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Scalar products of state-vectors of the integrable models and their combinatorial interpretation

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Abstract

The representation of the Bethe wave functions of certain integrable models via the Schur functions allows to apply the well-developed theory of the symmetric functions to the calculation of the thermal correlation functions. The algebraic relations arising in the calculation of the scalar products and the correlation functions are based on the Binet-Cauchy formula adapted for the Schur functions. We provide a combinatorial interpretation of the formula for the scalar products of the Bethe state-vectors in terms of nests of the self-avoiding lattice paths constituting the so-called watermelon configurations. The interpretation proposed is, in its turn, related to the enumeration of the boxed plane partitions.

Keywords: Schur functions, self-avoiding lattice paths, boxed plane partitions

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1 Introduction

The symmetric functions, the Young diagrams, the boxed plane partitions, and the vicious walkers [1–4] play an important role in the contemporary theoretical physics [5–9]. The N-particle wave functions of a certain class of integrable models on a chain are expressed in terms of Schur functions [10–16]. The Schur functions are defined by the Jacobi-Trudi relation:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}) \equiv S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) \equiv \frac{\det(x_j^{\lambda_k+N-k})_{1\le j,k\le N}}{\mathcal{V}_N(\mathbf{x})}\,. \tag{1}$$

Here the Vandermonde determinant, $\mathcal{V}_N(\mathbf{x}) \equiv \det(x_j^{N-k})_{1\le j,k\le N}$, is used:

$$\mathcal{V}_N(\mathbf{x}) = \prod_{1\le m<l\le N} (x_l - x_m)\,. \tag{2}$$

Besides, $\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2, \dots \lambda_N)$ in (1) is a partition, i.e., a nonincreasing sequence of nonnegative integers, $M \ge \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N \ge 0$, called the parts of $\boldsymbol{\lambda}$. Partition $\boldsymbol{\lambda}$ can be represented by Young diagram as an arrangement of squares with the coordinates (i, j) so that $1 \le j \le \lambda_i$.

For the bosonic models defined on a chain of $M + 1$ sites there is one-to-one correspondence between a set of occupation numbers $\{n_M, n_{M-1} \dots, n_1, n_0\}$ and the partition $\boldsymbol{\lambda} = (M^{n_M}, (M-1)^{n_{M-1}}, \dots, 1^{n_1}, 0^{n_0})$, where notation S^{n_S} expresses that the integer number S appears n_S times in $\boldsymbol{\lambda}$. For the Heisenberg spin-$\frac{1}{2}$ chains of $M + N$ sites the coordinates of the spin "down" states ("particles") constitute a strict decreasing partition $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N)$, where $M + N - 1 \ge \mu_1 > \mu_2 > \dots > \mu_N \ge 0$. The parts of $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are related: $\mu_j = \lambda_j + N - j$.

The Young diagram corresponding to $\boldsymbol{\lambda}$ is an arrangement of squares with λ_i squares in the j^{th} row [1]. The bijection between the particle coordinates encoded in $\boldsymbol{\lambda}$ and the cells of the Young diagram corresponding to $\boldsymbol{\lambda}$ is demonstrated on Fig. 1.

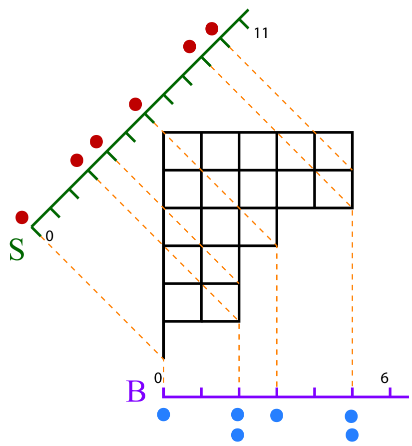

Figure 1: Configurations of spins and bosons on chains and the corresponding Young diagram of the related partition $\boldsymbol{\lambda} = (5, 5, 3, 2, 2, 0)$.

Calculation of the correlation functions of integrable models of special type, [10–13, 15, 16], is based on the Binet-Cauchy formula adapted for the Schur functions:

$$\sum_{\boldsymbol{\lambda}\subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{x})S_{\boldsymbol{\lambda}}(\mathbf{y}) \;=\; \frac{\det(M_{kj})_{1\le k,j\le N}}{\mathcal{V}_N(\mathbf{x})\mathcal{V}_N(\mathbf{y})}\,, \tag{3}$$

where summation is over all partitions $\boldsymbol{\lambda}$ with at most N parts, each of which is less than or equal to M, and $\mathcal{V}_N(\mathbf{x})$ is the Vandermonde determinant (2). The entries M_{kj} in (3) are:

$$M_{kj} = \frac{1-(x_k y_j)^{M+N}}{1-x_k y_j}\,. \tag{4}$$

In particular, Eq. (3) is related to the calculation of the scalar products of two N-particle Bethe state-vectors [15, 16].

We shall denote the box of the size $L\times N\times P$ as the set of integer lattice points:

$$\mathcal{B}(L,N,P) = \bigl\{(i,j,k)\in\mathbb{N}^3\big|\, 0\le i\le L,\ 0\le j\le N,\ 0\le k\le P\bigr\}\,.$$

We put $\mathbf{y} = \mathbf{q} \equiv (q, q^2, \dots, q^N)$, $\mathbf{x} = \mathbf{q}/q \equiv (1, q, \dots, q^{N-1})$ in (3) and obtain the q-parameterized Binet-Cauchy relation,

$$\sum_{\boldsymbol{\lambda}\subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q})S_{\boldsymbol{\lambda}}(\mathbf{q}/q) = \mathcal{V}_N^{-1}(\mathbf{q})\mathcal{V}_N^{-1}(\mathbf{q}/q)\det\left(\frac{1-q^{(M+N)(j+k-1)}}{1-q^{j+k-1}}\right)_{1\le j,k\le N}. \tag{5}$$

The relation (5) is used in calculation of the amplitudes of the low temperature asymtotics of the correlation functions in the limit when the total number of sites is large enough, $M\gg 1$, while the number of particles N is moderate: $1\ll N\ll M$ [15]. The framework of Quantum Inverse Scattering Method [17, 18] enabled [10] to establish the connection of (5) with enumeration of the plane partitions in $\mathcal{B}(N,N,M)$. The determinant in right-hand side of Eq. (5) was expressed as the Kuperberg determinant [19], what led to the answer:

$$\mathcal{V}_N^{-1}(\mathbf{q})\mathcal{V}_N^{-1}(\mathbf{q}/q)\det\left(\frac{1-q^{(M+N)(j+k-1)}}{1-q^{j+k-1}}\right)_{1\le j,k\le N} = \prod_{k=1}^{N}\prod_{j=1}^{N}\frac{1-q^{M+j+k-1}}{1-q^{j+k-1}}. \tag{6}$$

This formula is the MacMahon generating function for the boxed plane partitions [3].

As it follows from [16], the sum of the Schur functions in left-hand side of (5) may be expressed through the *q-binomial determinant*:

$$\sum_{\boldsymbol{\lambda}\subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q})S_{\boldsymbol{\lambda}}(\mathbf{q}/q) = q^{\frac{NM}{2}(1-M)}\,\det\left(\begin{bmatrix}2N+i-1\\ N+j-1\end{bmatrix}\right)_{1\le i,j\le M}. \tag{7}$$

The entries in (7) are the *q-binomial coefficients*, [20], defined as

$$\begin{bmatrix}R\\ r\end{bmatrix} \;\equiv\; \frac{[R]!}{[r]!\,[R-r]!}\,, \tag{8}$$

where $[n]$ is the *q-number* being q-analogue of a positive integer $n\in\mathbb{Z}^+$,

$$[n] \;\equiv\; \frac{1-q^n}{1-q}\,,$$

and the *q-factorial* $[n]!$ is: $[n]! \equiv [1]\,[2]\,\dots\,[n]$, $[0]! \equiv 1$. The determinant in right-hand side of (7) is independently calculated in [16], and the answer agrees with (5), (6).

2 The Schur functions and the lattice paths

In this Letter we shall give the combinatorial interpretation of Eq. (5) appearing in the integrable models of strongly correlated bosons, [10], and of free fermions, [15]. It is well-known that a combinatorial description of the Schur functions may be given in terms of *semistandard Young tableaux.* A filling of the cells of the Young diagram of $\boldsymbol{\lambda}$ with positive integers $n \in \mathbb{N}^+$ is called a *semistandard tableau of shape* $\boldsymbol{\lambda}$ provided it is weakly increasing along rows and strictly increasing along columns. The weight $\mathbf{x}^T$ of a tableau T is defined as

$$\mathbf{x}^T \equiv \prod_{i,j} x_{T_{ij}} \,,$$

where the product is over all entries T_{ij} of the tableau T. An equivalent definition of the Schur function is given by

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_m) = \sum_T \mathbf{x}^T \,, \tag{9}$$

where $m \geq N$, and the sum is over all tableaux T of shape $\boldsymbol{\lambda}$ with the entries being numbers from the set $\{1, 2, \dots, m\}$.

There is a natural way of representing each semistandard tableau of shape $\boldsymbol{\lambda}$ with entries not exceeding N as a nest of self-avoiding lattice paths with prescribed start and end points. Let T_{ij} be an entry in $i^{\rm th}$ row and $j^{\rm th}$ column of the semistandard tableau T. The $i^{\rm th}$ lattice path of the nest C (counted from the top of the nest) encodes the $i^{\rm th}$ row of the tableau ($i = 1, \dots, N$). It goes from $C_i = (N - i + 1, N - i)$ to $(1, \mu_i = \lambda_i + N - i)$ (see Fig. 2). It makes λ_i steps to the north so that the step along the line x_j corresponds to the occurrences of the letter $N - j + 1$ in the $i^{\rm th}$ row of T. The power l_j of x_j in the weight of any particular nest of paths is the number of steps to north taken along the vertical line x_j. Thus, an equivalent representation of the Schur function takes the form:

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) = \sum_C \prod_{j=1}^N x_j^{l_j}, \tag{10}$$

where summation is over all admissible nests C. This representation of the Schur functions is natural in the Quantum Inverse Scattering Method approach to the solution of the models. The $k^{\rm th}$ path is contained in a rectangle of the size $\lambda_k \times (N - k)$, $k = 1, \dots, N$. The starting point of each path is the lower left vertex. We define the volume of the path as the number of squares below it in the corresponding rectangle. The volume of the nest of lattice paths is equal to the volume of the lattice paths:

$$\mid \zeta \mid_C = \sum_{j=1}^N (j-1) l_j .$$

Therefore, the q-parametrized Schur function is a partition function of the described nest:

$$S_{\boldsymbol{\lambda}}(\mathbf{q}/q) = \sum_C q^{|\zeta|_C} \,,$$

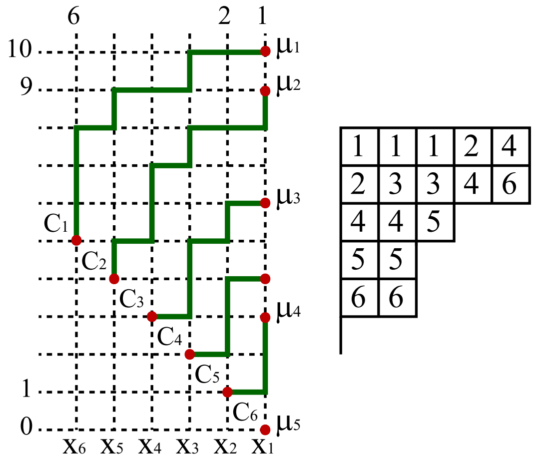

Figure 2: A semistandard tableau of shape $\boldsymbol{\lambda} = (5, 5, 3, 2, 2, 0)$ is represented as a nest of lattice paths. Vertical steps along the line x_j represent occurrences of letter $N - j + 1$, $N = 6$, in the tableau.

where summation is over all admissible nests C. Adding the weight of partition $|\boldsymbol{\lambda}| = \sum_{k=1}^{N} \lambda_k$ to the volume of the nest, we obtain that

$$\mid \xi \mid_C = \mid \lambda \mid + \mid \zeta \mid_C = \sum_{j=1}^{N} j \, l_j \, ,$$

and

$$S_{\boldsymbol{\lambda}}(\mathbf{q}) = \sum_C q^{|\xi|_C} = q^{|\lambda|} \sum_C q^{|\zeta|_C} = q^{|\lambda|} S_{\boldsymbol{\lambda}}(\mathbf{q}/q) \, .$$

Consider a *conjugated* nest of self-avoiding lattice paths (see Fig. 3) from $(1, \mu_i = \lambda_i + N - i)$ to $B_i = (i, N + M - i)$. The i^{th} path consists of $M - \lambda_i$ steps to the north. The representation of the Schur function corresponding to the described nest is:

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) = \sum_B \prod_{j=1}^{N} x_j^{M - l_j}, \tag{11}$$

where summation is over all admissible nests B of N self-avoiding lattice paths. The k^{th} path is contained in a rectangle of the size $(k-1) \times M$, $k = 1, \ldots, N$. The ending point of each path is the top right vertex. The volume of the path is the number of squares below it in the corresponding rectangle. The volume of the nest of the lattice paths is equal to the volume of the paths:

$$\mid \zeta \mid_B = \sum_{j=1}^{N} (j-1)(M - l_j).$$

In the limit $q \to 1$, the Schur function is equal to the number of nests of self-avoiding lattice paths of the types either B or C:

$$S_{\boldsymbol{\lambda}}(1, \ldots, 1) = \sum_B 1 = \sum_C 1 \, .$$

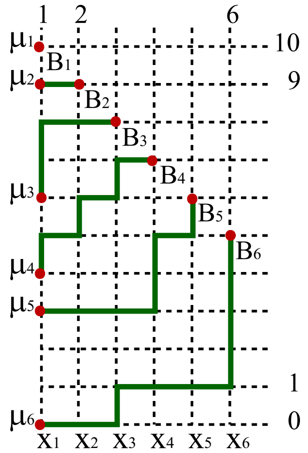

Figure 3: Conjugated nest of lattice paths.

The summand of the scalar product (3), being the product of two Schur functions, may be graphically expressed as a nest of N self-avoiding lattice paths starting at the equidistant points C_i and terminating at the equidistant points B_i ($i = 1, \ldots, N$). This configuration, known as *watermelon*, is presented on Fig. 4. The scalar product (3) is the sum of all such watermelons. Repeating the arguments used above to derive the lattice paths volumes, it

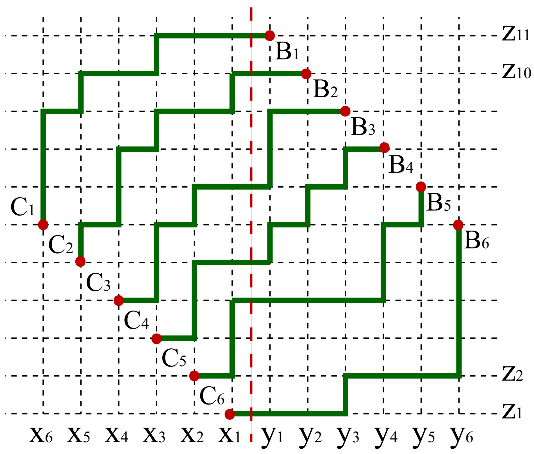

Figure 4: Watermelon configuration

is straightforward to find that the volume of the watermelon is equal to:

$$| \, w \, |=| \, \xi \, |_C + | \, \zeta \, |_B \ .$$

The partition function of watermelons (the generating function of watermelons) is equal to left-hand side of (5):

$$\mathsf{W}(N, M) = \sum_{W} q^{|w|} = \sum_{\boldsymbol{\lambda} \subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q) \,, \tag{12}$$

where the sum $\sum_W$ is taken over all watermelons with the fixed endpoints C_i, B_i, $1 \leq i \leq N$.

To connect watermelon with a semistandard tableaux, let us now read the watermelon configuration with the endpoints $C_i = (N-i+1, N-i)$, $B_i = (i, N+M-i)$ in the following way. The i^{th} path (counted from the bottom) makes $\lambda_i = N$ steps to the east. The power m_j of z_j in the weight is the number of steps to the east taken along the horizontal line z_j. The Young tableau of such configuration is rectangle of the size $N \times N$. The Schur function of the watermelon is:

$$S_{\mathbf{N}}(z_1, z_2, \dots, z_{N+M}) = \sum_W \prod_{j=1}^{N+M} z_j^{m_j} \,, \tag{13}$$

where summation is over all admissible watermelons, and $\mathbf{N}$ is the partition $(N, N, \dots, N)$ of the length N, i.e., $\mathbf{N} \equiv N^N$ in our notations. The volume of watermelon is equal to

$$\mid w \mid = \sum_{j=1}^{M+N} (j-1)m_j - \frac{N^2(N-1)}{2}.$$

The partition function of watermelons is expressed through the Schur function (13):

$$\mathsf{W}(N, M) = q^{-\frac{N^2}{2}(N-1)}\, S_{\mathbf{N}}(1, q^2, \dots, q^{N+M-1})\,. \tag{14}$$

This function is easy to calculate with the help of well known formula (see [1], Chapter 1, Example 1):

$$S_{\boldsymbol{\lambda}}(1, q^2, \dots, q^{m-1}) = q^{n(\lambda)} \prod_{1 \le i < j \le m} \frac{1 - q^{\lambda_i - \lambda_j - i + j}}{1 - q^{j-i}}, \tag{15}$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$. Moreover, if $m > N$, then $\lambda_i = 0$ for $i > N$. We obtain from (15) that

$$\mathsf{W}(N, M) = \prod_{i=1}^{N} \prod_{j=N+1}^{N+M} \frac{1 - q^{N-i+j}}{1 - q^{j-i}}\,. \tag{16}$$

Replacing the indices $j \to N + j$ and $i \to N + 1 - i$, we put (16) into the form:

$$\mathsf{W}(N, M) = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1 - q^{N+i+j-1}}{1 - q^{j+i-1}} = \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1 - q^{M+i+j-1}}{1 - q^{j+i-1}}\,. \tag{17}$$

Eventually, it is seen that Eqs. (12) and (17) are in agreement with Eqs. (5) and (6).

The watermelon with *deviation* k may be obtained by imposing the boundary condition $l_N = \ldots = l_{N-k+1} = 0$ in (10). The starting points D_i of the watermelon with deviation will be shifted to the east by k steps with respect to C_i. The watermelon with deviation is presented on Fig. 5. The boundary condition introduced is equivalent to the following property of the Schur function. Consider a partition $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{N-k}, \lambda_{N-k+1}, \dots, \lambda_N)$ with the last k parts equal to zero, $\lambda_{N-k+1} = \ldots = \lambda_N = 0$. Then the limiting relation is valid:

$$\lim_{x_N \to 0} \cdots \lim_{x_{N-k+1} \to 0} S_{\boldsymbol{\lambda}}(x_1, \dots, x_{N-k}, x_{N-k+1}, \dots, x_N) = S_{\tilde{\boldsymbol{\lambda}}}(x_1, \dots, x_{N-k})\,, \tag{18}$$

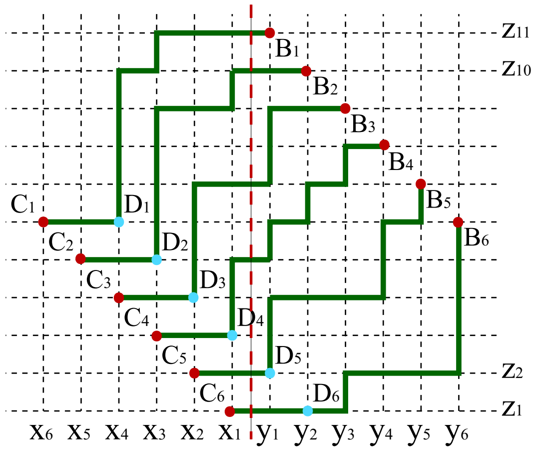

Figure 5: Watermelon with deviation $k = 2$. Starting points are D_i, endpoints are B_i.

where the parts of $\tilde{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2, \dots, \lambda_{N-k})$ satisfy $M \geq \lambda_1 \geq \lambda_2 \dots \lambda_{N-k} \geq 0$. Taking the limit (18) in (3), we obtain:

$$\sum_{\tilde{\boldsymbol{\lambda}} \subseteq M^{N-k}} S_{\tilde{\boldsymbol{\lambda}}}(x_1, \dots, x_{N-k}) S_{\hat{\boldsymbol{\lambda}}}(y_1, \dots, y_N) = \left(\prod_{l=1}^{N-k} x_l^{-k} \right) \frac{\det(\tilde{M}_{kj})_{1 \leq k,j \leq N}}{\mathcal{V}_{N-k}(\mathbf{x}) \mathcal{V}_N(\mathbf{y})},$$

where summation is over all partitions $\tilde{\boldsymbol{\lambda}}$ with at most $N - k$ parts, each of which is less than or equal to M. The partition $\hat{\boldsymbol{\lambda}}$ of the length N contains extra zeros $\hat{\lambda}_{N-k+1} = \hat{\lambda}_{N-k+2} = \dots \hat{\lambda}_N = 0$, and the entries $\tilde{M}_{kj}$ are:

$$\begin{aligned} \tilde{M}_{kj} &= M_{kj}\,, \quad 1 \leq k \leq N, \qquad 1 \leq j \leq N - k\,, \\ \tilde{M}_{kj} &= y_j^{N-k}\,, \quad 1 \leq k \leq N, \qquad N - k + 1 \leq j \leq N\,, \end{aligned}$$

where the entries M_{kj} are given by (4).

The semistandard tableau corresponding to the watermelon with deviation consists of N rows of the length $L = N - k$. The volume of the watermelon with deviation is

$$\mid w \mid = \sum_{j=1}^{M+N} (j-1) m_j - \frac{NM(M-1)}{2}. \tag{19}$$

In the case of the watermelon with deviation we obtain the representation analogous to (14):

$$\begin{aligned} \mathsf{W}(N, L, M) &= q^{-\frac{NM(M-1)}{2}} S_{\mathbf{L}}(1, q, \dots, q^{N+M-1}) \qquad (20) \\ &= \sum_{\tilde{\boldsymbol{\lambda}} \subseteq M^{N-k}} S_{\tilde{\boldsymbol{\lambda}}}(q, \dots, q^{N-k}) S_{\hat{\boldsymbol{\lambda}}}(1, \dots, q^{N-1})\,, \end{aligned}$$

where $\mathbf{L} = L^N$ for the partition $\mathbf{L}$. Calculating the Schur function $S_{\mathbf{L}}$ with the help of (15), we obtain:

$$\mathsf{W}(N, L, M) = \prod_{i=1}^{N} \prod_{j=N+1}^{N+M} \frac{1 - q^{L-i+j}}{1 - q^{j-i}} = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1 - q^{L+i+j-1}}{1 - q^{j+i-1}}\,. \tag{21}$$

In the limit $q \to 0$, this formula gives the number of the watermelons with deviation:

$$A(N, L, M) = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{L+i+j-1}{j+i-1} \,. \tag{22}$$

The Schur function can be expressed in a polynomial form through the complete symmetric functions, [1]: $S_{\boldsymbol{\lambda}}(\mathbf{x}) = \det(h_{\lambda_i - i + j}(\mathbf{x}))_{1\le i,j\le N}$. This expression agrees, [3], with the definition (9). Under the q-parametrization, the complete symmetric functions are the q-binomial coefficients (8):

$$h_r(\mathbf{q}/q) \;=\; \begin{bmatrix} N+r-1 \\ r \end{bmatrix}, \qquad 1 \le r \le N. \tag{23}$$

The following determinant with the q-binomial entries was calculated in [21]:

$$\det \left(q^{(j-1)(\lambda_i + j - i)} \begin{bmatrix} \lambda_i + m - i \\ m-j \end{bmatrix} \right)_{1\le i,j\le N} = S_{\boldsymbol{\lambda}}(1, q, \ldots, q^{m-1}) \,, \qquad m \ge N \,. \tag{24}$$

Using (23) and the Pascal formula for the q-binomial coefficients,

$$\begin{bmatrix} R \\ r \end{bmatrix} \;=\; \begin{bmatrix} R-1 \\ r-1 \end{bmatrix} \,+\, q^r \begin{bmatrix} R-1 \\ r \end{bmatrix} , \tag{25}$$

one can re-express left-hand side of (24) so that the following equation holds:

$$\det \left(h_{\lambda_i - i + j}(1, q, \ldots, q^{m-1}) \right)_{1\le i,j\le N} = S_{\boldsymbol{\lambda}}(1, q, \ldots, q^{m-1}) \,. \tag{26}$$

The partition function of the watermelon with deviation given by (20) and (21) may be rewritten with regard to the determinantal formulas (24) and (26):

$$\begin{aligned} \mathsf{W}(N, L, M) \;&=\; q^{-\frac{NM(M-1)}{2}} \det \left(q^{(j-1)(L+j-i)} \begin{bmatrix} L+M+N-i \\ M+N-j \end{bmatrix} \right)_{1\le i,j\le N} && (27) \\ &=\; q^{-\frac{NM(M-1)}{2}} \det \left(h_{L+j-i}(\mathbf{q}/q) \right)_{1\le i,j\le N} \,. && (28) \end{aligned}$$

The number of the watermelons with deviation (22) is expressed:

$$\begin{aligned} A(N, L, M) \;&=\; \det \left(\binom{L+M+N-i}{M+N-j} \right)_{1\le i,j\le N} && (29) \\ &=\; \det \left(\binom{L+M+N+j-i-1}{L+j-i} \right)_{1\le i,j\le N} , && (30) \end{aligned}$$

where the determinant (29) is the *binomial determinant*, [22], while the coincidence of (29) and (30) can independently be checked by means of (25) at $q = 1$.

In the limit $q \to 1$, the Schur function (11) may be expressed with the help of (24):

$$\det \left(\binom{\lambda_i + N - i}{N-j} \right)_{1\le i,j\le N} = S_{\boldsymbol{\lambda}}(1, \ldots, 1) \;=\; \sum_{B} 1 \;=\; \sum_{C} 1 \,. \tag{31}$$

Equation (31) expresses the statement of the Gessel-Viennot theorem, [22], connecting the binomial determinant in left-hand side of (31) with the number of nests of self-avoiding lattice paths of the types either B or C.

There exists bijection between watermelons and plane partitions confined in a box of finite size [23]. A plane partition is an array $(\pi_{ij})_{1\le i,j}$ of non-negative integers that are non-increasing as functions both of i and j [1,3]. The integers π_{ij} are called the parts of the plane partition, and $|\boldsymbol{\pi}| = \sum_{i,j} \pi_{ij}$ is its volume. Each plane partition has a three-dimensional diagram which can be interpreted as a stack of unit cubes (three-dimensional Young diagram). The height of stack with coordinates (i, j) is equal to π_{ij}. It is said that the plane partition corresponds to a box $\mathcal{B}(N, L, M)$ provided that $j \le N$, $i \le L$ and $\pi_{ij} \le M$ for all cubes of the Young diagram. The generating function of plane partitions

$$Z_q(N, L, M) = \sum_{\mathcal{B}(N,L,M)} q^{|\boldsymbol{\pi}|}\,, \tag{32}$$

where the sum is taken over all plane partitions contained in a box $\mathcal{B}(N, L, M)$.

Projection of gradient lines of plane partition (see Fig. 6) form a nest of lattice paths that correspond to watermelons (see Fig. 4 and Fig. 5, respectively). By its construction,

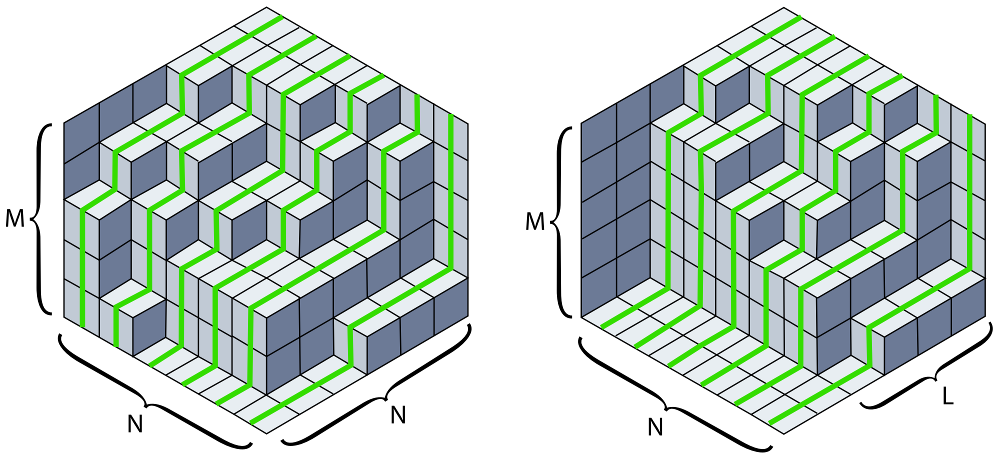

Figure 6: Plane partitions with gradient lines embedded into a symmetric box $\mathcal{B}(N, N, M)$ and into an arbitrary one $\mathcal{B}(N, L, M)$, obtained as a special limit of symmetric box.

the volume of watermelon (19) coincides with the volume of plane partition $|\boldsymbol{\pi}|$, and thus

$$Z_q(N, L, M) = \mathsf{W}(N, L, M)\,.$$

3 Discussion

The algebraic relations arising in calculation of the scalar products and the correlation functions of certain integrable models, [10,16], are based on the Binet-Cauchy formula (3) adapted for the Schur functions. There exists an interesting combinatorial interpretation for Eq. (5) in terms of nests of the self-avoiding lattice paths, which, in turn, are related to enumeration of the boxed plane partitions. All these combinatorial objects arise in the investigation of the asymptotical behavior of the thermal correlation functions [16]. In the limit when the total number of sites is large enough while the occupation is moderate, these correlation functions are related with the partition functions of the matrix models and hopefully give evidence of a third order phase transition [24] in the integrable models in question. Further investigations are in progress.

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