

ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

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**В.М.Бабич, Н.А.Вавилов, А.М.Вершик, М.А.Всемирнов, А.И.Генералов, И.А.Ибрагимов,
Л.Ю.Колотилина, Б.Б.Лурье, Ю.В.Матиясевич, Н.Ю.Нецветаев, С.И.Репин, Г.А.Серегин**

**Учредитель: Федеральное государственное бюджетное учреждение науки
Санкт-Петербургское отделение Математического института
им. В. А. Стеклова Российской академии наук**

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Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

телефоны: (812)312-40-58; (812) 571-57-54

e-mail: admin@pdmi.ras.ru

<http://www.pdmi.ras.ru/preprint/>

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SMOOTH AND NON-SMOOTH AF -ALGEBRAS AND PROBLEM ON INVARIANT MEASURES

A. M. Vershik

St.Petersburg Department of V. A. Steklov Institute
of Mathematics of Russian Academy of Sciences
vershik@pdmi.ras.ru

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ABSTRACT:

We separate the AF -algebras (correspondingly action of the countable groups on Cantor sets) onto two classes — “completely smooth” for which the set of all indecomposable traces (correspondingly list of all invariant ergodic measures) has nice parametrization, and the second class — “completely non-smooth” for which the set of all traces (correspondingly set of all invariant measures) is Poulsen simplex and therefore there is no suitable parametrization of indecomposable traces, (ergodic measures) e.g. Choquet boundary. Important example of the first type of AF -algebra is group algebra of infinite symmetric group, and of the second type — group algebra of some locally finite solvable group. The questions of recognition of those cases are closely related to the orbit theory of dynamical systems and theory of filtrations of sigma-fields in measure spaces and Borel spaces.

Key words: Smooth actions on the Cantor space, smooth AF -algebras, simplex of measures, standardness, distributive lattices.

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Ю.В.Матиясевич, Н.Ю.Нецветаев, С.И.Репин, Г.А.Серегин, В.Н.Судаков,
О.М.Фоменко

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1 INTRODUCTION

This is the detailed version of the part of my talk on the School on Probability and Statistical Physics at St.Petersburg (June 2012). I have discussed several problems about the combinatorial and probabilistic structures which are related to representation theory of the locally finite groups and locally semi-simple and AF -algebras, as well of asymptotic combinatorics and dynamics. In this note I discuss only one problem of that list, which I consider as the main one; this problem has deal with the description of the characters of locally finite groups, traces of AF -algebras, and with invariant measures for ergodic equivalence relations. It connected with various mathematical areas. Most known example of this type is the problem about the characters of infinite symmetric group. More general setting is the theory of the central measures on the set of paths of Bratteli diagram (f.e. Young graph) and the the problem about description of invariant probability measures with respect to the action of the countable group (for example adic transformation)

The goal this text is to distinguish so called “completely smooth” and “completely non-smooth” cases in the problems and correspondingly distinguish the AF -algebras onto several classes. Our classes can be formally expressed in terms of K -theory, namely, of K_0 -functor and its dual K^0 ([3]). But we will not use this language and try to formulate the tools which can be easily calculated. We show some typical examples of those classes and formulate a Conjecture which perhaps will help to solve some old problems. In particular we give the prediction which concerns to the list of characters for many dimensional Young graphs.

More concretely, our goal is to investigate the structure of the invariant measures on the Cantor sets, rather than explicit formulas for them. We explain the difference between two classes: for non-smooth case there are no reasons to expect regular formulas for the traces or invariant measures. But it is important to know whether given case is smooth or not and to give a criteria for that. I have started to think about two-dimensional (ordinary) Young graph) just after I became familiar with Thoma’s result and tried to understand when and why such nice answer exists.

In particular I want to emphasize that the pure combinatorial (or probabilistic) proof even for $AF(\mathfrak{S}_\infty)$ -algebra (=existence of borel parametrization of the list of indecomposable characters) does not exist up-to now. I mean the question here is not about precise formulas for invariant (central) measures or traces for AF -algebras, which is rather concrete problem, but about the reason why such formula can exist and what are the sufficient conditions for existence.

More deep features of the problem is discovered if we connect it with the orbit theory and with the theory of decreasing sequences of sigma-algebras (filtration). Shortly, the difference between smooth and non-smooth cases look as a difference between standard and nonstandard semi-homogeneous filtrations.

We conclude the Introduction with the claim that the problem of the description of the traces of AF -algebra (characters of the locally finite groups) is the analog of Fourier duality in classical analysis.

In the second paragraph we explain the problem and the links between traces of AF -algebras and invariant measures. Then we give the main definitions and in fourth and fifth paragraphs consider non-trivial examples - smooth — the case of group algebra of symmetric group (Young graph as Bratteli diagram) and non-smooth — group algebra of second order of solvability locally finite group. In the last paragraph we formulate our conjecture about smoothness of the case of Bratteli diagram which are Hasse diagram of the countable distributive lattices in particular - many dimensional Young graphs.

2 CENTRAL MEASURES, TRACES, CHARACTERS, ACTIONS.

We will mention briefly some notions: Bratteli diagrams, space of the paths, AF -algebra, skew-product structure, central measures, traces etc. It is easy to find this in the literature (see f.e.[3] and references their).

Recall that Bratteli diagram is a locally finite \mathbb{N} -graded graph Γ with one initial vertex \emptyset , with no final vertices and edges (could be multiple with finite multiplicity) which join the vertices of successive levels. The important object is the space $T(\Gamma)$ of all infinite pathes of the graph Γ - this sequence of edges (not vertices!) started in the from initial vertex without breaks. This space has natural structure of inverse limit of the finite sets and consequently has topology of Cantor space. A cylindric complex functions on $T(\Gamma)$ are the functions which depend on finite parts of the path. The algebra of all cylindric functions on $T(\Gamma)$ with pointwise multiplication called Gel'fand-Zetlin $GZ(\Gamma)$ this is commutative algebra (see [11]) which is maximal if there is no multiplicity of edges. Because Γ is \mathbb{N} -graded graph it supplied with so called *tail equivalence relation*, and correspondingly, *tail partition* — two paths are equivalent if they coincided at infinity, more exactly, both paths have the same vertices on the sufficiently large levels. Denote this tail partition on the space of the paths as ξ . In the interesting cases this is not measurable partition. It is possible to define a transformation of the path space, whose orbit partition coincides with tail partition -I mean *adic transformation* which depend on ordering of the set of edges coming to the each vertex. For our goals instead of Bratteli diagrams we can consider Cantor set with filtration of all cylindric functions and tail equivalence relation.

Thus instead of Bratteli diagram and the space of paths we can consider a standard Cantor space with a filtration in the space of all cylindric functions corresponding tail partition and invariant (homogeneous) measures on that partition. It happened that we are in the situation with the decreasing sequences of cylindric (measurable) partitions in the Cantor set (as topological space), and we can use the analogy with measure-theoretical results about decreasing sequences of the partitions (see [12]). We use this analogy below.

But first of all recall the connection of combinatorics with AF -algebras. It is well-known that each Bratteli diagram Γ generates canonically a locally-semi simple \mathbb{C} -algebra $\mathbb{C}(\Gamma)$. By theorem which independently was proved in [2, 3] this algebra has natural structure of skew-product of Gel'fand-Zetlin subalgebra $GZ(\Gamma)$ and some group G of the tail-preserving transformations of space $T(\Gamma)$. We can choose as that group the group \mathbb{Z} , generated by one transformation which is defined on the set of almost all paths — adic transformation [4]. The completion of this algebra $C(\Gamma)$ with respect to C^* -norm is so-called *approximately finite-dimensional algebra* — AF -algebra. We denote it as $AF(\Gamma)$. The theory of AF -algebras considers the connections between properties of the graph Γ itself and algebra $AF(\Gamma)$, including K -theory, classification and so on [5, 16]. But here we consider only very rough difference between AF -algebras.

Definition 1. *The central measure on the space of paths $T(\Gamma)$ of the graded graph Γ is a Borel probability measure on $T(\Gamma)$ which is invariant with respect to partition ξ . In another words, this is a measure induces a homogeneous conditional measure on the any measurable subpartition of ξ . This is equivalent to the fact that this measure is invariant under the adic transformation (see [4]) or any group of the transformations whose partition on the orbits coincides with partition ξ .*

Recall that the trace on C^* -algebra A is positive definite linear functional $\varphi : \varphi(hh^*) \geq 0$, which has property: $\varphi(hgh^*) = \varphi(g)$, $\varphi(e) = 1$ for all $g, h \in A$. If A is a group algebra then trace define a normalized character on the group.

Theorem 1. ([2, 3]) *There is a canonical one-to-one correspondence between the traces on the algebra $C^*(\Gamma)$ and central measure on $T(\Gamma)$.*

This identification is defined by the restriction of the trace on $C^*(\Gamma)$ to the Gelfand-Zetlin subalgebra $GZ(\Gamma)$, and trace on this commutative subalgebra is a Borel probability measure on the spectra e.g. on the space $T(\Gamma)$. It is easy to check that this measure must be central because of invariance of the trace with respect to inner automorphisms. And vice versa, each central measure generates trace as measure-type trace skew-product algebras.

The $C^*(\Gamma)$ — group algebra of the locally-finite group — is AF -algebra. In this we can apply the theorem. For example the infinite symmetric group has Young graph of the Young diagrams as branching graph of the simple modules and then the central measures are normalized character of the group and vice versa.

We have two identical problems:

To describe the set of ergodic central measures for given Bratteli diagram, or equivalently to describe all invariant measures for the tail partition of a cylindric filtration of the Cantor set.

and

To describe the set of indecomposable traces for given AF -algebra.

As a special case of the last problem we have the following problem:

To describe the set of the normalized indecomposable characters of the given locally finite group.

From point of view of representation theory the normalized traces on the C^* -algebras generate the representations of the C^* -algebra (group) of finite type in the sense of von Neumann: either type I_n or type II_1 . The same is true for normalized characters on the countable groups: if the trace or character is indecomposable (which means that the central measure is ergodic), then it is factor-representation of the group is of type I_n or type II_1 and vice versa.

The decomposable trace or character decomposable can be represent as integral over indecomposable ones, this decomposition corresponds to central decomposition of the von Neumann algebra of finite type; in terms of invariant measure it corresponds to ergodic decomposition of invariant measures.

3 DEFINITIONS OF COMPLETELY SMOOTH AND COMPLETELY NON-SMOOTH AF -ALGEBRAS AND GROUP ACTIONS.

3.1 Main definition

The set of all normalized traces of the C^* -algebra is convex compact in the weak topology on the state space, (=normalized linear positive functionals on the algebra), moreover it is affine simplex. The extremal points of this compact is indecomposable traces which means can't be proper convex combination of the other traces. The set of invariant probability Borel measures on the topological compact X under the action of the countable group Γ is also affine compact simplex and the set of ergodic measures coincide with its Choquet boundary.

Recall that Poulsen simplex is affine compact simplex in which the Choquet boundary (=the set of extremal points) is every where dense in the simplex. Such a simplex is unique up-to affine isomorphism (universality of Poulsen simplex). The simplex with closed Choquet boundary called as Bauer simplex, and there are uncountable many non-isomorphic Bauer simplicies.

Denote as $Tr(\mathcal{A})$ the simplex of all traces on the C^* -algebra \mathcal{A} and the set of indecomposable traces as $ExTr(\mathcal{A})$ traces (=Choquet boundary of that simplex). In parallel, denote the set of invariant measure on the Cantor set K with hyperfinite Borel equivalence relation with countable blocks – ξ — as $Inv(K, \xi)$; denote the set of the ergodic measures of $Inv(K, \xi)$ as $Erg(K, \xi)$. Partial case - ξ is the orbit equivalence relation for the measure preserving action of the countable group. We consider only hyper-finite ξ which means that group (if acts freely) is amenable.

Definition 2. *AF-algebra \mathcal{A} is called completely non-smooth if the set of normalized traces $Tr(\mathcal{A})$ as affine compact is Poulsen simplex. We say that \mathcal{A} is completely smooth if the Choquet boundary $ExTr(\mathcal{F})$ is open subset in its weak closure.*

The hyper-finite equivalence relation ξ of the Cantor space K called completely non-smooth if $Inv(K, \xi)$ is Poulsen simplex. We say that equivalence relation ξ (or action of the group with this orbit equivalence relation) is completely smooth if the set of ergodic measures (Choquet boundary) $Erg(K, \xi)$ is open subset in its weak closure in $Inv(K, \xi)$.¹

3.2 Some remarks

The condition: “The set $ExTr(\mathcal{A})$ (correspondingly $Erg(K, \xi)$) is weakly closed in $Tr(\mathcal{A})$ (correspondingly in $Inv(K, \xi)$)” e.g. the simplex is Bauer simplex, — seems too strong for our goals. In another words we allow the cases which is often ha been appeared when there are some (not so many) decomposable traces (correspondingly-non-ergodic measures) which is a limit of indecomposable traces (ergodic measures). In our Conjecture — see below — we assume just the case of the weak closure. I do not know name for the simplex with that property weaker that Bauer property.

The definition above put the main problem: how to distinguish those two cases in both situations in concrete examples, how to verify that given AF-algebra (given equivalence relation or action) belongs to one of two classes. We will illustrate such kind of problems below. The most interesting example concerns of Young diagrams of different dimensions, distributive lattices and special graphs.

¹We omit some time the word “completely” in all definitions. It needs strictly speaking only if we consider also the intermediate case -when AF-algebra action of the group) has simultaneously both parts, but we will not consider intermediate cases here.

The smoothness of the description above means that we can give a good parametrization of the set of indecomposable traces (correspondingly - ergodic invariant measures). But of course, to find such parametrization is nontrivial problem.

3.3 Connection with other topics: boundary, classification of equivalence relations

Remark that the definition of the ergodic central measures includes the information about *co-transition probability of the Markov chain which is uniform distribution*. It does not include the transition probabilities of Markov chain and those probabilities could be very different. In this sense our problem is NOT the problem of the calculation of Poisson-Furstenberg boundary (see [10]), rather -close to calculation of Martin boundary. We can say that the problem about description of central measure is the problem about description of all transition probabilities. But the attempt of direct calculation of it is not fruitful approach because it is too cumbersome. Centrality of measures (or invariance) can be generalized to the problem “to find all measures with given cocycle” - in our case this cocycle equal to 1. ²

In measure theoretical category all ergodic homogeneous hyper-finite partitions are isomorphic (Dye’s theorem and lacunary isomorphism theorem). In topological category it is not true: we have various tail partitions which are not (even Borel) isomorphic; they have the different structure of the set of invariant measures. The result in [14] claim that the cardinality of the number of invariant measures is complete invariant of Borel hyperfinite minimal equivalence relation upto Borel equivalence ³. So we need in formulation intermediate category between measure-theoretical and borel in which we can take in account some properties of the simplex of traces (invariant measures). K -theory by definition gave us such possibility for AF -algebras, but without concrete tools and this is the main difficulty in our problem.

3.4 Filtration and standardness

It seems that more important link of the problem concerns to the theory of filtration. In our situation we have the following useful fact:

Proposition 1. ([3]) *Each ergodic central measure on the N -graded graph Γ is a Markov measure with respect to the graded structure of graph.*

So we must consider the tail (hyperfinite) equivalence relation and its Markov approximation. This is the question of the theory of the decreasing sequence of finite

²In the theory of the stationary Markov compact the central measures called “measure with maximal entropy”

³I am grateful to professor A.Kechris for this reference.

partitions (filtration). More exactly we have as a Cantor set (space of paths of the Bratelli diagram) a Markov (non-stationary) compact \mathcal{X} with finite state spaces on each coordinate:

$$\mathcal{X} = \{\{x_n\}_{n=0}^\infty; x_n \in X_n, |X_n| < \infty, m_{x_n, x_{n+1}} = 1,$$

$$M_n = \{m_{i,k}^n\} \in Mat_n\{0;1\}, n \in \mathbb{N}.\}$$

We have a filtration of the sigma-fields: $\{\mathfrak{A}_N\}$ (which are generated by coordinates with number greater than N) Conjecturally we can formulate the answer on the question about smoothness of the tail equivalence relation in terms standardness of that filtration. Recall [12], that stationary Markov filtration is standard. The same is true for the case of group algebra of \mathfrak{S}_∞ -see theorem 2.

3.5 Link with general theory of C^* -algebras

The notions which was defined make sense for the arbitrary C^* -algebras and arbitrary equivalence relations as actions. So, we can compare our definition with analog in the theory of C^* -algebras — of the sharing of the C^* -algebras on liminaire and anti-liminaire (see [9] corresponds to the sharing of the AF -algebras onto

1) AF -algebras for which description of the indecomposable traces is “smooth problem” (this class contains all liminaire AF -algebras)

and

2) other AF -algebras (which must be anti-liminaire). Equivalently, we divide the problem of the description of ergodic invariant measures onto two classes.

It is naturally to compare this question with the question about classification of irreducible representations of groups or algebras. It is well-known (see[9]) that the problem of the classification of all irreducible $*$ -representation for C^* -algebra or group up-to unitary equivalence or? equivalently – the description of the pure states of the algebra up-to equivalence, — is tame or “smooth” iff this algebra so called “liminaire”. By classical results by Glimm the problem is wild (not smooth) for “anti-liminaire” algebras. For countable groups the analog of this by Thoma’s theorem is the following: for the groups which are finite extension of commutative groups (=virtually commutative) and only those groups the problem of the description of the characters is tame (smooth). The notion of the smooth and non-smooth AF -algebras allow to consider the further classification of “anti-liminaire” algebras. So the meaningful question is: for what kind of anti-liminaire algebras (or non virtually commutative groups) the description of the traces (characters) is smooth problem?

4 EXAMPLES OF THE COMPLETELY SMOOTH CASE: YOUNG DIAGRAMS AND YOUNG GRAPH.

It is very easy to give example of completely smooth AF -algebra or smooth hyperfinite equivalence relation - For example Glimm algebra (UHF) like $\bigotimes_1^\infty M_2\mathbb{C}$ has only one trace, and tail equivalence relation in the Cantor space $\prod_1^\infty \mathbb{Z}/2$ has only one invariant measure (Haar measure). In this case the filtration of tail partitions is standard diadic sequence ([12]). The Pascal graph ([13]) as Bratteli diagram defines well-known AF -algebra which is also smooth. Perhaps the Pascal filtration is also standard. Here we give more complicate examples and to prove smoothness. Namely we consider the group algebra symmetric group which is very popular now and became years ago the starting point for further investigations. The cases of infinite unitary and orthogonal groups are also smooth in our sense, — for all these groups we also have full list of the characters.

Concerning to Young graph that problem about characters of infinite symmetric group had several solutions: the first one was pure analytical solution by E.Thoma ([1]) who posed firstly the question. His solution contained noting about combinatorial or group-type feature of the problem. The second proof by A.Vershik-S.Kerov ([3]) used group approximation; the third approach due to A.Okounkov (1992), based on operator theory method. It is interesting that the problem itself in all this cases had considered from different point of view: Thoma's formulate it as a problem about description of the positive definite functions on infinite symmetric group, Vershik-Kerov approach was about central measures on the space of the infinite paths of Young graphs, the approach suggested by Ol'shansky [8] and realized by A.Okounkov [6] concerned so called admissible representations of the that group. We have used with S.Kerov my ergodic method which gives a general approach to similar problems but needs in complicate calculations.

We start with canonical example of the completely smooth case -group algebra of infinite symmetric group. The group algebra is of course anti-liminaire (no smooth classification of the irreducible representations) but completely smooth -list of traces has nice parametrization.

Consider Young graph \mathbf{Y} as Bratteli diagram of the group algebra of the infinite symmetric group $= \mathbb{C}(\mathfrak{S}_\infty)$. The vertices of the Young graph \mathbf{Y} is Young diagrams and number of cells is graduation; the finite path in Young graph \mathbf{Y} is ordinary Young tableaux; we will consider infinite paths which are infinite Young tableau. On the space of all infinite paths $T(\mathbf{Y})$ we have tail equivalence relation, and we can define the set of central measures on $T(\mathbf{Y})$, which, as we saw, in the canonical correspondence with space of traces on $\mathbb{C}(\mathfrak{S}_\infty)$, or characters on \mathfrak{S}_∞ . Ergodic

measures corresponds to indecomposable traces (characters).

As we have mentioned the list of indecomposable characters $Char(\mathfrak{S}_\infty)$ of infinite symmetric group was found by Thoma [1] in pure analytical way (see formula below). Ergodic method [3] in direct formulation for this case asserts that any ergodic central measure μ on the space $T(\mathbf{Y})$ is weak limit of the sequence of the elementary invariant measures $m_n(\lambda_n)$ where the last measure is uniform distribution on the set of all Young tableaux with given Young diagrams λ_n with n cells. In order to find all ergodic central measures we must consider all possible sequences of Young diagrams $\{\lambda_n\}_{n=1}^\infty$. The problem is to choose enough such sequences of λ_n in order to obtain all central measure, and then to make calculations of the weak limits for the obtaining of final Thoma's formula.

The key argument in the proof of [3] was that it is enough to consider the sequences of λ_n which have frequencies of the growth of rows and columns in the diagrams:

$$\alpha_k = \lim_n \frac{r_k(\lambda_n)}{n}; \quad k \in \mathbb{Z} \setminus 0,$$

where $r_k(\lambda)$ is length of k -th row (when $k > 0$) and length of $-k$ -column (when $k < 0$) of the diagram λ .

The explicit formula of the characters (traces) includes only sequence $\alpha = \{\alpha_k\}_{k \in \mathbb{Z} \setminus 0}$, and looks as a function χ on the group as follow:

$$\chi_\alpha(g) = \prod_{n>1} s_n(\alpha)^{c_n(g)},$$

where c_n is (finite) number of cycles of length $n > 1$ in the (finite) permutation g , and

$$s_n = \sum_{k \in \mathbb{Z} \setminus 0} sgn(k)^{(n-1)} \cdot \alpha_k^n, \quad n > 1$$

(super newtonian sum).

Here

$$\alpha = \{\alpha_k\}_{k \in \mathbb{Z} \setminus 0} : \alpha_1 \geq \alpha_2 \geq \dots \geq 0, \quad \alpha_{-1} \geq \alpha_{-2} \geq \dots \geq 0; \quad \sum_{k \in \mathbb{Z} \setminus 0} \alpha_k \leq 1$$

Interpretation of the value $\chi_\alpha(g)$ is a measure of fixed points of the action of the element g in appropriate measure space depending of α (see [13]). Most important case is the case of $\alpha_k \equiv 0$; in the case $\chi_\alpha(g) = \delta_e$ (delta function at identity element of the group), and corresponding central measure on the space of Young tableaux $T(\mathbf{Y})$ is *Plancherel measure*. The corresponding representation is regular representation of the group \mathfrak{S}_∞ .

An important observation.

Theorem 2. *Consider any central measure as measure on Markov compact of the paths (Young tableaux) and look at the sequence of the sigma-field $\mathfrak{F}_n, n = 1 \dots$ (or measurable partitions) generated by the coordinate of the paths with number more than n . Then this sequence (as homogenous partitions) is standard in the sense of [12].*

It is difficult to explain all the notion here, but the notion of the standardness is easy to formulate when we consider it in the framework of measure theory: the sequence of homogeneous partition is standard if normalized entropy equal to zero. As we had noticed in this case it is possible to connect the standardness with smoothness.

5 NON-SMOOTH AF -ALGEBRAS AND ACTIONS.

The following skew-product gives the example of the “non-smooth” situation: let us consider the group

$$G = \sum_1^\infty \mathbb{Z}/2; \quad G = \bigcup G_n; \quad G_n = \sum_1^n \mathbb{Z}/2$$

and its Bernoulli action group G on $X = 2^G = \prod_{g \in G} \mathbb{Z}/2$. The semi-product of the group G and space X is a group

$$\hat{G} = G \ltimes 2^G$$

which is locally compact solvable group. More convenient instead of X to consider its dense countable subgroup $X_0 = \sum_{g \in G} \mathbb{Z}/2$ and the semi-product of the group G and space X_0 , e.g. group

$$\bar{G} = G \ltimes \sum_{g \in G} \mathbb{Z}/2$$

with natural action of G on X_0 .

Of course \bar{G} is locally finite countable solvable group, its group algebra $C(\bar{G})$ is locally semi-simple algebra, and $C^*(\bar{G})$ is its C^* -group-algebra, which is an AF -algebra. Each finite dimensional simple subalgebra is a matrix algebra generated by the irreducible representations of G_n on 2^{G_n} for some $n \in \mathbb{N}$. Remark that matrix algebra is semidirect product of the action of G_n on the group of character 2^{G_n} . The result of this is the following theorem:

Theorem 3. *The Bratteli diagram $\Gamma(\bar{G})$ of the group algebra $C(\bar{G})$ has the following form; the set of vertices (simple modules) of the graph of the level n are parameterized by the orbits of the natural action of group G_n on the space 2^{G_n} . The edges of the graph between orbits O of the level n and orbit O' of the level $n+1$ of the diagram has multiplicity 0,1 or 2 as follow: recall that the orbit O' is non ordered union of two orbits of the level n ; so multiplicity equal zero, if orbit O is not part of the orbit of O' ; if O is a part of the orbit of O' , say $O = O' \cup O''$ then the multiplicity equal 1 if $O' = O''$, and 2 if $O' \neq O''$.*

It is easy to conclude from this that the space of all paths in the graph $\Gamma(\bar{G})$ coincides with group $X = 2^G$ and partition on the orbits of the action of G on X are the same as the tail partition on the paths. The combinatorics of these graph in terms of combinatorics of diadic decreasing sequences of measurable partitions has been studied in [12].

This description show us what is the Gelfand-Zetlin algebra GZ of $C(\hat{G})$. Observe that the canonical structure of semi-direct product of AF -algebra does not coincide in general with the structure of semi-direct product on the group, but in our case this is true. More exactly Gelfand-Zetlin subalgebra GZ of $C(\hat{G})$ does coincide with Group algebra of the group $X = 2^G$.

Theorem 4. *There is a bijection between the list of the traces of the AF -group algebra $C(\bar{G})$ of the group \bar{G} and the list of Borel probability measures on the space $X = \prod_{g \in G} Z/2$ with respect to the natural action of the group G .*

Corollary 1. *The AF -algebra $C(\hat{G})$ of the group \hat{G} is non-smooth in the sense of the definition which was given above.*

Proof. The simplex of the invariant measures on the space $X = \prod_{g \in G} Z/2$ with respect to action of the group G is Poulsen simplex because there are everywhere dense set δ of the discrete invariant measures with finite support, namely, measures whose support is finite set of the characteristic functions of the cosets over subgroups of finite index. \square

The fact that the action of the for any countable group G which has no Kazhdan property T the simplex of the invariant measures with respect to action G in 2^G is Poulsen simplex was proved in [15]. For the group $G = \sum_1^\infty Z/2$ this is evident. But now we need to identify that simplex with the simplex of the traces on group algebra $C(\hat{G})$. Let us identify the central measures as the invariant measures on the group X with traces and characters on the algebra and group. It is evident that the regular representation of the group \bar{G} corresponds to Haar (Lebesgue) measure on X and the action of the group G is Bernoulli action of this group with Bernoulli measure

μ . Moreover the regular representation in usual formulation (in the space $l^2(\bar{G})$) is isomorphic to von Neumann model of II_1 factor - as representation of semi-direct product $G \ltimes 2^G$ (with left and right action of the group \bar{G}). A very good exercise is to calculate the *Plancherel measure* as measure on the space of orbit, and its when order of orbits tends to infinity.

Now let us interpret this measure in terms of Markov compact of the sequences of orbits (= edges of Bratteli diagram; remember that central measure is a measure on the paths e.g. sequences of edges of that diagram). Crucial fact is the following:

Theorem 5. *Consider the image of invariant Bernoulli measure μ on the space $X = 2^G$ as a measure on the paths of the Markov compact (see definition in paragraph 2). Consider the decreasing sequences $\{\xi_n\}_{n=1}^\infty$ of the sigma-fields (measurable partitions) where ξ_n is sigma-algebra generated by the coordinates with numbers more than nn (correspondingly partition on the paths with the same n -tail – coordinates with numbers more than n) Then decreasing sequences $\{\xi_n\}_{n=1}^\infty$ is not standard in the sense of [12]. Moreover, this is an example of the decreasing sequences with positive entropy in that sense.*

It is enough to mention that this example is literally the same as in the paper [12], but it was considered at that paper in the framework of completely different reasons (theory of diadic sequences of measurable partitions, no any groups, AF -algebras and so on).

6 THE CONJECTURES AND THE PROBLEMS.

Classification of AF -algebras by the theorem by Elliott and others ([17, 18]) was reduced to the calculation of the K_0 -functor (Grothendick group of the classes of projective modules) with additional structure of Riesz group (ordering by the cone of true modules), and fixed element (one-dimensional free module). Sometimes it is called as “dimension group”). Unfortunately, it is difficult to apply this very important theorem because the calculations are very complicated. F.e. for group algebra of infinite symmetric group the answer which was done in [19], is rather cumbersome. The answer on the question whether AF -algebra is completely smooth or not of course is contained in the K_0 -functor, but we need to extract it and it is not so easy.⁴

⁴Remember that the list of traces (invariant measures) gives only a part of invariants of AF -algebra, because K_0 -functor could have the infinitesimal modules part of which does not separate by the traces.

Perhaps, it is difficult to give necessary and sufficient conditions for AF -algebras (or for actions of the groups) which guarantee the belonging to completely smooth class or to other classes. It is important to have at least sufficient conditions for smooth case. Hopefully such a condition gives us the hope to find precise formula for all characters and invariant measures.

K -theoretical approach to AF -algebra can be considered as *approximating approach* with finite dimensional algebras. But there is a dual (“co-approximating”) approach, which means that instead of increasing sequences of finite dimensional algebras we consider decreasing sequence (filtration) of infinite dimensional algebras.

In combinatorial term it means that we consider the sequence of the tail sigma-fields of sets of the paths which coincide after $n - th$ level, $n = 1, \dots$. For given Bratteli diagram Γ define this filtration (=decreasing sequence of the co-finite partitions) as $\mathfrak{A}_n(\Gamma)$, $n = 1, \dots$. For two different Bratteli diagrams of the given AF -algebra it is easy to formulate what kind of isomorphism of corresponding filtration. Let us call it *lacunary isomorphism*; and consider all such filtrations as filtrations on the standard Cantor space. The intersection of the partitions is *hyperfinite tail equivalence relation* about which we have discussed in the previous paragraphs.

PROBLEM To classify the filtrations on the Cantor space up to lacunary Borel equivalence.

More or less clear that this problem is “wild” in the sense of classification theory. But K_0 -theory shows that the classification can be reformulate in other terms. More important that wildness does not contradict to have constructive answer on some natural questions like completely smoothness or non-smoothness.

The analog of this Problem in measure theoretical category was done in the papers by author ([12]) one of the main result is lacunary theorem which is claim that two ergodic diadic filtrations are lacunary equivalent up to measure preserving transformation. But in Borel category it is not true. Moreover as we mentioned (in paragraph 3.3) that even intersection of the filtration (corresponding equivalence relation) has additional invariant (number of invariant measures) see [14]. But the isomorphism of the equivalence relation is too rough because the set of invariant measures could be infinite, nevertheless the A -algebras could be smooth or non-smooth cases which are not isomorphic in our sense.

So we need to develop the analog of the measure-theoretical approach to the filtration in Borel category. The main tool in that theory is notion of standardness and criteria of standardness (see [12]) That approach was developed mainly for homogeneous partition (like diadic) which means that the blocks of $n - th$ partitions in the filtration has the same number of points in almost all blocks and the conditional measure is uniform. It is important to generalize the main tools to the

case semi-homogeneous partitions - the uniform conditional measure on the blocks (centrality), but number of points is arbitrary finite. Hopefully, these tools can help to obtain the information about invariant measures and traces of AF -algebras. Our example of the completely non-smooth action (paragraph 5) shows the connection of non-standardness of the tail sequence and non-smoothness of AF -algebra.

I will consider the analog of measure-theoretical approach elsewhere but here I formulate a Conjecture of this type for very concrete case.

Consider a countable distributive lattice Γ ; by well-known G.Birkhoff's theorem there exist such a poset (=partial ordered set) ζ for which Γ is the set of all finite ideals with usual order $\Gamma = ID_f(\zeta)$. F.e. Young graph \mathbb{Y} is a Hasse diagram for the distributive lattice of finite ideals of the poset \mathbb{Z}_+^2 ⁵. The generalization of Young graph is the lattice of finite ideals of the poset $\mathbb{Z}_+^n, n = 3, \dots$ -its Hasse diagram is "Young" graphs of higher order - \mathbb{Y}^n . We can consider the AF -algebra which corresponds to is $\mathbb{Y}^n - A(\mathbb{Y}^n)$. For $n > 2$ this algebra is not group algebra of any group. Suppose $A(\Gamma)$ is AF -algebra corresponded to the Hasse-Bratteli diagram of distributive lattice Γ .

Conjecture 1. *The list of indecomposable traces of AF -Algebra $A(\Gamma)$ is weakly closed and consequently AF -algebra is completely smooth; In particular the same is true for $A(\mathbb{Y}^n)$.*

Possible proof of this Conjecture must use the list of infinite minimal ideals of the corresponding posets \mathbb{Z}_+^n which give the "frequencies" of minimal ideals for these measures. It is not difficult to prove the existence of frequencies, but the problem is to prove that the set of frequencies uniquely defines central measure and consequently the trace. One can apply combinatorial technique of ordinary Young diagram (contents etc.) but there is no tools like Symmetric Functions and no good description of the corresponding AF -algebra.

For $n = 2$ the Conjecture about Young Graph Y is true - this is equivalent to the Thoma's theorem [1] in our formulation [3]. For $n = 3$ the equivalent conjecture was considered by me with S.Kerov in 80-th.

The last remark - all this problems concern to the limit behavior of the product of adjacent matrices of Markov compact of the paths in Bratteli diagrams; this is the link with the typical problems in statistical physics.

⁵Each distributive lattices is a \mathbb{N} -graded poset; Hasse diagram of the graded poset is graded graph vertices of which is elements of poset and edges joins to elements on of which directly preceded to the second element; Hasse diagram of the graded poset can be considered as Bratteli diagram of some AF -algebra.

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