

ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

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РЕДКОЛЛЕГИЯ

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Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16
октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых
коммуникаций

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Заведующая информационно-издательским сектором Симонова В.Н

Elements of noncommutative geometry in inverse problems on manifolds

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May 22, 2013

ABSTRACT:

We deal with two dynamical systems associated with a Riemannian manifold with boundary. The first one is a system governed by the scalar wave equation, the second is governed by the Maxwell equations. Both of the systems are controlled from the boundary. The inverse problem is to recover the manifold via the relevant measurements at the boundary (inverse data).

We show that the inverse data determine a C^* -algebras, whose (topologized) spectra are identical to the manifold. By this, to recover the manifold is to determine a proper algebra from the inverse data, find its spectrum, and provide the spectrum with a Riemannian structure.

The paper develops an algebraic version of the boundary control method, which is an approach to inverse problems based on their relations to control theory.

Key words: wave equation and Maxwell system on a Riemannian manifold, reconstruction of manifolds via boundary inverse data, C^* -algebras, BC-method.

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1 Introduction

About the paper

One of the basic theses of noncommutative geometry is that a topological space can be characterized via an algebra associated with it [8], [11], [16]. In other words, a space can be encoded into an algebra. As was recognized in [2] and [4], such a coding is quite relevant and efficient for solving inverse problems on manifolds. In particular, it enables one to reconstruct a Riemannian manifold via its dynamical or spectral boundary inverse data.

Namely, it is shown that a Riemannian manifold Ω can be identified with the (topologized) spectrum $\widehat{\mathfrak{A}(\Omega)}$ of an appropriate Banach algebra $\mathfrak{A}(\Omega)$, the algebra being determined by the inverse data up to isometric isomorphism. Therefore, one can reconstruct Ω by the scheme:

- extract an isometric copy $\tilde{\mathfrak{A}}(\Omega)$ of $\mathfrak{A}(\Omega)$ from the data
- find its spectrum $\widehat{\tilde{\mathfrak{A}}(\Omega)} =: \tilde{\Omega}$, which is homeomorphic to $\widehat{\mathfrak{A}(\Omega)}$ by virtue of $\tilde{\mathfrak{A}}(\Omega) \stackrel{\text{isom}}{=} \mathfrak{A}(\Omega)$. Thus, we have $\tilde{\Omega} \stackrel{\text{hom}}{=} \Omega$
- endow $\tilde{\Omega}$ with a proper Riemannian structure.

As a result, we get a Riemannian manifold $\tilde{\Omega}$ isometric to the original Ω by construction. It is $\tilde{\Omega}$, which solves the reconstruction problem.

Our paper keeps this scheme and extends it to the inverse problem of electrodynamics.

Content

We deal with a smooth compact Riemannian manifold Ω with boundary.

Eikonals. We introduce the *eikonals*, which play the role of main instrument for reconstruction. An eikonal $\tau_\sigma(\cdot) = \text{dist}(\cdot, \sigma)$ is a distance function on Ω with the base $\sigma \subset \partial\Omega$. The eikonals determine the Riemannian structure on Ω .

With each eikonal one associates a self-adjoint operator $\tilde{\tau}_\sigma$ in $L_2(\Omega)$, which multiplies functions by τ_σ . Its representation via the Spectral Theorem is $\tilde{\tau}_\sigma = \int_0^\infty s dX_\sigma^s$, where X_σ^s is the projection onto the subspace $L_2(\Omega^s[\sigma])$ of functions supported in the metric neighborhood $\Omega^s[\sigma] \subset \Omega$ of σ of radius s .

For an oriented 3d-manifold Ω , by analogy with the scalar case, we introduce the *solenoidal eikonals* $\varepsilon_\sigma = \int_0^\infty s dY_\sigma^s$, which act in the space

$$\mathcal{C} = \{\text{curl } h \mid h, \text{curl } h \in \vec{L}_2(\Omega)\}$$

relevant to electrodynamics. Here Y_σ^s projects vector-fields onto the subspace of curls supported in $\Omega^s[\sigma]$.

Algebras. Eikonals $\{\tau_\sigma \mid \sigma \subset \partial\Omega\}$ generate the Banach algebra $C(\Omega)$ of real continuous functions. By the Gelfand theorem, its Gelfand spectrum (the set of characters) $\widehat{C(\Omega)}$ is homeomorphic to Ω [13], [14].

Operator eikonals $\{\tilde{\tau}_\sigma \mid \sigma \subset \partial\Omega\}$ generate an operator algebra \mathfrak{T} , which is a commutative C^* -subalgebra of the bounded operator algebra $\mathfrak{B}(L_2(\Omega))$. The algebras \mathfrak{T} and $C(\Omega)$ are isometrically isomorphic (via $\tilde{\tau}_\sigma \mapsto \tau_\sigma$). By this, their spectra are homeomorphic, and we have $\widehat{\mathfrak{T}} \stackrel{\text{hom}}{=} \widehat{C(\Omega)} \stackrel{\text{hom}}{=} \Omega$.

Solenoidal eikonals generate an operator algebra \mathfrak{E} , which is a C^* -subalgebra of $\mathfrak{B}(\mathcal{C})$. In contrast to \mathfrak{T} , the algebra \mathfrak{E} is *noncommutative*. However, the factor-algebra $\dot{\mathfrak{E}} = \mathfrak{E}/\mathfrak{K}$ over the ideal of compact operators $\mathfrak{K} \in \mathfrak{E}$ turns out to be commutative. Moreover, one has $\dot{\mathfrak{E}} \stackrel{\text{isom}}{=} C(\Omega)$ that implies $\widehat{\dot{\mathfrak{E}}} \stackrel{\text{hom}}{=} \widehat{C(\Omega)} \stackrel{\text{hom}}{=} \Omega$.

Inverse problems. Following [4], we begin with a dynamical system, which is governed by the scalar *wave equation* in Ω and controlled from the boundary $\partial\Omega$. The input \rightarrow output correspondence is realized by a *response operator* R , which plays the role of inverse data. A reconstruction (inverse) problem is to recover the manifold Ω via given R .

Solving this problem, we construct (via R) an operator algebra $\tilde{\mathfrak{T}}$ isometric to \mathfrak{T} , find its spectrum $\tilde{\Omega} := \widehat{\tilde{\mathfrak{T}}} \stackrel{\text{hom}}{=} \widehat{\mathfrak{T}} \stackrel{\text{hom}}{=} \Omega$, endow it with the Riemannian structure by the use of images of eikonals, and eventually turn $\tilde{\Omega}$ into an isometric copy of the original manifold Ω . The copy $\tilde{\Omega}$ provides the solution to the reconstruction problem.

In electrodynamics, the corresponding system is governed by the *Maxwell equations* and also controlled from the boundary. The relevant response operator R plays the role of inverse data for the reconstruction problem. To solve this problem, we repeat all the steps of the above described procedure. The only additional step is the factorization $\mathfrak{E} \mapsto \dot{\mathfrak{E}}$, which eliminates noncommutativity.

Appendix. Here the basic lemmas on the eikonals ε_σ and algebra \mathfrak{E} are proven.

Comments

What is "to recover a manifold"? Setting the goal to determine Ω from R , one has to take into account the evident nonuniqueness of such a determination. Indeed, if two manifolds Ω and Ω' are isometric and have the mutual boundary $\partial\Omega = \partial\Omega'$ then their boundary inverse data (in particular, the response operators) turn out to be identical. Hence, the correspondence $\Omega \mapsto R$ is not injective and to recover the original Ω via R is impossible.

From the physical viewpoint, the inverse data formalize the measurements, which the external observer implements at the boundary. The above mentioned nonuniqueness means that the observer is not able to distinguish Ω from Ω' in principle. In such a situation, the only reasonable understanding of the reconstruction problem is the following: *to construct a manifold $\tilde{\Omega}$, which possesses the prescribed inverse data.* It is the above mentioned isometric copy $\tilde{\Omega}$, which satisfies this requirement: we have $\tilde{R} = R$ by construction.

Remark Reconstruction via algebras is known in Noncommutative Geometry: see [8], [11], [16]. However, there is a principle difference: in the mentioned papers the starting point for reconstruction is the so-called *spectral triple* $\{\mathcal{A}, \mathcal{H}, \mathcal{D}\}$, which consists of a commutative algebra, a Hilbert space, and a self-adjoint (Dirac-like) operator. So, an algebra *is given*.

In our case, we at first have to extract an algebra from R . Then we deal with this algebra imposed by inverse data, whereas its "good" properties are not guaranteed. For instance, a metric graph is a "commutative space" but its eikonal algebra \mathfrak{T} turns out to be strongly noncommutative¹. The latter leads to difficulties in reconstruction problem, which are not overcome yet.

Reconstruction via algebras in inverse problems was originated in [2] and developed in [4]. It represents an algebraic version of the *boundary control method*, which is an approach to inverse problems based on their relations to control theory [1], [3]. We hope for further applications of this version to inverse problems of mathematical physics.

Acknowledgements The authors thank B.A.Plamenevskii for kind and useful consultations. The work is supported by the grants RFBR 11-01-00407A, RFBR 12-01-31446, SPbGU 11.38.63.2012, 6.38.670.2013 and RF Government grant 11.G34.31.0026.

2 Eikonals

We deal with a real smooth² compact Riemannian manifold Ω with the boundary Γ , g is the metric tensor, $\dim \Omega = n \geq 2$.

For a set $A \subset \Omega$, by

$$\Omega^r[A] := \{x \in \Omega \mid \text{dist}(x, A) < r\}, \quad r > 0$$

¹no factorization turns \mathfrak{T} into a commutative algebra

²everywhere in the paper, "smooth" means C^∞ -smooth

we denote its metric r -neighborhood. Compactness implies $\text{diam } \Omega := \sup\{\text{dist}(x, y) \mid x, y \in \Omega\} < \infty$ and

$$\Omega^r[A] = \Omega \quad \text{as } r > \text{diam } \Omega. \quad (2.1)$$

2.1 Scalar eikonals

Let us say a subset $\sigma \subset \Gamma$ to be *regular* and write $\sigma \in \mathcal{R}(\Gamma)$ if σ is diffeomorphic to a "disk" $\{p \in \mathbb{R}^{n-1} \mid \|p\| \leq 1\}$.

By a (scalar) *eikonal* we name a distant function of the form

$$\tau_\sigma(x) := \text{dist}(x, \sigma), \quad x \in \Omega \quad (\sigma \in \mathcal{R}(\Gamma)).$$

The set σ is said to be a *base*. Eikonals are Lipschitz functions: $\tau_\sigma \in \text{Lip}(\Omega) \subset C(\Omega)$. Moreover, eikonals are smooth almost everywhere and

$$|\nabla \tau_\sigma(x)| = 1 \quad \text{a.a. } x \in \Omega \quad (2.2)$$

holds. Also, note the following simple geometric facts.

Proposition 1. *For any $x \in \Omega$ there is $\sigma \in \mathcal{R}(\Gamma)$ such that $\tau_\sigma(x) \neq 0$. For any different $x, x' \in \Omega$ there is a $\sigma \in \mathcal{R}(\Gamma)$ such that $\tau_\sigma(x) \neq \tau_\sigma(x')$ (i.e., the eikonals distinguish points of Ω). The equality $\sigma = \{\gamma \in \Gamma \mid \tau_\sigma(\gamma) = 0\}$ holds.*

Copy $\tilde{\Omega}$

As functions on Ω , eikonals are determined by the Riemannian structure of Ω . The converse is also true in the following sense.

Assume that we are given with a topological space $\tilde{\Omega}$, which is homeomorphic to Ω (with the Riemann metric topology) via a homeomorphism $\eta : \Omega \rightarrow \tilde{\Omega}$; let $\tilde{\tau}_\sigma := \tau_\sigma \circ \eta^{-1}$. Also, assume that η is *unknown* but we are given with the map

$$\mathcal{R}(\Gamma) \ni \sigma \mapsto \tilde{\tau}_\sigma \in C(\tilde{\Omega}). \quad (2.3)$$

Then one can endow $\tilde{\Omega}$ with the Riemannian structure, which turns it into a manifold *isometric* to Ω . Roughly speaking, the way is the following³.

For a fixed point $p \in \tilde{\Omega}$ one can find its neighborhood $\omega \subset \tilde{\Omega}$ and the sets $\sigma_1, \dots, \sigma_n \in \mathcal{R}(\Gamma)$ such that the functions $x^1 = \tilde{\tau}_{\sigma_1}(\cdot), \dots, x^n = \tilde{\tau}_{\sigma_n}(\cdot)$ constitute a coordinate chart $\phi : \omega \ni p \mapsto \{x^k(p)\}_{k=1}^n \in \mathbb{R}^n$. The coordinates endow

³see [5] for detail

ω with tangent spaces. These spaces can be provided with the metric tensor $\tilde{g} = \eta_*g$: one can determine its components \tilde{g}^{ij} from the equations

$$\tilde{g}^{ij}(x) \frac{\partial \tilde{\tau}_\sigma \circ \phi^{-1}}{\partial x^i}(x) \frac{\partial \tilde{\tau}_\sigma \circ \phi^{-1}}{\partial x^j}(x) = 1, \quad x \in \phi(\omega), \quad \sigma \in \mathcal{R}(\Gamma) \quad (2.4)$$

which are just (2.2) written in coordinates. Choosing here $\sigma = \sigma_i$, we get $\tilde{g}^{ii} = 1$. Choosing (a finite number of) additional sets σ , we can determine the functions $\frac{\partial \tilde{\tau}_\sigma \circ \phi^{-1}}{\partial x^i}$ and then find all other components $\tilde{g}^{ij}(x)$ by solving the system (2.4) with respect to them.

So, although the homeomorphism η is unknown, we are able to endow $\tilde{\Omega}$ with the metric tensor $\tilde{g} = \eta_*g$, which turns it into a Riemannian manifold $(\tilde{\Omega}, \tilde{g})$ isometric to (Ω, g) by construction.

Moreover, there is a natural way to identify the boundaries $\tilde{\Gamma} := \partial\tilde{\Omega}$ and $\Gamma = \partial\Omega$. At first, we can select the boundary points in $\tilde{\Omega}$ by

$$\tilde{\Gamma} = \bigcup_{\sigma \in \mathcal{R}(\Gamma)} \tilde{\sigma}, \quad \text{where } \tilde{\sigma} := \{\tilde{\gamma} \in \tilde{\Omega} \mid \tilde{\tau}_\sigma(\tilde{\gamma}) = 0\}.$$

Then we identify $\Gamma \ni \gamma \equiv \tilde{\gamma} \in \tilde{\Gamma}$ if $\gamma \in \sigma$ implies $\tilde{\gamma} \in \tilde{\sigma}$ for all regular σ containing γ .

As a result, we get the manifold $(\tilde{\Omega}, \tilde{g})$ isometric to (Ω, g) , these manifolds having the mutual boundary Γ . In what follows we refer to $(\tilde{\Omega}, \tilde{g})$ as a canonical copy of the original manifold Ω (shortly: the *copy* $\tilde{\Omega}$).

The aforesaid is summarized as follows.

Proposition 2. *A space $\tilde{\Omega}$ along with the map (2.3) determine the copy $\tilde{\Omega}$ and, hence, determine Ω up to isometry of Riemannian manifolds.*

2.2 Operator eikonals

Introduce the space $\mathcal{H} := L_2(\Omega)$ with the inner product

$$(u, v)_{\mathcal{H}} = \int_{\Omega} u(x)v(x) dx.$$

Let $A \subset \Omega$ be a measurable subset, $\chi_A(\cdot)$ its indicator (a characteristic function). By

$$\mathcal{H}\langle A \rangle := \{\chi_A y \mid y \in \mathcal{H}\}$$

we denote the subspace of functions supported on A . The (orthogonal) projection X_A in \mathcal{H} onto $\mathcal{H}\langle A \rangle$ multiplies functions by χ_A , i.e., cuts off functions on A .

Let $\mathfrak{B}(\mathcal{H})$ be the normed algebra of bounded operators in \mathcal{H} . With a scalar eikonal τ_σ one associates an operator $\tilde{\tau}_\sigma \in \mathfrak{B}(\mathcal{H})$, which acts in \mathcal{H} by

$$(\tilde{\tau}_\sigma y)(x) := \tau_\sigma(x) y(x), \quad x \in \Omega$$

and is bounded since Ω is compact. Moreover, one has

$$\|\tilde{\tau}_\sigma\| = \max_{x \in \Omega} |\tau_\sigma(x)| = \|\tau_\sigma\|_{C(\Omega)} \leq \text{diam } \Omega. \quad (2.5)$$

With a slight abuse of terms, we also call $\tilde{\tau}_\sigma$ an *eikonal*.

Each eikonal is a self-adjoint positive operator, which is represented by the Spectral Theorem in the well-known form.

Proposition 3. *The representation*

$$\tilde{\tau}_\sigma = \int_0^\infty s dX_\sigma^s \quad (2.6)$$

is valid, where the projections $X_\sigma^s := X_{\Omega^s[\sigma]}$ cut off functions on the metric neighborhoods of σ .

Note that the integration interval is in fact $0 \leq s \leq \|\tilde{\tau}_\sigma\|$.

The eikonals corresponding to different bases do commute. This follows from commutation of X_σ^s and $X_{\sigma'}^{s'}$ for all $\sigma, \sigma' \in \mathcal{R}(\Gamma)$ and $s, s' \geq 0$.

2.3 Solenoidal operator eikonals

Here we introduce an analog of $\tilde{\tau}_\sigma$ relevant to electrodynamics.

3d-manifold

Now, let $\dim \Omega = 3$. Also, let Ω be orientable and endowed with a volume 3-form dv . On such a manifold, the intrinsic operations of vector analysis \wedge (vector product), ∇ , div , curl , are well defined on smooth functions and vector fields (sections of the tangent bundle $T\Omega$): see, e.g., [17].

Solenoidal spaces

The class of smooth fields $\vec{C}^\infty(\Omega)$ is dense in the space $\vec{\mathcal{H}}$ of square-summable fields with the product

$$(a, b)_{\vec{\mathcal{H}}} = \int_{\Omega} a(x) \cdot b(x) dx,$$

where \cdot is the inner product in $T\Omega_x$. This space contains the (sub)spaces

$$\mathcal{J} := \{y \in \vec{\mathcal{H}} \mid \operatorname{div} y = 0 \text{ in } \Omega\}, \quad \mathcal{C} := \{\operatorname{curl} h \in \vec{\mathcal{H}} \mid h, \operatorname{curl} h \in \vec{\mathcal{H}}\} \subset \mathcal{J}$$

of solenoidal fields and curls. Note that the smooth classes $\mathcal{J} \cap \vec{C}^\infty(\Omega)$ and $\mathcal{C} \cap \vec{C}^\infty(\Omega)$ are dense in \mathcal{J} and \mathcal{C} respectively.

Recall the well-known decompositions

$$\vec{\mathcal{H}} = \mathcal{G}_0 \oplus \mathcal{J} = \mathcal{G}_0 \oplus \mathcal{C} \oplus \mathcal{D}, \quad (2.7)$$

where $\mathcal{G}_0 := \{\nabla q \mid q \in H_0^1(\Omega)\}$ is the space of potential fields, $\mathcal{D} := \{y \in \mathcal{J} \mid \operatorname{curl} h = 0, \nu \wedge y = 0 \text{ on } \Gamma\}$ is a finite-dimensional subspace of harmonic Dirichlet fields [17].

For an $A \subset \Omega$ we denote by

$$\begin{aligned} \vec{\mathcal{H}}\langle A \rangle &:= \{\chi_{Ay} \mid y \in \vec{\mathcal{H}}\}, \quad \mathcal{J}\langle A \rangle := \overline{\{y \in \mathcal{J} \mid \operatorname{supp} y \subset A\}}, \\ \mathcal{C}\langle A \rangle &:= \overline{\{\operatorname{curl} h \mid h \in \vec{C}^\infty(\Omega), \operatorname{supp} h \subset A\}} \end{aligned}$$

(the closure in $\vec{\mathcal{H}}$) the subspaces of fields supported in A .

Eikonals ε_σ

Fix a $\sigma \in \mathcal{R}(\Gamma)$ and take $A = \Omega^s[\sigma]$. Let Y_σ^s be the projection in \mathcal{C} onto the subspace $\mathcal{C}\langle \Omega^s[\sigma] \rangle$. Note that the action of Y_σ^s is not reduced to cutting off fields on $\Omega^s[\sigma]$, it acts in more complicated way (see [3], [5]).

By analogy with (2.6), define a *solenoidal operator eikonal*

$$\varepsilon_\sigma := \int_0^\infty s dY_\sigma^s, \quad (2.8)$$

which is an operator in \mathcal{C} . We omit a simple proof of the following result.

Proposition 4. *The eikonal ε_σ is a bounded self-adjoint positive operator, the equalities*

$$\|\varepsilon_\sigma\| = \|\tau_\sigma\|_{C(\Omega)} \stackrel{(2.5)}{=} \|\check{\tau}_\sigma\| \quad (2.9)$$

being valid.

An important fact is that, in contrast to the cutting off projections X_σ^s , the projections Y_σ^s and $Y_{\sigma'}^{s'}$ do not commute in general. As a consequence, the eikonals ε_σ and $\varepsilon_{\sigma'}$ also *do not commute*.

Multiplying a field $h \in \mathcal{C}$ by a bounded function φ , one takes the field out of the subspace of curls: $\varphi h \in \vec{\mathcal{H}}$ but $\varphi h \notin \mathcal{C}$ in general. However, a map $h \mapsto \varphi h$ is a well defined bounded operator from \mathcal{C} to $\vec{\mathcal{H}}$. For instance, understanding $\check{\tau}_\sigma$ as an operator, which multiplies vector fields by the scalar eikonal τ_σ , we have $\check{\tau}_\sigma \in \mathfrak{B}(\mathcal{C}; \vec{\mathcal{H}})$.

The following result is of crucial character for future application to inverse problems. By $\mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}}) \subset \mathfrak{B}(\mathcal{C}; \vec{\mathcal{H}})$ we denote the set of compact operators.

Lemma 1. *For any $\sigma \in \Gamma$ the relation $\varepsilon_\sigma - \check{\tau}_\sigma \in \mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}})$ holds.*

In the proof (see Appendix) we use the technique developed in [9].

3 Algebras

3.1 Handbook

We begin with minimal information about algebras: for detail see, e.g., [13], [14]. The abbreviations BA and CBA mean a Banach and commutative Banach algebra respectively.

1. A BA is a (complex or real) Banach space \mathcal{A} equipped with the multiplication operation ab satisfying $\|ab\| \leq \|a\| \|b\|$ $a, b \in \mathcal{A}$. We deal with algebras with the unit $e \in \mathcal{A}$: $ea = ae = a$.

A BA \mathcal{A} is called commutative if $ab = ba$ for all $a, b \in \mathcal{A}$. *Example:* the algebra $C(X)$ of continuous functions on a topological space X with the norm $\|a\| = \sup_X |a(\cdot)|$. The subalgebras of $C(X)$ are called function algebras.

A CBA is said to be uniform if $\|a^2\| = \|a\|^2$ holds. All function algebras are uniform.

2. Let \mathcal{A}' be the space of linear continuous functionals on a CBA \mathcal{A} . A functional $\delta \in \mathcal{A}'$ is called multiplicative if $\delta(ab) = \delta(a)\delta(b)$. *Example:* a Dirac measure $\delta_{x_0} \in C'(X)$: $\delta_{x_0}(a) = a(x_0)$ ($x_0 \in X$). Each multiplicative functional is of the norm 1.

The set of multiplicative functionals endowed with $*$ -weak topology (in \mathcal{A}') is called a *spectrum* of \mathcal{A} and denoted by $\widehat{\mathcal{A}}$. A spectrum is a compact Hausdorff space.

3. The *Gelfand transform* acts from a CBA \mathcal{A} to $C(\widehat{\mathcal{A}})$ by the rule $G: a \mapsto a(\cdot)$, $a(\delta) := \delta(a)$, $\delta \in \widehat{\mathcal{A}}$. It represents \mathcal{A} as a function algebra. The passage from \mathcal{A} to $G\mathcal{A} \subset C(\widehat{\mathcal{A}})$ is referred to as a geometrization of \mathcal{A} .

Theorem 1. *(I.M.Gelfand) If \mathcal{A} is a uniform CBA, then G is an isometric isomorphism from \mathcal{A} onto $G\mathcal{A}$, i.e., $G(\alpha a + \beta b + cd) = \alpha Ga + \beta Gb + Gc Gd$ and $\|Ga\|_{C(\widehat{\mathcal{A}})} = \|a\|_{\mathcal{A}}$ holds for all $a, b, c, d \in \mathcal{A}$ and numbers α, β .*

4. If two CBA \mathcal{A} and \mathcal{B} are isometrically isomorphic (we write $\mathcal{A} \stackrel{\text{isom}}{=} \mathcal{B}$) via an isometry j , then the dual isometry $j^* : \mathcal{B}' \rightarrow \mathcal{A}'$ provides a homeomorphism of their spectra: $j^* \widehat{\mathcal{B}} = \widehat{\mathcal{A}}$. Also, one has $G\mathcal{A} \stackrel{\text{isom}}{=} G\mathcal{B}$ via the map $j_{\sharp} : Ga \mapsto (Ga) \circ j^*$.

5. Let $\mathcal{A}(X) \subset C(X)$ be a closed function algebra. For each $x_0 \in X$, the Dirac measure δ_{x_0} belongs to $\widehat{\mathcal{A}(X)}$. Therefore, the map $x_0 \mapsto \delta_{x_0}$ provides a canonical embedding $X \subset \widehat{\mathcal{A}(X)}$.

If X is a compact Hausdorff space, then the Dirac measures exhaust the spectrum of $C(X)$, whereas the map $x_0 \mapsto \delta_{x_0}$ provides a canonical homeomorphism from X onto $\widehat{C(X)}$ (we write $X \stackrel{\text{hom}}{=} \widehat{C(X)}$). Also, one has $C(X) \stackrel{\text{isom}}{=} GC(X)$.

The trick, which is used in inverse problems for reconstruction of manifolds, is the following. Assume that we are given with an "abstract" CBA \mathfrak{A} , which is known to be isometrically isomorphic to $C(X)$, but neither the (compact Hausdorff) space X nor the isometry map is given. Then, by determining the spectrum $\widehat{\mathfrak{A}}$, we in fact recover the space X up to a homeomorphism: $X \stackrel{\text{hom}}{=} \widehat{C(X)} \stackrel{\text{hom}}{=} \widehat{\mathfrak{A}}$, whereas $C(X) \stackrel{\text{isom}}{=} GC(X) \stackrel{\text{isom}}{=} G\mathfrak{A}$ does hold. Thus, \mathfrak{A} provides a homeomorphic copy $\widehat{\mathfrak{A}}$ of the space X and a concrete isometric copy $C(\widehat{\mathfrak{A}})$ of the algebra $C(X)$.

6. A C^* -algebra is a BA endowed with an involution $(*)$ satisfying $(\alpha a + \beta b + cd)^* = \bar{\alpha} a^* + \bar{\beta} b^* + d^* c^*$ and $\|a^* a\| = \|a\|^2$ for all elements a, b, c, d and numbers α, β . In the real case, we have just $\bar{\alpha} = \alpha$. *Example:* the algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators in a Hilbert space \mathcal{H} with the operator norm and conjugation.

7. Let \mathcal{I} be a norm-closed two-side ideal in a C^* -algebra \mathcal{A} . Then $a \sim b \Leftrightarrow a - b \in \mathcal{I}$ is an equivalence. The factor \mathcal{A}/\mathcal{I} is endowed with a C^* -structure via the projection $\pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$ (element $a \mapsto$ equivalence class of a). Namely, one sets $\|\pi a\| := \inf\{\|b\|_{\mathcal{A}} \mid b \in \pi a\}$, $\alpha \pi a + \beta \pi b + \pi c \pi d := \pi(\alpha a + \beta b + cd)$, $(\pi a)^* := \pi(a^*)$ for elements $a, b, c, d \in \mathcal{A}$ and numbers α, β . Thus, π is a homomorphism of C^* -algebras.

3.2 Algebra \mathfrak{T}

Now let X be our Riemannian manifold Ω , which is definitely a compact Hausdorff space. Let $C(\Omega)$ be the CBA of real continuous functions on Ω .

The eikonals τ_{σ} generate $C(\Omega)$ in the following sense. For a Banach algebra \mathcal{A} and a subset $S \subset \mathcal{A}$, by $\vee S$ we denote the *minimal norm-closed subalgebra of \mathcal{A} , which contains S* . The following fact is a straightforward consequence of the separating properties of eikonals (Proposition 1) and the Stone-Weierstrass theorem [14].

Proposition 5. *The equality $\vee\{\tau_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} = C(\Omega)$ is valid.*

Recall that $\mathcal{H} = L_2(\Omega)$, $\mathfrak{B}(\mathcal{H})$ is the bounded operator algebra, $\check{\tau}_\sigma \in \mathfrak{B}(\mathcal{H})$ is the multiplication by τ_σ (see sec 2.2). Introduce the (sub)algebra

$$\mathfrak{T} := \vee\{\check{\tau}_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} \subset \mathfrak{B}(\mathcal{H}) \quad (3.1)$$

generated by scalar operator eikonals. As easily follows from (2.5) and Proposition 5, the map $C(\Omega) \ni \tau_\sigma \mapsto \check{\tau}_\sigma \in \mathfrak{T}$, which connects the generators, is extended to an isometric isomorphism of CBA $C(\Omega)$ and \mathfrak{T} . With regard to items 4, 5 of sec 3.1, the isometry implies

$$\Omega \stackrel{\text{hom}}{=} \widehat{C(\Omega)} \stackrel{\text{hom}}{=} \widehat{\mathfrak{T}}. \quad (3.2)$$

On reconstruction

Here we prepare a fragment of the procedure, which will be used for solving inverse problems.

Assume that we are given with a Hilbert space $\tilde{\mathcal{H}} = U\mathcal{H}$, where U is a unitary operator. Also assume that we know the map

$$\mathcal{R}(\Gamma) \times [0, T] \ni \{\sigma, s\} \mapsto \tilde{X}_\sigma^s \in \mathfrak{B}(\tilde{\mathcal{H}}) \quad (T > \text{diam } \Omega), \quad (3.3)$$

where $\tilde{X}_\sigma^s := UX_\sigma^s U^*$, but the operator $U : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ is *unknown*⁴. Show that this map determines the manifold Ω up to isometry. Indeed,

1. using the map, one can construct the operators

$$\tau'_\sigma := \int_0^T s d\tilde{X}_\sigma^s = \int_0^T s d[UX_\sigma^s U^*] \stackrel{(2.6)}{=} U\check{\tau}_\sigma U^*$$

2. determine the algebra $\tilde{\mathfrak{T}} = \vee\{\tau'_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} \subset \mathfrak{B}(\tilde{\mathcal{H}})$, which is isometric to $\mathfrak{T} \subset \mathfrak{B}(\mathcal{H})$ (via the *unknown* U)
3. applying the Gelfand transform to $\tilde{\mathfrak{T}}$, find its spectrum $\widehat{\tilde{\mathfrak{T}}} =: \tilde{\Omega}$ and the functions $\tilde{\tau}_\sigma := G\tau'_\sigma$ on $\tilde{\Omega}$.

Since $\tilde{\mathfrak{T}} \stackrel{\text{isom}}{=} \mathfrak{T}$, one has $\tilde{\Omega} := \widehat{\tilde{\mathfrak{T}}} \stackrel{\text{hom}}{=} \widehat{\mathfrak{T}} \stackrel{\text{hom}}{=} \Omega$ (see (3.2)). Hence, we get a homeomorphic copy $\tilde{\Omega}$ of the original Ω along with the images $\tilde{\tau}_\sigma$ of the original

⁴in other words, we are given with a representation of the projection family $\{X_\sigma^s\}_{\sigma \in \mathcal{R}(\Gamma)}$ in a space $\tilde{\mathcal{H}}$

eikonals τ_σ on Ω ⁵. Thus, we have a version of the map (2.3), which determines the copy $\tilde{\Omega}$ (see Proposition 2).

Summarizing, we arrive at the following assertion.

Proposition 6. *The map (3.3) determines the copy $\tilde{\Omega}$ and, hence, determines Ω up to isometry of Riemannian manifolds.*

Moreover, the procedure 1.– 3. provides the copy $\tilde{\Omega}$.

3.3 Algebra \mathfrak{E}

Recall that the eikonals ε_σ are introduced on a 3d-manifold Ω by (2.8).

An operator (sub)algebra

$$\mathfrak{E} := \vee\{\varepsilon_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} \subset \mathfrak{B}(\mathcal{C}) \quad (3.4)$$

is a "solenoidal" analog of the algebra \mathfrak{T} defined by (3.1). It is a real algebra generated by self-adjoint operators. As such, \mathfrak{E} is a C^* -algebra. In contrast to \mathfrak{T} , the algebra \mathfrak{E} is not commutative (see the remark below Proposition 4). However, this non-commutativity is weak in the following sense.

Let $\mathfrak{K} \subset \mathfrak{B}(\mathcal{C})$ be the ideal of compact operators. Denote $\mathfrak{K}[\mathfrak{E}] := \mathfrak{K} \cap \mathfrak{E}$ and $\dot{\mathfrak{E}} := \mathfrak{E}/\mathfrak{K}[\mathfrak{E}]$; let $\pi : \mathfrak{B}(\mathcal{C}) \rightarrow \mathfrak{B}(\mathcal{C})/\mathfrak{K}$ be the canonical projection. By (3.4), the latter factor-algebra is generated by the equivalence classes of eikonals:

$$\dot{\mathfrak{E}} := \vee\{\pi\varepsilon_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\}.$$

Recall that the eikonals τ_σ generate the algebra $C(\Omega)$: see Proposition 5.

Theorem 2. *$\dot{\mathfrak{E}}$ is a commutative C^* -algebra. The map*

$$C(\Omega) \ni \tau_\sigma \mapsto \pi\varepsilon_\sigma \in \dot{\mathfrak{E}} \quad (\sigma \in \mathcal{R}(\Gamma)),$$

which relates the generators, can be extended to an isometric isomorphism from $C(\Omega)$ onto $\dot{\mathfrak{E}}$.

Proof. Define a map

$$\dot{\pi} : C(\Omega) \rightarrow \mathfrak{B}(\mathcal{C})/\mathfrak{K}$$

in the following way. Let Y be the projection on \mathcal{C} acting in $\vec{\mathcal{H}}$. With a function $f \in C(\Omega)$ we associate an operator $Y[f] \in \mathfrak{B}(\mathcal{C})$ acting by

$$Y[f]y := Y(fy), \quad y \in \mathcal{C}.$$

⁵by construction, $\tilde{\tau}_\sigma$ turns out to be a pull-back function of τ_σ via the homeomorphism $\tilde{\Omega} \rightarrow \Omega$

Now, define

$$\tilde{\pi}(f) := \pi(Y[f]).$$

For $f \in C(\Omega)$ we denote by \check{f} the operator in $\vec{\mathcal{H}}$, which multiplies fields by f . The following two Lemmas are proved in Appendix.

Lemma 2. *For any $f \in C(\Omega)$ we have*

$$\check{f} - Y[f] \in \mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}}).$$

Lemma 3. *The mapping $\tilde{\pi}$ is an injective homomorphism of C^* -algebras.*

To prove Theorem 2 it suffices to show that the map $\tilde{\pi}$ is an extension of the map $\tau_\sigma \mapsto \pi\varepsilon_\sigma$. Toward this end, let us show that $\varepsilon_\sigma - Y[\tau_\sigma] \in \mathfrak{K}$. Indeed, we have

$$\varepsilon_\sigma - Y[\tau_\sigma] = \varepsilon_\sigma - \check{\tau}_\sigma + \check{\tau}_\sigma - Y[\tau_\sigma]$$

and, due to Lemmas 1 and 2, there is a sum of two compact operators from $\mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}})$ in the right hand side. Now Theorem 2 follows from Lemma 3 and the fact that algebra \mathfrak{E} is generated by elements $\pi\varepsilon_\sigma$. \square

With regard to items 4, 5 of sec 3.1, the relation $C(\Omega) \stackrel{\text{isom}}{=} \mathfrak{E}$ established by Theorem 2 implies

$$\Omega \stackrel{\text{hom}}{=} \widehat{C(\Omega)} \stackrel{\text{hom}}{=} \widehat{\mathfrak{E}}. \quad (3.5)$$

Remark Examples, in which factorization eliminates noncommutativity, are well known. For instance, let X be a compact smooth manifold (without boundary) and let $\mathfrak{A} \subset \mathfrak{B}(L_2(X))$ be a C^* -algebra generated by a certain class of pseudo-differential operators of order 0. Then the factor-algebra $\mathfrak{A}/\mathfrak{K}$ is commutative and isomorphic to the algebra of continuous functions on the cosphere bundle of X (see [15]).

On reconstruction

Here we provide an analog of the procedure described in sec 3.2. This analog is relevant to inverse problems of electrodynamics. Recall that Y_σ^s is the projection in \mathcal{C} onto the subspace $\mathcal{C}\langle\Omega^s[\sigma]\rangle$.

Assume that we are given with a Hilbert space $\tilde{\mathcal{C}} = U\mathcal{C}$, where U is a unitary operator. Also assume that we know the map

$$\mathcal{R}(\Gamma) \times [0, T] \ni \{\sigma, s\} \mapsto \tilde{Y}_\sigma^s \in \mathfrak{B}(\tilde{\mathcal{C}}) \quad (T > \text{diam } \Omega), \quad (3.6)$$

where $\tilde{Y}_\sigma^s := UY_\sigma^sU^*$, but the operator $U : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ is *unknown*. Show that this map determines the manifold Ω up to isometry. Indeed,

1. using the map, one can construct the operators

$$\varepsilon'_\sigma := \int_0^T s d\tilde{Y}_\sigma^s = \int_0^T s d[UY_\sigma^s U^*] \stackrel{(2.8)}{=} U\tilde{\varepsilon}_\sigma U^*$$

2. determine the algebra $\mathfrak{E}' = \vee\{\varepsilon'_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} \subset \mathfrak{B}(\tilde{\mathcal{C}})$, which is isometric to $\mathfrak{E} \subset \mathfrak{B}(\mathcal{C})$ (via *unknown* U)
3. construct the factor-algebra $\tilde{\mathfrak{E}} := \mathfrak{E}'/\mathfrak{K}[\mathfrak{E}']$ over the compact operator ideal in \mathfrak{E}' . By construction, one has $\tilde{\mathfrak{E}} \stackrel{\text{isom}}{=} \mathfrak{E}/\mathfrak{K}[\mathfrak{E}] =: \hat{\mathfrak{E}}$.
4. applying the Gelfand transform to $\tilde{\mathfrak{E}}$, find its spectrum $\tilde{\Omega} =: \hat{\tilde{\mathfrak{E}}}$ and the functions $\tilde{\tau}_\sigma := G\pi\varepsilon'_\sigma$ on $\tilde{\Omega}$.

Since $\tilde{\mathfrak{E}} \stackrel{\text{isom}}{=} \hat{\mathfrak{E}}$, one has

$$\tilde{\Omega} := \hat{\tilde{\mathfrak{E}}} \stackrel{\text{hom}}{=} \hat{\mathfrak{E}} \stackrel{\text{hom}}{=} \Omega$$

(see (3.5)). So, we get a homeomorphic copy $\tilde{\Omega}$ of the original Ω along with the images $\tilde{\tau}_\sigma$ of the original eikonals τ_σ on Ω . Thus, we have a version of the map (2.3). This map determines the Riemannian structure on $\tilde{\Omega}$, which turns it into an isometric copy of Ω (see Proposition 2).

Summarizing, we arrive at the following.

Proposition 7. *The map (3.6) determines the copy $\tilde{\Omega}$ and, hence, determines Ω up to isometry of Riemannian manifolds.*

Moreover, the procedure 1.– 4. enables one to construct the copy $\tilde{\Omega}$. This procedure differs from its scalar analog by one additional step that is factorization.

4 Inverse problems

4.1 Acoustical system

With the manifold Ω one associates a dynamical system α^T of the form

$$u_{tt} - \Delta u = 0 \quad \text{in } (\Omega \setminus \Gamma) \times (0, T) \quad (4.1)$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \Omega \quad (4.2)$$

$$u = f \quad \text{on } \Gamma \times [0, T], \quad (4.3)$$

where Δ is the (scalar) Beltrami–Laplace operator, $t = T > 0$ is a final time, f is a *boundary control*, $u = u^f(x, t)$ is a solution. For controls of the smooth class

$$\mathcal{M}^T := \{f \in C^\infty(\Gamma \times [0, T]) \mid \text{supp } f \subset \Gamma \times (0, T)\}$$

problem (4.1)–(4.3) has a unique classical (smooth) solution u^f . Note that the condition on $\text{supp } f$ means that f vanishes near $t = 0$.

From the physical viewpoint, u^f can be interpreted as an acoustical *wave*, which is initiated by the boundary sound source f and propagates into a domain Ω filled with an inhomogeneous medium.

Attributes

- The space of controls $\mathcal{F}^T := L_2(\Gamma \times [0, T])$ is said to be an *outer space* of the system α^T . The smooth class \mathcal{M}^T is dense in \mathcal{F}^T .

The outer space contains the subspaces

$$\mathcal{F}_\sigma^{T,s} := \{f \in \mathcal{F}^T \mid \text{supp } f \subset \sigma \times [T - s, T]\}, \quad \sigma \in \mathcal{R}(\Gamma).$$

Such a subspace consists of controls, which are located on σ and switched on with delay $T - s$ (the value s is an action time).

- An *inner space* of the system is $\mathcal{H} = L_2(\Omega)$. The waves $u^f(\cdot, t)$ are time dependent elements of \mathcal{H} .
- In the system α^T , the input \mapsto state correspondence is realized by a *control operator* $W^T : \mathcal{F}^T \rightarrow \mathcal{H}$, $\text{Dom } W^T = \mathcal{M}^T$

$$W^T f := u^f(\cdot, T).$$

A specifics of the system governed by the *scalar* wave equation (4.1) is that W^T is a bounded operator. Therefore one can extend it from \mathcal{M}^T onto \mathcal{F}^T by continuity that we assume to be done.

- The input \mapsto output map is represented by a *response operator* $R^T : \mathcal{F}^T \rightarrow \mathcal{F}^T$, $\text{Dom } R^T = \mathcal{M}^T$,

$$R^T f := \frac{\partial u^f}{\partial \nu} \Big|_{\Gamma \times [0, T]},$$

where $\nu = \nu(\gamma)$ is an outward normal at $\gamma \in \Gamma$.

The following evident fact was already mentioned in Introduction.

Proposition 8. *If two Riemannian manifolds have the mutual boundary and are isometric (the isometry being identity at the boundary), then their (acoustical) response operators coincide. In particular, for the manifold Ω and its copy $\tilde{\Omega}$ one has $R^{2T} = \tilde{R}^{2T}$ for any $T > 0$.*

- A *connecting operator* $C^T : \mathcal{F}^T \rightarrow \mathcal{F}^T$ is defined by

$$C^T := (W^T)^* W^T. \quad (4.4)$$

By the definition, we have

$$(C^T f, g)_{\mathcal{F}^T} = (W^T f, W^T g)_{\mathcal{H}} = (u^f(\cdot, T), u^g(\cdot, T))_{\mathcal{H}},$$

i.e., C^T connects the Hilbert metrics of the outer and inner spaces. A significant fact is that the connecting operator is determined by the response operator of the system α^{2T} through an explicit formula

$$C^T = \frac{1}{2} (S^T)^* R^{2T} J^{2T} S^T, \quad (4.5)$$

where the map $S^T : \mathcal{F}^T \rightarrow \mathcal{F}^{2T}$ extends the controls from $\Gamma \times [0, T]$ to $\Gamma \times [0, 2T]$ as odd functions (of time t) with respect to $t = T$; $J^{2T} : \mathcal{F}^{2T} \rightarrow \mathcal{F}^{2T}$ is an integration: $(J^{2T} f)(\cdot, t) = \int_0^t f(\cdot, s) ds$ (see [1], [3]).

Controllability

The set $\mathcal{U}_\sigma^s := \{u^f(\cdot, s) \mid f \in \mathcal{F}_\sigma^T\}$ is said to be *reachable* (from σ , at the moment $t = s$).

The operator Δ , which governs the evolution of the system α^T , does not depend on time. By this, a time delay of controls implies the same delay of the waves. As a result, one has

$$\mathcal{U}_\sigma^s = W^T \mathcal{F}_\sigma^{T,s}, \quad 0 \leq s \leq T.$$

Problem (4.1)–(4.3) is hyperbolic and the finiteness of domains of influence does hold for its solutions: for the delayed controls one has

$$\text{supp } u^f(\cdot, T) \subset \overline{\Omega^s[\sigma]}, \quad f \in \mathcal{F}_\sigma^{T,s}. \quad (4.6)$$

The latter means that in the system α^T the waves propagate with the unit velocity. As a result, the embedding $\mathcal{U}_\sigma^s \subset \mathcal{H}(\Omega^s[\sigma])$ is valid. The character of this embedding is of principal importance: it turns out to be *dense*. The following result is based upon the fundamental Holmgren–John–Tataru uniqueness theorem (see [1], [3] for detail).

Proposition 9. *For any $s > 0$ and $\sigma \in \mathcal{R}(\Gamma)$, the relation $\overline{\mathcal{U}_\sigma^s} = \mathcal{H}(\Omega^s[\sigma])$ is valid (the closure in \mathcal{H}). In particular, for $s = T > \text{diam } \Omega$ one has $\overline{\mathcal{U}_\sigma^T} = \mathcal{H}$.*

In control theory this property is referred to as a *local approximate boundary controllability* of the system α^T . It shows that the reachable sets are rich enough: any function supported in the neighborhood $\Omega^s[\sigma]$ can be approximated (in \mathcal{H} -metric) by a wave $u^f(\cdot, T)$ by means of the proper choice of the control $f \in \mathcal{F}_\sigma^{T,s}$.

By P_σ^s we denote the projection in \mathcal{H} onto the reachable subspace $\overline{\mathcal{U}_\sigma^s}$ and call it a *wave projection*. Recall that X_σ^s is the projection in \mathcal{H} onto $\mathcal{H}\langle\Omega^s[\sigma]\rangle$, which cuts off functions onto the neighborhood $\Omega^s[\sigma]$. As a consequence of the Proposition 9 we obtain

$$P_\sigma^s = X_\sigma^s, \quad s > 0, \quad \sigma \in \mathcal{R}(\Gamma). \quad (4.7)$$

4.2 IP of acoustics

Setup

A dynamical inverse problem (IP) for the system (4.1)–(4.3) is set up as follows: *given for a fixed $T > \text{diam}\Omega$ the response operator R^{2T} , to recover the manifold Ω .*

A physical meaning of the condition $T > \text{diam}\Omega$ is that the waves u^f , which prospect the manifold from the parts σ of its boundary, need big enough time to fill the whole Ω : see (4.6) and (2.1).

As was clarified in Introduction, *to recover Ω* means to construct (via given R^{2T}) a Riemannian manifold, which has the same boundary Γ , and possesses the response operator, which is equal to R^{2T} . Speaking in advance, it will be shown that R^{2T} determines the copy $\tilde{\Omega}$. Thus, $\tilde{\Omega}$ provides the solution to the IP.

Model

As an operator connecting two Hilbert spaces, the control operator $W^T : \mathcal{F}^T \rightarrow \mathcal{H}$ can be represented in the form of a *polar decomposition*

$$W^T = \Phi^T |W^T|,$$

where

$$|W^T| := \left[(W^T)^* W^T \right]^{\frac{1}{2}} \stackrel{(4.4)}{=} (C^T)^{\frac{1}{2}}$$

and $\Phi^T : |W^T|f \mapsto W^T f$ is an isometry from $\text{Ran } |W^T| \subset \mathcal{F}^T$ onto $\text{Ran } W^T \subset \mathcal{H}$ (see, e.g., [7]). In what follows we assume that Φ^T is extended by continuity to an isometry from $\overline{\text{Ran } |W^T|}$ onto $\overline{\text{Ran } W^T}$.

Recall that $\mathcal{U}_\sigma^s := W^T \mathcal{F}_\sigma^{T,s}$ are the reachable sets of the system α^T and P_σ^s is the projection in \mathcal{H} onto $\overline{\mathcal{U}_\sigma^s}$.

Let us say the (sub)space $\tilde{\mathcal{H}} := \overline{\text{Ran } |W^T|} \subset \mathcal{F}^T$ to be a *model inner space*, $\tilde{\mathcal{U}}_\sigma^s := |W^T| \mathcal{F}_\sigma^{T,s} \subset \tilde{\mathcal{H}}$ a *model reachable set*. By \tilde{P}_σ^s we denote the projection in $\tilde{\mathcal{H}}$ onto $\tilde{\mathcal{U}}_\sigma^s$ and call it a *model wave projection*.

The model and original objects are related through the isometry Φ^T . In particular, the definitions imply $\Phi^T \tilde{P}_\sigma^s = P_\sigma^s \Phi^T$.

Now let $T > \text{diam } \Omega$, so that $\Omega^T[\sigma] = \Omega$ holds for any σ . By Proposition 9, one has $\overline{\text{Ran } W^T} = \mathcal{H}$. By this, the isometry Φ^T turns out to be a unitary operator from $\tilde{\mathcal{H}}$ onto \mathcal{H} . Its inverse $U := (\Phi^T)^*$ maps \mathcal{H} onto $\tilde{\mathcal{H}}$ isometrically and $UP_\sigma^s = \tilde{P}_\sigma^s U$ holds.

Let $\tilde{X}_\sigma^s := UX_\sigma^s U^*$ be the image (in $\tilde{\mathcal{H}}$) of the cutting off projection. The property (4.7) implies

$$\tilde{P}_\sigma^s = \tilde{X}_\sigma^s, \quad s > 0, \quad \sigma \in \mathcal{R}(\Gamma). \quad (4.8)$$

Solving IP

It suffices to show that the operator R^{2T} determines the copy $\tilde{\Omega}$. The procedure is the following.

1. Find the connecting operator by (4.5). Determine the operator $|W^T| = (C^T)^{\frac{1}{2}}$ and the subspace $\tilde{\mathcal{H}} = \overline{\text{Ran } |W^T|} \subset \mathcal{F}^T$.
2. Fix a $\sigma \in \mathcal{R}(\Gamma)$ and $s \in (0, T]$. In $\tilde{\mathcal{H}}$ recover the model reachable set $\tilde{\mathcal{U}}_\sigma^s = |W^T| \mathcal{F}_\sigma^{T,s} \subset \tilde{\mathcal{H}}$ and determine the corresponding projection \tilde{P}_σ^s . By (4.8), we get the projection \tilde{X}_σ^s . Thus, the map (3.3) is at our disposal.
3. By Proposition 6, this map determines the copy $\tilde{\Omega}$. Its response operator \tilde{R}^{2T} coincides with the given R^{2T} : see Proposition 8.

The acoustical IP is solved.

4.3 Maxwell system

Here Ω is a smooth compact oriented Riemannian 3d-manifold.

Propagation of electromagnetic waves in a curved space is described by the dynamical Maxwell system α_M^T

$$e_t = \text{curl } h, \quad h_t = -\text{curl } e \quad \text{in } (\Omega \setminus \Gamma) \times (0, T) \quad (4.9)$$

$$e|_{t=0} = 0, \quad h|_{t=0} = 0 \quad \text{in } \Omega \quad (4.10)$$

$$e_\theta = f \quad \text{on } \Gamma \times [0, T], \quad (4.11)$$

where $e_\theta := e - e \cdot \nu \nu$ is a tangent component of e at the boundary, f is a time-dependent tangent field on Γ (*boundary control*), e and h are the electric and magnetic components of the solution. For controls of the smooth class

$$\mathcal{M}^T := \left\{ f \in \vec{C}^\infty(\Gamma \times [0, T]) \mid \nu \cdot f = 0, \text{ supp } f \subset \Gamma \times (0, T] \right\},$$

problem (4.9)-(4.11) has a unique classical smooth solution $\{e^f(x, t), h^f(x, t)\}$. Note that the condition on $\text{supp } f$ means that f vanishes near $t = 0$.

Since a divergence is an integral of motion of the Maxwell system, one has

$$\text{div } e^f(\cdot, t) = 0, \quad \text{div } h^f(\cdot, t) = 0, \quad t \geq 0.$$

Attributes

- An *outer space* of the system α_M^T is the space

$$\mathcal{F}^T := \left\{ f \in \vec{L}_2(\Gamma \times [0, T]) \mid \nu \cdot f = 0 \right\}.$$

The smooth class \mathcal{M}^T is dense in \mathcal{F}^T .

The outer space contains the subspaces

$$\mathcal{F}_\sigma^{T,s} := \{ f \in \mathcal{F}^T \mid \text{supp } f \subset \sigma \times [T - s, T] \}, \quad \sigma \in \mathcal{R}(\Gamma)$$

of controls, which are located on σ and switched on with delay $T - s$ (the value s is an action time).

- An *inner space* of the system is the space $\mathcal{C} \oplus \mathcal{C}$. By (4.9), the solutions $\{e^f(\cdot, t), h^f(\cdot, t)\}$ are time dependent elements of this space. Also, we select its electric part $\mathcal{C} \oplus \{0\} \ni e^f(\cdot, t)$.

- The input \mapsto state correspondence is realized by a *control operator* $W_M^T : \mathcal{F}^T \rightarrow \mathcal{C} \oplus \mathcal{C}$, $\text{Dom } W_M^T = \mathcal{M}^T$, $W_M^T f := \{e^f(\cdot, T), h^f(\cdot, T)\}$. Its electric part is $W^T : \mathcal{F}^T \rightarrow \mathcal{C}$,

$$W^T : f \mapsto e^f(\cdot, T).$$

In contrast to the acoustical (scalar) system, W_M^T and W^T are unbounded (but closable) operators.

A reason to select an electric part of the system α_M^T is that it is the electric component, which is controlled at the boundary: see (4.11). By this, e^f and h^f are not quite independent. Moreover, for $T < \inf\{r > 0 \mid \Omega^r[\Gamma] = \Omega\}$ the operator W^T is injective and, hence, $e^f(\cdot, T)$ determines $h^f(\cdot, T)$ [3], [5].

- The input \mapsto output map of the system α_M^T is represented by a *response operator* $R^T : \mathcal{F}^T \rightarrow \mathcal{F}^T$, $\text{Dom } R^T = \mathcal{M}^T$,

$$R^T f := \nu \wedge h^f \Big|_{\Gamma \times [0, T]}.$$

The following fact is quite evident.

Proposition 10. *If two Riemannian manifolds have the mutual boundary and are isometric (the isometry being identity at the boundary), then their Maxwell response operators coincide. In particular, for the manifold Ω and its canonical copy $\tilde{\Omega}$ one has $R^{2T} = \tilde{R}^{2T}$ for any $T > 0$.*

• An electric connecting operator $C^T : \mathcal{F}^T \rightarrow \mathcal{F}^T$ is introduced via a connecting form c^T , $\text{Dom } c^T = \mathcal{M}^T \times \mathcal{M}^T$,

$$c^T[f, g] := (e^f(\cdot, T), e^g(\cdot, T))_c = (W^T f, W^T g)_c .$$

It is a Hermitian nonnegative bilinear form. As such, it is closable, the closure \bar{c}^T being defined on $\mathcal{N}^T \times \mathcal{N}^T$, where \mathcal{N}^T is a linear in \mathcal{F}^T , $\mathcal{N}^T \supset \mathcal{M}^T$. The form \bar{c}^T determines a unique self-adjoint operator C^T by the relation

$$(C^T f, g)_{\mathcal{F}^T} = \bar{c}^T[f, g], \quad f \in \text{Dom } C^T, g \in \mathcal{N}^T$$

(see, e.g., [7]). In fact, to close c^T is to close W^T , and one has $\mathcal{N}^T = \text{Dom } \bar{W}^T = \text{Dom } (C^T)^{\frac{1}{2}}$. Hence, the knowledge of c^T enables one to extend W^T from \mathcal{M}^T to \mathcal{N}^T . In what follows this extension (closure) is assumed to be done and denoted by the same symbol W^T . The images $W^T f$ for $f \in \mathcal{N}^T$ are regarded as the generalized solutions $e^f(\cdot, T)$.

As a result, one has the relations

$$\bar{c}^T[f, g] = \left((C^T)^{\frac{1}{2}} f, (C^T)^{\frac{1}{2}} g \right)_{\mathcal{F}^T} = (W^T f, W^T g)_c, \quad f, g \in \mathcal{N}^T. \quad (4.12)$$

A key fact is that the connecting form is determined by the response operator of the system α_M^{2T} through an explicit formula

$$c^T[f, g] = (2^{-1}(S^T)^* R^{2T} J^{2T} S^T f, g)_{\mathcal{F}^T}, \quad f, g \in \mathcal{M}^T, \quad (4.13)$$

where the map $S^T : \mathcal{F}^T \rightarrow \mathcal{F}^{2T}$ extends the controls from $\Gamma \times [0, T]$ to $\Gamma \times [0, 2T]$ as odd functions (of time t) with respect to $t = T$; $J^{2T} : \mathcal{F}^{2T} \rightarrow \mathcal{F}^{2T}$ is an integration: $(J^{2T} f)(\cdot, t) = \int_0^t f(\cdot, s) ds$ (see [3]).

Resuming the aforesaid, we can claim that R^{2T} determines the operator $(C^T)^{\frac{1}{2}}$ by the scheme

$$R^{2T} \stackrel{(4.13)}{\Rightarrow} c^T \Rightarrow \bar{c}^T \Rightarrow C^T \Rightarrow (C^T)^{\frac{1}{2}}. \quad (4.14)$$

Controllability

The set $\mathcal{E}_\sigma^s := \{e^f(\cdot, s) \mid f \in \mathcal{F}_\sigma^T \cap \mathcal{M}^T\}$ is said to be *reachable* (from σ , at the moment $t = s$).

The operators curl, which govern the evolution of the system α_M^T , does not depend on time. By this, a time delay of controls implies the same delay of the waves. As a result, one can represent

$$\mathcal{E}_\sigma^s = W^T [\mathcal{F}_\sigma^{T,s} \cap \mathcal{M}^T] .$$

The Maxwell system (4.9)–(4.11) obeys the finiteness of domains of influence principle: for the delayed controls one has

$$\text{supp } e^f(\cdot, T) \subset \overline{\Omega^s[\sigma]}, \quad f \in [\mathcal{F}_\sigma^{T,s} \cap \mathcal{M}^T] . \quad (4.15)$$

The latter means that electromagnetic waves propagate with the unit velocity. As a consequence, the embedding $\mathcal{E}_\sigma^s \subset \mathcal{C}\langle\Omega^s[\sigma]\rangle$ is valid. Moreover, this embedding is *dense*. This fact is derived from a vectorial version of the Holmgren–John–Tataru uniqueness theorem (see [3] for detail).

Proposition 11. *For any $s > 0$ and $\sigma \in \mathcal{R}(\Gamma)$, the relation $\overline{\mathcal{E}_\sigma^s} = \mathcal{C}\langle\Omega^s[\sigma]\rangle$ is valid (the closure in \mathcal{C}). In particular, for $s = T > \text{diam } \Omega$ one has $\overline{\mathcal{E}_\sigma^T} = \mathcal{C}$.*

This property is interpreted as a *local approximate boundary controllability* of the electric subsystem of α_M^T .

By E_σ^s we denote the projection in \mathcal{C} onto the reachable subspace $\overline{\mathcal{E}_\sigma^s}$ and call it a *wave projection*. Recall that Y_σ^s is the projection in \mathcal{C} onto $\mathcal{C}\langle\Omega^s[\sigma]\rangle$. As a consequence of the Proposition 11 we obtain

$$E_\sigma^s = Y_\sigma^s, \quad s > 0, \quad \sigma \in \mathcal{R}(\Gamma) . \quad (4.16)$$

4.4 IP of electrodynamics

Setup

A dynamical inverse problem (IP) for the system (4.9)–(4.11) is set up as follows: *given for a fixed $T > \text{diam } \Omega$ the response operator R^{2T} , to recover the manifold Ω .*

A physical meaning of the condition $T > \text{diam } \Omega$ is the same as in the acoustical case: the electromagnetic waves need big enough time to prospect the whole Ω : see (4.15) and (2.1).

As before, *to recover Ω* means to construct (via given R^{2T}) a Riemannian manifold, which has the same boundary Γ , and possesses the response operator, which is equal to R^{2T} . As well as in the scalar case, we will show that R^{2T} determines the copy $\tilde{\Omega}$. Thus, $\tilde{\Omega}$ will provide the solution to the IP.

Model

Representing the (closed) control operator $W^T : \mathcal{F}^T \rightarrow \mathcal{C}$ in the polar decomposition form, one has $W^T = \Psi^T |W^T|$, where $|W^T| := [(W^T)^* W^T]^{\frac{1}{2}}$ and $\Psi^T : |W^T|f \mapsto W^T f$ is an isometry from $\text{Ran } |W^T| \subset \mathcal{F}^T$ onto $\text{Ran } W^T \subset \mathcal{C}$ [7]. In what follows Ψ^T is assumed to be extended by continuity to an isometry from $\overline{\text{Ran } |W^T|}$ onto $\overline{\text{Ran } W^T}$. Also note that (4.12) implies $|W^T| = (C^T)^{\frac{1}{2}}$.

Recall that $\mathcal{E}_\sigma^s := W^T[\mathcal{F}_\sigma^{T,s} \cap \mathcal{M}^T]$ is an electric reachable set and E_σ^s is the (wave) projection in \mathcal{C} onto $\overline{\mathcal{E}_\sigma^s}$.

Let us say the (sub)space $\tilde{\mathcal{C}} := \overline{\text{Ran } |W^T|} \subset \mathcal{F}^T$ to be a *model inner space*, $\tilde{\mathcal{E}}_\sigma^s := |W^T|[\mathcal{F}_\sigma^{T,s} \cap \mathcal{M}^T] \subset \tilde{\mathcal{C}}$ the *model reachable sets*. By \tilde{E}_σ^s we denote the projection in $\tilde{\mathcal{C}}$ onto $\overline{\tilde{\mathcal{E}}_\sigma^s}$ and call it a *model wave projection*.

The model and original objects are related through the isometry Ψ^T . In particular, the definitions imply $\Psi^T \tilde{E}_\sigma^s = E_\sigma^s \Psi^T$.

Now, let $T > \text{diam } \Omega$. By Proposition 11, one has $\overline{\text{Ran } W^T} = \mathcal{C}$. Therefore the isometry Ψ^T turns out to be a unitary operator from $\tilde{\mathcal{C}}$ onto \mathcal{C} . Its inverse $U := (\Psi^T)^*$ maps \mathcal{C} onto $\tilde{\mathcal{C}}$ isometrically and $UE_\sigma^s = \tilde{E}_\sigma^s U$ holds.

Let $\tilde{Y}_\sigma^s := UY_\sigma^s U^*$. The property (4.16) implies

$$\tilde{E}_\sigma^s = \tilde{Y}_\sigma^s, \quad s > 0, \quad \sigma \in \mathcal{R}(\Gamma). \quad (4.17)$$

Solving IP

Let us show that the operator R^{2T} determines the copy $\tilde{\Omega}$.

1. Find the connecting form c^T by (4.13). Determine the model control operator $|W^T| = (C^T)^{\frac{1}{2}}$ (see (4.14)) and the model inner space $\tilde{\mathcal{C}} = \overline{\text{Ran } |W^T|} \subset \mathcal{F}^T$.
2. Fix a $\sigma \in \mathcal{R}(\Gamma)$ and $s \in (0, T)$. In $\tilde{\mathcal{C}}$ recover the model reachable set $\tilde{\mathcal{E}}_\sigma^s = |W^T|[\mathcal{F}_\sigma^{T,s} \cap \mathcal{M}^T] \subset \tilde{\mathcal{C}}$ and determine the corresponding projection \tilde{E}_σ^s . By (4.17), we get the projection \tilde{Y}_σ^s . Thus, the map (3.6) is at our disposal.
3. By Proposition 7, this map determines the copy $\tilde{\Omega}$. Its Maxwell response operator \tilde{R}^{2T} coincides with the given R^{2T} (see Proposition 10).

The IP of electrodynamics is solved.

4.5 Comments

- In this paper, the condition $T > \text{diam}\Omega$ is imposed for the sake of simplicity. It provides the embedding $\check{\tau}_\sigma C(\Omega) \subset C(\Omega)$, which is convenient just by technical reasons. However, there is a *time-optimal* setup of the reconstruction problem, which takes into account a local character of dependence of the acoustical and Maxwell response operators on a near-boundary part of the manifold. Namely, by the finiteness of the domain of influence, for an arbitrary fixed $T > 0$ the operator R^{2T} is determined by the submanifold $\Omega^T[\Gamma]$ (does not depend on the part $\Omega \setminus \Omega^T[\Gamma]$). Therefore, the natural setup is: *given for a fixed $T > 0$ the operator R^{2T} , to recover $\Omega^T[\Gamma]$* . In such a stronger form the problem is solved in [3] and [6].
- In reconstruction via a spectral triple $\{\mathcal{A}, \mathcal{H}, \mathcal{D}\}$ (see [8], [16]), the algebra provides a topological space (that is $\widehat{\mathcal{A}}$), whereas the operator \mathcal{D} encodes a Riemannian metric on $\widehat{\mathcal{A}}$. The metric is recovered (via \mathcal{D}) by means of the *Connes distance formula*. In our scheme, the object responsible for the metric is a selected family of generators of the algebra (that is the eikonals).
- Dealing with the reconstruction problem for a graph, one can introduce the straightforward analog of the eikonal algebra \mathfrak{T} . However, this algebra turns out to be noncommutative. By this, we have to deal with its *Jacobson spectrum* $\widehat{\mathfrak{T}}$, which is the topologized set of the primitive ideals of \mathfrak{T} [13]. As the known examples show, its structure is related with geometry of the graph but the relation is of rather implicit character. This challenging problem is open yet. An intriguing fact is that in some examples the space $\widehat{\mathfrak{T}}$ is non-Hausdorff. It contains "clusters", which are the groups of nonseparable points. Presumably, the clusters of $\widehat{\mathfrak{T}}$ are related with interior vertices of the graph.

5 Appendix

Here we give proof of Lemmas 1, 2, 3.

The standard operations on vector fields on the manifold ∇ , div , curl are understood in the generalized sense. Here are standard formulas of vector analysis:

$$\text{div}(\varphi u) = \nabla\varphi \cdot u + \varphi \text{div} u, \quad (5.1)$$

$$\text{div}(u \wedge v) = \text{curl} u \cdot v - u \cdot \text{curl} v, \quad (5.2)$$

$$\text{curl}(\varphi u) = \nabla\varphi \wedge u + \varphi \text{curl} u. \quad (5.3)$$

In (5.1) and (5.3) a function φ is Lipschitz; a field u is locally integrable and its divergence is also locally integrable. In (5.2) we may suppose that u or v is Lipschitz, and the other field is locally integrable and has locally integrable curl.

5.1 Proof of Lemma 1

Let the field $z \in \vec{\mathcal{H}}$ satisfy $\text{curl } z \in \vec{\mathcal{H}}$. Following [12], we say that the field z satisfies the condition

$$z_\theta|_\Gamma = 0, \quad (5.4)$$

if for any field $v \in \vec{\mathcal{H}}$, such that $\text{curl } v \in \vec{\mathcal{H}}$, we have

$$(z, \text{curl } v)_\Omega = (\text{curl } z, v)_\Omega.$$

Here and further in this section $(\cdot, \cdot)_U$ and $\|\cdot\|_U$ means the inner product and the norm in $L_2(U)$ or $\vec{L}_2(U)$. It can be shown, that due to smoothness of the boundary Γ it suffices to check this condition only for $v \in \vec{C}^\infty(\Omega)$.

Introduce the space

$$F := \{u \in \vec{\mathcal{H}} : \text{div } u \in L_2(\Omega), \text{curl } u \in \vec{\mathcal{H}}, u_\theta|_\Gamma = 0\}$$

with the norm

$$\|u\|_F^2 := \|u\|_\Omega^2 + \|\text{div } u\|_\Omega^2 + \|\text{curl } u\|_\Omega^2.$$

The following result is valid for an $\Omega \subset \mathbb{R}^3$ (see [12], section 8.4) and can be easily generalized on a smooth manifold.

Theorem 3. *The embedding of the space F to $\vec{\mathcal{H}}$ is compact.*

Actually, the stronger fact holds true: the space F coincides with vector Sobolev space $\vec{H}^1(\Omega)$, which is compactly embedded to $\vec{\mathcal{H}}$. However, Theorem 3 will suffice for our purposes. Theorem 3 is used in spectral analysis of the Maxwell operator on compact manifolds (see, e.g., [10]).

Let us outline the scheme of the proof of Lemma 1. We obtain estimates for L_2 -norms of curl and divergence of the difference $\check{\tau}_\sigma u - \varepsilon_\sigma u$ by L_2 -norm of $u \in \mathcal{C}$ (inequalities (5.13), (5.15)), and establish the boundary condition (5.4) on Γ for this difference. This means that the field $\check{\tau}_\sigma u - \varepsilon_\sigma u$ belongs to F with the corresponding norm estimate, which implies that the operator $\check{\tau}_\sigma - \varepsilon_\sigma$ restricted to \mathcal{C} is compact (by compactness of the embedding $F \subset \vec{\mathcal{H}}$).

In what follows we consider X_σ^s as the projections in $\vec{\mathcal{H}}$, which cut off fields on $\Omega^s[\sigma]$.

We will use the following relations, which are valid for any $T > 0$:

$$\begin{aligned} \int_{[0,T]} s dX_\sigma^s &= TX_\sigma^T - \int_{[0,T]} X_\sigma^s ds, \\ \int_{[0,T]} s dY_\sigma^s &= TY_\sigma^T - \int_{[0,T]} Y_\sigma^s ds. \end{aligned}$$

Along with (2.6) this implies that for $T > \text{diam } \Omega$ we have

$$(\varepsilon_\sigma - \tilde{\tau}_\sigma) y = \left(\int_{[0,T]} (X_\sigma^s - Y_\sigma^s) ds \right) y, \quad y \in \mathcal{C}. \quad (5.5)$$

To prove Lemma 1 we need to establish a compactness of the operator, which acts from \mathcal{C} to $\vec{\mathcal{H}}$ by

$$K_\sigma := \int_0^T (X_\sigma^\xi - Y_\sigma^\xi) d\xi$$

(this integral is the same for any $T > \text{diam } \Omega$). Define a family of operators acting from \mathcal{C} to $\vec{\mathcal{H}}$ by

$$K_\sigma^s := \int_0^s (X_\sigma^\xi - Y_\sigma^\xi) d\xi, \quad 0 \leq s < \infty.$$

One can easily check the following relation

$$\left(\int_0^s X_\sigma^\xi d\xi y \right) (x) = \max\{s - \tau_\sigma(x), 0\} y(x), \quad x \in \Omega. \quad (5.6)$$

Lemma 4. *Choose $\sigma \subset \Gamma$ and $s > 0$. Let a field $\beta \in \vec{\mathcal{H}}\langle \Omega^s[\sigma] \rangle$ be smooth in $\Omega^s[\sigma]$ (in particular, smooth on the boundary $\Omega^s[\sigma] \cap \Gamma$) and orthogonal to $\mathcal{C}\langle \Omega^s[\sigma] \rangle$. Then for any $z \in \vec{C}^\infty(\Omega)$ one has*

$$(\beta, K_\sigma^s \text{curl } z)_{\Omega^s[\sigma]} = (\beta, \nabla \tau_\sigma \wedge z)_{\Omega^s[\sigma]}.$$

Proof. Let $0 < s' < s$. By the absolute continuity of Lebesgue integral we have

$$(\beta, K_\sigma^{s'} \text{curl } z)_{\Omega^{s'}[\sigma]} \rightarrow (\beta, K_\sigma^s \text{curl } z)_{\Omega^s[\sigma]}, \quad s' \rightarrow s - 0. \quad (5.7)$$

As is evident, β is orthogonal to $\mathcal{C}\langle \Omega^\xi[\sigma] \rangle$ for $\xi \leq s$; therefore

$$\begin{aligned} (\beta, K_\sigma^{s'} \text{curl } z)_{\Omega^{s'}[\sigma]} &= \int_0^{s'} d\xi (\beta, (X_\sigma^\xi - Y_\sigma^\xi) \text{curl } z)_{\Omega^\xi[\sigma]} = \\ &= \int_0^{s'} d\xi (\beta, X_\sigma^\xi \text{curl } z)_{\Omega^\xi[\sigma]} \stackrel{(5.6)}{=} (\beta, (s' - \tau_\sigma) \text{curl } z)_{\Omega^{s'}[\sigma]} = \\ &= ((s' - \tau_\sigma) \beta, \text{curl } z)_{\Omega^{s'}[\sigma]}. \end{aligned}$$

Define a Lipschitz function h in Ω as follows

$$h(x) := \max\{s' - \tau_\sigma(x), 0\}$$

We have

$$((s' - \tau_\sigma) \beta, \operatorname{curl} z)_{\Omega^{s'[\sigma]}} = (h\beta, \operatorname{curl} z)_\Omega \quad (5.8)$$

(the field $h\beta$ is defined in Ω since h vanishes outside of $\Omega^{s'[\sigma]} \subset \Omega^s[\sigma]$). The field $h\beta$ is Lipschitz, as function h is Lipschitz, and the field β is smooth in the neighborhood of $\operatorname{supp} h$, so we can apply a formula of integration by parts to the right hand side in (5.8). Orthogonality of β to $\mathcal{C}\langle \Omega^s[\sigma] \rangle$ implies

$$\operatorname{curl} \beta|_{\Omega^s[\sigma]} = 0, \quad \beta_\theta|_{\Omega^s[\sigma] \cap \Gamma} = 0. \quad (5.9)$$

Due to the second equality we have $(h\beta)_\theta|_\Gamma = 0$. So the integral over Γ in integration by parts vanishes. Applying the first equality in (5.9) and formula (5.3), we obtain:

$$\begin{aligned} (h\beta, \operatorname{curl} z)_\Omega &= (\operatorname{curl}(h\beta), z)_\Omega = (\nabla h \wedge \beta, z)_\Omega = ((-\nabla \tau_\sigma) \wedge \beta, z)_{\Omega^{s'[\sigma]}} = \\ &= (\beta, \nabla \tau_\sigma \wedge z)_{\Omega^{s'[\sigma]}}. \end{aligned}$$

The latter term tends to $(\beta, \nabla \tau_\sigma \wedge z)_{\Omega^s[\sigma]}$ as $s' \rightarrow s$. Taking into account (5.7), we obtain the required equality. \square

Note that Lemma 4 holds true if $\Omega^s[\sigma] = \Omega$.

Lemma 5. *Let $\sigma \subset \Gamma$. For a field $z \in \vec{C}^\infty(\Omega)$ we have*

$$(K_\sigma \operatorname{curl} z, K_\sigma \operatorname{curl} z)_\Omega = 2(K_\sigma \operatorname{curl} z, \nabla \tau_\sigma \wedge z)_\Omega. \quad (5.10)$$

Proof. We have

$$\begin{aligned} (K_\sigma \operatorname{curl} z, K_\sigma \operatorname{curl} z)_\Omega &= \int_0^T ds ((X_\sigma^s - Y_\sigma^s) \operatorname{curl} z, K_\sigma \operatorname{curl} z)_\Omega = \\ &= \int_0^T ds \int_0^T d\xi ((X_\sigma^s - Y_\sigma^s) \operatorname{curl} z, (X_\sigma^\xi - Y_\sigma^\xi) \operatorname{curl} z)_\Omega = \\ &= 2 \int_0^T ds \int_0^s d\xi ((X_\sigma^s - Y_\sigma^s) \operatorname{curl} z, (X_\sigma^\xi - Y_\sigma^\xi) \operatorname{curl} z)_\Omega = \\ &= 2 \int_0^T ds ((X_\sigma^s - Y_\sigma^s) \operatorname{curl} z, K_\sigma^s \operatorname{curl} z)_{\Omega^s[\sigma]}. \end{aligned} \quad (5.11)$$

As is clear, the field $\beta := (X_\sigma^s - Y_\sigma^s) \operatorname{curl} z$ is orthogonal to $\mathcal{C}\langle \Omega^s[\sigma] \rangle$. Moreover, it is smooth in $\Omega^s[\sigma]$, since it is solenoidal and satisfies (5.9). So we can apply Lemma 4 to the integrand:

$$((X_\sigma^s - Y_\sigma^s) \operatorname{curl} z, K_\sigma^s \operatorname{curl} z)_{\Omega^s[\sigma]} = ((X_\sigma^s - Y_\sigma^s) \operatorname{curl} z, \nabla \tau_\sigma \wedge z)_{\Omega^s[\sigma]}.$$

Substituting this to (5.11), we obtain

$$\begin{aligned} (K_\sigma \operatorname{curl} z, K_\sigma \operatorname{curl} z)_\Omega &= 2 \int_0^T ds ((X_\sigma^s - Y_\sigma^s) \operatorname{curl} z, \nabla \tau_\sigma \wedge z)_{\Omega^s[\sigma]} = \\ &= 2 (K_\sigma \operatorname{curl} z, \nabla \tau_\sigma \wedge z)_\Omega. \end{aligned}$$

□

Applying (5.10) to $z \in \vec{C}^\infty(\Omega)$, we obtain

$$\|K_\sigma \operatorname{curl} z\|_\Omega^2 = 2 (K_\sigma \operatorname{curl} z, \nabla \tau_\sigma \wedge z)_\Omega \leq C \|K_\sigma \operatorname{curl} z\|_\Omega \cdot \|z\|_\Omega.$$

Therefore,

$$\|K_\sigma \operatorname{curl} z\|_\Omega \leq C \|z\|_\Omega. \quad (5.12)$$

Lemma 6. *For any field $u \in \mathcal{C}$ the relations*

$$\|\operatorname{curl} (K_\sigma u)\|_\Omega \leq C \|u\|_\Omega \quad (5.13)$$

and

$$(K_\sigma u)_\theta|_\Gamma = 0 \quad (5.14)$$

are valid.

Proof. Let $z \in \vec{C}^\infty(\Omega)$. Operator K_σ is self-adjoint by (5.12) and we have

$$\begin{aligned} |(K_\sigma u, \operatorname{curl} z)_\Omega| &= |(u, K_\sigma \operatorname{curl} z)_\Omega| \leq \|u\|_\Omega \cdot \|K_\sigma \operatorname{curl} z\|_\Omega \leq \\ &\leq C \|u\|_\Omega \cdot \|z\|_\Omega. \end{aligned}$$

Since z is arbitrary this estimate implies (5.13). Since z is not necessarily compactly supported, the equality (5.14) holds true. □

Lemma 7. *Let $\sigma \subset \Gamma$. For any field $u \in \mathcal{C}$ we have*

$$\|\operatorname{div} (K_\sigma u)\|_\Omega \leq C \|u\|_\Omega. \quad (5.15)$$

Proof. By the definition of K_σ , for large enough T we have

$$K_\sigma u = \left(\int_0^T X_\sigma^s ds \right) u - \left(\int_0^T E_\sigma^s ds \right) u.$$

The second term belongs to \mathcal{C} and thus is solenoidal in Ω . By (5.6) the first term is equal to $(T - \tau_\sigma) u$. Then by formula (5.1) we have

$$\operatorname{div} (K_\sigma u) = \operatorname{div} ((T - \tau_\sigma) u) = -\nabla \tau_\sigma \wedge u.$$

This completes the proof. □

Proof of Lemma 1. Suppose $u \in \mathcal{C}$. It follows from the estimates (5.13), (5.15) and boundary condition (5.14) that

$$\|K_\sigma u\|_F \leq \tilde{C} \|u\|_\Omega.$$

Then by compactness of the embedding $F \subset \vec{\mathcal{H}}$ (Theorem 3) we conclude that $K_\sigma \in \mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}})$. In view of (5.5) this completes the proof. \square

5.2 Proof of Lemma 2

At first we prove Lemma for $f \in C^\infty(\Omega)$.

Choose a finite open cover $\{U_j\}$ of the support of f such that every set of this cover is C^∞ -diffeomorphic to a ball in case $U_j \cap \Gamma = \emptyset$ or to a semi-ball $\{x \in \mathbb{R}^3 : |x| < 1, x^3 \geq 0\}$ otherwise. Choose a partition of unity $\zeta_j \in C_0^\infty(U_j)$ such that

$$0 \leq \zeta_j \leq 1, \quad \sum_j \zeta_j \Big|_{\text{supp } f} = 1.$$

It is clear that

$$\check{f} - Y[f] = \sum_j (\zeta_j \check{f} - Y[\zeta_j f]),$$

and the functions $\zeta_j f$ belong to $C_0^\infty(U_j)$. Thus, it is necessary to prove the Lemma for a function f supported in some open set U C^∞ -diffeomorphic to a ball or a semiball. In this case, for any $y \in \mathcal{C}$ we have

$$(fy - Y[f]y)|_U = \nabla p_y, \quad p_y \in H^1(U), \quad (5.16)$$

and if the set U intersects with Γ , then the following equality holds true

$$p_y|_{U \cap \Gamma} = \text{const.}$$

This can be easily obtained with the help of the Helmholtz decomposition in U .

The function p_y in (5.16) is uniquely determined up to additive constant, which can be chosen so that

$$p_y|_{U \cap \Gamma} = 0 \quad (5.17)$$

if $U \cap \Gamma \neq \emptyset$, and

$$\int_U p_y dx = 0$$

otherwise. The Friedrichs and Poincaré inequalities imply that, in the both cases, there is a constant C such that

$$\|p_y\|_U \leq C \|\nabla p_y\|_U = \|fy - Y[f]y\|_U \leq C \|\check{f} - Y[f]\| \cdot \|y\|.$$

Therefore, the mapping $y \mapsto p_y$ is continuous from \mathcal{C} to $H^1(U)$.

Now assume that a sequence y_n weakly converges to zero in \mathcal{C} . Then the sequence p_{y_n} weakly converges to zero in $H^1(U)$, and due to compactness of the embedding $H^1(U) \subset L_2(U)$ this implies

$$\|p_{y_n}\|_U \rightarrow 0, \quad n \rightarrow \infty. \quad (5.18)$$

Next, we have

$$\|fy_n - Y[f]y_n\|_\Omega^2 = (fy_n, fy_n - Y[f]y_n)_\Omega = (fy_n, \nabla p_{y_n})_\Omega.$$

In the last equality we used (5.16) and the inclusion $\text{supp } f \subset U$. Integrating by parts in this inner product, and applying formula (5.1) and equality $\text{div } y_n = 0$, we arrive at

$$(fy_n, \nabla p_{y_n})_\Omega = - \int_U \nabla f \cdot y_n p_{y_n} dx \leq M \|y_n\|_\Omega \cdot \|p_{y_n}\|_U$$

(M depends only on f). Integral over ∂U vanishes since f vanishes on $\partial U \setminus \Gamma$ and in the case $U \cap \Gamma \neq \emptyset$ we have (5.17). The right hand side of the latter inequality tends to zero because the norms of y_n are bounded and (5.18) takes place. Then, with regard to the result of the previous calculation, we get the relation

$$\|fy_n - Y[f]y_n\|_\Omega \rightarrow 0, \quad n \rightarrow \infty,$$

which shows that the operator $\tilde{f} - Y[f]$ is compact.

Now let us consider the case $f \in C(\Omega)$. The function f can be approximated in $C(\Omega)$ by functions $f_n \in C^\infty(\Omega)$. Operators of multiplication by f_n tend to the operator of multiplication by f in the operator norm. Hence, the operator $\tilde{f} - Y[f]$ is compact as a limit of compact operators.

5.3 Proof of Lemma 3

Here we prove the following properties:

$$\begin{aligned} \dot{\pi}(\alpha f + \beta g) &= \alpha \dot{\pi}(f) + \beta \dot{\pi}(g), \\ \dot{\pi}(fg) &= \dot{\pi}(f) \dot{\pi}(g), \\ \|\dot{\pi}(f)\| &= \|f\|, \end{aligned}$$

where $f, g \in C(\Omega)$, $\alpha, \beta \in \mathbb{R}$. The first and second relations follow from Lemma 2. For example, consider the second one. We show that

$$Y[f]Y[g] - Y[fg] \in \mathfrak{K}. \quad (5.19)$$

By Lemma 2 we have

$$Y[f]Y[g] = (f + K_1)Y[g] = fY[g] + K = f(g + K_2) + K = fg + \tilde{K},$$

where $K_1, K_2, K, \tilde{K} \in \mathfrak{K}(\mathcal{C}, \vec{\mathcal{H}})$. Applying Lemma 2 to the function fg , we obtain (5.19).

Consider the fourth property. We can restrict ourselves with smooth f since the mapping $\hat{\pi}$ is bounded. The latter follows from the obvious inequality

$$\|\hat{\pi}(f)\| \leq \|f\|.$$

Let us establish the opposite inequality. We need to show that for any compact operator $K \in \mathfrak{K}$ we have

$$\|Y[f] + K\| \geq \|f\|. \quad (5.20)$$

Fix a point $x_0 \in \Omega \setminus \Gamma$ such that $\nabla f(x_0) \neq 0$ (the case of a constant f is trivial). Choose a sequence of functions $\varphi_j \in C_0^\infty(\Omega \setminus \Gamma)$ such that $\text{supp } \varphi_j$ shrink to x_0 as $j \rightarrow \infty$. Introduce the fields

$$y_j := \nabla f \wedge \nabla \varphi_j.$$

Functions φ_j can be chosen such that every field y_j does not vanish identically. Owing to (5.2) we have $\text{div } y_j = 0$. Since $\text{supp } y_j$ tend to x_0 as $j \rightarrow \infty$, for sufficiently large j the fields y_j belong to \mathcal{C} . Further, we have

$$f y_j = f \nabla f \wedge \nabla \varphi_j = \frac{1}{2} \nabla(f^2) \wedge \nabla \varphi_j,$$

so by (5.2) $\text{div}(f y_j) = 0$ and for large j the fields $f y_j$ also belong to \mathcal{C} . Hence

$$Y[f]y_j = Y(f y_j) = f y_j. \quad (5.21)$$

Consider a normed sequence

$$\tilde{y}_j = y_j / \|y_j\|.$$

Obviously, the sequence \tilde{y}_j weakly converges to zero in \mathcal{C} . Therefore $K \tilde{y}_j \rightarrow 0$ in \mathcal{C} . With regard to (5.21) this yields

$$\|(Y[f] + K) \tilde{y}_j\| = \|f \tilde{y}_j + K \tilde{y}_j\| \rightarrow |f(x_0)|, \quad j \rightarrow \infty.$$

Since $\|\tilde{y}_j\| = 1$ we arrive at the inequality $\|Y[f] + K\| \geq |f(x_0)|$. This occurs for all points x_0 , at which f has nonzero gradient. So (5.20) holds true.

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