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L_p-THEORY OF FREE BOUNDARY PROBLEMS OF MAGNETOHYDRODYNAMICS IN SIMPLY CONNECTED DOMAINS

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Dedicated to the memory of O. A. Ladyzhenskaya

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ABSTRACT:

We present L_p-theory of solvability of free boundary problems of magnetohydrodynamics for a viscous incompressible fluid constructed in the paper [1] for $p = 2$. We consider the simplest problem studied in this paper.

Key words: Sobolev spaces, free boudaries, magnetohydrodynamics

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1 Introduction.

The paper is concerned with the problem of finding a bounded variable domain $\Omega_{1t} \subset \mathbb{R}^3$ with the boundary Γ_t, $t > 0$, that is a strictly interior subdomain of another bounded fixed domain $\Omega \subset \mathbb{R}^3$ with the boundary S. In addition, it is required to find the vector fields $\boldsymbol{H}(x,t) = (H_1, H_2, H_3)$, $x \in \Omega_{1t} \cup \Omega_{2t}$, $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$, and a scalar function $p(x,t)$, $x \in \Omega_{1t}$, satisfying the relations

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla \cdot T(\boldsymbol{v}, p) - \nabla \cdot T_M(H) = 0, \\ \nabla \cdot \boldsymbol{v}(x,t) = 0, \quad x \in \Omega_{1t}, \quad t > 0, \\ \mu_1 \boldsymbol{H}_t + \alpha^{-1} rot rot \boldsymbol{H} - \mu_1 rot(\boldsymbol{v} \times \boldsymbol{H}) = 0, \\ \nabla \cdot \boldsymbol{H}(x,t) = 0, \quad x \in \Omega_{1t}, \\ rot \boldsymbol{H} = 0, \quad \nabla \cdot \boldsymbol{H}(x,t) = 0, \quad x \in \Omega_{2t}, \\ \big(T(\boldsymbol{v}, p) + [T_M(\boldsymbol{H})]\big)\boldsymbol{n} = \sigma \boldsymbol{n} H, \quad V_n = \boldsymbol{v} \cdot \boldsymbol{n}, \quad x \in \Gamma_t, \\ [\mu \boldsymbol{H} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad x \in \Gamma_t, \\ \boldsymbol{H} \cdot \boldsymbol{n} = 0, \quad x \in S, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_{10}, \quad \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x), \quad x \in \Omega_{10} \cup \Omega_{20}, \end{cases} \tag{1.1}$$

where $\Omega_{2t} = \Omega \backslash \overline{\Omega}_{1t}$, $T(\boldsymbol{v}, p)$ is the viscous stress tensor: $T(\boldsymbol{v}, p) = -pI + \nu S(\boldsymbol{v})$, $S(\boldsymbol{v}) = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T$ is the doubled rate-of-strain tensor, $T_M(\boldsymbol{H}) = \mu(\boldsymbol{H} \otimes \boldsymbol{H} - \frac{1}{2}|\boldsymbol{H}|^2 I)$, $x \in \Omega_{it}$, is the magnetic stress tensor, μ is a piece-wise constant function equal to μ_i in Ω_{it}, $\boldsymbol{n}$ is the exterior normal to Γ_t and to S, V_n is the velocity of evolution of Γ_t in the direction $\boldsymbol{n}$, $\boldsymbol{H}_\tau = \boldsymbol{H} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{H})$ is the tangential component of $\boldsymbol{H}$, H is the doubled mean curvature of Γ_t negative for convex domains. The parameters ν, μ_i, α, σ are positive constants. By $[u]$ we mean the jump on Γ_t of the function defined in Ω_{it}: $[u] = u^{(1)} - u^{(2)}$, and $u^{(i)} = u(x,t)|_{x \in \Omega_{it}}$.

Both Ω_{1t} and Ω are assumed to be simply connected.

The problem (1.1) is studied in the paper [1] where the local in time existence theorem of the problem is proved in the Sobolev-Slobodetskii spaces $W_2^{2+l,1+l/2}$, $l \in (1/2, 1)$. In the present paper the solution belonging to $W_p^{2,1}$ with $p > 3$ is obtained. Working in these spaces allows one to get rid of some technical difficulties, in particular, in the estimates of non-linear terms.

In the case of fixed domain Ω_1 filled with the liquid the problem has been studied in the joint papers [2,3] of the author and his teacher O.A.Ladyzhenskaya to whose memory the present article is dedicated.

We assume that Γ_0 is located in the neighborhood of a smooth connected surface $\mathcal{G}$ of arbitrary shape (but such that the domain $\mathcal{F}_1$ bounded by $\mathcal{G}$ is simply connected). Then Γ_0 can be regarded as a normal perturbation of $\mathcal{G}$:

$$\Gamma_0 = \{x = y + \boldsymbol{N}(y)\rho_0(y), \quad y \in \mathcal{G}\},$$

where ρ_0 is a given small function and $\boldsymbol{N}(y)$ is the exterior normal to $\mathcal{G}$. Moreover, we assume that also for $t > 0$

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho(y,t), \quad y \in \mathcal{G}\},$$

with an unknown function $\rho(y,t)$ such that $\rho(y,0) = \rho_0(y)$. We extend $\boldsymbol{N}(y)$ and $\rho(y,t)$ from $\mathcal{G}$ into Ω in such a way that the extension $\boldsymbol{N}^*$ of $\boldsymbol{N}$ is a smooth non-zero regular function in Ω and ρ^* vanishes near S and satisfies the inequalities (5.7), (5.8) (hence ρ^* is small for small ρ).

The transformation

$$x \equiv e_\rho(y,t) = y + \boldsymbol{N}^*(y)\rho^*(y,t), \quad y \in \Omega \tag{1.2}$$

maps $\mathcal{F}_1$ on Ω_{1t}, $\mathcal{F}_2 = \Omega \setminus \overline{\mathcal{F}_1}$ on Ω_{2t} and, as shown in [1], it converts (1.1) in

$$\begin{cases} \boldsymbol{u}_t - \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u} + (\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \widetilde{\nabla}\cdot\widetilde{T}(\boldsymbol{u},q) - \widetilde{\nabla}\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h}) = 0, \\ \nabla\cdot\widehat{\mathcal{L}}\boldsymbol{u} = 0, \quad y \in \mathcal{F}_1, \ \ t > 0, \\ \mu_1\Big(\boldsymbol{h}_t - \frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h} - \rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h}\Big) \\ \ + \alpha^{-1}rot\mathcal{P}rot\mathcal{P}\boldsymbol{h} - \mu_1 rot(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}) = 0, \\ \nabla\cdot\boldsymbol{h} = 0, \quad y \in \mathcal{F}_1, \\ rot\mathcal{P}\boldsymbol{h} = 0, \quad \nabla\cdot\boldsymbol{h} = 0, \quad y \in \mathcal{F}_2, \\ \widetilde{T}(\boldsymbol{u},q)\boldsymbol{n}(e_\rho) + [T_M(\frac{1}{L}\mathcal{L}h)]\boldsymbol{n}(e_\rho) = \sigma H(e_\rho)\boldsymbol{n}(e_\rho), \\ \rho_t = \dfrac{\boldsymbol{u}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}{\boldsymbol{N}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}, \quad y \in \mathcal{G}, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \Big(\dfrac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2} - \boldsymbol{N}\Big)[\boldsymbol{h}\cdot\boldsymbol{N}], \quad y \in \mathcal{G}, \\ \boldsymbol{h}\cdot\boldsymbol{n} = 0, \quad y \in S, \\ \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y \in \mathcal{F}_1, \quad \boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y \in \mathcal{F}_1\cup\mathcal{F}_2, \\ \rho(y,0) = \rho_0(y), \quad y \in \mathcal{G}, \end{cases} \tag{1.3}$$

where $\mathcal{L} = \mathcal{L}(y,\rho^*) = (\delta_{ij} + \frac{\partial}{\partial x_j}N_i^*(y)\rho^*(y,t))_{i,j=1,2,3}$ is the Jacobi matrix of the transformation (1.2), $L = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$ is the co-factors matrix of $\mathcal{L}$, $\mathcal{P}(y,\rho) = L^{-1}\mathcal{L}^T\mathcal{L}$,
$\boldsymbol{u}(y,t) = \boldsymbol{v}(e_\rho(y,t),t)$, $q(y,t) = p(e_\rho,t)$, $\boldsymbol{h}(y,t) = \widehat{\mathcal{L}}(y,\rho^*)\boldsymbol{H}(e_\rho,t)$,
$\widetilde{\nabla} = \mathcal{L}^{-T}(y,\rho^*)\nabla_y$ is the transformed gradient $\nabla_x = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big)$,
$\mathcal{L}^{-T} = (\mathcal{L}^{-1})^T$ (the sign T means transposition),
$\widetilde{T}(\boldsymbol{u},q)$ is the transformed stress tensor: $\widetilde{T} = -qI + \nu\widetilde{S}(\boldsymbol{u})$,
$\widetilde{S}(\boldsymbol{u}) = (\widetilde{\nabla}\boldsymbol{u}) + (\widetilde{\nabla}\boldsymbol{u})^T$ is the transformed rate-of-strain tensor,
ρ is an additional unknown function,
$\boldsymbol{u}_0(y) = \boldsymbol{v}_0(e_{\rho_0}(y))$, $\boldsymbol{h}_0(y) = \widehat{\mathcal{L}}(y,\rho_0^*)\boldsymbol{H}_0(e_{\rho_0}(y))$.

The vectors $\boldsymbol{n}(e_\rho)$ and $\boldsymbol{N}(y)$ are connected by

$$\boldsymbol{n}(e_\rho) = \frac{\widehat{\mathcal{L}}^T \boldsymbol{N}(y)}{|\widehat{\mathcal{L}}^T \boldsymbol{N}|}, \quad y \in \mathcal{G}.$$

We consider the problem (1.3) in the spaces

$$W_p^{2,1}(\mathcal{D}_T) = W_p^{2,0}(\mathcal{D}_T) \cap W_p^{0,1}(\mathcal{D}_T), \quad p > 3, \quad \mathcal{D}_T = \mathcal{D} \times (0,T), \quad \mathcal{D} \subset \mathbb{R}^n,$$

where $W_p^{2,0}(\mathcal{D}_T) = L_p(0,T; W_p^2(\mathcal{D}))$, $W_2^{0,1}(\mathcal{D}_T) = W_p^1(0,T; L_p(\mathcal{D}))$. The norm in $W_p^{2,1}(\mathcal{D}_T)$ is defined by

$$\|u\|^p_{W_p^{2,1}(\mathcal{D}_T)} = \int_0^T \|u(\cdot,t)\|^p_{W_p^2(\mathcal{D})} dt + \int_0^T (\|u_t(\cdot,t)\|^p_{L_p(\mathcal{D})} + \|u(\cdot,t)\|^p_{L_p(\mathcal{D})}) dt, \qquad (1.4)$$

and the integrals in the right-hand side represent the norms in $W_p^{2,0}(\mathcal{D}_T)$ and $W_p^{0,1}(\mathcal{D}_T)$, respectively. By $W_p^l(\mathcal{D})$ we mean the space of functions $u(x)$, $x \in \mathcal{D}$, with finite norm

$$\|u\|_{W_p^l(\mathcal{D})} = \Big(\sum_{|j|\le l} \|D^j u\|^p_{L_p(\mathcal{D})}\Big)^{1/p},$$

if l is an integer, and

$$\|u\|_{W_p^l(\mathcal{D})} = \Big(\|u\|^p_{W_p^{[l]}(\mathcal{D})} + \sum_{|j|=[l]} \int_{\mathcal{D}}\int_{\mathcal{D}} \frac{|D^j u(x) - D^j u(y)|^p}{|x-y|^{n+p\lambda}} dx dy\Big)^{1/p},$$

if $l = [l] + \lambda$, $0 < \lambda < 1$. The anisotropic space $W_p^{l,l/2}(\mathcal{D}_T)$ is defined as $W_p^{l,0}(\mathcal{D}_T) \cap W_p^{0,l/2}(\mathcal{D}_T) = L_p(0,T; W_p^l(\mathcal{D})) \cap W_p^{l/2}(0,T; L_p(\mathcal{D}))$.

The W_p^l-spaces on manifolds, in particular, on the surfaces $\mathcal{G}$ and S, are introduced with the help of local maps and partition of unity.

We recall some imbedding theorems for the space $W_p^{2,1}(\mathcal{D}_T)$:

1. *If* $u \in W_p^{2,1}(\mathcal{D}_T)$, *then* $u|_{t=t_0} \in W_p^{2-2/p}(\mathcal{D})$, $u|_{x\in\Sigma} \in W_p^{2-1/p,1-1/(2p)}(\Sigma_T)$, $\Sigma_T = \Sigma \times (0,T)$, *where* $\Sigma = \partial\mathcal{D}$, *and*

$$\|u(\cdot,t_0)\|_{W_p^{2-2/p}(\mathcal{D})} \le c\|u\|_{W_p^{2,1}(\mathcal{D}_T)},$$

$$\|u\|_{W_p^{2-1/p,1-1/(2p)}(\Sigma_T)} \le c\|u\|_{W_p^{2,1}(\mathcal{D}_T)}.$$

2. *For arbitrary* $\varphi \in W_p^{2-2/p}(\mathcal{D})$ *there exists such* $u \in W_p^{2,1}(\mathcal{D}_T)$ *that* $u(x,0) = \varphi(x)$ *and*

$$\|u\|_{W_p^{2,1}(\mathcal{D}_T)} \le c\|\varphi\|_{W_p^{2-2/p}(\mathcal{D})};$$

for arbitrary $\psi \in W_p^{2-1/p,1-1/(2p)}(\Sigma_T)$ *there exists such* $u \in W_p^{2,1}(\mathcal{D}_T)$ *that* $u|_{x\in\Sigma} = \psi(x,t)$ *and*

$$\|u\|_{W_p^{2,1}(\mathcal{D}_T)} \le c\|\psi\|_{W_p^{2-1/p,1-1/(2p)}(\Sigma_T)}.$$

3. *If* $\rho \in W_p^{3-1/p,0}(\Sigma_T) \cap W_p^1(0,T;W_p^{2-1/p}(\Sigma))$, *then* $\rho(\cdot,t_0) \in W_p^{3-2/p}(\Sigma)$ *and*

$$\|\rho(\cdot,t_0)\|_{W_p^{3-2/p}(\Sigma)} \le c(\|\rho\|_{W_p^{3-1/p,0}(\Sigma_T)} + \|\rho_t\|_{W_p^{2-1/p,0}(\Sigma_T)}).$$

For arbitrary $\rho_0 \in W_p^{3-2/p}(\Sigma)$ *there exists* $\rho \in W_p^{3-1/p,0}(\Sigma_T)$ *with* $\rho_t \in W_p^{2-1/p,1-1/2p}(\Sigma_T)$ *such that* $\rho(x,0) = \rho_0(x)$ *and*

$$\|\rho\|_{W_p^{3-1/p,0}(\Sigma_T)} + \|\rho_t\|_{W_p^{2-1/p,1-1/2p}(\Sigma_T)} \le c\|\rho_0\|_{W_p^{3-2/p}(\Sigma)}$$

(cf. Proposition 4.1 in [4] for $p=2$ *and* [10]*).*

4. *If* $lp > n$, *then*

$$\sup_{\mathcal{D}} |u(x)| \le c\|u\|_{W_p^l(\mathcal{D})}.$$

Now we present the main result of the paper.

Theorem 1. *Let* $\boldsymbol{u}_0 \in W_p^{2-2/p}(\mathcal{F}_1)$, $\rho_0 \in W_p^{3-2/p}(\mathcal{G})$, $\boldsymbol{h}_0 \in W_r^{2-2/r}(\mathcal{F}_i)$, $i=1,2$, *with*

$$3 < r < p, \quad 1/r - 1/p \le 1/5(1-2/p) \tag{1.5}$$

and let the compatibility conditions

$$\begin{aligned}
&\nabla\cdot\widehat{\mathcal{L}}(y,\rho_0^*)\boldsymbol{u}_0 = 0, \quad y\in\mathcal{F}_1, \quad \widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n}_0(e_{\rho_0}) - \boldsymbol{n}_0(\boldsymbol{n}_0\cdot\widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n}_0) = 0, \quad y\in\mathcal{G},\\
&\nabla\cdot\boldsymbol{h}_0 = 0, \quad y\in\mathcal{F}_i, \quad i=1,2,\\
&rot\mathcal{P}(y,\rho_0^*)\boldsymbol{h}_0 = 0, \quad y\in\mathcal{F}_2,\\
&[\mu\boldsymbol{h}_0\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_{0\tau}] = \Big(\frac{\widehat{\mathcal{L}}(y,\rho_0)\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2} - \boldsymbol{N}\Big)[\boldsymbol{h}_0\cdot\boldsymbol{N}], \quad y\in\mathcal{G},\\
&\boldsymbol{h}_0\cdot\boldsymbol{n} = 0, \quad y\in S,
\end{aligned} \tag{1.6}$$

where $\boldsymbol{n}_0$ *is the normal to* Γ_0, *and the smallness condition*

$$\|\rho_0\|_{W_p^{2-1/p}(\mathcal{G})} \le \epsilon \ll 1 \tag{1.7}$$

be satisfied. Then the problem (1.3) has a unique solution defined in a certain (small) time interval $(0,T)$ *with the following regularity properties:*

$$\begin{aligned}
&\boldsymbol{u}\in W_p^{2,1}(Q_T^1), \quad \nabla q\in L_p(Q_T^1), \quad q\in W_p^{l-1/p,0}(G_T), \quad \rho\in W_p^{3-1/p,0}(G_T)\\
&\rho_t\in W_p^{2-1/p,1-1/2p}(G_T), \quad \boldsymbol{h}^{(i)}\in W_r^{2,1}(Q_T^i), \quad i=1,2,
\end{aligned}$$

where $Q_T^i = \mathcal{F}_i \times (0,T)$, $G_T = \mathcal{G} \times (0,T)$, $\boldsymbol{h}^{(i)} = \boldsymbol{h}|_{Q_T^i}$, $i = 1,2$. *The solution satisfies the inequality*

$$\begin{aligned}
&\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)} + \|\nabla q\|_{L_p(Q_T^1)} + \|q\|_{W_p^{1-1/p,0}(G_T)} + \sup_{t<T} \|\boldsymbol{u}\|_{W_p^{2-2/p}(\mathcal{F}_1)} \\
&+ \|\rho\|_{W_p^{3-1/p,0}(G_T)} + \|\rho_t\|_{W_p^{2-1/p,1-1/2p}(G_T)} + \sup_{t<T} \|\rho\|_{W_p^{3-2/p}(\mathcal{G})} \\
&+ \sum_{i=1}^{2} \big(\|\boldsymbol{h}^{(i)}\|_{W_r^{2,1}(Q_T^i)} + \sup_{t<T} \|\boldsymbol{h}^{(i)}\|_{W_r^{2-2/r}(\mathcal{F}_i)}\big) \\
&\leq c\Big(\|\boldsymbol{f}\|_{L_p(Q_T^1)} + \|\mathcal{H}\|_{W_p^{1-1/p}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_p^{2-2/p}(\mathcal{F}_1)} + \|\rho_0\|_{W_p^{3-2/p}(\mathcal{G})} \\
&+ \sum_{i=1}^{2} \|\boldsymbol{h}_0\|_{W_r^{2-2/r}(\mathcal{F}_i)}\Big),
\end{aligned} \tag{1.8}$$

where $\mathcal{H}$ *is the doubled mean curvature of* $\mathcal{G}$.

For the problem (1.1) this means that it is solvable in the time interval $(0,T)$ and

$$\begin{aligned}
&\boldsymbol{v} \circ e_\rho \in W_p^{2.1}(Q_T^1), \quad \nabla p \circ e_\rho \in L_p(Q_T^1), \quad p \circ e_\rho \in W_p^{1-1/p,0}(G_T), \\
&\rho \in W_2^{3-1/p,0}(G_T), \quad \rho_t \in W_p^{2-1/p,1-1/2p}(G_T), \quad \boldsymbol{H}^{(i)} \circ e_\rho \in W_r^{2,1}(Q_T^i). \quad i = 1,2.
\end{aligned}$$

Once the solution with the above-mentioned properties is obtained, it can be shown that

$$\boldsymbol{h}^{(i)} \in W_p^{2,1}(Q_T^i), \quad i = 1,2,$$

provided $\boldsymbol{h}_0 \in W_p^{2-2/p}(\Omega_{i0})$, $i = 1,2$ (see Sec. 5).

2 Linear problems

The proof of Theorem 1 is based on the analysis of the following non-homogeneous linear problems:

1. Find $(\boldsymbol{v}, p, \rho)$ such that

$$\begin{cases}
\boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(y,t), \\
\nabla \cdot \boldsymbol{v} = f(y,t) = \nabla \cdot \boldsymbol{F}(y,t), \quad y \in \mathcal{F}_1, \ \ t > 0, \\
T(\boldsymbol{v},p)\boldsymbol{N}(y) + \sigma\boldsymbol{N}(y)\mathfrak{B}\rho = \boldsymbol{d}(y,t), \\
\rho_t + \boldsymbol{V}(y) \cdot \nabla_\tau \rho - \boldsymbol{v} \cdot \boldsymbol{N}(y) = g(y,t), \quad y \in \mathcal{G}, \\
\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y \in \mathcal{F}_1, \quad \rho(y,0) = \rho_0(y), \quad y \in \mathcal{G},
\end{cases} \tag{2.1}$$

where $\mathfrak{B}\rho = -\Delta_{\mathcal{G}}\rho - b(y)\rho$, $b = (\mathcal{H}^2 - 2\mathcal{K})$, $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively, $\boldsymbol{V}(x)$ is a given vector field from $W_p^{2-1/p}(\mathcal{G})$.

2. Find the vector field $\boldsymbol{H}(y,t)$, satisfying the equations

$$\begin{cases} \mu_1 \boldsymbol{H}_t + \alpha^{-1} rotrot\boldsymbol{H} = \boldsymbol{G}(y,t), \\ \nabla\cdot\boldsymbol{H} = 0, \quad y\in\mathcal{F}_1, \\ rot\boldsymbol{H} = rot\boldsymbol{\ell}(y,t), \quad \nabla\cdot\boldsymbol{H}=0, \quad y\in\mathcal{F}_2, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{H}_\tau] = \boldsymbol{a}(y,t), \quad y\in\mathcal{G}, \quad \boldsymbol{H}\cdot\boldsymbol{n}(y) = 0, \quad y\in S, \\ \boldsymbol{H}(y,0) = \boldsymbol{H}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2, \end{cases} \tag{2.2}$$

In addition, we need to consider the auxiliary problem
3.

$$\begin{cases} rot\boldsymbol{h}(y,t) = \boldsymbol{k}(y,t), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2, \\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \boldsymbol{a}, \quad y\in\mathcal{G}, \\ \boldsymbol{h}\cdot\boldsymbol{n}(y) = 0, \quad y\in S. \end{cases} \tag{2.3}$$

Theorem 2 [5]. *Let $p>3$. Assume that $\boldsymbol{f}\in L_p(Q_T^1)$, $f\in W_p^{1,0}(Q_T^1)$, $f=\nabla\boldsymbol{F}$, $\boldsymbol{F}\in W_p^{0,1}(Q_T^1)$, $\boldsymbol{d}\cdot\boldsymbol{N}\in W_2^{1-1/p,0}(G_T)$, $\boldsymbol{d}-\boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})\equiv\boldsymbol{d}_\tau\in W_p^{1-1/p,1/2-1/(2p)}(G_T)$, $g\in W_p^{2-1/p,1-1/2p}(G_T)$, $\boldsymbol{v}_0\in W_p^{2-2/p}(\mathcal{F}_1)$, $\rho_0\in W_p^{3-2/p}(\mathcal{G})$, $\boldsymbol{V}\in W_p^{2-1/p}(\mathcal{G})$ and let the compatibility conditions*

$$\nabla\cdot\boldsymbol{v}_0(x) = f(x,0), \quad x\in\mathcal{F}_1, \quad \nu(S(\boldsymbol{v}_0)\boldsymbol{N})_\tau = \boldsymbol{d}_\tau(x,0), \quad x\in\mathcal{G}, \tag{2.4}$$

be satisfied. Then the problem (2.1) has a unique solution $\boldsymbol{v},p,\rho$ such that $\boldsymbol{v}\in W_p^{2,1}(Q_T^1)$, $\nabla p\in L_p(Q_T^1)$, $p\in W_p^{1-1/p,0}(G_T)$, $\rho\in W_p^{3-1/p,0}(G_T)$, $\rho_t\in W_p^{2-1/p,1-1/2p}(G_T)$, $\rho(\cdot,t)\in W_p^{3-2/p}(\mathcal{G})$, $\forall t\in(0,T)$, and the solution satisfies the inequality

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_p^{2,1}(Q_T^1)} + \|\nabla p\|_{L_p(Q_T^1)} + \|p\|_{W_p^{1-1/p,0}(G_T)} \\ &+ \|\rho\|_{W_p^{3-1/p,0}(G_T)} + \|\rho_t\|_{W_2^{2-1/p,1-1/2p}(G_T)} + \sup_{t<T}\|\rho(\cdot,t)\|_{W_p^{3-2/p}(\mathcal{G})} \\ &\le c(T)\Big(\|\boldsymbol{f}\|_{L_p(Q_T^1)} + \|f\|_{W_p^{1,0}(Q_T^1)} + \|\boldsymbol{F}\|_{W_p^{0,1}(Q_T^1)} \\ &+ \|\boldsymbol{d}_\tau\|_{W_2^{1-1/p,l/2-1/(2p)}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1-1/p,0}(G_T)} \\ &+ \|g\|_{W_p^{2-1/p,1-1/2p}(G_T)} + \|\boldsymbol{v}_0\|_{W_p^{2-2/p}(\mathcal{F})} + \|\rho_0\|_{W_p^{3-2/p}(\mathcal{G})}\Big). \end{aligned} \tag{2.5}$$

The compatibility condition $\nabla\cdot\boldsymbol{v}_0(x) = f(x,0), \quad x\in\mathcal{F}_1$ can be understood in a weak sense as

$$\int_{\mathcal{F}_1}(\boldsymbol{v}_0(x)-\boldsymbol{F}(x,0))\cdot\nabla\eta(x)dx = 0$$

for arbitrary smooth $\eta(x)$ vanishing on $\mathcal{G}$.

Theorem 3. *Assume that $\boldsymbol{k} = rot\boldsymbol{K}(y,t)$, $\boldsymbol{a} = [\boldsymbol{A}]$, $\boldsymbol{K},\boldsymbol{A}\in W_p^{2,1}(Q_T^i)$, $i=1,2$,*

$$\boldsymbol{K} = \boldsymbol{A} = 0, \quad y\in S, \quad [\boldsymbol{K}_\tau] = \boldsymbol{a}, \quad \boldsymbol{A}^{(1)}\cdot\boldsymbol{N}(y) = \boldsymbol{A}^{(2)}\cdot\boldsymbol{N}(y) = 0, \quad y\in\mathcal{G}. \tag{2.6}$$

Then the problem (2.3) has a unique solution $\boldsymbol{h} \in W_p^{2,1}(\mathcal{F}_i)$, $i = 1, 2$, *that satisfies the inequality*

$$\sum_{i=1}^{2} \|\boldsymbol{h}\|_{W_p^{2,1}(\mathcal{F}_i)} \leq c \sum_{i=1}^{2} (\|\boldsymbol{K}\|_{W_p^{2,1}(Q_T^i)} + \|\boldsymbol{A}\|_{W_p^{2,1}(Q_T)^i}). \tag{2.7}$$

Proof. Following [1], Theorem 3, we seek the solution in the form

$$\boldsymbol{h} = \boldsymbol{A} + \nabla\phi + \boldsymbol{X},$$

where ϕ and $\boldsymbol{X}$ are solutions to the problems

$$\begin{cases} \nabla^2\phi(x) = -\nabla \cdot \boldsymbol{A}(x), \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad \dfrac{\partial\phi}{\partial n} = 0, \quad x \in S, \\ [\phi] = 0, \quad [\mu \dfrac{\partial\phi}{\partial N}] = 0, \quad x \in \mathcal{G}, \end{cases} \tag{2.8}$$

$$\begin{cases} rot\boldsymbol{X} = rot(\boldsymbol{K}(x,t) - \boldsymbol{A}(x,t)), \quad \nabla \cdot \boldsymbol{X} = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \\ [\mu\boldsymbol{X} \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{X}_\tau] = 0, \quad x \in \mathcal{G}, \quad \boldsymbol{X} \cdot \boldsymbol{n} = 0, \quad x \in S. \end{cases}$$

The problem (2.8) can be written in a weak form as

$$\int_\Omega \mu\nabla\phi \cdot \nabla\eta dx = -\int_\Omega \mu\boldsymbol{A} \cdot \nabla\eta dx$$

with arbitrary smooth $\eta(x)$. It is uniquely solvable, up to a constant, and

$$\|\nabla\phi\|_{L_p(\Omega)} \leq c\|\boldsymbol{A}\|_{L_p(\Omega)}, \quad \sum_{i=1}^{2} \|\nabla\phi\|_{W_p^2(\mathcal{F}_i)} \leq c \sum_{i=1}^{2} \|\boldsymbol{A}\|_{W_p^2(\mathcal{F}_i)}, \tag{2.9}$$

which implies

$$\sum_{i=1}^{2} \|\nabla\phi\|_{W_p^{2,1}(Q_T^i)} \leq c \sum_{i=1}^{2} \|\boldsymbol{A}\|_{W_p^{2,1}(Q_T^i)}. \tag{2.10}$$

As for $\boldsymbol{X}$, we represent it as the sum

$$\boldsymbol{X}(x,t) = \boldsymbol{X}_1(x,t) + \nabla U(x,t),$$
$$\boldsymbol{X}_1(x,t) = \frac{1}{4\pi} rot \int_\Omega \frac{rot(\boldsymbol{K}(y,t) - \boldsymbol{A}(y,t))}{|x-y|} dy,$$

$$\begin{cases} \nabla^2 U(x,t) = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad [U(x,t)] = 0, \quad [\mu \dfrac{\partial U}{\partial N}] = -[\mu]\boldsymbol{X}_1 \cdot \boldsymbol{N}, \quad x \in \mathcal{G}, \\ \dfrac{\partial U}{\partial n} = -\boldsymbol{X}_1 \cdot \boldsymbol{n}, \quad x \in S, \end{cases}$$

or, in a weak form,

$$\int_\Omega \mu\nabla U \cdot \nabla\eta dx = \int_{\mathcal{G}} [\mu]\boldsymbol{X}_1 \cdot \boldsymbol{N}\eta dS + \int_S \mu_2\boldsymbol{X}_1 \cdot \boldsymbol{n}\eta dS = \int_\Omega \mu\boldsymbol{X}_1 \cdot \nabla\eta dx.$$

Since $[\boldsymbol{N}\times(\boldsymbol{K}-\boldsymbol{A})]|_{x\in\mathcal{G}}=0$, $\boldsymbol{K}_\tau=\boldsymbol{A}_\tau|_{x\in S}=0$, it holds $rot\boldsymbol{X}_1=\boldsymbol{K}-\boldsymbol{A}$,

$$\boldsymbol{X}_1(x,t)=\frac{1}{4\pi}rot\int_\Omega\nabla\frac{1}{|x-y|}\times(\boldsymbol{K}(y,t)-\boldsymbol{A}(y,t))dy,$$

$$\begin{aligned}&\|\boldsymbol{X}_1(\cdot,t)\|_{L_p(\Omega)}\le c(\|\boldsymbol{K}(\cdot,t)\|_{L_p(\Omega)}+\|\boldsymbol{A}(\cdot,t)\|_{L_p(\Omega)}),\\&\sum_{i=1}^2\|\boldsymbol{X}_1(\cdot,t)\|_{W_p^2(\mathcal{F}_i)}\le c\sum_{i=1}^2(\|\boldsymbol{K}(\cdot,t)\|_{W_p^2(\mathcal{F}_\rangle)}+\|\boldsymbol{A}(\cdot,t)\|_{W_p^2(\mathcal{F}_i)})\end{aligned}$$

and, as a consequence,

$$\sum_{i=1}^2\|\boldsymbol{X}_1\|_{W_p^{2,1}(Q_T^i)}\le c\sum_{i=1}^2(\|\boldsymbol{K}\|_{W_p^{2,1}(Q_T^i)}+\|\boldsymbol{A}\|_{W_p^{2,1}(\mathcal{F}_i)}).\tag{2.11}$$

In addition, we have

$$\begin{aligned}&\|\nabla U(\cdot,t)\|_{L_p(\Omega)}\le c\|\boldsymbol{X}_1(\cdot,t)\|_{L_p(\Omega)},\\&\|\nabla U(\cdot,t)\|_{W_p^2(\Omega)}\le c\sum_{i=1}^2\|\boldsymbol{X}_1(\cdot,t)\|_{W_p^2(\mathcal{F}_i)}\\&\sum_{i=1}^2\|\nabla U\|_{W_p^{2,1}(Q_T^i)}\le c\sum_{i=1}^2\|\boldsymbol{X}_1\|_{W_p^{2,1}(Q_T^i)}.\end{aligned}\tag{2.12}$$

Inequality (2.7) follows from (2.9)-(2.12). If $\boldsymbol{k}=0$, $\boldsymbol{a}=0$, then $\boldsymbol{h}=0$, hence the solution of (2.3) constructed above is unique. The theorem is proved.

Theorem 4. *Assume that the data of the problem (2.2) possess the following properties:* $\boldsymbol{G}\in L_p(Q_T^1)$, $\boldsymbol{H}_0\in W_p^{2-2/p}(\mathcal{F}_1)\cap W_p^{2-2/p}(\mathcal{F}_2)$, $\boldsymbol{a}\in W_p^{2-1/p,1-1/2p}(G_T)$, $\boldsymbol{\ell}\in W_p^{2,1}(Q_T^2)$, $\boldsymbol{\ell}|_{x\in S}=0$, *moreover,* $\boldsymbol{a}=[\boldsymbol{A}]$ *with* $\boldsymbol{A}^{(i)}\in W_p^{2,1}(Q_T^i)$, $\boldsymbol{A}^{(i)}\cdot\boldsymbol{N}(x)|_{x\in\mathcal{G}}=0$, $i=1,2$, *and the compatibility conditions*

$$\begin{aligned}&\nabla\cdot\boldsymbol{G}(x,t)=0,\quad x\in\mathcal{F}_1,\quad\nabla\cdot\boldsymbol{H}_0(x)=0,\quad x\in\mathcal{F}_1\cup\mathcal{F}_2,\\&\mathrm{rot}\boldsymbol{H}_0(x)=\mathrm{rot}\boldsymbol{\ell}(x,0),\quad x\in\mathcal{F}_2,\\&[\mu\boldsymbol{H}_0\cdot\boldsymbol{N}]=0,\quad[\boldsymbol{H}_{0\tau}]=\boldsymbol{a}(x,0)=[\boldsymbol{A}(x,0)],\quad x\in\mathcal{G},\quad\boldsymbol{H}_0\cdot\boldsymbol{n}=0,\quad x\in S\end{aligned}\tag{2.13}$$

are satisfied. Then the problem (2.2) has a unique solution $\boldsymbol{H}\in W_p^{2,1}(Q_T^1)\cap W_p^{2,1}(Q_T^2)$, *and*

$$\begin{aligned}&\sum_{i=1}^2\|\boldsymbol{H}^{(i)}\|_{W_p^{2,1}(Q_T^i)}\le c(T)\Big(\|\boldsymbol{G}\|_{L_p(Q_T^1)}+\|\boldsymbol{\ell}\|_{W_p^{2,1}(Q_T^2)}\\&+\sum_{i=1}^2(\|\boldsymbol{H}_0\|_{W_p^{2-2/p}(\mathcal{F}_i)}+\|\boldsymbol{A}^{(i)}\|_{W_p^{2,1}(Q_T^i)})\Big)\end{aligned}\tag{2.14}$$

Proof. We reduce the problem to the particular case $\boldsymbol{\ell}=0$, $\boldsymbol{a}=0$. We extend $\boldsymbol{\ell}$ into Ω with the preservation of class, i.e., so that the extended function $\boldsymbol{\ell}^*$ satisfies

$$\|\boldsymbol{\ell}^*\|_{W_p^{2,1}(Q_T)} \le c\|\boldsymbol{\ell}\|_{W_p^{2,1}(Q_T^2)}, \tag{2.15}$$

and we define

$$\begin{aligned}&\boldsymbol{a}^*(x,t)=\boldsymbol{A}^{(1)}(x,t)-\boldsymbol{A}^{(2)*}(x,t), \quad x\in\mathcal{F}_1,\\ &\boldsymbol{a}^*(x,t)=0, \quad x\in\mathcal{F}_2,\end{aligned}$$

where $\boldsymbol{A}^{(2)*}$ is the extension of $\boldsymbol{A}^{(2)}$ such that

$$\|\boldsymbol{A}^{(2)*}\|_{W_p^{2,1}(Q_T)} \le c\|\boldsymbol{A}^{(2)}\|_{W_p^{2,1}(Q_T^2)}.$$

It is easily verified that

$$[\boldsymbol{a}^*]=[\boldsymbol{A}]=\boldsymbol{a}, \quad [\boldsymbol{\ell}^*+\boldsymbol{a}^*]=\boldsymbol{a}, \quad x\in\mathcal{G}.$$

Now we define $\boldsymbol{h}_1(x,t)$ as a solution of the problem (2.3) with $\boldsymbol{k}(x,t)=rot(\boldsymbol{\ell}^*+\boldsymbol{a}^*)$. By (2.7),

$$\begin{aligned}&\sum_{i=1}^2\|\boldsymbol{h}_1\|_{W_p^{2,1}(Q_T^i)} \le c(\sum_{i=1}^2\|\boldsymbol{\ell}^*\|_{W_p^{2,1}(Q_T^i)}+\|\boldsymbol{a}^*\|_{W_p^{2,1}(Q_T^i)})\\ &\le c(\|\boldsymbol{\ell}\|_{W_p^{2,1}(Q_T^2)}+\sum_{i=1}^2\|\boldsymbol{A}\|_{W_p^{2,1}(Q_T^i)}).\end{aligned} \tag{2.16}$$

For $\boldsymbol{h}=\boldsymbol{H}-\boldsymbol{h}_1$ we obtain the problem

$$\begin{cases}\mu_1\boldsymbol{h}_t+\alpha^{-1}\mathrm{rot}\mathrm{rot}\boldsymbol{h}=\boldsymbol{G}'(y,t),\\ \nabla\cdot\boldsymbol{h}=0, \quad y\in\mathcal{F}_1,\\ \mathrm{rot}\boldsymbol{h}=0, \quad \nabla\cdot\boldsymbol{h}=0, \quad y\in\mathcal{F}_2,\\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}]=0, \quad [\boldsymbol{h}_\tau]=0, \quad y\in\mathcal{G}, \quad \boldsymbol{h}\cdot\boldsymbol{n}(y)=0, \quad y\in S,\\ \boldsymbol{h}(y,0)=\boldsymbol{h}_0(y)=\boldsymbol{H}_0(x)-\boldsymbol{h}_1(y,0), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2,\end{cases} \tag{2.17}$$

where $\boldsymbol{G}'=\boldsymbol{G}-\mu_1\boldsymbol{h}_{1t}-\alpha^{-1}rotrot\boldsymbol{h}_1$.

The equations $rot\boldsymbol{h}(y,t)=0$, $\nabla\cdot\boldsymbol{h}(y,t)=0$ in $\mathcal{F}_2$ imply

$$\begin{aligned}&\boldsymbol{h}(y,t)=\nabla\varphi(y,t),\\ &\nabla^2\varphi(y,t)=0, \quad y\in\mathcal{F}_2,\\ &\frac{\partial\varphi}{\partial n}\Big|_{y\in S}=0, \quad \mu_2\frac{\partial\varphi}{\partial n}-\mu_1\boldsymbol{h}^{(1)}\cdot\boldsymbol{N}\Big|_{y\in\mathcal{G}}=0,\\ &\boldsymbol{h}_\tau^{(1)}-\nabla_\tau\varphi|_{y\in\mathcal{G}}=0,\end{aligned} \tag{2.18}$$

so (2.17) can be written in the form

$$
\begin{cases}
\mu_1 \boldsymbol{h}_t + \alpha^{-1} rotrot \boldsymbol{h} = \boldsymbol{G}'(y,t), \quad \nabla \cdot \boldsymbol{h} = 0, \quad y \in \mathcal{F}_1, \\
\nabla^2 \varphi(y,t) = 0, \quad y \in \mathcal{F}_2, \quad \dfrac{\partial \varphi}{\partial n}\Big|_{y \in S} = 0, \\
\mu_2 \dfrac{\partial \varphi}{\partial N} - \mu_1 \boldsymbol{h}^{(1)} \cdot \boldsymbol{N}\Big|_{y \in \mathcal{G}} = 0, \quad \boldsymbol{h}_\tau^{(1)} - \nabla_\tau \varphi|_{y \in \mathcal{G}} = 0, \\
\boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2,
\end{cases}
\tag{2.19}
$$

As shown in [6], this problem is uniquely solvable, and

$$
\|\boldsymbol{h}\|_{W_p^{2,1}(Q_T^1)} + \|\nabla \varphi\|_{W_p^{2,1}(Q_T^2)} \leq c(\|\boldsymbol{G}'\|_{L_p(Q_T^1)} + \sum_{i=1}^{2} \|\boldsymbol{h}_0\|_{W_2^{2-2/p}(\mathcal{F}_i)}).
$$

Together with (2.16), this inequality implies (2.14). The theorem is proved.

Equations (2.18) show that $\boldsymbol{h}^{(2)} = \nabla \varphi$ is completely determined by $\boldsymbol{h}^{(1)} \cdot \boldsymbol{N}|_{y \in \mathcal{G}}$ and

$$
\begin{aligned}
&\|\boldsymbol{h}^{(2)}\|_{W_p^{2,1}(Q_T^2)} \leq c \|\boldsymbol{h}^{(1)}\|_{W_p^{2,1}(Q_T^1)}, \\
&\|\boldsymbol{h}_0^{(2)}\|_{W_p^{2-2/p}(\mathcal{F}_2)} \leq c \|\boldsymbol{h}_0^{(1)}\|_{W_p^{2-2/p}(\mathcal{F}_1)}.
\end{aligned}
$$

3 Nonlinear problem

In this section we outline main ideas of the proof of Theorem 1. Following [1], we write the problem (1.3) in the form

$$
\begin{cases}
\boldsymbol{u}_t(y,t) - \nu \nabla^2 \boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q, \boldsymbol{h}, \rho), \\
\nabla \cdot \boldsymbol{u} = l_2(\boldsymbol{u}, \rho), \quad y \in \mathcal{F}_1, \ t > 0, \\
\Pi_{\mathcal{G}} S(\boldsymbol{u}) \boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}, \rho), \\
-q + \nu \boldsymbol{N} \cdot S(\boldsymbol{u}) \boldsymbol{N}(y) + \sigma \mathfrak{B} \rho = l_4(\boldsymbol{u}, \boldsymbol{h}, \rho) + l_5(\rho) + \sigma \mathcal{H}(y), \\
\rho_t + \boldsymbol{V}(x) \cdot \nabla_\tau \rho - \boldsymbol{u} \cdot \boldsymbol{N}(y) = l_6(\boldsymbol{u}, \rho), \quad y \in \mathcal{G}, \\
\mu_1 \boldsymbol{h}_t + \alpha^{-1} \mathrm{rot}\,\mathrm{rot} \boldsymbol{h} = \boldsymbol{l}_7(\boldsymbol{h}, \boldsymbol{u}, \rho), \quad \nabla \cdot \boldsymbol{h} = 0, \quad y \in \mathcal{F}_1, \\
\mathrm{rot} \boldsymbol{h} = \mathrm{rot} \boldsymbol{l}_8(\boldsymbol{h}, \rho), \quad \nabla \cdot \boldsymbol{h} = 0, \quad y \in \mathcal{F}_2, \\
[\mu \boldsymbol{h} \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \boldsymbol{l}_9(\boldsymbol{h}, \rho), \quad y \in \mathcal{G}, \quad \boldsymbol{h} \cdot \boldsymbol{n} = 0, \quad y \in S, \\
\boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y \in \mathcal{F}_1, \quad \boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2, \\
\rho(y,0) = \rho_0(y), \quad y \in \mathcal{G},
\end{cases}
\tag{3.1}
$$

where

$$
\begin{cases}
\boldsymbol{l}_1(\boldsymbol{u},q,\boldsymbol{h},\rho)=\nu(\widetilde{\nabla}^2-\nabla^2)\boldsymbol{u}+(\nabla-\widetilde{\nabla})q+\rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u}\\
-(\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\widetilde{\nabla}\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h}),\\
l_2(\boldsymbol{u},\rho)=(I-\widehat{\mathcal{L}}^T)\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{L}(\boldsymbol{u},\rho),\\
\boldsymbol{L}(\boldsymbol{u},\rho)=(I-\widehat{\mathcal{L}})\boldsymbol{u},\quad y\in\mathcal{F}_1,\\
\boldsymbol{l}_3(\boldsymbol{u},\rho)=\Pi_{\mathcal{G}}(\Pi_{\mathcal{G}}S(\boldsymbol{u})\boldsymbol{N}(y))-\Pi\widetilde{S}(\boldsymbol{u})\boldsymbol{n}(e_\rho(y)),\\
l_4(\boldsymbol{u},\boldsymbol{h},\rho)=\nu(\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}-\boldsymbol{n}\cdot\widetilde{S}(\boldsymbol{u})\boldsymbol{n})-[\boldsymbol{n}\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h})\boldsymbol{n}],\\
l_5(\rho)=-\int_0^1(1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla\cdot\frac{\mathcal{L}^T(y,s\rho)\boldsymbol{N}}{|\mathcal{L}^T(y,s\rho)\boldsymbol{N}|}ds,\\
l_6(\boldsymbol{u},\boldsymbol{h},\rho)=(\frac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{\boldsymbol{N}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}+\nabla_\tau\rho-\boldsymbol{N})\cdot\boldsymbol{u}+(\boldsymbol{V}-\boldsymbol{u})\cdot\nabla_\tau\rho,\quad y\in\mathcal{G},\\
\boldsymbol{l}_7(\boldsymbol{h},\boldsymbol{u},\rho)=\alpha^{-1}\mathrm{rot}(\mathrm{rot}\boldsymbol{h}-\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h})+\mu_1(\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h}\\
\qquad+\rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h})+\mu_1\mathrm{rot}(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}),\quad y\in\mathcal{F}_1,\\
\boldsymbol{l}_8(\boldsymbol{h},\rho)=(I-\mathcal{P})\boldsymbol{h},\quad y\in\mathcal{F}_2,\\
\boldsymbol{l}_9(\boldsymbol{h},\rho)=(\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2}-\boldsymbol{N})[\boldsymbol{h}\cdot\boldsymbol{N}]=[\boldsymbol{A}(\boldsymbol{h},\rho)],\quad y\in\mathcal{G},\\
\boldsymbol{A}^{(i)}(\boldsymbol{h},\rho^*)=\Big(\frac{\widehat{\mathcal{L}}(y,\rho^*)\widehat{\mathcal{L}}^T(y,\rho^*)\boldsymbol{N}^*}{|\widehat{\mathcal{L}}^T(y,\rho^*)\boldsymbol{N}^*|^2}-\frac{\boldsymbol{N}^*}{|\boldsymbol{N}^*|^2}\Big)\boldsymbol{h}^{(i)}\cdot\boldsymbol{N}^*,\quad y\in\mathcal{F}_i,\quad i=1,2.
\end{cases}
\tag{3.2}
$$

$$
\Pi\boldsymbol{f}=\boldsymbol{f}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{f}),\quad \Pi_{\mathcal{G}}\boldsymbol{g}=\boldsymbol{g}-\boldsymbol{N}(\boldsymbol{g}\cdot\boldsymbol{N}).
$$

We have used the formula for the variation of the mean curvature under the normal perturbation of the surface:

$$
H(e_\rho)-\mathcal{H}(y)=-\mathfrak{B}\rho-\int_0^1(1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla\cdot\frac{\mathcal{L}^T(y,s\rho)\boldsymbol{N}}{|\mathcal{L}^T(y,s\rho)\boldsymbol{N}|}ds.
$$

The vector field $\boldsymbol{V}\in W_p^{2-1/p}(\mathcal{G})$ should be close to $\boldsymbol{u}_0$ in the norm $W_p^{1-1/p}(\mathcal{G})$ (see (3.6)) and provide good estimates for the non-linear expression $l_6(\boldsymbol{u},\boldsymbol{h},\rho)$.

The solvability of the problem (3.1), (3.2) is proved by successive approximations, according to a standard scheme:

$$\begin{cases} \boldsymbol{u}_{m+1,t}(y,t)-\nu\nabla^2\boldsymbol{u}_{m+1}+\nabla q_{m+1}=\boldsymbol{l}_1(\boldsymbol{u}_m,q_m,\boldsymbol{h}_m,\rho_m), \\ \nabla\cdot\boldsymbol{u}_{m+1}=l_2(\boldsymbol{u}_m,\rho_m)=\nabla\cdot\boldsymbol{L}(\boldsymbol{u}_m,\rho_m), \quad y\in\mathcal{F}_1,\ t>0, \\ \Pi_{\mathcal{G}}S(\boldsymbol{u}_{m+1})\boldsymbol{N}=\boldsymbol{l}_3(\boldsymbol{u}_m,\rho_m), \\ -q_{m+1}+\nu\boldsymbol{N}\cdot S(\boldsymbol{u}_{m+1})\boldsymbol{N}(y)+\sigma\mathfrak{B}\rho_{m+1}=l_4(\boldsymbol{u}_m,\boldsymbol{h}_m,\rho_m)+l_5(\rho_m)+\sigma\mathcal{H}(y), \\ \rho_{m+1,t}+\boldsymbol{V}(y)\cdot\nabla_\tau\rho_{m+1}-\boldsymbol{u}_{m+1}\cdot\boldsymbol{N}(y)=l_6(\boldsymbol{u}_m,\rho_m), \quad y\in\mathcal{G}, \\ \mu_1\boldsymbol{h}_{m+1,t}+\alpha^{-1}\mathrm{rot}\,\mathrm{rot}\boldsymbol{h}_{m+1}=\boldsymbol{l}_7(\boldsymbol{h}_m,\boldsymbol{u}_m,\rho_m), \\ \nabla\cdot\boldsymbol{h}_{m+1}=0, \quad y\in\mathcal{F}_1, \\ \mathrm{rot}\boldsymbol{h}_{m+1}=\mathrm{rot}\boldsymbol{l}_8(\boldsymbol{h}_m,\rho_m), \quad \nabla\cdot\boldsymbol{h}_{m+1}=0, \quad y\in\mathcal{F}_2, \\ [\mu\boldsymbol{h}_{m+1}\cdot\boldsymbol{N}]=0, \quad [\boldsymbol{h}_{m+1\tau}]=\boldsymbol{l}_9(\boldsymbol{h}_m,\rho_m), \quad y\in\mathcal{G}, \\ \boldsymbol{h}_{m+1}\cdot\boldsymbol{n}=0, \quad y\in S, \\ \boldsymbol{u}_{m+1}(y,0)=\boldsymbol{u}_0(y), \quad y\in\mathcal{F}_1, \quad \boldsymbol{h}_{m+1}(y,0)=\boldsymbol{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2, \\ \rho_{m+1}(y,0)=\rho_0(y), \quad y\in\mathcal{G}, \qquad m=1,2,... \end{cases} \tag{3.3}$$

The first approximation, $(\boldsymbol{u}_1,q_1,\rho_1,\boldsymbol{h}_1)$, is defined in the following way: $q_1=0$, $\boldsymbol{u}_1$ and ρ_1 satisfy the initial conditions

$$\boldsymbol{u}_1(y,0)=\boldsymbol{u}_0(y), \quad y\in\mathcal{F}_1, \quad \rho_1(y,0)=\rho_0(y), \quad y\in\mathcal{G}$$

and the inequalities

$$\begin{aligned} &\|\boldsymbol{u}_1\|_{W_p^{2,1}(Q_T^1)}\le c\|\boldsymbol{u}_1\|_{W_P^{2,1}(Q_\infty^1)}\le c\|\boldsymbol{u}_0\|_{W_p^{2-2/p}(\mathcal{F}_1)}, \\ &\|\rho_1\|_{W_p^{3-1/p,0}(G_\infty)}+\|\rho_{1,t}\|_{W_p^{2-1/p,0}(G_\infty)}\le c\|\rho_0\|_{W_p^{3-2/p}(\mathcal{G})}, \end{aligned} \tag{3.4}$$

and $\boldsymbol{h}_1$ is defined as a divergence free vector field satisfying the equations (2.18), initial conditions

$$\boldsymbol{h}_1(y,0)=\boldsymbol{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2,$$

and the inequality

$$\sum_{i=1}^2\|\boldsymbol{h}_1^{(i)}\|_{W_r^{2,1}(Q_\infty^i)}\le c\|\boldsymbol{h}_0^{(1)}\|_{W_r^{2-2/r}(\mathcal{F}_1)}. \tag{3.5}$$

The existence of such $\boldsymbol{u}_1$, ρ_1 follows from the inverse trace theorems (see Sec. 1) and $\boldsymbol{h}_1$ can be constructed exactly as in the proof of Theorem 3 in [1].

The proof of the solvability of (3.3) is based on Theorems 2, 4 (with the exponent r instead of p) and on the estimates of nonlinear terms (3.2) obtained in Sec. 5.

Theorem 5. *Assume that* $\boldsymbol{u}_m\in W_p^{2,1}(Q_T^1)$, $\nabla q_m\in L_p(Q_T^1)$, $\rho_m\in W_p^{3-1/p,0}(G_T)$,

$\rho_{m,t} \in W_p^{2-1/p,1-1/2p}(G_T)$, $\boldsymbol{h}_m \in W_r^{2,0}(Q_T^i)$, $i = 1,2$, *and the conditions (1.5),*

$$\begin{aligned} &\|\rho_m(\cdot,t)\|_{W_p^{1-1/p}(\mathcal{G})} \leq \delta \ll 1, \\ &\|\boldsymbol{V} - \boldsymbol{u}_0\|_{W_p^{1-1/p}(\mathcal{G})} \leq \delta, \\ &T^{1/2-1/p}\|\boldsymbol{V}\|_{W_p^{2-1/p}(\mathcal{G})} \leq \delta \end{aligned} \tag{3.6}$$

are satisfied. Then

$$Z_m(T) \leq \vartheta(\delta,T)(Y_m(T) + Y_m^2(T)), \tag{3.7}$$

where

$$\begin{aligned} Z_m(T) &= \|\boldsymbol{l}_1(\boldsymbol{u}_m, q_m, \boldsymbol{h}_m, \rho)\|_{L_p(Q_T^1)} + \|l_2(\boldsymbol{u}_m,\rho_m)\|_{W_p^{1,0}(Q_T^1)} + \|\boldsymbol{L}(\boldsymbol{u}_m,\rho_m)\|_{W_p^{0,1}(Q_T^1)} \\ &+ \|\boldsymbol{l}_3(\boldsymbol{u}_m,\rho_m)\|_{W_p^{1-1/p,1/2-1/2p}(G_T)} + \|l_4(\boldsymbol{u}_m,\boldsymbol{h}_m,\rho_m)\|_{W_p^{1-1/p,0}(G_T)} \\ &+ \|l_5(\rho_m)\|_{W_p^{1-1/p,0}(G_T)} + \|l_6(\boldsymbol{u}_m,\rho_m)\|_{W_p^{2-1/p,1-1/2p}(G_T)} + \|\boldsymbol{l}_7(\boldsymbol{h}_m,\boldsymbol{u}_m,\rho_m)\|_{L_r(Q_T^1)} \\ &+ \|\boldsymbol{l}_8(\boldsymbol{h}_m,\rho_m)\|_{W_r^{2,1}(Q_T^2)} + \sum_{i=1}^{2}\|\boldsymbol{A}^{(i)}\|_{W_r^{2,1}(Q_T^i)}\Big), \end{aligned}$$

and $\vartheta(\delta,T)$ is a constant that can be made arbitrarily small by the choice of small δ and T.

Estimate (3.8) is a direct consequence of the inequalities (5.15)-(5.20), (5.22), (5.23), (5.26), (5.27), (5.32), (5.34)-(5.38) obtained below in Sec.5.

If $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$ are found, then, by Theorems 2 and 4, the problem (3.3) is uniquely solvable and

$$Y_{m+1}(T) \leq c\theta(\delta,T)(Y_m(T) + Y_m^2(T)) + c_1 N, \tag{3.8}$$

where

$$\begin{aligned} Y_m(T) &= \|\boldsymbol{u}_m\|_{W_p^{2,1}(Q_T^1)} + \sup_{t<T}\|\boldsymbol{u}_m\|_{W_p^{2-2/p}(\mathcal{F}_1)} + \|\nabla q_m\|_{L_p(Q_T^1)} + \|q_m\|_{W_p^{1-1/p,0}(G_T)} \\ &+ \|\rho_m\|_{W_p^{3-1/p,0}(G_T)} + \|\rho_{m,t}\|_{W_p^{2-1/p,1-1/2p}(G_T)} + \sup_{t<T}\|\rho_m(\cdot,t)\|_{W_p^{3-2/p}(\mathcal{G})} \\ &+ \sum_{i=1}^{2}(\|\boldsymbol{h}_m\|_{W_r^{2,1}(Q_T^i)} + \sup_{t<T}\|\boldsymbol{h}_m\|_{W_r^{2-2/r}(\mathcal{F}_i)}) \equiv Y(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m), \end{aligned} \tag{3.9}$$

$$N = \|\boldsymbol{u}_0\|_{W_p^{2-2/p}(\mathcal{F}_1)} + \|\rho_0\|_{W_p^{3-2/p}(\mathcal{G})} + \|\mathcal{H}\|_{W_p^{1-1/p}(\mathcal{G})} + \sum_{i=1}^{2}\|\boldsymbol{h}_0\|_{W_r^{2-2/r}(\mathcal{F}_i)}. \tag{3.10}$$

It follows that

$$Y_{m+1}(T) \leq 2c_1 N, \tag{3.11}$$

if $Y_m(T) \leq 2c_1 N$ and ϑ is so small that

$$\vartheta(\delta,T)(2c_1 N + 4(c_1 N)^2) \leq c_1 N$$

i.e.,

$$\vartheta(\delta, T) \le \frac{1}{2(1+2c_1N)} \le \frac{1}{2}; \tag{3.12}$$

in addition,

$$\begin{aligned}\|\rho_{m+1}(\cdot,t)\|_{W_p^{2-1/p}(\mathcal{G})} &\le \|\rho_0\|_{W_p^{2-1/p}(\mathcal{G})} + \int_0^t \|\rho_{m+1,\tau}(\cdot,\tau)\|_{W_p^{2-1/p}(\mathcal{G})} d\tau \\ &\le \epsilon + 2c_1 T^{1-1/p} N \le \delta.\end{aligned} \tag{3.13}$$

Hence if c_1 is chosen in such a way that (3.11) holds for $m = 1$, then (3.11) and (3.13) are satisfied for all m.

In order to prove the convergence of the sequence $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$, we estimate the differences

$$\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \;\; q_{m+1} - q_m, \;\; \rho_{m+1} - \rho_m, \;\; \boldsymbol{h}_{m+1} - \boldsymbol{h}_m.$$

We show that

$$\begin{aligned}&Y(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, q_{m+1} - q_m, \rho_{m+1} - \rho_m, \boldsymbol{h}_{m+1} - \boldsymbol{h}_m) \\ &\le \vartheta_1 Y(\boldsymbol{u}_m - \boldsymbol{u}_{m-1}, q_m - q_{m-1}, \rho_m - \rho_{m-1}, \boldsymbol{h}_m - \boldsymbol{h}_{m-1}),\end{aligned}$$

$m > 1$, with a small ϑ_1, which guarantees the boundedness of

$$\sum_{m=1}^{\infty} Y(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, q_{m+1} - q_m, \rho_{m+1} - \rho_m, \boldsymbol{h}_{m+1} - \boldsymbol{h}_m)$$

and the convergence of $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$. This is made in the same way as in [1,7] in the case $p, r = 2$, and we omit the details.

It is clear that $(\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m)$ tend to the solution of the problem (3.1), as $m \to \infty$ and the solution satisfies (3.11):

$$Y(T) \le 2c_1 N. \tag{3.14}$$

Remark. In the proof of (3.11) we have used the fact that the constants $c(T)$ in (2.5) and (2.14) are bounded for small T. This follows from the assumption $p > 3$, which implies $1/2 - 1/2p > 1/p$. As a consequence, the functions $d \in W_p^{k-1/p, k/2-1/2p}(G_T)$, $k = 1, 2$, vanishing for $t = 0$ (like $\boldsymbol{d}_\tau$ in (2.1) in the case of zero initial data) admit zero extension in the domain $t < 0$, and the extended functions $d^{(0)}$ satisfy the inequality

$$\|d^{(0)}\|_{W_p^{k-1/p,k/2-1/2p}(\mathcal{G}\times(-\infty,T))} \le c\|d\|_{W_p^{k-1/p,k/2-1/2p}(G_T)}, \quad \forall t < T,$$

(see [8, Sec.4, §4]) without any additional terms of the type

$$\frac{1}{T^\alpha}\|d\|_{L_p(\mathcal{G})}, \quad \alpha > 0,$$

as it was in the case $p = 2$ (see details in [7]).

4 Construction of the electric field

The complete system of equations of magnetohydrodynamics contains the electric field $\boldsymbol{E}(x,t)$; in particular, the problem (1.1) consists of finding the free boundary Γ_t and the functions $\boldsymbol{v}, p, \boldsymbol{H}$ and $\boldsymbol{E}$ such that

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla\cdot T(\boldsymbol{v},p) - \nabla\cdot T_M(\boldsymbol{H}) = 0, \\ \nabla\cdot\boldsymbol{v}(x,t) = 0, \quad x\in\Omega_{1t}, \;\; t>0, \end{cases} \tag{4.1}$$

$$\begin{cases} \mu\boldsymbol{H}_t = -\mathrm{rot}\boldsymbol{E}, \quad \nabla\cdot\boldsymbol{H} = 0, \qquad x\in\Omega_{1t}\cup\Omega_{2t}, \\ \mathrm{rot}\boldsymbol{H} = \alpha(\boldsymbol{E}+\mu(\boldsymbol{v}\times\boldsymbol{H})), \quad x\in\Omega_{1t}, \;\; t>0, \\ \mathrm{rot}\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{E} = 0, \quad x\in\Omega_{2t}. \end{cases} \tag{4.2}$$

$$\begin{cases} \boldsymbol{H}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{E}_\tau = 0, \quad x\in S, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \\ \big(T(\boldsymbol{v},p) + [T_M(\boldsymbol{H})]\big)\boldsymbol{n} = \sigma\boldsymbol{n}H, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x\in\Gamma_t, \end{cases} \tag{4.3}$$

$$\mathfrak{n}_t[\mu\boldsymbol{H}] + [\boldsymbol{\mathfrak{n}}_x\times\boldsymbol{E}] = 0, \quad x\in\Gamma_t, \tag{4.4}$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x\in\Omega_{10}, \quad \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x), \quad x\in\Omega_{10}\cup\Omega_{20}, \tag{4.5}$$

where $\boldsymbol{\mathfrak{n}}_x = (\mathfrak{n}_1,\mathfrak{n}_2,\mathfrak{n}_3)$ and $\mathfrak{n}_t$ are components of the normal vector $\boldsymbol{\mathfrak{n}}$ to the surface $\mathfrak{G} = \{x\in\Gamma_t, t>0\}$ in $\mathbb{R}^4$. In particular, if $\mu_1=\mu_2$, then the condition (4.4) takes a standard form $[\boldsymbol{\mathfrak{n}}_x\times\boldsymbol{E}] = 0$, i.e., $[\boldsymbol{E}_\tau] = 0$.

The condition (4.4) can be deduced from the assumption that the first equation in (4.2) is satisfied in the sense of the distributions theory, i.e.,

$$\int_{t_0}^{t_0+\tau}\int_K (-\boldsymbol{B}\cdot\boldsymbol{\varphi}_t + \boldsymbol{E}\cdot\mathrm{rot}\boldsymbol{\varphi})\,dxdt = 0, \tag{4.6}$$

where $\boldsymbol{B} = \mu\boldsymbol{H}$, $\boldsymbol{\varphi}\in C_0^\infty(K)$, $K\subset\Omega$, $K\cap\Gamma_t\neq\emptyset$. From (4.6) it follows that

$$\begin{aligned} &\boldsymbol{B}_t + \mathrm{rot}\boldsymbol{E} = 0, \quad x\in K, \\ &\mathfrak{n}_t[\boldsymbol{B}] + [\boldsymbol{\mathfrak{n}}_x\times\boldsymbol{E}] = 0, \quad x\in K\cup\Gamma_t, \quad t\in(t_0,t_0+\tau). \end{aligned}$$

The following proposition is important for the construction of $\boldsymbol{E}$.

Theorem 6. *If the relations*

$$\begin{aligned} &\nabla\cdot\boldsymbol{B}(x,t) = 0, \quad x\in K\cap(\Omega_1\cup\Omega_2), \quad [\boldsymbol{B}\cdot\boldsymbol{n}] = 0, \quad x\in K\cap\Gamma_t, \\ &\mathfrak{n}_t[\boldsymbol{B}] + [\boldsymbol{n}_x\times\boldsymbol{E}] = 0, \quad x\in K\cap\Gamma_t \end{aligned} \tag{4.7}$$

are satisfied, then

$$[\boldsymbol{n}\cdot\mathbf{rot}\boldsymbol{E}] = -[\boldsymbol{n}\cdot\boldsymbol{B}_t], \quad x\in\Gamma_t\cap K. \tag{4.8}$$

Proof. Suppose the surface $\mathfrak{G}_\tau = \mathfrak{G}\cap(K\times(t_0,t_0+\tau))$ is given by the equation $\xi_3 = z(\xi_1,\xi_2,t)$, where (ξ_1,ξ_2,ξ_3) are Cartesian coordinates in $\mathbb{R}^3$. The functions given on $\mathfrak{G}_\tau$ can be considered as functions of ξ_1, ξ_2, t. The normal $\mathfrak{n}$ to $\mathfrak{G}_\tau$ is parallel to the vector $(\widetilde{\boldsymbol{n}}, z_t)$, where $\widetilde{\boldsymbol{n}} = (z_{\xi_1}, z_{\xi_2}, -1)$.

It is easily verified that

$$\begin{aligned} &-[\widetilde{\boldsymbol{n}} \cdot \mathbf{rot}\boldsymbol{E}] = [z_{\xi_1}(E_{2,3} - E_{3,2}) + z_{\xi_2}(E_{3,1} - E_{1,3}) - (E_{1,2} - E_{2,1})] \\ &= \big[\frac{\partial}{\partial \xi_1}(z_{\xi_2} E_3 + E_2) + \frac{\partial}{\partial \xi_2}(-E_1 - z_{\xi_1} E_3)\big] = \frac{\partial}{\partial \xi_1}[\widetilde{\boldsymbol{n}} \times \boldsymbol{E}]_1 + \frac{\partial}{\partial \xi_2}[\widetilde{\boldsymbol{n}} \times \boldsymbol{E}]_2, \end{aligned} \tag{4.9}$$

where $E_{i,j}$ is the partial derivative of E_i with respect to ξ_j and

$$\frac{\partial \boldsymbol{E}}{\partial \xi_\alpha} = \boldsymbol{E}_{,\alpha} + z_{\xi_\alpha} \boldsymbol{E}_{,3}, \quad \alpha = 1, 2,$$

are derivatives calculated taking the dependence of $\boldsymbol{E}$ on ξ_3 into account. By (4.7), the equation (4.9) is equivalent to

$$\begin{aligned} &-[\widetilde{\boldsymbol{n}} \cdot \mathbf{rot}\boldsymbol{E}] = -\frac{\partial}{\partial \xi_1} z_t [B_1] - \frac{\partial}{\partial \xi_2} z_t [B_2] \\ &= -[z_{\xi_1,t} B_1 + z_t B_{1,\xi_1} + z_{\xi_1} z_t B_{1,3} + z_{\xi_2,t} B_2 + z_t B_{2,\xi_2} + z_{\xi_2} z_t B_{2,3}]. \end{aligned} \tag{4.10}$$

Now we differentiate $[\widetilde{\boldsymbol{n}} \cdot \boldsymbol{B}] = 0$ with respect to t, which leads to

$$[z_{\xi_1,t} B_1 + z_{\xi_1} B_{1,t} + z_t z_{\xi_1} B_{1,3} + z_{\xi_2,t} B_2 + z_{\xi_2} B_{2,t} + z_{\xi_2} z_t B_{2,3} - B_{3,t} - z_t B_{3,t}] = 0. \tag{4.11}$$

When we add (4.11) to (4.10) and make use of the equation $\nabla \cdot \boldsymbol{B} = 0$, we obtain

$$-[\widetilde{\boldsymbol{n}} \cdot \mathbf{rot}\boldsymbol{E}] = [z_{\xi_1} B_{1,t} + z_{\xi_2} B_{2,t} - B_{3,t}] = [\widetilde{\boldsymbol{n}} \cdot \boldsymbol{B}_t],$$

q.e.d.

The proof of the theorem is due to Dr. N.Filonov.

The transformation (1.2) converts (4.1)-(4.4) in

$$\begin{cases} \boldsymbol{u}_t - \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y) \cdot \nabla)\boldsymbol{u} + (\mathcal{L}^{-1}\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \\ \quad - \widetilde{\nabla} \cdot \widetilde{T}(\boldsymbol{u}, q) - \widetilde{\nabla} \cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h}) = 0, \\ \nabla \cdot \widehat{\mathcal{L}}\boldsymbol{u} = 0, \quad y \in \mathcal{F}_1, \quad t > 0, \end{cases} \tag{4.12}$$

$$\begin{cases} \mu(\boldsymbol{h}_t - \frac{1}{L}\widehat{\mathcal{L}}_t \mathcal{L}\boldsymbol{h} - \rho_t^* \widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}^*(y) \cdot \nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h}) = -\mathrm{rot}\mathcal{P}\boldsymbol{e}, \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2, \\ \mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h} = \alpha(\mathcal{P}\boldsymbol{e} + \mu(\mathcal{L}^{-1}\boldsymbol{u} \times \boldsymbol{h})), \quad \nabla \cdot \boldsymbol{h} = 0, \quad y \in \mathcal{F}_1, \\ \mathrm{rot}\mathcal{P}\boldsymbol{h} = 0, \quad \nabla \cdot \boldsymbol{h} = 0, \quad \nabla \cdot \boldsymbol{e} = 0, \quad x \in \mathcal{F}_2, \end{cases} \tag{4.13}$$

$$\begin{cases} \boldsymbol{h} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{e}_\tau = 0, \quad y \in S, \\ [\mu \boldsymbol{h} \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{h} - \dfrac{\widetilde{\mathcal{L}}\widetilde{\mathcal{L}}^T \boldsymbol{N}}{|\widetilde{\mathcal{L}}^T \boldsymbol{N}|^2}(\boldsymbol{h} \cdot \boldsymbol{N})] = 0, \\ \widetilde{T}(\boldsymbol{u}, q)\boldsymbol{n}(e_\rho) + [T_M(\dfrac{\mathcal{L}}{L}\boldsymbol{h})\boldsymbol{n}(e_\rho)] = \sigma H \boldsymbol{n}, \quad \rho_t = \dfrac{\boldsymbol{u} \cdot \widehat{\mathcal{L}}^T \boldsymbol{N}}{\Lambda(y, \rho)}, \\ -\Lambda \rho_t [\mu \boldsymbol{h}] + L[\boldsymbol{N} \times \mathcal{P}\boldsymbol{e}] = 0, \quad y \in \mathcal{G}, \end{cases} \tag{4.14}$$

and (4.7), (4.8) (with $K=\Omega$) in

$$\begin{aligned}&\nabla\cdot\boldsymbol{b}=0,\quad y\in\mathcal{F}_1\cup\mathcal{F}_2,\quad [\boldsymbol{b}\cdot\boldsymbol{N}]=0,\quad y\in\mathcal{G},\\ &[\boldsymbol{N}\times\mathcal{P}\boldsymbol{e}]=\boldsymbol{\Psi}(\boldsymbol{b},\rho),\quad y\in\mathcal{G},\end{aligned}\tag{4.15}$$

$$[\boldsymbol{N}\times\mathrm{rot}\mathcal{P}\boldsymbol{e}]=\boldsymbol{\Phi}(\boldsymbol{b},\rho),\tag{4.16}$$

where $\boldsymbol{e}=\widehat{\mathcal{L}}\boldsymbol{E}(e_\rho,t)$, $\boldsymbol{b}=\widehat{\mathcal{L}}\boldsymbol{B}(e_\rho,t)$, $\boldsymbol{h}=\widehat{\mathcal{L}}\boldsymbol{H}(e_\rho,t)$, $y\in\mathcal{F}_1\cup\mathcal{F}_2$,

$$\begin{aligned}&\boldsymbol{\Phi}(\boldsymbol{b},\rho)=\frac{1}{L}\widehat{\mathcal{L}}_t^T\mathcal{L}\boldsymbol{b}+\rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{b},\quad \boldsymbol{\Psi}(\boldsymbol{b},\rho)=\frac{\Lambda\rho_t}{L}[\boldsymbol{b}],\\ &\Lambda(y,\rho)=\boldsymbol{N}(y)\cdot\widehat{\mathcal{L}}^T(y,\rho)\boldsymbol{N}(y)=1-\rho\mathcal{H}+\rho^2\mathcal{K},\quad y\in\mathcal{G}.\end{aligned}\tag{4.17}$$

Hence (4.15) imply (4.16). We observe that the vector field $\boldsymbol{\Phi}(\boldsymbol{h},\rho)$ is divergence free in $\mathcal{F}_1\cup\mathcal{F}_2$ [1].

Now we pass to the construction of $\boldsymbol{e}$ satisfying (4.13), (4.14) together with $\boldsymbol{u},p,\rho,\boldsymbol{h}$ whose existence is established above. We introduce auxiliary vector field $\boldsymbol{e}_1(y,t)$ such that

$$\begin{aligned}&\nabla\cdot\boldsymbol{e}_1(y,t)=0,\quad y\in\mathcal{F}_1\cup\mathcal{F}_2,\\ &[\boldsymbol{N}(y)\times\mathcal{P}\boldsymbol{e}_1]=\boldsymbol{\Psi}(\mu\boldsymbol{h},\rho),\quad [\boldsymbol{N}\cdot\boldsymbol{e}_1]=0,\quad y\in\mathcal{G},\quad \boldsymbol{e}_1(y,t)=0,\quad y\in S,\\ &\sum_{i=1}^2\|\boldsymbol{e}_1\|_{W_r^{1,0}(Q_T^i)}\leq c\|\boldsymbol{\Psi}\|_{W_r^{1-1/r,0}(G_T)},\end{aligned}\tag{4.18}$$

and we solve the problem

$$\mathrm{rot}\boldsymbol{\mathcal{E}}(y,t)=-\mu\boldsymbol{h}_t+\boldsymbol{\Phi}(\mu\boldsymbol{h},\rho)-\mathrm{rot}\mathcal{P}\boldsymbol{e}_1(y,t),\quad \nabla\cdot\mathcal{P}^{-1}\cdot\boldsymbol{\mathcal{E}}(y,t)=0,\quad y\in\Omega,\quad \boldsymbol{\mathcal{E}}_\tau=0,\quad y\in S.\tag{4.19}$$

In view of Theorem 6, we have

$$\begin{aligned}&[\boldsymbol{N}(y)\cdot(\mu(\boldsymbol{h}_t-\boldsymbol{\Phi})+\mathrm{rot}\mathcal{P}\boldsymbol{e}_1)]=0,\quad [\mu\boldsymbol{h}_t\cdot\boldsymbol{N}]=0,\quad y\in\mathcal{G},\\ &\boldsymbol{n}\cdot(\mu(\boldsymbol{h}_t-\boldsymbol{\Phi})+\mathrm{rot}\mathcal{P}\boldsymbol{e}_1(y,t))=0,\quad y\in S,\end{aligned}$$

hence the problem (4.19) is solvable. The solution has the form $\boldsymbol{\mathcal{E}}=\boldsymbol{\mathcal{E}}_1+\nabla Z(y,t)$, where

$$\begin{aligned}&\boldsymbol{\mathcal{E}}_1(y,t)=\frac{1}{4\pi}\mathrm{rot}\int_\Omega\frac{-\mu\boldsymbol{h}_t+\mu\boldsymbol{\Phi}-\mathrm{rot}\mathcal{P}\boldsymbol{e}_1(z,t)}{|y-z|}dz,\\ &\nabla\cdot\mathcal{P}^{-1}\nabla Z=-\nabla\mathcal{P}^{-1}\boldsymbol{\mathcal{E}}_1,\quad y\in\Omega,\quad Z(y,t)=-g(y,t),\quad y\in S.\end{aligned}$$

It is easily seen that

$$\mathrm{rot}\boldsymbol{\mathcal{E}}_1=-\mu\boldsymbol{h}_t+\mu\boldsymbol{\Phi}-\mathrm{rot}\mathcal{P}\boldsymbol{e}_1(y,t),\quad y\in\Omega.$$

The function g is defined as follows. Since $(\mu\boldsymbol{h}_t-\mu\boldsymbol{\Phi}+\mathrm{rot}\mathcal{P}\boldsymbol{e}_1)\cdot\boldsymbol{n}|_S=0$, we have $\int_\Sigma\mathrm{rot}\boldsymbol{\mathcal{E}}_1\cdot\boldsymbol{n}dS=0$ for arbitrary $\Sigma\subset S$. Hence by the Stokes formula,

$$\int_\gamma\boldsymbol{\mathcal{E}}_1\cdot d\boldsymbol{l}=0$$

for arbitrary closed contour $\gamma = \partial\Sigma \subset S$, which implies $\boldsymbol{\mathcal{E}}_1|_S = \nabla_\tau g(y,t)$ with a certain single-valued g. Now the equations (4.19) are easily verified.

By the Calderon-Zygmund theorem,

$$\|\boldsymbol{\mathcal{E}}_1\|_{W_r^{1,0}(Q_T)} \leq c\|\mu \boldsymbol{h}_t - \mu\boldsymbol{\Phi} + \mathrm{rot}\mathcal{P}\boldsymbol{e}_1\|_{L_r(Q_T)} \leq c(\|\boldsymbol{h}_t\|_{L_r(Q_T)} + \|\boldsymbol{\Phi}\|_{L_r(Q_T)} + \|\boldsymbol{\Psi}\|_{W_r^{1-1/r,0}(G_T)}), \tag{4.20}$$

in addition,

$$\|\nabla Z\|_{W_r^{1,0}(Q_T)} \leq c\|\boldsymbol{\mathcal{E}}_1\|_{W_r^{1,0}(Q_T)},$$

hence

$$\|\boldsymbol{\mathcal{E}}\|_{W_r^{1,0}(Q_T)} \leq c(\|\boldsymbol{h}_t\|_{L_r(Q_T)} + \|\boldsymbol{\Phi}\|_{L_r(Q_T)} + \|\boldsymbol{\Psi}\|_{W_r^{1-1/r,0}(G_T)}). \tag{4.21}$$

The norms of $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are estimated below in (5.37), (5.38).

Now we define $\boldsymbol{e}^{(1)}$ by

$$\mathcal{P}\boldsymbol{e}^{(1)} = \alpha_1^{-1}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h} - \mu(\mathcal{L}^{-1}\boldsymbol{u} \times \boldsymbol{h}). \tag{4.22}$$

Then

$$\begin{aligned} \mathrm{rot}\mathcal{P}\boldsymbol{e}^{(1)} &= \mathrm{rot}(\alpha_1^{-1}\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h} - \mu(\mathcal{L}^{-1}\boldsymbol{u} \times \boldsymbol{h})) \\ &= -\mu\boldsymbol{h}_t + \mu\boldsymbol{\Phi} = \mathrm{rot}\boldsymbol{\mathcal{E}} + \mathrm{rot}\mathcal{P}\boldsymbol{e}_1, \quad y \in \mathcal{F}_1, \end{aligned}$$

which implies

$$\mathcal{P}\boldsymbol{e}^{(1)} = \boldsymbol{\mathcal{E}}^{(1)} + \mathcal{P}\boldsymbol{e}_1^{(1)} + \nabla\chi_1(y,t), \quad y \in \mathcal{F}_1 \tag{4.23}$$

with a certain single-valued $\chi_1(y,t)$. Finally we set

$$\mathcal{P}\boldsymbol{e}^{(2)} = \boldsymbol{\mathcal{E}}^{(2)} + \mathcal{P}\boldsymbol{e}_1^{(2)} + \nabla\chi_2(y,t) + C(t)\nabla w(y), \quad y \in \mathcal{F}_2, \tag{4.24}$$

where χ_2 and $\boldsymbol{w}$ are solutions to the problems

$$\begin{aligned} &\nabla\cdot\mathcal{P}^{-1}\nabla\chi_2(y,t) = 0, \quad y \in \mathcal{F}_2, \quad \chi_2(y,t) = \chi_1(y,t), \quad y \in \mathcal{G}, \quad \chi_2(y,t) = 0, \quad y \in S, \\ &\nabla\cdot\mathcal{P}^{-1}\nabla w(y) = 0, \quad y \in \mathcal{F}_2, \quad w(y) = 1, \quad y \in \mathcal{G}, \quad w(y,t) = 0, \quad y \in S. \end{aligned} \tag{4.25}$$

The vector field $\boldsymbol{e}$ defined in this way satisfies (4.13), (4.14). Indeed,

$$\begin{aligned} &\mu\boldsymbol{h}_t - \boldsymbol{\Phi} + \mathrm{rot}\mathcal{P}\boldsymbol{e}^{(i)} = \mu\boldsymbol{h}_t - \boldsymbol{\Phi} + \mathrm{rot}\boldsymbol{\mathcal{E}}^{(i)} + \mathrm{rot}\mathcal{P}\boldsymbol{e}_1^{(i)} = 0, \quad y \in \mathcal{F}_i, \quad i = 1,2, \\ &\nabla\cdot\boldsymbol{e}^{(2)} = \nabla\cdot\mathcal{P}^{-1}\boldsymbol{\mathcal{E}}^{(2)} + \nabla\cdot\boldsymbol{e}_1^{(2)} = 0, \quad y \in \mathcal{F}_2, \\ &[\boldsymbol{N}\times\mathcal{P}\boldsymbol{e}] = [\boldsymbol{N}\times\boldsymbol{\mathcal{E}}] + [\boldsymbol{N}\times\mathcal{P}\boldsymbol{e}_1] = \boldsymbol{\Psi}(\mu\boldsymbol{h},\rho), \quad y \in \mathcal{G}, \quad \boldsymbol{e}_\tau|_S = \mathcal{E}_\tau + \boldsymbol{e}_{1\tau}|_S = 0. \end{aligned}$$

In order to find $C(t)$, we need to impose on $\boldsymbol{e}^{(2)}$ a normalization restriction, for instance of the form

$$\int_S \mathcal{P}\boldsymbol{e}^{(2)}(y,t)\cdot\boldsymbol{n}(y)dS = 0, \tag{4.26}$$

which coincides with

$$\int_S \boldsymbol{E}^{(2)}(x,t)\cdot\boldsymbol{n}(x)dS = 0.$$

Since $I=\int_S \frac{\partial w}{\partial n}dS\neq 0$, we have

$$C(t)=-I^{-1}\int_S(\boldsymbol{\mathcal{E}}^{(2)}+\mathcal{P}\boldsymbol{e}_1^{(2)}+\nabla\chi_2(y,t))\cdot\boldsymbol{n}(y)dS. \tag{4.27}$$

As for the estimates of $\boldsymbol{e}$, we have, by (4.21)-(4.24),

$$\begin{aligned}
&\|\boldsymbol{e}^{(1)}\|_{W_r^{1,0}(Q_T^1)}\leq c(\|\mathcal{P}\boldsymbol{h}\|_{W_r^{2,0}(Q_T^1)}+\|(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})\|_{W_r^{1,0}(Q_T^1)}),\\
&\|\boldsymbol{e}^{(2)}\|_{W_r^{1,0}(Q_T^2)}\leq c(\|\boldsymbol{\mathcal{E}}^{(2)}\|_{W_r^{1,0}(Q_T^2)}+\|\boldsymbol{e}_1\|_{W_r^{1,0}(Q_T^2)}\\
&+\|\nabla\chi_2\|_{W_r^{1,0}(Q_T^2)}+\|C\|_{L_r(0,T)}),\\
&\|\nabla\chi_2\|_{W_r^{1,0}(Q_T^2)}\leq c\|\nabla\chi_1\|_{W_r^{1,0}(Q_T^2)},\\
&\|\nabla\chi_1\|_{W_r^{1,0}(Q_T^1)}\leq c(\|\boldsymbol{e}^{(1)}\|_{W_r^{1,0}(Q_T^1)}+\|\boldsymbol{\mathcal{E}}^{(1)}\|_{W_r^{1,0}(Q_T^1)}+\|\boldsymbol{e}_1\|_{W_r^{1,0}(Q_T^1)}),\\
&\|C\|_{L_p(0,T)}\leq c(\|\boldsymbol{e}_1\|_{W_r^{1,0}(Q_T^2)}+\|\boldsymbol{\mathcal{E}}^{(2)}\|_{W_r^{1,0}(Q_T^2)}+\|\nabla\chi_2\|_{W_r^{1,0}(Q_T^2)}).
\end{aligned} \tag{4.28}$$

Taking (4.21), (4.18), (2.5), (2.15) into account, we conclude that $\boldsymbol{e}\in W_r^{1,0}(Q_T^i)$, $i=1,2$.

5 Estimates of nonlinear terms

In this section we estimate the expressions (3.2). For this we need some auxiliary propositions.

Proposition 1. *Arbitrary functions $u(x),v(x)$ given in a domain $\mathcal{D}\subset\mathbb{R}^n$ satisfy the following inequalities:*

$$\|uv\|_{W_p^1(\mathcal{D})}\leq c(\sup_{\mathcal{D}}|v(x)|\|u\|_{W_p^1(\mathcal{D})}+\sup_{\mathcal{D}}|u(x)|\|v\|_{W_p^1(\mathcal{D})})\leq c\|u\|_{W_p^1(\mathcal{D})}\|v\|_{W_p^1(\mathcal{D})}, \tag{5.1}$$

$$\|uv\|_{W_p^2(\mathcal{D})}\leq c(\|u\|_{W_p^2(\mathcal{D})}\|v\|_{W_p^1(\mathcal{D})}+\|v\|_{W_p^2(\mathcal{D})}\|u\|_{W_p^1(\mathcal{D})}), \tag{5.2}$$

if $p>n$,

$$\|uv\|_{W_p^l(\mathcal{D})}\leq c(\|u\|_{W_p^l(\mathcal{D})}\sup_{\mathcal{D}}|v(x)|+\|v\|_{W_p^l(\mathcal{D})}\sup_{\mathcal{D}}|u(x)|), \tag{5.3}$$

if $pl>n$ and l is not an integer.

Proof. The inequality (5.1) is obvious, and (5.2) follows from (5.1) and

$$\|\nabla(uv)\|_{W_p^1(\mathcal{D})}\leq\|v\nabla u\|_{W_p^1(\mathcal{D})}+\|u\nabla v\|_{W_p^1(\mathcal{D})}\leq c(\|u\|_{W_p^2(\mathcal{D})}\|v\|_{W_p^1(\mathcal{D})}+\|v\|_{W_p^2(\mathcal{D})}\|u\|_{W_p^1(\mathcal{D})}). \tag{5.4}$$

To prove (5.3), we estimate the semi-norm

$$\|w\|_{\dot W_p^l(\mathbb{R}^n)}=\Big(\int_{\mathbb{R}^n}\|\Delta^m(z)w\|^p_{L_p(\mathbb{R}^n)}\frac{dz}{|z|^{n+pl}}\Big)^{1/p},\quad m>l, \tag{5.5}$$

of the product $w(x)=u(x)v(x)$, assuming that $u,v\in C_0^\infty(\mathbb{R}^n)$.By $\Delta^m(z)w(x)$ we mean a finite difference

$$\Delta^m(z)w(x)=\sum_{j=0}^m(-1)^{m-j}C_m^jw(x+jz)$$

and we notice that the semi-norms (5.5) are equivalent to each other for different $m>l$ [9]. We assume that $m>2l$; then in every term of the sum

$$\Delta^m(z)u(x)v(x)=\sum_{j=0}^m C_m^j(\Delta^j(z)u(x))\Delta^{m-j}(z)v(x+(m-j)z)$$

one of the numbers j or $m-j$ is greater than l. Hence

$$\|\Delta^m(z)uv\|_{L_p(\mathbb{R}^n)}\le c(\sup_{\mathbb{R}^n}|v(x)|\|\Delta^k(z)u\|_{L_p(\mathbb{R}^n)}+\sup_{\mathbb{R}^n}|u(x)|\|\Delta^k(z)v\|_{L_p(\mathbb{R}^n)}),$$

$k>l$, and

$$\|uv\|_{\dot W_p^l(\mathbb{R}^n)}\le c(\sup_{\mathbb{R}^n}|v(x)|\|u\|_{\dot W_p^l(\mathbb{R}^n)}+\sup_{\mathbb{R}^n}|u(x)|\|v\|_{\dot W_p^l(\mathbb{R}^n)}).$$

From this inequality and

$$\|uv\|_{L_p(\mathbb{R}^n)}\le c(\sup_{\mathbb{R}^n}|v(x)|\|u\|_{L_p(\mathbb{R}^n)}+\sup_{\mathbb{R}^n}|u(x)|\|v\|_{L_p(\mathbb{R}^n)})$$

it follows that

$$\|uv\|_{\widetilde W_p^l(\mathbb{R}^n)}\le c(\sup_{\mathbb{R}^n}|v(x)|\|u\|_{\widetilde W_p^l(\mathbb{R}^n)}+\sup_{\mathbb{R}^n}|u(x)|\|v\|_{\widetilde W_p^l(\mathbb{R}^n)}),\tag{5.6}$$

where

$$\|u\|_{\widetilde W_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\|u\|_{\dot W_p^l(\mathbb{R}^n)}.$$

This norm is equivalent to the norm in $W_p^l(\mathbb{R}^n)$ defined in Sec 1, if l is not an integer. Since every function from $W_p^l(\mathcal{D})$ can be extended in $\mathbb{R}^n$ with preservation of class, (5.6) implies (5.3). The proposition is proved.

Now we formulate our assumptions concerning the extensions $\boldsymbol{N}^*$ and ρ^*. We assume that $\boldsymbol{N}^*(y)$ is a sufficiently regular non-zero vector field in Ω and $\rho^*=E\rho$ where E is a linear extension operator with the following properties:

$$\begin{gathered}\frac{\partial\rho^*(x,t)}{\partial N}\Big|_{\mathcal{G}}=0,\quad \rho^*(y,t)=0\quad\text{near}\quad \mathrm{S},\\ \|\rho^*(\cdot,t)\|_{W_p^r(\Omega)}\le c\|\rho\|_{W_p^{r-1/p}(\mathcal{G})},\quad r\in(1/p,3].\end{gathered}\tag{5.7}$$

It follows that the time derivative of ρ^* satisfies similar inequalities:

$$\|\rho_t^*(\cdot,t)\|_{W_p^r(\Omega)}\le c\|\rho_t\|_{W_p^{r-1/p}(\mathcal{G})},\quad r\in(1/p,2].\tag{5.8}$$

Proposition 2. *Let* $\boldsymbol{R}(x,t)=\left(\frac{\partial N_i^*\rho^*}{\partial x_j}\right)_{i,j=1,2,3}$ *and let the conditions* $p>3$ *and*

$$\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}\le\delta\ll 1 \tag{5.9}$$

be satisfied. For arbitrary smooth function $f(\widetilde{\boldsymbol{R}})$ *defined for* $|\widetilde{\boldsymbol{R}}|\le\delta_0$, $\delta_0>\delta$, *the inequalities*

$$\begin{aligned}&\|\boldsymbol{R}f(\boldsymbol{R})\|_{L_p(\Omega)}\le c\|\boldsymbol{R}\|_{L_p(\Omega)},\\&\|\boldsymbol{R}f(\boldsymbol{R})\|_{W_p^1(\Omega)}\le c\|\boldsymbol{R}\|_{W_p^1(\Omega)}\le c\|\rho\|_{W_p^{2-1/p}(\mathcal{G})},\\&\|f(\boldsymbol{R})\|_{W_p^1(\Omega)}\le c,\end{aligned} \tag{5.10}$$

$$\|\boldsymbol{R}f(\boldsymbol{R})\|_{W_p^{1-1/p}(\mathcal{G})}\le c\|\rho\|_{W_p^{1-1/p}(\mathcal{G})} \tag{5.11}$$

hold.

Proof. Inequalities (5.10) are obvious, and (5.11) follows from (5.10):

$$\|\boldsymbol{R}f(\boldsymbol{R})\|_{W_p^{1-1/p}(\mathcal{G})}\le c\|\boldsymbol{R}f(\boldsymbol{R})\|_{W_p^1(\Omega)}\le c\|\boldsymbol{R}\|_{W_p^1(\Omega)}\le c\|\rho\|_{W_p^{2-1/p}(\mathcal{G})},$$

q.e.d.

We also have

$$\sup_\Omega|\boldsymbol{R}(x)|\le c\|\boldsymbol{R}\|_{W_p^1(\Omega)}\le c\|\rho\|_{W_p^{2-1/p}(\Omega)}\le c\delta. \tag{5.12}$$

Examples of functions satisfying the assumptions of Proposition 2 are provided by the elements of the matrices $\mathcal{L}(x,\rho^*)$, $\mathcal{L}^{-1}(x,\rho^*)$, $\widehat{\mathcal{L}}(x,\rho^*)$, whereas $\boldsymbol{n}(e_\rho)$ and $\widehat{\mathcal{L}}(x,\rho)\boldsymbol{N}(x)(\Lambda(x,\rho)^{-1}$ are smooth functions of $\boldsymbol{R}$, ρ and x. They satisfy the inequality

$$\|\boldsymbol{R}f(\boldsymbol{R},\rho,\cdot)\|_{W_p^1(\Omega)}+\|\rho^*f(\boldsymbol{R},\rho,\cdot)\|_{W_p^1(\Omega)}\le c\Big(\|\boldsymbol{R}\|_{W_p^1(\Omega)}+\|\rho^*\|_{W_p^1(\mathcal{F})}\Big)\le c\delta. \tag{5.13}$$

It is obtained in the same way as (5.10).

We shall use similar estimate in the two-dimensional case:

$$\|\boldsymbol{R}f(\boldsymbol{R},\rho,\cdot)\|_{W_p^{1-1/p}(\mathcal{G})}+\|\rho f(\boldsymbol{R},\rho,\cdot)\|_{W_p^{1-1/p}(\mathcal{G})}\le c\Big(\|\boldsymbol{R}\|_{W_p^{1-1/p}(\mathcal{G})}+\|\rho\|_{W_p^{1-1/p}(\mathcal{G})}\Big)\le c\delta. \tag{5.14}$$

Now we estimate all the terms in (3.2), step by step. We always assume that $p>3$.

1. *Estimate of* $(\nabla-\widetilde{\nabla})q$.

Since

$$(\nabla-\widetilde{\nabla})q=(I-\mathcal{L}^{-T})\nabla q$$

and $|I-\mathcal{L}^{-T}|\le c|\boldsymbol{R}|$, we have

$$\|(\nabla-\widetilde{\nabla})q\|_{L_p(Q_T^1)}\le c\sup_{Q_T^1}|\boldsymbol{R}(x,t)|\|\nabla q\|_{L_p(Q_T)}\le c\delta\|\nabla q\|_{L_p(Q_T)}. \tag{5.15}$$

2. *Estimate of* $(\widetilde{\nabla}^2-\nabla^2)\boldsymbol{u}$.

It is easily seen that

$$(\widetilde{\nabla}^2-\nabla^2)\boldsymbol{u}=\widetilde{\nabla}\cdot(\widetilde{\nabla}-\nabla)\boldsymbol{u}+(\widetilde{\nabla}-\nabla)\cdot\nabla\boldsymbol{u}=(\mathcal{L}^{-T}\nabla\cdot(\mathcal{L}^{-T}-I)\nabla)\boldsymbol{u}$$
$$+(\mathcal{L}^{-T}-I)\nabla\cdot\nabla\boldsymbol{u}=(\mathcal{L}^{-1}(\mathcal{L}^{-T}-I):D^2)\boldsymbol{u}+(\mathcal{L}^{-T}-I)\nabla\cdot\nabla\boldsymbol{u}+(\boldsymbol{M}\cdot\nabla)\boldsymbol{u},$$

where $\boldsymbol{M}$ is the vector field whose components $M_j=\sum_{k,i=1}^3 l^{ki}\frac{\partial}{\partial x_k}l^{ji}$ are linear combinations of $\nabla\boldsymbol{R}g(\boldsymbol{R})$ (l^{ij} are elements of $\mathcal{L}^{-1}$). The first two terms on the right hand side are estimated in the same way as $(\widehat{\nabla}-\nabla)q$:

$$\|(\mathcal{L}^{-T}-I)\nabla\cdot\nabla\boldsymbol{u}\|_{L_p(Q_T^1)}+\|(\mathcal{L}^{-1}(\mathcal{L}^{-T}-I):D^2)\boldsymbol{u}\|_{L_p(Q_T^1)}$$
$$\le c\delta\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)},$$

and

$$\|(\boldsymbol{M}\cdot\nabla)\boldsymbol{u}\|_{L_p(Q_T^1)}\le c\Big(\int_0^T\sup_{x\in\mathcal{F}_1}|\nabla\boldsymbol{u}|^p dt\Big)^{1/p}\sup_{t<T}\|\boldsymbol{M}\|_{L_p(\mathcal{F}_1)}$$
$$\le c\sup_{t<T}\|\rho^*\|_{W_p^2(\mathcal{F}_1)}\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)}\le c\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)}\le c\delta\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)}. \tag{5.16}$$

3. *Estimate of* $(\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$.

$$\|(\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{L_p(Q_T^1)}\le c\sup_{Q_T^1}|\boldsymbol{u}(x,t)|\|\nabla\boldsymbol{u}\|_{L_p(Q_T^1)}\le cT^{1/p}(\sup_{t<T}\|\boldsymbol{u}\|_{W_p^1(\mathcal{F}_1)})^2 \tag{5.17}$$

4.*Estimate of* $\rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*\cdot\nabla)\boldsymbol{u}$.

$$\|\rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*\cdot\nabla)\boldsymbol{u}\|_{L_p(Q_T^1)}\le c\sup_{Q_T^1}|\rho_t^*(x,t)|\|\nabla\boldsymbol{u}\|_{L_p(Q_T^1)}$$
$$\le c\sup_{t<T}\|\rho_t^*\|_{W_p^1(\mathcal{F}_1)}\|\boldsymbol{u}\|_{W_p^{1,0}(Q_T^1)}\le cT^{1/p}\sup_{t<T}\|\boldsymbol{u}\|_{W_p^1(\mathcal{F}_1)}\sup_{t<T}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}. \tag{5.18}$$

5. *Estimate of* $(I-\widehat{\mathcal{L}}^T)\nabla\cdot\boldsymbol{u}=\nabla\cdot(I-\widehat{\mathcal{L}})\boldsymbol{u}$.

$$\|(I-\widehat{\mathcal{L}}^T)\nabla\boldsymbol{u}\|_{W_p^{1,0}(Q_T^1)}\le c\sup_{t<T}\|I-\widehat{\mathcal{L}}^T\|_{W_p^1(\mathcal{F}_1)}\|\nabla\boldsymbol{u}\|_{W_p^{1,0}(Q_T^1)}$$
$$\le c\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)}\le c\delta\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)}, \tag{5.19}$$

$$\|(I-\widehat{\mathcal{L}}^T)\boldsymbol{u}\|_{L_p(Q_T^1)}+\|(I-\widehat{\mathcal{L}}^T)\boldsymbol{u}_t\|_{L_p(Q_T^1)}\le c\delta\|\boldsymbol{u}\|_{W_p^{0,1}(Q_T)},$$
$$\|\widehat{\mathcal{L}}_t^T\boldsymbol{u}\|_{L_p(Q_T^1)}\le c\sup_{Q_T^1}|\boldsymbol{u}|\|\rho_t\|_{W_p^{1-1/p}(G_T)}\le cT^{1/p}\sup_{t<T}\|\boldsymbol{u}\|_{W_p^1(\mathcal{F}_1)}\sup_{t<T}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}. \tag{5.20}$$

6. *Estimate of* $\nabla\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h})$.

If (1.5) holds, then

$$\|\boldsymbol{h}\|_{W^i_p(\mathcal{F}_1)} \le c\|\boldsymbol{h}\|_{W^{2-2/r}_r(\mathcal{F}_i)}, \tag{5.21}$$

hence

$$\begin{aligned}
&\|\nabla\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h})\|_{L_p(Q^i_T)} \le c(\|\boldsymbol{h}\|_{W^{1,0}_p(Q^i_T)} + \|\rho^*\|_{W^{2,0}_p(Q^i_T)}\sup_{Q^i_T}|\boldsymbol{h}(y,t)|)\sup_{Q^i_T}|\boldsymbol{h}(y,t)|)\\
&\le cT^{1/p}(\sup_{t<T}\|\boldsymbol{h}\|_{W^{2-2/r}_r(\mathcal{F}_i)} + \sup_{t<T}\|\rho\|_{W^{2-1/p}_p(\mathcal{G})}\sup_{t<T}\|\boldsymbol{h}\|_{W^1_r(\mathcal{F}_i)})\sup_{t<T}\|\boldsymbol{h}\|_{W^1_r(\mathcal{F}_i)}).
\end{aligned} \tag{5.22}$$

It follows that

$$\begin{aligned}
&\|[T_M(\frac{\mathcal{L}}{L}\boldsymbol{h})]\|_{W^{1-1/p,0}_p(G_T)} \le c\sum_{i=1}^2(\|\boldsymbol{h}\|_{W^{1,0}_p(Q^i_T)} + \|\rho^*\|_{W^{2,0}_p(Q^i_T)}\sup_{Q^i_T}|\boldsymbol{h}(y,t)|)\sup_{Q^i_T}|\boldsymbol{h}(y,t)|\\
&\le c\sum_{i=1}^2 T^{1/p}(\sup_{t<T}\|\boldsymbol{h}\|_{W^{2-2/r}_r(\mathcal{F}_i)} + \sup_{t<T}\|\rho\|_{W^{2-1/p}_p(\mathcal{G})}\sup_{t<T}\|\boldsymbol{h}\|_{W^1_r(\mathcal{F}_i)})\sup_{t<T}\|\boldsymbol{h}\|_{W^1_r(\mathcal{F}_i)}.
\end{aligned} \tag{5.23}$$

7. *Estimates of* $\widetilde{S}(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{N}$.

Since $\boldsymbol{n}(e_\rho) = \frac{\widetilde{\mathcal{L}}^T\boldsymbol{N}(y)}{|\widetilde{\mathcal{L}}^T\boldsymbol{N}(y)|}$, we have

$$\sup_{\mathcal{G}}|\boldsymbol{n}-\boldsymbol{N}| \le c\|\boldsymbol{n}-\boldsymbol{N}\|_{W^{1-1/p}_p(\mathcal{G})} \le c\|\rho\|_{W^{2-1/p}_p(\mathcal{G})} \le c\delta,$$

hence

$$\begin{aligned}
&\|\widetilde{S}(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{N}\|_{W^{1-1/p,0}_p(G_T)} \le \|(\widetilde{S}(\boldsymbol{u}) - S(\boldsymbol{u}))\boldsymbol{N}\|_{W^{1-1/p,0}_p(G_T)}\\
&+\|\widetilde{S}(\boldsymbol{u})(\boldsymbol{n}-\boldsymbol{N})\|_{W^{1-1/p,0}_p(G_T)} \le c\delta\|\boldsymbol{u}\|_{W^{2-1/p,0}_p(G_T)}.
\end{aligned} \tag{5.24}$$

Now we pass to the estimate of

$$J = \int_0^T\frac{dh}{h^{1/2+p/2}}\int_h^T\|\Delta_t(-h)(\widetilde{S}(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{N})\|^p_{L_p(\mathcal{G})}dt,$$

where $\Delta_t(-h)f(t) = f(t-h) - f(t)$. We have

$$\begin{aligned}
&\|\Delta_t(-h)(\widetilde{S}(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{N})\|_{L_p(\mathcal{G})} \le \|\Delta_t(-h)(\widetilde{S}(\boldsymbol{u}) - S(\boldsymbol{u}))\boldsymbol{N}\|_{L_p(\mathcal{G})}\\
&+\|(\Delta_t(-h)\widetilde{S}(\boldsymbol{u}))(\boldsymbol{n}-\boldsymbol{N})\|_{L_p(\mathcal{G})} + \|S(\boldsymbol{u})\Delta_t(-h)\boldsymbol{n}\|_{L_p(\mathcal{G})}\\
&\le c(\delta\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{L_p(\mathcal{G})} + \|\nabla\boldsymbol{u}\|_{L_p(\mathcal{G})}\int_0^h\|\rho_t(\cdot,t-\xi)\|_{W^{2-1/p}_p(\mathcal{G})}d\xi),
\end{aligned}$$

which implies

$$J^{1/p} \le c(\delta\|\nabla\boldsymbol{u}\|_{W^{0,1/2-1/2p}_p(G_T)} + cT^{1/2-1/2p}\sup_{t<T}\|\nabla\boldsymbol{u}\|_{L_p(\mathcal{G})}\|\rho_t\|_{W^{2-1/p,0}_p(G_T)},$$

hence

$$\begin{aligned}&\|\widetilde{S}(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{N}\|_{W_p^{1-1/p,1/2-1/2p}(G_T)}\\&\leq c\delta\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)}+cT^{1/2-1/2p}\sup_{t<T}\|\nabla\boldsymbol{u}\|_{L_p(\mathcal{G})}\|\rho_t\|_{W_p^{2-1/p,0}(G_T)}\end{aligned}\tag{5.25}$$

It follows that

$$\|\boldsymbol{l}_3(\boldsymbol{u},\rho)\|_{W_p^{1-1/p,1/2-1/2p}(G_T)}\leq c(\delta\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)}+T^{1/2-1/2p}\sup_{t<T}\|\nabla\boldsymbol{u}\|_{L_p(\mathcal{G})}\|\rho_t\|_{W_p^{2-1/p,0}(G_T)}),$$

$$\begin{aligned}&\|l_4(\boldsymbol{u},\boldsymbol{h},\rho)\|_{W_p^{1-1/p,0}(G_T)}\leq c(\delta\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)}\\&+\sum_{i=1}^{2}(T^{1/p}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^{2-2/r}(\mathcal{F}_i)}+T^{1/p}\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)})\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)}).\end{aligned}\tag{5.26}$$

The next step is the estimate of $l_5(\rho)$ and $l_6(\boldsymbol{u},\rho)$.

8. *Estimate of* $\frac{d^2}{ds^2}\mathcal{L}^{-T}(x,s\rho)\nabla\cdot\frac{\widehat{\mathcal{L}}^T(x,s\rho)\boldsymbol{N}(x)}{|\widehat{\mathcal{L}}^T(x,s\rho)\boldsymbol{N}(x)|}$ *and* $\frac{d^2}{ds^2}\frac{\widehat{\mathcal{L}}^T(x,s\rho)\boldsymbol{N}}{\Lambda(x,s\rho)}$.

The function $\frac{d^2}{ds^2}\mathcal{L}^{-T}(x,s\rho)\nabla\cdot\frac{\widehat{\mathcal{L}}^T(x,s\rho)\boldsymbol{N}(x)}{|\widehat{\mathcal{L}}^T(x,s\rho)\boldsymbol{N}(x)|}$ is a linear combination of the expressions $\nabla\boldsymbol{R}\boldsymbol{R}f(s\boldsymbol{R},x)$ where $\boldsymbol{R}=\Big(\frac{\partial N_i(x)\rho(x,t)}{\partial x_j}\Big)_{i,j=1,2,3}$, $s\in(0,1)$ and f is the function satisfying the assumptions of Proposition 2 but depending also on x. Making use of this proposition (in the two-dimensional case), we obtain

$$\|\nabla\boldsymbol{R}\boldsymbol{R}f(s\boldsymbol{R},\cdot)\|_{W_p^{1-1/p}(\mathcal{G})}\leq c\|\nabla\boldsymbol{R}\|_{W_p^{1-1/p}(\mathcal{G})}\|\boldsymbol{R}\|_{W_p^{1-1/p}(\mathcal{G})}.$$

This inequality implies

$$\|\nabla\boldsymbol{R}\boldsymbol{R}f(s\boldsymbol{R},\cdot)\|_{W_p^{1-1/p,0}(G_T)}\leq c\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}\|\rho\|_{W_p^{3-1/p,0}(G_T)}\leq c\delta\|\rho\|_{W_p^{3-1/p,0}(G_T)},$$

hence

$$\|l_5\|_{W_p^{1-1/p,0}(G_T)}\leq c\delta\|\rho\|_{W_p^{3-1/p,0}(G_T)}.\tag{5.27}$$

The function $\frac{d^2}{ds^2}\frac{\widehat{\mathcal{L}}^T(x,s\rho)\boldsymbol{N}}{\Lambda(x,s\rho)}$ is a linear combination of the expressions

$$\boldsymbol{R}\boldsymbol{R}f_1(s\boldsymbol{R},s\rho,x),\quad \rho\boldsymbol{R}f_2(s\boldsymbol{R},s\rho,x),\quad \rho^2f_3(s\boldsymbol{R},s\rho,x)$$

with f_i as in Proposition 2, but depending also on ρ and x. We estimate $\boldsymbol{R}\boldsymbol{R}f_1$, since two other functions are estimated in a similar way, and ρ^* is more regular than $\mathcal{R}$. As above, we have

$$\|\boldsymbol{R}\boldsymbol{R}f_1(s\boldsymbol{R},s\rho,\cdot)\|_{W_p^{2-1/p,0}(G_T)}\leq c\delta\|\rho\|_{W_p^{3-1/p,0}(G_T)},$$

$$\Delta_t(-h)|\boldsymbol{R}\boldsymbol{R}f_1(s\boldsymbol{R},s\rho,\cdot)\|_{L_p(\mathcal{G})}\leq c\sup_{\mathcal{G}}|\mathcal{R}|\int_0^h\|\rho_t(\cdot,t-\xi)\|_{W_p^{2-1/p}(\mathcal{G})}d\xi,$$

$$(\int_0^T\frac{dh}{h^{1/2+p}}\int_h^T\|\Delta_t(-h)\boldsymbol{R}\boldsymbol{R}f_1\|^p_{L_p(\mathcal{G})}dt)^{1/p}\leq cT^{1/2p}\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}\|\rho_t\|_{W_p^{2-1/p,0}(G_T)}.$$

The same inequalities hold for $\frac{d^2}{ds^2}\frac{\widehat{\mathcal{L}}^T(x,s\rho)\boldsymbol{N}}{\Lambda(x,s\rho)}$.

It follows that

$$\boldsymbol{D}=\frac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{\Lambda(y,\rho)}-\boldsymbol{N}+\nabla_\tau\rho=\int_0^1(1-s)\frac{d^2}{ds^2}\frac{\widehat{\mathcal{L}}^T(x,s\rho)\boldsymbol{N}}{\Lambda(x,s\rho)}ds$$

satisfies

$$\|\boldsymbol{D}\|_{W_p^{2-1/p,1-1/2p}(G_T)}\le c\delta(\|\rho\|_{W_p^{3-1/p,0}(G_T)}+T^{1/2p}\|\rho_t\|_{W_p^{2-1/p,0}(G_T)}),$$
$$\sup_{\mathcal{G}}|\boldsymbol{D}(y,t)|\le csup_{\mathcal{G}}|\mathcal{R}\mathcal{R}|\le c\delta\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}$$

hence by (5.3)

$$\begin{aligned}&\|\boldsymbol{D}\cdot\boldsymbol{u}\|_{W_p^{2-1/p,0}(G_T)}\le c(\|\boldsymbol{D}\|_{W_p^{2-1/p,0}(G_T)}\sup_{t<T}\|\boldsymbol{u}(\cdot,t)\|_{W_p^1(\mathcal{F}_1)}\\
&+\|u\|_{W_p^{2-1/p,0}(G_T)}\sup_{t<T}\delta\|\rho\|_{W_p^{2-1/p}(\mathcal{G})})\\
&\le c\delta(\|\rho\|_{W_p^{3-1/p,0}(G_T)}\sup_{t<T}\|\boldsymbol{u}\|_{W_p^1(\mathcal{F}_1)}+\|\boldsymbol{u}\|_{W_p^{2-1/p,0}(G_T)}\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}),\\
&\|\boldsymbol{D}\cdot\boldsymbol{u}\|_{W_p^{0,1-1/2p}(G_T)}\\
&\le c\delta(T^{1/2p}\|\rho_t\|_{W_p^{2-1/p,0}(G_T)}\sup_{t<T}\|\boldsymbol{u}\|_{W_p^1(\mathcal{F}_1)}+\|\boldsymbol{u}\|_{W_p^{0,1-1/2p}(G_T)}\sup_{t<T}\|\rho\|_{W_p^{2-2/p}(\mathcal{G})}).\end{aligned}\tag{5.28}$$

9. *Estimate of* $(\boldsymbol{V}-\boldsymbol{u})\cdot\nabla_\tau\rho$.

We have, by (5.3),

$$\begin{aligned}&\|(\boldsymbol{V}-\boldsymbol{u})\cdot\nabla_\tau\rho\|_{W_p^{2-1/p,0}(G_T)}\le c\|\nabla_\tau\rho\|_{W_p^{2-1/p,0}(G_T)}\sup_{t<T}\|\boldsymbol{V}-\boldsymbol{u}\|_{W_p^{1-1/p}(\mathcal{G})}\\
&+\Big(\int_0^T\|\boldsymbol{V}-\boldsymbol{u}\|^p_{W_p^{2-1/p}(\mathcal{G})}\|\nabla_\tau\rho\|^p_{W_p^{1-1/p}(\mathcal{G})}dt\Big)^{1/p}.\end{aligned}\tag{5.29}$$

We require that

$$\|\boldsymbol{V}-\boldsymbol{u}_0\|_{W_p^{1-1/p}(\mathcal{G})}\le\delta,\tag{5.30}$$

then

$$\sup_{t<T}\|\boldsymbol{V}-\boldsymbol{u}\|_{W_p^{1-1/p}(\mathcal{G})}\le\delta+\sup_{t<T}\|\boldsymbol{u}-\boldsymbol{u}_0\|_{W_p^{1-1/p}(\mathcal{G})}\le(\delta+cT^\beta\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)}),\quad\beta=1/2-1/p,$$

because $\boldsymbol{u}\in W_p^{1/2}(0,T;W_p^1(\mathcal{F}_1))$. Hence the first term in (5.29) does not exceed

$$c(\delta+T^\beta\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T)})\|\rho\|_{W_p^{3-1/p,0}(G_T)},$$

and the second term is not greater than

$$\begin{aligned}&\|\boldsymbol{V}\|_{W_p^{2-1/p}(\mathcal{G})}\|\nabla_\tau\rho\|_{W_p^{1-1/p,0}(G_T)}+\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)}\sup_{t<T}\|\nabla_\tau\rho\|_{W_p^{1-1/p}(\mathcal{G})}\\
&\le c(T^{1/p}\sup_{t<T}\|\nabla_\tau\rho\|_{W_p^{1-1/p}(\mathcal{G})}\|\boldsymbol{V}\|_{W_p^{2-1/p}(\mathcal{G})}+\delta\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)})\\
&\le c\delta(\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}+\|\boldsymbol{u}\|_{W_p^{2,0}(Q_T^1)}),\end{aligned}$$

if

$$T^{1/p}\|\boldsymbol{V}\|_{W_p^{2-1/p}(\mathcal{G})}\le\delta. \tag{5.31}$$

We also have

$$\begin{aligned}&\|(\Delta_t(-h)\boldsymbol{u}))\cdot\nabla_\tau\rho\|_{L_p(\mathcal{G})}+\|(\boldsymbol{V}-\boldsymbol{u})\cdot\Delta_t(-h)\nabla_\tau\rho\|_{L_p(\mathcal{G})}\\&\le c\|\Delta_t(-h)\boldsymbol{u}\|_{L_p(\mathcal{G})}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}+c\|\boldsymbol{V}-\boldsymbol{u}\|_{W_p^{1-1/p}(\mathcal{G})}\|\Delta_t(-h)\nabla_\tau\rho\|_{L_p(\mathcal{G})},\end{aligned}$$

$$\begin{aligned}&\Big(\int_0^T\frac{dh}{h^{1/2+p}}\int_h^T\|\Delta_t(-h)\boldsymbol{u}\|^p_{L_p(\mathcal{G})}dt\Big)^{1/p}\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}\\&+c\sup_{t<T}\|\boldsymbol{V}-\boldsymbol{u}\|_{W_p^{1-1/p}(\mathcal{G})}\Big(\int_0^T\frac{dh}{h^{1/2+p}}\int_h^T\|\Delta_t(-h)\nabla_\tau\rho\|^p_{L_p(\mathcal{G})}dt\Big)^{1/p}\\&\le c\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}\|\boldsymbol{u}\|_{W_p^{0,1-1/2p}(G_T)}\\&+cT^{1/2p}(\delta+cT^\beta\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)})\|\rho_t\|_{W_p^{2-1/p,0}(G_T)}\end{aligned}$$

It follows that

$$\begin{aligned}&\|l_6\|_{W_p^{2-1/p,1-1/2p}(G_T)}\le c\delta(\|\rho\|_{W_p^{3-1/p,0}(G_T)}+\sup_{t<T}\|\rho\|_{W_p^{2-1/p}(\mathcal{G})}+\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)})\\&+cT^{1/2p}(\delta+cT^\beta\|\boldsymbol{u}\|_{W_p^{2,1}(Q_T^1)})\|\rho_t\|_{W_p^{2-1/p,0}(G_T)}.\end{aligned} \tag{5.32}$$

Now we pass to the estimate of $\boldsymbol{l}_8$, $\boldsymbol{l}_9$, $\boldsymbol{A}$.

10. *Estimate of* $(I-\mathcal{P})\boldsymbol{h}$.

We have

$$\begin{aligned}&\|(I-\mathcal{P})\boldsymbol{h}\|_{W_r^{2,0}(Q_T^i)}\le c\big(\delta\|\boldsymbol{h}\|_{W_r^{2,0}(Q_T^i)}+\|\rho^*\|_{W_r^{3,0}(Q_T^i)}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)}\big),\\&\|\rho^*\|_{W_r^{3,0}(Q_T^i)}=\Big(\int_0^T\|\rho^*\|^r_{W_r^3(\mathcal{F}_i)}dt\Big)^{1/r}\le T^{1/r-1/p}\Big(\int_0^T\|\rho^*\|^p_{W_r^3(\mathcal{F}_i)}dt\Big)^{1/p}\\&\le cT^{1/r-1/p}\|\rho\|_{W_p^{3-1/p,0}(Q_T^i)};\end{aligned}$$

hence

$$\|(I-\mathcal{P})\boldsymbol{h}\|_{W_r^{2,0}(Q_T^i)}\le c\big(\delta\|\boldsymbol{h}\|_{W_r^{2,0}(Q_T^i)}+T^{1/r-1/p}\|\rho\|_{W_p^{3-1/p,0}(Q_T^i)}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)}\big).$$

Moreover,

$$\|((I-\mathcal{P})\boldsymbol{h})_t\|_{L_r(Q_T^i)}\le c(\delta\|\boldsymbol{h}_t\|_{L_r(Q_T^i)}+T^{1/r}\sup_{t<T}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)},$$

and finally

$$\begin{aligned}&\|(I-\mathcal{P})\boldsymbol{h}\|_{W_r^{2,1}(Q_T^i)}\le c\big(\delta\|\boldsymbol{h}\|_{W_r^{2,1}(Q_T^i)}+T^{1/r-1/p}\|\rho\|_{W_p^{3-1/p,0}(Q_T^i)}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)}\\&+T^{1/r}\sup_{t<T}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)}\big),\quad i=1,2,\end{aligned} \tag{5.33}$$

The same inequalities hold for $\boldsymbol{l}_8(\boldsymbol{h},\rho)$. Moreover, since the elements of the matrix

$$\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T}{|\widehat{\mathcal{L}}^T\boldsymbol{N}^*(y)|^2}-\frac{I}{|\boldsymbol{N}^*|^2}\equiv f(y,\boldsymbol{R})$$

in (3.2) satisfy the assumptions of Proposition 2 (just as the elements of $I-\mathcal{P}$), (5.33) holds also for $\|\boldsymbol{A}^{(i)}\|_{W_r^{2,1}(Q_T^i)}$.

11. *Estimate of* $\boldsymbol{l}_7(\boldsymbol{u},\boldsymbol{h},\rho)$ *and* $\boldsymbol{\Phi}(\boldsymbol{h},\rho)=\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h}+\rho^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h}$.

We estimate the expression

$$\mathrm{rot}\boldsymbol{h}-\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h}=\mathcal{P}\mathrm{rot}(\boldsymbol{h}-\mathcal{P}\boldsymbol{h})+(1-\mathcal{P})\mathrm{rot}\boldsymbol{h}.$$

We have

$$\begin{aligned}&\|(I-\mathcal{P})\mathrm{rot}\boldsymbol{h}\|_{W_r^{1,0}(Q_T^1)}\le c\delta\|\boldsymbol{h}\|_{W_r^{2,0}(Q_T^1)},\\&\|\mathcal{P}\mathrm{rot}(I-\mathcal{P})\boldsymbol{h}\|_{W_r^{1,0}(Q_T^1)}\le c\|\mathrm{rot}(I-\mathcal{P})\boldsymbol{h}\|_{W_r^{1,0}(Q_T^1)},\end{aligned}$$

and this norm has been estimated above in (5.32). Hence

$$\|\mathrm{rot}\boldsymbol{h}-\mathcal{P}\mathrm{rot}\mathcal{P}\boldsymbol{h}\|_{W_r^{1,0}(Q_T^1)}\le c\big(\delta\|\boldsymbol{h}\|_{W_r^{2,0}(Q_T^1)}+T^{1/r-1/p}\|\rho\|_{W_p^{3-1/p,0}(Q_T^1)}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_1)}\big).\tag{5.34}$$

Other terms in $\boldsymbol{l}_7$ are estimated as follows:

$$\begin{aligned}&\|\rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h}\|_{L_r(Q_T^1)}\le c\sup_{t<T}\|\rho_t^*\|_{W_p^1(\mathcal{F}_1)}\|\frac{\mathcal{L}}{L}\boldsymbol{h}\|_{W_r^{1,0}(\mathcal{F}_1)}\\&\le cT^{1/r}\sup_{t<T}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_1)},\\&\|\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h}\|_{L_r(Q_T^1)}\le c\sup_{Q_T^1}|\boldsymbol{h}(x,t)|\|\nabla\rho_t^*\|_{L_r(Q_T^1)}\\&\le cT^{1/r}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_1)}\sup_{t<T}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})},\end{aligned}\tag{5.35}$$

$$\begin{aligned}&\|\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}\|_{W_r^{1,0}(Q_T^1)}\le c(\|\boldsymbol{u}\|_{W_r^{1,0}(Q_T^1)}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_1)}\\&+\|\boldsymbol{h}\|_{W_r^{1,0}(Q_T^1)}\sup_{t<T}\|\boldsymbol{u}\|_{W_r^1(\mathcal{F}_1)})\le cT^{1/r}\sup_{t<T}\|\boldsymbol{u}\|_{W_r^1(\mathcal{F}_1)}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_1)},\end{aligned}\tag{5.36}$$

hence

$$\|\boldsymbol{\Phi}(\boldsymbol{h},\rho)\|_{L_r(Q_T)}\le cT^{1/r}\sum_{i=1}^{2}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)}\sup_{t<T}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}.\tag{5.37}$$

12. *Estimate of* $\boldsymbol{\Psi}(\boldsymbol{h},\rho)$.

$$
\begin{aligned}
\|\boldsymbol{\Psi}\|_{W_r^{1-1/r,0}(G_T)} &= \|\frac{\Lambda(\rho)\rho_t}{L}[\mu\boldsymbol{h}]\|_{W_r^{1-1/r,0}(G_T)} \le c\sum_{i=1}^{2}\|\frac{\Lambda(\rho^*)\rho_t^*}{L(\cdot,\rho^*)}\mu\boldsymbol{h}\|_{W_r^{1,0}(Q_T^i)} \\
&\le cT^{1/r}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}\sum_{i=1}^{2}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^1(\mathcal{F}_i)}.
\end{aligned}
\tag{5.38}
$$

In conclusion, we show that Theorem 1 holds for $r = p$, i.e., that the solution of the problem under consideration possess the property $\boldsymbol{h} \in W_p^{2,1}(Q_T)$, if $\boldsymbol{h}_0 \in W_p^{2-1/p}(\mathcal{F}_i)$, $i = 1, 2$ We restrict ourselves with obtaining the a-priori estimate. It suffices to estimate the norms

$$
\|l_7(\boldsymbol{u}, \boldsymbol{h}, \rho)\|_{L_p(Q_T^i)}, \quad \|\boldsymbol{l}_8\|_{W_p^{2,1}(Q_T^i)}, \quad \|\boldsymbol{A}\|_{W_p^{2,1}(Q_T^i)}, \quad \|\boldsymbol{\Psi}\|_{W_p^{1-1/p,0}(G_T)}.
$$

Instead of (5.33), we have

$$
\begin{aligned}
&\|(I - \mathcal{P})\boldsymbol{h}\|_{W_p^{2,1}(Q_T^i)} \le c\big(\delta\|\boldsymbol{h}\|_{W_p^{2,1}(Q_T^i)} \\
&+ (\|\rho\|_{W_p^{3-1/p,0}(Q_T^i)} + T^{1/p}\sup_{t<T}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}) sup_{t<T}\|\boldsymbol{h}\|_{W_p^1(\mathcal{F}_i)}\big).
\end{aligned}
\tag{5.39}
$$

With the help of this inequality and the estimate (5.21) we obtain

$$
\|\boldsymbol{l}_8\|_{W_p^{2,1}(Q_T^2)} + \sum_{i=1}^{2}\|\boldsymbol{A}\|_{W_p^{2,1}(Q_T^i)} \le c\sum_{i=1}^{2}(\delta\|\boldsymbol{h}\|_{W_p^{2,1}(Q_T^i)} + C_1(\rho)\sup_{t<T}\|\boldsymbol{h}\|_{W_r^{2-2/r}(\mathcal{F}_i)}) \tag{5.40}
$$

with $C_1(\rho)$ depending on the norms of ρ estimated in (3.14).

In view of (5.35), (5.36), (5.38), $\|\boldsymbol{l}_7\|_{L_p(Q_T^i)}$ also satisfies (5.40), and

$$
\|\boldsymbol{\Psi}\|_{W_p^{1-1/p,0}(G_T)} \le cT^{1/p}\|\rho_t\|_{W_p^{1-1/p}(\mathcal{G})}\sum_{i=1}^{2}\sup_{t<T}\|\boldsymbol{h}\|_{W_r^{2-2/r}(\mathcal{F}_i)}.
$$

Hence by Theorem 4

$$
\sum_{i=1}^{2}\|\boldsymbol{h}\|_{W_p^{2,1}(Q_T^1)} \le c\sum_{i=1}^{3}(\delta\|\boldsymbol{h}\|_{W_p^{2,1}(Q_T^i)} + C_2(\rho, \boldsymbol{u})\sup_{t<T}\|\boldsymbol{h}\|_{W_r^{2-2/r}(\mathcal{F}_i)} + c\|\boldsymbol{h}_0\|_{W_p^{2-2/p}(\mathcal{F}_i)})
$$

with a certain $C_2(\rho, \boldsymbol{u}) < \infty$, which yields the desired estimate in the case of small δ.

Inequalities (4.18), (4.20), (4.21), (4.28) with p instead of r show that $\boldsymbol{e} \in W_p^{1,0}(Q_T^i)$, $i = 1, 2$.

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