(\mathcal{F}_3)$ satisfying the equations $\mathrm{rot}\psi(y) = 0$, $\nabla \cdot \boldsymbol{\psi}(y) = 0$ in $\mathcal{F}_2$ and the relations (2.16), (2.17) with $k_i = const$ (cf. [12], [9]). Then (4.10) follows from Theorem 3 in [2] (with $\boldsymbol{a} = 0$, $\boldsymbol{j}(y) = 0$ in $\mathcal{F}_2$), that shows that $\mathrm{rot}\boldsymbol{\psi}|_{\mathcal{F}_i}$, $i = 1, 3$, coincides with the space of solenoidal vector fields in $\mathcal{F}_i$ with vanishing normal components on the boundaries $\mathcal{G}$ and S_3; see also Theorem 3.3 in [9].

Let $\mathcal{H}_0^{(1)}(\Omega)$ be the subspace of $\mathcal{H}^{(1)}(\Omega)$ such that $\boldsymbol{\psi}(y) = \nabla\varphi(y)$, $y \in \mathcal{F}_2$ and $k_i = 0$ in (2.16). The remaining finite dimensional subspace of $\mathcal{H}^{(1)}(\Omega)$ is a linear combination of vector fields $\boldsymbol{\psi}_q(y)$, $q = 1, ..., b_1(\mathcal{F}_2)$ such that

$$\begin{aligned} &\boldsymbol{\psi}_q(y) = \boldsymbol{u}_q^{\mathcal{F}}(y), \quad \boldsymbol{u}_q^{\mathcal{F}} \in U_n(\mathcal{F}_2), \quad y \in \mathcal{F}_2, \\ &\nabla \cdot \boldsymbol{\psi}_q(y) = 0, \quad y \in \mathcal{F}_1, \quad \boldsymbol{\psi}_q(y) \cdot \boldsymbol{N}(y) = 0, \quad (\boldsymbol{\psi}_q)_\tau = (\boldsymbol{u}_q)_\tau(y), \quad y \in \mathcal{G}, \\ &\nabla \cdot \boldsymbol{\psi}_q(y) = 0, \quad y \in \mathcal{F}_3, \quad \boldsymbol{\psi}_q(y) \cdot \boldsymbol{n}(y) = 0, \quad (\boldsymbol{\psi}_q)_\tau = (\boldsymbol{u}_q)_\tau(y), \quad y \in S_3. \end{aligned}$$

Let $\boldsymbol{\psi} \in \mathcal{H}_0^{(1)}(\Omega)$ and let $\Phi(y)$, $y \in \mathcal{F}_1 \cup \mathcal{F}_3$ be the solution of

$$\begin{aligned} &\nabla^2\Phi(y) = 0, \quad y \in \mathcal{F}_1, \quad \Phi(y) = \varphi(y), \quad y \in \mathcal{G}, \\ &\nabla^2\Phi(y) = 0, \quad y \in \mathcal{F}_3, \quad \Phi(y) = \varphi(y), \quad y \in S_3. \end{aligned}$$

Since $(\varphi(y)-\nabla\Phi(y))_\tau=0, \quad y\in\mathcal{G}\cup S_3$, we have

$$\begin{aligned}
&\int_{\mathcal{F}_1}(\mathcal{P}\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\mathcal{P}\boldsymbol{e}_1)\cdot \mathrm{rot}\boldsymbol{\psi}(y)dy+\int_{\mathcal{F}_3}(\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\boldsymbol{e}_1)\cdot\mathrm{rot}\boldsymbol{\psi}(y)dy\\
&=\int_{\mathcal{F}_1}(\mathcal{P}\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\mathcal{P}\boldsymbol{e}_1)\cdot\mathrm{rot}(\boldsymbol{\psi}(y)-\nabla\Phi)dy+\int_{\mathcal{F}_3}(\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\boldsymbol{e}_1)\cdot\mathrm{rot}(\boldsymbol{\psi}(y)-\nabla\Phi(y)dy\\
&=\int_{\mathcal{F}_1}\mathrm{rot}(\mathcal{P}\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\mathcal{P}\boldsymbol{e}_1)\cdot(\boldsymbol{\psi}(y)-\nabla\Phi(y))dy\\
&+\int_{\mathcal{F}_3}\mathrm{rot}(\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\boldsymbol{e}_1)\cdot(\boldsymbol{\psi}(y)-\nabla\Phi(y))dy=0,
\end{aligned} \tag{4.12}$$

$\forall\psi\in\mathcal{H}_0^{(1)}(\Omega)$. In addition, if $\psi=\psi_q$, then

$$\begin{aligned}
&\int_{\mathcal{F}_1}(\mathcal{P}\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\mathcal{P}\boldsymbol{e}_1)\cdot\mathrm{rot}\boldsymbol{\psi}_q(y)dy+\int_{\mathcal{F}_3}(\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\boldsymbol{e}_1)\cdot\mathrm{rot}\boldsymbol{\psi}_q(y)dy\\
&=\int_{\mathcal{F}_1}(\alpha^{-1}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h}-\boldsymbol{J})\cdot\mathrm{rot}\boldsymbol{\psi}_q(y)dy+\int_{\mathcal{F}_3}\alpha^{-1}(\mathrm{rot}\boldsymbol{h}-\boldsymbol{j})\cdot\mathrm{rot}\boldsymbol{\psi}_q(y)dy\\
&+\int_{\mathcal{F}_1\cup\mathcal{F}_3}\mu(\boldsymbol{h}_t-\boldsymbol{\Phi})\cdot\boldsymbol{\psi}_q dy\\
&+\int_{\mathcal{G}}\boldsymbol{N}\times(\boldsymbol{\mathcal{E}}+\mathcal{P}\boldsymbol{e}_1^{(1)})\cdot\boldsymbol{\psi}_q(y)dS+\int_{S_3}\boldsymbol{n}\times(\boldsymbol{\mathcal{E}}+\boldsymbol{e}_1^{(3)})\cdot\boldsymbol{\psi}_q(y)dS,
\end{aligned} \tag{4.13}$$

and since the last two surface integrals are equal to

$$-\int_{\mathcal{F}_2}\mathrm{rot}(\mathcal{E}+\mathcal{P}\boldsymbol{e}_1)\cdot\boldsymbol{\psi}_q(y)dy=\int_{\mathcal{F}_2}\mu(\boldsymbol{h}_t-\boldsymbol{\Phi})\cdot\boldsymbol{\psi}_q dy+\int_{\mathcal{G}}\boldsymbol{\Psi}\cdot\boldsymbol{\psi}_q dS,$$

we can conclude from (4.10) and (1.16) that

$$\int_{\mathcal{F}_1}(\mathcal{P}\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\mathcal{P}\boldsymbol{e}_1)\cdot\mathrm{rot}\boldsymbol{\psi}_q(y)dy+\int_{\mathcal{F}_3}(\boldsymbol{e}-\boldsymbol{\mathcal{E}}-\boldsymbol{e}_1)\cdot\mathrm{rot}\boldsymbol{\psi}_q(y)dy=0,$$

q.e.d.

Finally, we set

$$\mathcal{P}\boldsymbol{e}=\boldsymbol{\mathcal{E}}+\mathcal{P}\boldsymbol{e}_1+\nabla\chi_2+\sum_{j=1}^{b_2(\mathcal{F}_2)}D_j(t)\nabla W_j(y), \quad y\in\mathcal{F}_2, \tag{4.14}$$

where W_j and χ_2 are solutions to the problems (4.6) and

$$\begin{aligned}
&\nabla\cdot\mathcal{P}^{-1}\nabla\chi_2(y,t)=0, \quad y\in\mathcal{F}_2,\\
&\chi_2(y,t)=\chi_1(y,t), \quad y\in\mathcal{G}, \quad \chi_2(y,t)=\chi_3(y,t),\\
&y\in S_3, \quad \chi_2(y,t)=0, \quad y\in S.
\end{aligned} \tag{4.15}$$

.

It is easily verified that the vector field $\boldsymbol{e}$ defined by (4.10), (4.14) satisfies (1.7), (1.8). In order to find $D_j(t)$, we need to impose on $\boldsymbol{e}^{(2)}$ some normalization restrictions, for instance of the form

$$\int_{S'_k}\mathcal{P}\boldsymbol{e}^{(2)}(y,t)\cdot\boldsymbol{n}(y)dS=\int_{S'_k}\boldsymbol{e}^{(2)}(y,t)\cdot\boldsymbol{n}(y)dS=0, \quad k=1,...,b_2(\mathcal{F}_2). \tag{4.16}$$

Since the matrix $\mathcal{J}$ with the elements $\int_{S'_k} \frac{\partial W_j}{\partial n} dS$, $j,k = 1,...,b_2(\mathcal{F}_2)$ is non-degenerate, the equations (4.14), (4.16) define D_j in a unique way.

As in [3], the functions $\boldsymbol{e}$, $\nabla\chi_i$, D_j are easily estimated with the help of the relations (4.8), (4.14), and these estimates show that $\boldsymbol{e} \in W_r^{1,0}(Q_T^i)$.

But if $\boldsymbol{h}_0 \in W_p^{2-2/p}(\mathcal{F}_i)$, $\boldsymbol{j} \in W_p^{1,0}(\mathcal{F}_2)$, then $\boldsymbol{h} \in W_p^{2,1}(Q_T^i)$, $\boldsymbol{e} \in W_p^{1,0}(Q_T^i)$, $i = 1,2,3$ (cf. the remark at the end of Sec. 5 in [3]).

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