ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

РЕДКОЛЛЕГИЯ

В.М.Бабич, Н.А.Вавилов, А.М.Вершик, М.А.Всемирнов, А.И.Генералов, И.А.Ибрагимов, Л.Ю.Колотилина, Б.Б.Лурье, Ю.В.Матиясевич, Н.Ю.Нецветаев, С.И.Репин, Г.А.Серегин

Учредитель: Федеральное государственное бюджетное учреждение науки
Санкт-Петербургское отделение Математического института
им. В. А. Стеклова Российской академии наук

Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16 октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых коммуникаций

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

телефоны:(812)312-40-58; (812) 571-57-54

e-mail: admin@pdmi.ras.ru

http://www. pdmi.ras.ru /preprint/

Заведующая информационно-издательским сектором Симонова В.Н

PDMI preprint 19/2012

Correlation functions of the XXZ chain at zero anisotropy and enumeration of boxed plane partitions *†

N. M. Bogoliubov⋆, C. Malyshev⋄

Steklov Mathematical Institute, St.-Petersburg Department, RAS
Fontanka 27, St.-Petersburg, 191023, Russia

⋆ e-mail: *bogoliub@pdmi.ras.ru*
⋄ e-mail: *malyshev@pdmi.ras.ru*

Abstract

The temperature correlation functions of the XXZ Heisenberg chain at zero anisotropy are calculated over the base of N-particle Bethe states. The wave functions are expressed by means of the symmetric Schur functions. The correlation functions both of the ferromagnetic string and domain wall are studied. A relationship is demonstrated between the correlation functions in question and enumeration of the boxed plane partitions as well as of the lattice paths. The determinant formula of the *Kuperberg type* that provides a connection to (q-)binomial determinants is obtained. An approach is developed to obtain the asymptotics of the correlators in the limit of small temperature provided that the characteristic sizes of the system are large enough.

Keywords: XXZ Heisenberg chain, Schur function, plane partition, q-binomial determinant

---

*Part of this work has been presented at the *International Workshop on Classical and Quantum Integrable Systems CQIS-2012* (January 23-27, 2012, JINR, Dubna, Russia)

†Partially supported by RFBR (No. 10-01-00600) and by the Russian Academy of Sciences program 'Mathematical Methods in Non-Linear Dynamics'

ПРЕПРИНТЫ ПОМИ РАН

Учредитель

Санкт-Петербургское отделение

Математического института им. В. А. Стеклова

Российской академии наук

PDMI PREPRINTS

St.Petersburg Department of Steklov Institute of Mathematics

ГЛАВНЫЙ РЕДАКТОР

член-корреспондент РАН

С. В. Кисляков

РЕДКОЛЛЕГИЯ

В.М.Бабич, Н.А.Вавилов, А.М.Вершик, М.А.Всемирнов,
А.И.Генералов, И.А.Ибрагимов, Л.Ю.Колотилина, П.П.Кулиш,
Б.Б.Лурье, Ю.В.Матиясевич, Н.Ю.Нецветаев, С.И.Репин,
Г.А.Серегин, В.Н.Судаков, О.М.Фоменко

1 Introduction

The Heisenberg XXZ model continues to attract considerable attention since the calculation of its correlation functions in the approach of the *Algebraic Bethe Ansatz* is up-to-date problem both in theoretical and mathematical physics [1–10]. The limit of zero anisotropy (the free-fermion limit) of the XXZ model, so-called, XX Heisenberg chain, is also of considerable interest [11–14].

Random walks is a classical problem both for combinatorics and statistical physics. Since they were introduced in the paper by Fisher [15], the problem of enumeration of the paths of the *vicious* walkers is also actively investigated [16–28]. The random walks over one-dimensional periodic lattice are related to the correlation functions of the XX Heisenberg magnet [29, 30]. Certain averages taken over the ferromagnetic state of the XX model play the role of the generating functions of the paths traced by the vicious walkers. The correlation functions of the XX model have been used in [31–33] to study the problem of enumeration of the trajectories of the vicious walkers.

In this paper the approach of [29–34] will be further explored. It is crucial that the Bethe wave functions of the XXZ model are expressible in terms of the Schur functions [40] in two important limits: zero and infinite anisotropy [33, 34]. In the present paper we restrict ourselves with the limit of zero anisotropy to study the correlation functions and certain form-factors in the Schur function representation. The correlation functions of two types will be investigated: the correlation function describing the states with no excitations on the first n sites of the chain (would be called as *Survival Probability of Ferromagnetic String*), as well as the correlation function of the operators corresponding to creation of n excitations on the first n sites of the chain (would be called as *Survival Probability of Domain Wall*). We shall concentrate on the combinatorial aspects of the correlation functions. The formulas obtained will be studied from the viewpoint both of the random walks of the vicious walkers and enumeration of the boxed plane partitions. A relationship to certain lattice paths will also be provided. Similar problems appear in the case of the *phase model* (being a special limit of q-boson model) describing the so-called friendly walkers [35]: boxed plane partitions, three-dimensional Young diagrams in a box, lattice paths, etc.

The paper is organized as follows. Section 1 is introductive. The XXZ model and its solution at arbitrary anisotropy parameter Δ are presented shortly in Section 2. Section 3 contains the outline of the problem at zero value of Δ. Section 4 is concerned with calculation of the correlation functions in question. In Section 5 the combinatorial aspects of the problem are under consideration. The generating functions related to enumeration of the boxed plane partitions and the lattice paths are obtained. The asymptotical estimates of the Survival Probabilities are obtained in Section 6. Discussion in Section 7 concludes the paper. A proof of the determinantal formulas crucial for this paper is given in Appendix I. A connection of these formulas with the q-binomial determinants is established. Appendix II deals with the derivation of the asymptotical formulas.

2 XXZ Heisenberg model

The XXZ Heisenberg model is defined on one-dimensional lattice consisting of $M+1$ sites labelled by elements of the set $\mathcal{M} \equiv \{0, 1, \ldots, M\} \ni k$, and its Hamiltonian is of the

form:

$$\hat{H}_{XXZ} = -\frac{1}{2}\sum_{k=0}^{M}(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-} + \frac{\Delta}{2}\,(\sigma_{k+1}^{z}\sigma_{k}^{z} - 1))\,, \tag{1}$$

where $\Delta \in \mathbb{R}$ is the anisotropy. We assume that external magnetic field is absent, and the number of sites, $M+1$, is even. The local spin operators $\sigma_k^{\pm} = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$ and σ_k^z obey the commutation rules:

$$[\,\sigma_k^{+}, \sigma_l^{-}\,] \;=\; \delta_{k,l}\,\sigma_l^{z}\,, \quad [\,\sigma_k^{z}, \sigma_l^{\pm}\,] \;=\; \pm 2\,\delta_{k,l}\,\sigma_l^{\pm}\,.$$

The spin operators act in the space $\mathfrak{H}_{M+1}$ spanned over the state-vectors $\bigotimes_{k=0}^{M}\,|\,s\rangle_k$, where s implies either $\uparrow$ or $\downarrow$. The spin "up" and "down" states, $|\uparrow\rangle$ and $|\downarrow\rangle$, provide a natural basis of the linear space $\mathbb{C}^2$ so that

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The periodic boundary conditions $\sigma_{k+(M+1)}^{\#} = \sigma_k^{\#}$ are imposed.

To represent N-particle state-vectors of the model, $|\,\Psi_N(u_1,\ldots,u_N)\rangle$, let the sites with spin "down" states be labelled by the coordinates μ_i, $1 \le i \le N$. These coordinates form a strict partition $\boldsymbol{\mu} \equiv (\mu_1, \mu_2, \ldots, \mu_N)$, where $M \ge \mu_1 > \mu_2 > \ldots > \mu_N \ge 0$. There is a correspondence between each partition and an appropriate sequence of zeros and unities of the form: $\big\{e_k \equiv e_k(\boldsymbol{\mu})\big\}_{k\in\mathcal{M}}$, where $e_k \equiv \delta_{k,\mu_n}$, $1 \le n \le N$. Obviously, the relation $\sum_{k=0}^{M} e_k = N$ is respected. The Hamiltonian (1) is diagonalized by means of the ansatz:

$$|\,\Psi_N(\mathbf{u})\rangle \;=\; \sum_{\{e_k(\boldsymbol{\mu})\}_{k\in\mathcal{M}}} \chi_{\boldsymbol{\mu}}^{XXZ}(\mathbf{u}) \prod_{k=0}^{M} (\sigma_k^{-})^{e_k}\;|\Uparrow\rangle\,, \tag{2}$$

where the summation is over $C_M^N \equiv \frac{M!}{N!(M-N)!}$ strict partitions $\boldsymbol{\mu}$. The state $|\Uparrow\rangle$ in (2) is the fully polarized state with all spins "up": $|\Uparrow\rangle \equiv \bigotimes_{n=0}^{M}\,|\uparrow\rangle_n$. It is proposed to use bold-faced letters as short-hand notations for appropriate N-tuples of numbers: for instance, $\mathbf{u}$ instead of $(u_1,\ldots,u_N)$, etc. Therefore, the wave function $\chi_{\boldsymbol{\mu}}^{XXZ}(\mathbf{u})$ in (2) is of the form:

$$\chi_{\boldsymbol{\mu}}^{XXZ}(\mathbf{u}) \;=\; \sum_{S_{p_1,p_2,\ldots,p_N}} \mathcal{A}_S(\mathbf{u})\, u_{p_1}^{2\mu_1} u_{p_2}^{2\mu_2} \ldots u_{p_N}^{2\mu_N}\,, \tag{3}$$

where the summation is over all permutations $S_{p_1,p_2,\ldots,p_N} \equiv S\begin{pmatrix} 1, & 2, & \ldots, & N \\ p_1, & p_2, & \ldots, & p_N \end{pmatrix}$. The amplitude $\mathcal{A}_S$ is given by the product:

$$\mathcal{A}_S(\mathbf{u}) \;\equiv\; \prod_{1\le j<i\le N} \frac{1 - 2\Delta u_{p_i}^2 + u_{p_i}^2 u_{p_j}^2}{u_{p_i}^2 - u_{p_j}^2}\,.$$

The state-vectors (2) provide the eigen-states of the Hamiltonian (1),

$$\hat{H}_{XXZ}\,|\,\Psi_N(\mathbf{u})\rangle \;=\; E_N(\mathbf{u})\,|\,\Psi_N(\mathbf{u})\rangle\,,$$

with the corresponding eigen-values given by

$$E_N(\mathbf{u}) \;=\; -\frac{1}{2}\sum_{i=1}^{N}(u_i^2 + u_i^{-2} - 2\Delta)\,, \tag{4}$$

if and only if the variables u_l $(1 \le l \le N)$ satisfy the *Bethe equations*:

$$u_l^{2(M+1)} = (-1)^{N-1}\prod_{k=1}^{N}\frac{1 - 2\Delta u_l^2 + u_l^2 u_k^2}{1 - 2\Delta u_k^2 + u_l^2 u_k^2}\,. \tag{5}$$

3 Outline of the problem

The isotropic limit of XXZ model, so-called, XX model [11–14], is described by the Hamiltonian $\hat{H}_{XX}$, which arises from the Hamiltonian (1) at $\Delta \to 0$:

$$\hat{H}_{XX} \equiv -\frac{1}{2}\sum_{k=0}^{M}(\sigma_{k+1}^{-}\sigma_k^{+} + \sigma_{k+1}^{+}\sigma_k^{-})\,. \tag{6}$$

The Hamiltonian (6) corresponds to nearest-neighbor interactions of spin "up" and "down" states located on the sites of the periodic chain, and it admits a free fermion representation [13]. The XX magnet can also be deduced in the limit of infinite on-site repulsion in the boson Hubbard model [36]. The XX model is of interest for description of the Frenkel excitons [37], as well as concerning the quantum information and computation theory [38, 39].

The state-vector of the XX model is given by (2) at $\Delta \to 0$. In this limit, the wave function (3) is specified, up to irrelevant pre-factor, as follows:

$$\chi_{\boldsymbol{\mu}}^{XX}(\mathbf{u}) = \det(u_j^{2\mu_k})_{1\le j,k\le N}\prod_{1\le n<l\le N}(u_l^2 - u_n^2)^{-1}, \tag{7}$$

while the Bethe equations (5) take the form [13]:

$$u_j^{2(M+1)} = (-1)^{N-1}\,, \quad 1 \le j \le N\,. \tag{8}$$

The substitution $u_j^2 = e^{i\theta_j}$ brings these equations to the exponential form,

$$e^{i(M+1)\theta_j} = (-1)^{N-1}\,. \tag{9}$$

Equations (9) are solved by the "angles" θ_j,

$$\theta_j = \frac{2\pi}{M+1}\Big(I_j - \frac{N-1}{2}\Big)\,, \quad 1 \le j \le N\,, \tag{10}$$

where I_j are integers or half-integers depending on whether N is odd or even. It is sufficient to consider a set of N different numbers I_j satisfying the condition: $M \ge I_1 > I_2 > \cdots > I_N \ge 0$. The notation $\boldsymbol{\theta}$ for N-tuple $(\theta_1, \theta_2, \ldots, \theta_N)$ will be especially convenient for usage below in order to stress that one is concerned with the solution (10) of the Bethe equation.

Otherwise, it is appropriate to use $\mathbf{u}$ as an indication that arbitrary set of parameters is meant. It follows from (4) that the eigen-energy of the XX model is equal to

$$E_N^{XX}(\boldsymbol{\theta}) \;=\; -\sum_{j=1}^{N}\cos\theta_j \;=\; -\sum_{j=1}^{N}\cos\left(\frac{2\pi}{M+1}\Big(I_j-\frac{N-1}{2}\Big)\right). \tag{11}$$

The ground-state solution is given by the "angles" (10) at $I_j = N-j$:

$$\theta_j^{\mathrm{v}} \equiv \frac{2\pi}{M+1}\Big(N-j-\frac{N-1}{2}\Big), \quad 1\le j\le N\,, \tag{12}$$

and it corresponds to the eigen-energy of the form:

$$E_N^{XX}(\boldsymbol{\theta}^{\mathrm{v}}) \;=\; -\operatorname{cosec}\frac{\pi}{M+1}\,\sin\frac{\pi N}{M+1}\,. \tag{13}$$

The wave function (7) is expressible through the *Schur functions* given by the Jacobi-Trudy relation [40]:

$$\begin{aligned} S_{\boldsymbol{\lambda}}(\mathbf{x}) \;\equiv\; S_{\boldsymbol{\lambda}}(x_1,x_2,\ldots,x_N) \;&\equiv\; \frac{\det(x_j^{\lambda_k+N-k})_{1\le j,k\le N}}{\det(x_j^{N-k})_{1\le j,k\le N}} \\ &=\; \det(x_j^{\lambda_k+N-k})_{1\le j,k\le N}\prod_{1\le n<l\le N}(x_l-x_n)^{-1}\,, \end{aligned} \tag{14}$$

where $\boldsymbol{\lambda}$ denotes the partition $(\lambda_1,\lambda_2,\ldots,\lambda_N)$ being N-tuple of non-increasing non-negative integers: $L\ge\lambda_1\ge\lambda_2\ge\ \ldots\ \ge\lambda_N\ge 0$. Any strict partition $M\ge\mu_1>\mu_2>\ldots>\mu_N\ge 0$ and non-strict partition $M+1-N\ge\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_N\ge 0$ (denoted as $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$, respectively) can be related by means of the relation $\lambda_j=\mu_j-N+j$, where $1\le j\le N$. In other terms, $\boldsymbol{\lambda}=\boldsymbol{\mu}-\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the strict partition $(N-1,N-2,\ldots,1,0)$. So the wave function (7) may equivalently be written as

$$\chi_{\boldsymbol{\mu}}^{XX}(\mathbf{u}) = S_{\boldsymbol{\lambda}}(\mathbf{u}^2)\,. \tag{15}$$

Thermal correlation functions of two types will be investigated in the present paper. First of all, the correlator, would be called as *Survival Probability of Ferromagnetic String*, related to the states with no spins down on first n sites of the chain. Specifically, we shall consider the expectation value defined by the ratio:

$$\mathcal{T}(\boldsymbol{\theta}^{\mathrm{v}},n,\beta) \;\equiv\; \frac{\langle\Psi_N(\boldsymbol{\theta}^{\mathrm{v}})\mid\bar{\Pi}_n\,e^{-\beta\hat{H}}\,\bar{\Pi}_n\mid\Psi_N(\boldsymbol{\theta}^{\mathrm{v}})\rangle}{\langle\Psi_N(\boldsymbol{\theta}^{\mathrm{v}})\mid e^{-\beta\hat{H}}\mid\Psi_N(\boldsymbol{\theta}^{\mathrm{v}})\rangle}\,, \qquad \bar{\Pi}_n\equiv\prod_{j=0}^{n-1}\check{q}_j\,, \tag{16}$$

where $\hat{H}$ implies $\hat{H}_{XX}$ (6), $\boldsymbol{\theta}^{\mathrm{v}}$ is given by (12), and $\beta\in\mathbb{C}$. The projector $\bar{\Pi}_n$ is expressed by means of the operators $\check{q}_j$ dependent on the spin operators $\sigma_k^{\#}$ as follows:

$$\check{q}_k \;\equiv\; \frac{1}{2}\,(\sigma_k^0+\sigma_k^z)\,, \quad \hat{q}_k \;\equiv\; \frac{1}{2}\,(\sigma_k^0-\sigma_k^z)\,, \quad \check{q}_k+\hat{q}_k \;=\; \mathbb{I}, \qquad k\in\mathcal{M}\,. \tag{17}$$

The projector $\bar{\Pi}_n$ (16) annihilates the excited states on first n sites. Investigation of the correlator (16) has already been started in [33, 34]. We shall also consider the correlation function called as *Survival Probability of Domain Wall*, related to the operators $\bar{\mathsf{F}}_n$

corresponding to creation of n excitations on first n sites:

$$\mathcal{F}(\boldsymbol{\theta}^{\rm v}, n, \beta) \equiv \frac{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\rm v}) \mid \bar{\mathsf{F}}_n \, e^{-\beta \hat{H}} \, \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\boldsymbol{\theta}^{\rm v}) \rangle}{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\rm v}) \mid e^{-\beta \hat{H}} \mid \Psi_{N-n}(\boldsymbol{\theta}^{\rm v}) \rangle} , \qquad \bar{\mathsf{F}}_n \equiv \prod_{j=0}^{n-1} \sigma_j^- . \tag{18}$$

We shall use a relation for the Schur functions which is due to the Binet–Cauchy formula [41]:

$$\begin{aligned} & \sum_{\boldsymbol{\lambda} \subseteq \{(L/n)^N\}} S_{\boldsymbol{\lambda}}(x_1^2, \ldots, x_N^2) S_{\boldsymbol{\lambda}}(y_1^2, \ldots, y_N^2) \\ = \; & \det(T_{jk})_{1 \le j,k \le N} \prod_{1 \le k < j \le N} \left(y_j^2 - y_k^2\right)^{-1} \prod_{1 \le m < l \le N} \left(x_l^2 - x_m^2\right)^{-1} , \end{aligned} \tag{19}$$

where summation goes over all non-strict partitions $\boldsymbol{\lambda}$ into at most N parts so that each is less than L but greater than n: $L \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N \ge n$. The entries T_{jk} take the form:

$$T_{jk} = \frac{(x_k y_j)^{2n} - (x_k y_j)^{2(N+L)}}{1 - (x_k y_j)^2} . \tag{20}$$

4 The correlation functions

4.1 The Bethe states and form-factors

Before to proceed with calculation of $\mathcal{T}(\boldsymbol{\theta}^{\rm v}, n, \beta)$ (16), we use (15) to specialize the state-vector (2) and its conjugated:

$$\begin{aligned} |\, \Psi_N(\mathbf{u}) \rangle &= \sum_{\boldsymbol{\lambda} \subseteq \{(M+1-N)^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \prod_{k=0}^{M} (\sigma_k^-)^{e_k} \, |\Uparrow\rangle , \\ \langle \Psi_N(\mathbf{v}) \,| &= \sum_{\boldsymbol{\lambda} \subseteq \{(M+1-N)^N\}} \langle \Uparrow| \prod_{k=0}^{M} (\sigma_k^+)^{\tilde{e}_k} S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2}) , \end{aligned} \tag{21}$$

where summation goes over all non-strict partitions $\boldsymbol{\lambda}$, which are related to the non-strict partitions $\boldsymbol{\mu} = \boldsymbol{\lambda} + \boldsymbol{\delta}$. The orthogonality relation

$$\langle \Uparrow| \prod_{k=0}^{M} (\sigma_k^+)^{\tilde{e}_k} \prod_{l=0}^{M} (\sigma_l^-)^{e_l} |\Uparrow\rangle = \prod_{n=0}^{M} \delta_{\tilde{e}_n e_n} ,$$

as well as the relation (19), enable to obtain the scalar product of the state-vectors (21):

$$\langle \Psi_N(\mathbf{v}) \,|\, \Psi_N(\mathbf{u}) \rangle \; = \sum_{\boldsymbol{\lambda} \subseteq \{(M+1-N)^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \; = \; \frac{\det(T^{\rm o}_{kj})_{1 \le k,j \le N}}{\mathcal{V}(\mathbf{u}^2)\mathcal{V}(\mathbf{v}^{-2})} , \tag{22}$$

where the entries $T^{\rm o}_{kj}$ coincide with those given by (20) at $n = 0$:

$$T^{\rm o}_{kj} = \frac{1 - (u_k^2/v_j^2)^{M+1}}{1 - u_k^2/v_j^2} . \tag{23}$$

Besides, the notation for the Vandermonde determinant is used in (22):

$$\mathcal{V}(\mathbf{u}^2)\,(\equiv\,\mathcal{V}_N(\mathbf{u}^2))\,\equiv\,\prod_{1\le m<l\le N}(u_l^2-u_m^2)\,. \tag{24}$$

Define the squared norm of the states (21) parametrized arbitrarily: $\mathcal{N}^2(\mathbf{u})=\langle\Psi_N(\mathbf{u})\mid\Psi_N(\mathbf{u})\rangle$. Let us introduce the exponential parametrization for the solutions of the Bethe equations:

$$\mathbf{u}^2=e^{i\boldsymbol{\theta}}\,,\qquad e^{i\boldsymbol{\theta}}\equiv(e^{i\theta_1},e^{i\theta_2},\ldots,e^{i\theta_N})\,,$$

where the "angles" (10) are meant. The entries (23) taken on the solutions (10) are given as follows:

$$T^{\rm o}_{kj}=\frac{\sin\pi(I_k-I_j)}{\sin\frac{\pi}{M+1}(I_k-I_j)}\,e^{i\,\frac{\pi M}{M+1}(I_k-I_j)}\;=\;(M+1)\delta_{jk}\,. \tag{25}$$

Then, the corresponding square of the norm $\mathcal{N}^2(\boldsymbol{\theta})$ arises as follows:

$$\mathcal{N}^2(\boldsymbol{\theta})\;=\;\frac{(M+1)^N}{|\mathcal{V}(e^{i\boldsymbol{\theta}})|^2}=\frac{(M+1)^N}{\prod\limits_{1\le m<l\le N}2(1-\cos\frac{2\pi}{M+1}(I_l-I_m))}\,. \tag{26}$$

Let us now consider the form-factor given by the ratio:

$$\mathcal{T}(\mathbf{v},\mathbf{u},n)\;\equiv\;\frac{\langle\Psi_N(\mathbf{v})\mid\bar{\Pi}_n\mid\Psi_N(\mathbf{u})\rangle}{\mathcal{N}(\mathbf{v})\mathcal{N}(\mathbf{u})}\,, \tag{27}$$

where $\bar{\Pi}_n$ is defined by (16). With regard at (21), we calculate:

$$\bar{\Pi}_n\;|\Psi_N(\mathbf{u})\rangle\;=\;\sum_{\boldsymbol{\lambda}\subseteq\{(M-N+1/n)^N\}}S_{\boldsymbol{\lambda}}(\mathbf{u}^2)\left(\prod_{k=n}^{M}(\sigma_k^-)^{e_k}\right)|\Uparrow\rangle\,, \tag{28}$$

where summation goes over non-strict partitions $\boldsymbol{\lambda}$ bounded below: $M-N+1\ge\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_N\ge n$. Taking into account (19), (20), and (28), we calculate the nominator of (27) [1]:

$$\langle\Psi_N(\mathbf{v})\mid\bar{\Pi}_n\mid\Psi_N(\mathbf{u})\rangle\;=\;\sum_{\boldsymbol{\lambda}\subseteq\{(M-N+1/n)^N\}}S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2})S_{\boldsymbol{\lambda}}(\mathbf{u}^2)\;=\;\frac{\det(T_{kj})_{1\le k,j\le N}}{\mathcal{V}(\mathbf{u}^2)\mathcal{V}(\mathbf{v}^{-2})}\,, \tag{29}$$

$$T_{kj}\;=\;\frac{(u_k^2/v_j^2)^n-(u_k^2/v_j^2)^{M+1}}{1-u_k^2/v_j^2}\,. \tag{30}$$

Adding and subtracting unity in the nominator of T_{kj} (30), one obtains (29) on the solutions (10):

$$\langle\Psi_N(\boldsymbol{\theta})\mid\bar{\Pi}_n\mid\Psi_N(\boldsymbol{\theta})\rangle\;=\;\frac{1}{|\mathcal{V}(e^{i\boldsymbol{\theta}})|^2}\det\bigl(T^{\rm o}_{kj}-(M+1)K_n(\theta_k,\theta_j)\bigr)_{1\le k,j\le N}\,, \tag{31}$$

[1] In spite of the fact that $\bar{\Pi}_n$ (16) is defined at $n\ge1$, right-hand side of (28) admits $n=0$ and therefore right-hand side of Eq. (29) reproduces, formally, the scalar product (22).

where the matrix $K_n(\theta_k,\theta_j)$ is

$$K_n(\theta_k,\theta_j) \equiv \frac{e^{i(n-1)(\theta_k-\theta_j)/2}}{M+1}\,\frac{\sin\frac{n(\theta_k-\theta_j)}{2}}{\sin\frac{\theta_k-\theta_j}{2}}\,. \tag{32}$$

It is seen from (25), (26), (31), (32) that the weighted ratio (27) is related, in the thermodynamic limit, with the Fredholm determinant:

$$\mathcal{T}(e^{i\boldsymbol{\theta}/2},e^{i\boldsymbol{\theta}/2},n) = \det\big(\delta_{lm}-K_n(\theta_l,\theta_m)\big)_{1\le l,m\le N} \sim \mathrm{Det}\big(\delta(p,q)-K_n(p,q)\big)\,, \tag{33}$$

where $K_n(p,q)$ is the linear integral operator acting on functions on the segment $[-k_F,k_F]$ with the kernel corresponding to (32) where $M+1$ is replaced by 2π. In other words, appropriately parametrized average (27) is related to the *Emptiness Formation Probability* (EFP), $P(n)\equiv\langle GS|\bar{\Pi}_n|GS\rangle$, which is the probability of formation of a string of empty (i.e., spin "up") states on first n sites [7, 8, 13, 42, 43]. For zero magnetic field, the fully polarized state $|\Uparrow\rangle$ (see (2)) is not the ground state. The ground state $|GS\rangle$ in this case is constructed, in the thermodynamic limit, by filling the reference state $|\Uparrow\rangle$ with quasi-particles occupying all admissible positions in the Fermi band $[-k_F,k_F]\ni p$ (here, k_F is the Fermi momentum, $k_F=\arccos(h/2)\big|_{h=0}=\frac{\pi}{2}$) [14]. The probability $P(n)$ is constructed by means of the matrix determinants as follows [44]:

$$P(n) = \sum_{J=0}^{n}\frac{1}{J!}\int\limits_{-k_F}^{k_F}\dots\int\limits_{-k_F}^{k_F}\det\big(-K_n(p_l,p_m)\big)_{1\le l,m\le N}\,dp_1\dots dp_N\,. \tag{34}$$

The asymptotical behavior of EFP is investigated in [42, 43]. The multiple integral representation [8] is alternative to (34) (see [42] for equivalence of these two representations). However, we shall not touch the thermodynamic limit since it is assumed that the spin chain is long enough though remains finite.

Another form-factor to be considered is that corresponding to $\bar{\mathsf{F}}_n$ (18). Let us introduce an auxiliary operator $\mathsf{D}_{N-j}(\mathbf{u})$ which acts according to the rule:

$$\mathsf{D}_{N-j}(\mathbf{u})(\cdot) \equiv \lim_{u_{N-j}^2\to 0}\frac{1}{j\,!}\,\frac{\mathrm{d}^j}{\mathrm{d}(u_{N-j}^2)^j}\Big(\prod_{l_j=1}^{N-j-1}(u_{N-j}^2-u_{l_j}^2)\times(\cdot)\Big)\,. \tag{35}$$

The definition (35) implies that the corresponding argument (marked by the point between the brackets) is firstly multiplied by the product of the differences $u_{N-j}^2-u_{l_j}^2$. The expression obtained is differentiated j times over u_{N-j}^2, and, finally, $u_{N-j}^2\to 0$. This operation should be repeated subsequently n times. We introduce the operator $\mathsf{D}\equiv\mathsf{D}_{N-n+1,N-n+2,\dots,N}(\mathbf{u})$ to denote n-fold subsequent application of (35):

$$\mathsf{D}_{N-n+1,N-n+2,\dots,N}(\mathbf{u}) \equiv \prod_{j=0}^{n-1}\mathsf{D}_{N-j}(\mathbf{u})\,. \tag{36}$$

The usage of (36) enables derivation of the average for $\bar{\mathsf{F}}_n$ by means of differentiation of the corresponding representation $\langle\Psi_N(\mathbf{v})\mid\Psi_N(\mathbf{u})\rangle$ (22) either in the series or the determinant

form:

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle \;=\; \mathsf{D}_{N-n+1,N-n+2,\ldots,N}(\mathbf{u}) \Big(\sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{K}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \Big)$$

$$= \mathsf{D}_{N-n+1,N-n+2,\ldots,N}(\mathbf{u}) \left(\frac{\det(T_{kj})_{1 \le k,j \le N}}{\mathcal{V}(\mathbf{u}^2)\mathcal{V}(\mathbf{v}^{-2})} \right), \tag{37}$$

where we introduced $\mathcal{K} \equiv M + 1 - N$. It is tacitly assumed that $\mathbf{u}$ implies the sequence $(u_1, u_2, \ldots, u_{N-n})$ of the "length" $N - n$ in the case of the state-vector $\mid \Psi_{N-n}(\mathbf{u}) \rangle$ in left-hand side of (37). The auxiliary values $u_{N-n+1}^2, \ldots, u_N^2$ (which are arbitrary) are used in right-hand sides of (37) formally, only in order to express $\langle \Psi_N(\mathbf{v}) \mid \Psi_N(\mathbf{u}) \rangle$.

Using (22) in the series form, one can evaluate the form-factor (37) in terms of the Schur functions:

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle \;=\; \prod_{l=1}^{N-n} u_l^{2n} \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{K}^{N-n}\}} S_{\hat{\boldsymbol{\lambda}}}(\mathbf{v}^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}^2)\,. \tag{38}$$

The summation in (38) runs over the partitions $\boldsymbol{\lambda}$ of the "length" $N - n$:

$$M - N + 1 (= \mathcal{K}) \ge \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{N-n} \ge 0\,.$$

Besides, we used in (38) the notation for the partition $\hat{\boldsymbol{\lambda}}$ of the length N, $\hat{\boldsymbol{\lambda}} \equiv (\boldsymbol{\lambda}, \mathbf{0})$, where the 'hat' points out that $\hat{\lambda}_{N-n+1} = \hat{\lambda}_{N-n+2} = \cdots = \hat{\lambda}_N = 0$.

Using the explicit form of the entries T_{kj} (23), we directly evaluate (37) in the determinant form:

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle \;=\; \frac{\det(\bar{T}_{kj})_{1 \le k,j \le N}}{\mathcal{V}_{N-n}(\mathbf{u}^2)\mathcal{V}_N(\mathbf{v}^{-2})}\,, \tag{39}$$

where

$$\begin{aligned} &\bar{T}_{kj} = T_{kj}\,, && 1 \le k \le N - n, \quad 1 \le j \le N\,, \\ &\bar{T}_{kj} = v_j^{-2(N-k)}\,, && N - n + 1 \le k \le N,\ 1 \le j \le N\,. \end{aligned} \tag{40}$$

The notations $\mathcal{V}_N$ and $\mathcal{V}_{N-n}$ are chosen for the Vandermond determinants (24) so that the subscripts imply that in the second case the indices in the product are restricted as follows: $1 \le m < l \le N - n$. The matrix in right-hand side of (39) consists of two rectangular blocks. It is crucial that right-hand sides of (38) and (39) mutually coincide. This is a by-product of the calculation of the average $\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle$ in two ways.

Doing analogously, we obtain the average $\langle \Psi_{N-n}(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_N(\mathbf{u}) \rangle$ in the determinant form:

$$\langle \Psi_{N-n}(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_N(\mathbf{u}) \rangle \;=\; \mathsf{D}_{N-n+1,N-n+2,\ldots,N}(\mathbf{v}^{-1}) \big(\langle \Psi_N(\mathbf{v}) \mid \Psi_N(\mathbf{u}) \rangle \big)$$

$$= \frac{\det(\tilde{T}_{kj})_{1 \le k,j \le N}}{\mathcal{V}_{N-n}(\mathbf{v}^{-2})\mathcal{V}_N(\mathbf{u}^2)}\,, \tag{41}$$

where

$$\begin{aligned} &\tilde{T}_{kj} = T_{kj}\,, && 1 \le k \le N, \quad 1 \le j \le N - n\,, \\ &\tilde{T}_{kj} = u_j^{2(N-k)}\,, && 1 \le k \le N,\ N - n + 1 \le j \le N\,. \end{aligned} \tag{42}$$

The structure of the blockwise matrix $\tilde{T}$ is clear after the notations in (42). On the other hand, the following relation takes place:

$$\langle \Psi_{N-n}(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_N(\mathbf{u}) \rangle \;=\; \prod_{l=1}^{N-n} v_l^{-2n} \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{K}^{N-n}\}} S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2}) S_{\hat{\boldsymbol{\lambda}}}(\mathbf{u}^2)\,, \tag{43}$$

where the summation is the same as in (38). Right-hand sides of (41) and (43) coincide.

4.2 Survival Probability of Ferromagnetic String

Let us recall, following [33], the main relations concerning the survival probability of the ferromagnetic string and its relationship to the random walks. For instance, it is of interest to evaluate the nominator of (16) parametrized by arbitrary $\mathbf{u}$ and $\mathbf{v}$:

$$\begin{aligned} &\langle \Psi_N(\mathbf{v}) \mid \bar{\Pi}_n\, e^{-\beta \hat{H}}\, \bar{\Pi}_n \mid \Psi_N(\mathbf{u}) \rangle \\ &= \sum_{\boldsymbol{\lambda}^L,\, \boldsymbol{\lambda}^R \subseteq \{(M-N+1/n)^N\}} S_{\boldsymbol{\lambda}^L}(\mathbf{v}^{-2}) S_{\boldsymbol{\lambda}^R}(\mathbf{u}^2) F_{\boldsymbol{\mu}^L;\, \boldsymbol{\mu}^R}(\beta)\,, \end{aligned} \tag{44}$$

where (as well as below) $\hat{H}$ implies $\hat{H}_{XX}$. Summations in (44) run over two non-strict partitions $\boldsymbol{\lambda}^L$ and $\boldsymbol{\lambda}^R$ of the same kind as in (28). Superscripts L and R are to distinguish two independent summations. The corresponding strict partitions $\boldsymbol{\mu}^L$ and $\boldsymbol{\mu}^R$ are given as follows: $\boldsymbol{\mu}^{L,R} = \boldsymbol{\lambda}^{L,R} + \boldsymbol{\delta}$, where $\boldsymbol{\delta} \equiv (\delta_1, \delta_2, \ldots, \delta_N)$, $\delta_j = N - j$. The notation $F_{\boldsymbol{\mu}^L;\, \boldsymbol{\mu}^R}(\beta)$ implies the following average:

$$\begin{aligned} F_{\boldsymbol{\mu}^L;\, \boldsymbol{\mu}^R}(\beta) \;&\equiv\; F_{\mu_1^L, \mu_2^L, \ldots, \mu_N^L; \mu_1^R, \mu_2^R, \ldots, \mu_N^R}(\beta) \;= \\ &=\; \langle \Uparrow | \sigma^+_{\mu_1^L} \sigma^+_{\mu_2^L} \ldots \sigma^+_{\mu_N^L} e^{-\beta \hat{H}} \sigma^-_{\mu_1^R} \sigma^-_{\mu_2^R} \ldots \sigma^-_{\mu_N^R} | \Uparrow \rangle\,, \end{aligned} \tag{45}$$

which is nothing but $2N$-point correlation function over the fully polarized state $|\Uparrow\rangle$. The average $F_{\boldsymbol{\mu}^L;\, \boldsymbol{\mu}^R}(\beta)$ is related to enumeration of admissible trajectories which are traced by N vicious walkers travelling over sites of one-dimensional chain [29–33]. The relation (44) is clearly reduced to (30) at $\beta = 0$.

Furthermore, let $|P_K(\mu_1^R, \ldots, \mu_N^R \to \mu_1^L, \ldots, \mu_N^L)|$ be a number of trajectories consisting of K links made by N vicious walkers in the random turns model. Here, the initial and final positions of the walkers on the sites are given respectively by parts of the strict decreasing partitions $\mu_1^R > \mu_2^R > \cdots > \mu_N^R$ and $\mu_1^L > \mu_2^L > \cdots > \mu_N^L$. Let $\mathcal{D}_\ell^K$ be the differentiation operator of Kth order with respect to ℓ at $\ell = 0$ [32]. Then, the average (44) turns out to be the generating function of the polynomials dependent on $2N$ variables, $u_1^2, u_2^2, \ldots, u_N^2$ and $v_1^{-2}, v_2^{-2}, \ldots, v_N^{-2}$, as follows [33]:

$$\begin{aligned} &\mathcal{D}^K_{\beta/2} \Big[\langle \Psi_N(\mathbf{v}) \mid \bar{\Pi}_n\, e^{-\beta \hat{H}}\, \bar{\Pi}_n \mid \Psi_N(\mathbf{u}) \rangle \Big] \;= \\ &= \sum_{\boldsymbol{\lambda}^L,\, \boldsymbol{\lambda}^R \subseteq \{(M-N+1/n)^N\}} |P_K(\boldsymbol{\mu}^R \to \boldsymbol{\mu}^L)|\, S_{\boldsymbol{\lambda}^L}(\mathbf{v}^{-2})\, S_{\boldsymbol{\lambda}^R}(\mathbf{u}^2) \end{aligned} \tag{46}$$

(remind that $\boldsymbol{\mu}^{L,R} = \boldsymbol{\lambda}^{L,R} + \boldsymbol{\delta}$). The number of trajectories consisting of K links, which are traced by N vicious walkers on an axis, i.e, $|P_K(\boldsymbol{\mu}^R \to \boldsymbol{\mu}^L)|$, is expressible through

the number of trajectories of the same "length" K traced by a single walker travelling over N-dimensional lattice of infinite extension [32].

The correlator (45) respects the following equation:

$$\frac{d}{d\beta}\, F_{\mu_1^L,\mu_2^L,\ldots,\mu_N^L;\mu_1^R,\mu_2^R,\ldots,\mu_N^R}(\beta) \;= \\ = \frac{1}{2}\sum_{k=1}^{N}\Big(F_{\mu_1^L,\mu_2^L,\ldots,\mu_N^L;\mu_1^R,\mu_2^R,\ldots,\mu_k^R+1,\ldots,\mu_N^R}(\beta) \;+\; F_{\mu_1^L,\mu_2^L,\ldots,\mu_N^L;\mu_1^R,\mu_2^R,\ldots,\mu_k^R-1,\ldots,\mu_N^R}(\beta)\Big)\,. \tag{47}$$

Equation (47) has been considered in [30] for the case of the periodic boundary condition with respect to the lattice argument and with the initial condition:

$$F_{\mu_1^L,\mu_2^L,\ldots,\mu_N^L;\mu_1^R,\mu_2^R,\ldots,\mu_N^R}(0) = \prod_{k=1}^{N}\delta_{\mu_k^L,\mu_k^R}\,.$$

Solution to (47) can be expressed as the determinant of the matrix $\big(F_{k;\,l}(\beta)\big)_{1\le k,l\le N}$ [29,30]:

$$F_{\mu_1^L,\mu_2^L,\ldots,\mu_N^L;\mu_1^R,\mu_2^R,\ldots,\mu_N^R}(\beta) \;=\; \det\big(F_{\mu_k^L;\mu_l^R}(\beta)\big)_{1\le k,l\le N}\,, \tag{48}$$

where the entries respect the following difference-differential equation:

$$\frac{d}{d\beta}\,F_{k;\,l}(\beta) = \frac{1}{2}\left(F_{k+1;l}(\beta) \;+\; F_{k-1;l}(\beta)\right). \tag{49}$$

Similar equation can be also obtained for the fixed index l.

It can be checked that the transition amplitude $\langle\Uparrow|\sigma_k^+ e^{-\beta\hat{H}}\sigma_l^-|\Uparrow\rangle$ respects (49), [29–32]. This average can be considered as the generating function of the number of the random turn walks of a single pedestrian travelling between l^{th} and k^{th} sites of (periodic) chain [30, 32]. The corresponding solution to (49) can be written as the following sum:

$$F_{k;\,l}(\beta) \;\equiv\; \frac{1}{M+1}\sum_{s=0}^{M} e^{\beta\cos\phi_s}\, e^{i\phi_s(k-l)}\,, \tag{50}$$

where the parametrization $\phi_s = \dfrac{2\pi}{M+1}\Big(s-\dfrac{M}{2}\Big)$ is used. The periodicity condition with respect of the lattice argument and the "initial" condition $F_{k;\,l}(0)=\delta_{k,l}$ are imposed.

Using (48) and (50), we re-express (45) through the Schur functions (14) and the Vandermonde determinants (24) as follows [29]:

$$F_{\boldsymbol{\mu}^L;\,\boldsymbol{\mu}^R}(\beta) \;=\; \frac{1}{(M+1)^N}\sum_{M\ge k_1>k_2\cdots>k_N\ge 0} e^{\beta\sum\limits_{l=1}^{N}\cos(\phi_{k_l})} \\ \times|\mathcal{V}(e^{i\boldsymbol{\phi}})|^2\, S_{\boldsymbol{\lambda}^L}(e^{i\boldsymbol{\phi}})S_{\boldsymbol{\lambda}^R}(e^{-i\boldsymbol{\phi}})\,. \tag{51}$$

The same parametrization ϕ_s, $0\le s\le M$, are used both in (50) and (51). We continue to use bold-faced letters to denote N-tuples of numbers: for instance, $\boldsymbol{\phi}$ corresponds to

$(\phi_{k_1}, \phi_{k_2}, \ldots, \phi_{k_N})$. We substitute (51) into (44), and use (30) in order to calculate the sums:

$$\mathcal{P}(\mathbf{v}^{-2}, e^{i\boldsymbol{\phi}}) \equiv \sum_{\boldsymbol{\lambda}^L} S_{\boldsymbol{\lambda}^L}(\mathbf{v}^{-2})\, S_{\boldsymbol{\lambda}^L}(e^{i\boldsymbol{\phi}})\,, \quad \mathcal{P}(e^{-i\boldsymbol{\phi}}, \mathbf{u}^2) \equiv \sum_{\boldsymbol{\lambda}^R} S_{\boldsymbol{\lambda}^R}(e^{-i\boldsymbol{\phi}})\, S_{\boldsymbol{\lambda}^R}(\mathbf{u}^2)\,. \tag{52}$$

The range of summation in (52) is taken as follows: $\boldsymbol{\lambda}^L$, $\boldsymbol{\lambda}^R \subseteq \{(M-N+1/n)^N\}$. Then, we obtain:

$$\begin{aligned}
&\langle \Psi_N(\mathbf{v}) \mid \bar{\Pi}_n\, e^{-\beta\hat{H}}\, \bar{\Pi}_n \mid \Psi_N(\mathbf{u})\rangle \;= \\
&= \frac{1}{(M+1)^N} \sum_{M\geq k_1>k_2\cdots>k_N\geq 0} e^{-\beta E_N(\boldsymbol{\phi})} |\mathcal{V}(e^{i\boldsymbol{\theta}})|^2 \mathcal{P}(\mathbf{v}^{-2}, e^{i\boldsymbol{\theta}}) \mathcal{P}(e^{-i\boldsymbol{\theta}}, \mathbf{u}^2) \\
&= \frac{\prod_{l=1}^N u_l^{2n} v_l^{-2n}}{(M+1)^N\, \mathcal{V}(\mathbf{u}^2)\mathcal{V}(\mathbf{v}^{-2})} \sum_{M\geq k_1>k_2\cdots>k_N\geq 0} e^{-\beta E_N(\boldsymbol{\phi})} \\
&\times \det\Big(\frac{1-(e^{i\phi_{k_i}} v_j^{-2})^{M-n+1}}{1-e^{i\phi_{k_i}} v_j^{-2}}\Big)_{1\leq i,j\leq N} \det\Big(\frac{1-(u_p^2 e^{-i\phi_{k_l}})^{M-n+1}}{1-u_p^2 e^{-i\phi_{k_l}}}\Big)_{1\leq p,l\leq N}\,.
\end{aligned} \tag{53}$$

The Binet–Cauchy formula enables to evaluate (53):

$$\begin{aligned}
&\langle \Psi_N(\mathbf{v}) \mid \bar{\Pi}_n\, e^{-\beta\hat{H}}\, \bar{\Pi}_n \mid \Psi_N(\mathbf{u})\rangle \;= \\
&= \frac{1}{\mathcal{V}(\mathbf{u}^2)\mathcal{V}(\mathbf{v}^{-2})} \det\left(\sum_{k,l=n}^{M} F_{k;\,l}(\beta)\, \frac{u_i^{2l}}{v_j^{2k}}\right)_{1\leq i,j\leq N}\,,
\end{aligned} \tag{54}$$

where $F_{k;\,l}(\beta)$ is defined by (50). Clearly, the relation (54) is reduced, at $\beta = 0$, to (30).

We take into account that

$$\langle \Psi_N(\mathbf{v}) \mid e^{-\beta\hat{H}} \mid \Psi_N(\boldsymbol{\theta})\rangle \;=\; \langle \Psi_N(\mathbf{v}) \mid \Psi_N(\boldsymbol{\theta})\rangle\, e^{-\beta E_N(\boldsymbol{\theta})}\,, \tag{55}$$

where the energy $E_N(\boldsymbol{\theta})$ is given by (11). Further, expression for $\mathcal{T}(\boldsymbol{\theta}^{\mathrm{v}}, n, \beta)$ (16) can straightforwardly be obtained from (54):

$$\mathcal{T}(\boldsymbol{\theta}^{\mathrm{v}}, n, \beta) \;=\; \frac{e^{\beta E_N(\boldsymbol{\theta}^{\mathrm{v}})}}{(M+1)^N} \det\left(\sum_{k,l=n}^{M} F_{k;\,l}(\beta)\, e^{i(l\theta_i^{\mathrm{v}} - k\theta_j^{\mathrm{v}})}\right)_{1\leq i,j\leq N}\,, \tag{56}$$

where $E_N(\boldsymbol{\theta}^{\mathrm{v}})$ is given by (13).

4.3 Survival Probability of Domain Wall

With the representations (39) and (41) at hands, let us turn to the correlation functions of the one-dimensional domain wall (18). We begin with the nominator of (18) parametrized arbitrarily:

$$\langle \Psi_{N-n}(\mathbf{v}) \mid \bar{\mathsf{F}}_n\, e^{-\beta\hat{H}}\, \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u})\rangle\,. \tag{57}$$

The solutions of the Bethe equation (9) constitute a complete set of the eigen-states. Taking into account their orthogonality, one can consider the resolution of the identity operator:

$$\mathbb{I} \;=\; \sum_{\{\boldsymbol{\theta}\}} \mathcal{N}^{-2}(\boldsymbol{\theta}) \mid \Psi_N(\boldsymbol{\theta})\rangle\langle \Psi_N(\boldsymbol{\theta}) \mid\,, \tag{58}$$

where summation goes over all independent solutions of the Bethe equation (9), and the square of the norm $\mathcal{N}^2(\boldsymbol{\theta})$ is given by (26). We shall calculate (57) inserting (58) and using appropriately (37) and (39). We take into account (55) and re-express (57) as follows:

$$\langle \Psi_{N-n}(\mathbf{v}) \mid \bar{\mathsf{F}}_n \, e^{-\beta \hat{H}} \, \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle \; = \\ = \sum_{\{\boldsymbol{\theta}\}} \langle \Psi_{N-n}(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_N(\boldsymbol{\theta}) \rangle \langle \Psi_N(\boldsymbol{\theta}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle \, \frac{e^{-\beta E_N(\boldsymbol{\theta})}}{\mathcal{N}^2(\boldsymbol{\theta})} \, . \tag{59}$$

In order to evaluate (59) further, we use the differentiation representations (37) and (41) for each form-factor , and straightforwardly obtain:

$$\begin{aligned} {}_{N-n}\langle \bar{\mathsf{F}}_n \, e^{-\beta \hat{H}} \, \bar{\mathsf{F}}_n \rangle_{N-n} \; &= \; (M+1)^{-N} \, \mathsf{D}_{N-n+1,N-n+2,\ldots,N}(\mathbf{v}^{-1}) \, \mathsf{D}_{N-n+1,N-n+2,\ldots,N}(\mathbf{u}) \\ &\times \Bigg(\frac{1}{\mathcal{V}_N(\mathbf{u}^2)\mathcal{V}_N(\mathbf{v}^{-2})} \sum_{M \geq I_1 > I_2 \cdots > I_N \geq 0} e^{-\beta E_N(\boldsymbol{\theta})} \det\Big(\frac{1-(e^{i\theta_i} v_j^{-2})^{M+1}}{1-e^{i\theta_i} v_j^{-2}} \Big)_{1 \leq i,j \leq N} \\ &\times \; \det\Big(\frac{1-(u_p^2 e^{-i\theta_l})^{M+1}}{1-u_p^2 e^{-i\theta_l}} \Big)_{1 \leq p,l \leq N} \Bigg) \, . \end{aligned} \tag{60}$$

Let us introduce the operators:

$$\widetilde{\mathsf{D}}_{N-n+1,N-n+2,\ldots,N}(\mathbf{u}) \; \equiv \; \prod_{j=0}^{n-1} \Big(\lim_{u_{N-j}^2 \to 0} \frac{1}{j\,!} \, \frac{\mathrm{d}^j}{\mathrm{d}(u_{N-j}^2)^j} \Big) \, . \tag{61}$$

In terms of $\widetilde{\mathsf{D}}$ (61), we re-express (60):

$$\begin{aligned} {}_{N-n}\langle \bar{\mathsf{F}}_n \, e^{-\beta \hat{H}} \, \bar{\mathsf{F}}_n \rangle_{N-n} \; &= \; \frac{1}{(M+1)^N \mathcal{V}_{N-n}(\mathbf{u}^2)\mathcal{V}_{N-n}(\mathbf{v}^{-2})} \, \widetilde{\mathsf{D}}_{N,N-1,\ldots,N-n+1}(\mathbf{v}^{-1}) \\ &\times \widetilde{\mathsf{D}}_{N,N-1,\ldots,N-n+1}(\mathbf{u}) \Bigg(\sum_{M \geq I_1 > I_2 \cdots > I_N \geq 0} e^{-\beta E_N(\boldsymbol{\theta})} \det\Big(\frac{1-(e^{i\theta_i} v_j^{-2})^{M+1}}{1-e^{i\theta_i} v_j^{-2}} \Big)_{1 \leq i,j \leq N} \\ &\times \; \det\Big(\frac{1-(u_p^2 e^{-i\theta_l})^{M+1}}{1-u_p^2 e^{-i\theta_l}} \Big)_{1 \leq p,l \leq N} \Bigg) \, . \end{aligned} \tag{62}$$

Applying the Binet-Cauchy formula (just like to pass from (53) to (54)), we obtain from (62):

$$\begin{aligned} {}_{N-n}\langle \bar{\mathsf{F}}_n \, e^{-\beta \hat{H}} \, \bar{\mathsf{F}}_n \rangle_{N-n} \; &= \; \frac{1}{\mathcal{V}_{N-n}(\mathbf{u}^2)\mathcal{V}_{N-n}(\mathbf{v}^{-2})} \, \widetilde{\mathsf{D}}_{N-n+1,N-n+2,\ldots,N}(\mathbf{v}^{-1}) \\ &\times \widetilde{\mathsf{D}}_{N-n+1,N-n+2,\ldots,N}(\mathbf{u}) \det \left(\sum_{k,l=0}^{M} F_{k;\,l}(\beta) \, \frac{u_i^{2l}}{v_j^{2k}} \right)_{1 \leq i,j \leq N} \, . \end{aligned} \tag{63}$$

After the differentiations, the representation (63) takes the form:

$${}_{N-n}\langle \bar{\mathsf{F}}_n \, e^{-\beta \hat{H}} \, \bar{\mathsf{F}}_n \rangle_{N-n} \; = \; \frac{1}{\mathcal{V}_{N-n}(\mathbf{u}^2)\mathcal{V}_{N-n}(\mathbf{v}^{-2})} \det{}_N \begin{pmatrix} \mathbf{A} \, \mathbf{B} \\ \mathbf{C} \, \mathbf{D} \end{pmatrix} , \tag{64}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are the rectangle matrices with the entries dependent on the parameters $\mathbf{u}$ and $\mathbf{v}$:

$$A_{ij} \equiv \sum_{k,l=0}^{M} F_{k;\,l}(\beta)\,\frac{u_i^{2l}}{v_j^{2k}}\,, \quad 1 \le i,j \le N-n\,,$$

$$B_{ij} \equiv \sum_{l=0}^{M} F_{n-j;\,l}(\beta)\,u_i^{2l}\,, \quad 1 \le i \le N-n\,,\ 1 \le j \le n\,,$$

$$C_{ij} \equiv \sum_{l=0}^{M} F_{l;\,n-i}(\beta)\,v_j^{-2l}\,, \quad 1 \le i \le n\,,\ 1 \le j \le N-n\,,$$

$$D_{ij} \equiv F_{n-i;\,n-j}(\beta)\,, \qquad 1 \le i,j \le n\,.$$

Eventually, $\mathcal{F}(\boldsymbol{\theta}^{\rm v}, n, \beta)$ (18) acquires the form:

$$\mathcal{F}(\boldsymbol{\theta}^{\rm v}, n, \beta) \;=\; \frac{e^{\beta E_N(\boldsymbol{\theta}^{\rm v})}}{(M+1)^{N-n}}\,\det{}_N\begin{pmatrix} \mathbf{A}|_{\boldsymbol{\theta}^{\rm v}} & \mathbf{B}|_{\boldsymbol{\theta}^{\rm v}} \\ \mathbf{C}|_{\boldsymbol{\theta}^{\rm v}} & \mathbf{D}|_{\boldsymbol{\theta}^{\rm v}} \end{pmatrix}, \tag{65}$$

where $\mathbf{A}|_{\boldsymbol{\theta}^{\rm v}} = \lim_{\mathbf{u},\mathbf{v}\to\boldsymbol{\theta}^{\rm v}} \mathbf{A}$ (the same for $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$).

5 Boxed plane partitions

Let us demonstrate that the scalar products of the state vectors and the form-factors obtained are related to the generating functions of the boxed plane partitions. Recall that an array $(\pi_{i,j})_{i,j\,\ge 1}$ of non-negative integers that are non-increasing as functions both of i and j ($i,j \in \{1,2,\dots\}$ is called a *plane partition* $\boldsymbol{\pi}$ [40]. The integers $\pi_{i,j}$ are called the *parts* of the plane partition, and $|\boldsymbol{\pi}| = \sum_{i,j\,\ge 1} \pi_{i,j}$ is its volume. Each plane partition has a three dimensional diagram which can be interpreted as a stack of unit cubes (three-dimensional Young tableau). The height of a stack with coordinates (i,j) is equal to $\pi_{i,j}$. It is said that the plane partition corresponds to a box of the size $L \times N \times P$ provided $i \le L$, $j \le N$ and $\pi_{i,j} \le P$ for all cubes of the Young tableau. If $\pi_{i,j} > \pi_{i+1,j}$, i.e. if the parts of plane partition $\boldsymbol{\pi}$ are decaying along each column, then $\boldsymbol{\pi}$ is called the *column strict plane partition* (cspp). We shall call $\boldsymbol{\pi}$ as the *strict plane partition* (spp) provided the decaying occurs along each column and each raw ($\pi_{i,j} > \pi_{i+1,j}$ and $\pi_{i,j} > \pi_{i,j+1}$). The part $\pi_{1,1}$ of the strict plane partition $\boldsymbol{\pi}$ satisfies the condition $\pi_{1,1} \ge 2N-2$, if all $i,j \le N$.

An arbitrary plane partition corresponding to a box $N \times N \times P$ may be transferred into a column strict plane partition for a box $N \times N \times (P+N-1)$ by adding to an array $(\pi_{i,j})_{i,j\,\ge 1}$ the $N \times N$ matrix

$$\boldsymbol{\pi}_{\rm cspp} = \begin{pmatrix} N-1 & N-1 & \cdots & N-1 \\ N-2 & N-2 & \cdots & N-2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

which corresponds to a minimal column strict plane partition. The volumes of the column strict plane partition and correspondent plane partition are related:

$$|\boldsymbol{\pi}_{\rm cspp}| = |\boldsymbol{\pi}| + \frac{N^2(N-1)}{2}.$$

An arbitrary plane partition with a box $N \times N \times P$ may be transferred into a strict plane partition with a box $N \times N \times (P + 2N - 2)$ by adding to an array $(\pi_{i,j})_{i,j\geq 1}$ the $N \times N$ matrix

$$\boldsymbol{\pi}_{\rm spp} = \begin{pmatrix} 2N-2 & 2N-3 & \cdots & N-1 \\ 2N-3 & 2N-4 & \cdots & N-2 \\ \vdots & \vdots & & \vdots \\ N-1 & N-2 & \cdots & 0 \end{pmatrix},$$

which corresponds to a minimal strict plane partition. The volumes of the strict plane partition and correspondent plane partition are related:

$$|\boldsymbol{\pi}_{\rm spp}| = |\boldsymbol{\pi}| + N^2(N-1).$$

The partition function of the three-dimensional Young tableaux, or saying differently the generating function of plane partitions is equal to

$$Z_q(L,N,P) = \sum_{\{\boldsymbol{\pi}\}} q^{|\boldsymbol{\pi}|}, \tag{66}$$

where $q^{|\boldsymbol{\pi}|}$ is the Boltzmann weight, and summation is over all plane partitions confined in the $L \times N \times P$ box. The proof that

$$Z_q(L,N,P) \;=\; \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} \;=\; \prod_{j=1}^{L}\prod_{k=1}^{N}\frac{1-q^{P+j+k-1}}{1-q^{j+k-1}} \tag{67}$$

may be found in [40] (it is assumed that $L < N$ to justify the second equality in right-hand side of (67)). According to the classical formula of MacMahon, there are exactly

$$A(L,N,P) \;=\; \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P}\frac{i+j+k-1}{i+j+k-2} \;=\; \prod_{j=1}^{L}\prod_{k=1}^{N}\frac{P+j+k-1}{j+k-1} \tag{68}$$

plane partitions contained in the $L \times N \times P$ box. It is clear that right-hand side of (67) tends to $A(L,N,P)$ is the $q \to 1$ limit.

It is of interest for us here that the generating function of the column strict partitions placed into $N \times N \times P$ box is calculated as follows:

$$Z_q^{\rm cspp}(N,N,P) \;=\; q^{\frac{N^2}{2}(N-1)} \prod_{1\leq j,k\leq N}\frac{1-q^{P+1+j-k}}{1-q^{j+k-1}}\,. \tag{69}$$

The limit $q \to 1$ gives the number of the column strict partitions placed into $N \times N \times P$ box, $Z_{q=1}^{\rm cspp}(N,N,P)$, as follows:

$$\begin{aligned} Z_1^{\rm cspp}(N,N,P) \;&=\; \prod_{1\leq j,k\leq N}\frac{P+1+j-k}{j+k-1} \\ &= \prod_{j=1}^{N}\frac{\Gamma(j)\,\Gamma(j+P+1)}{\Gamma(j+N)\,\Gamma(j+P+1-N)}\,. \end{aligned} \tag{70}$$

Expression for $Z^{\rm cspp}_{q=1}(N,N,P)$ in terms of the gamma-functions [53] will be appropriately used below.

Let us demonstrate that the scalar product $\langle\Psi_N(\mathbf{v})\mid\Psi_N(\mathbf{u})\rangle$ (22) is related to $Z^{\rm cspp}_q(N,N,M)$. We use the parametrizations $v_j=q^{-\frac{j}{2}}$ and $u_j=q^{\frac{j-1}{2}}$, and obtain:

$$\begin{aligned}\langle\Psi_N(\mathbf{q}^{-\frac12})\,|\,\Psi_N((\mathbf{q}/q)^{\frac12})\rangle \;&=\; \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{K}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{q})\,S_{\boldsymbol{\lambda}}(\mathbf{q}/q)\;=\\ =\;\mathcal{V}^{-1}(\mathbf{q})\mathcal{V}^{-1}(\mathbf{q}/q)\,&\det\Big(\frac{1-q^{(M+1)(j+k-1)}}{1-q^{j+k-1}}\Big)_{1\le j,k\le N}\,,\end{aligned}\tag{71}$$

where $\mathbf{q}\equiv(q,q^2,\ldots,q^N)$. The product of the Vandermonde determinants is calculated:

$$\mathcal{V}(\mathbf{q})\mathcal{V}(\mathbf{q}/q)\;=\;q^{\frac{N}{6}(N-1)(2N-1)}\prod_{1\le k<j\le N}\big(1-q^{j-k}\big)^2\,.\tag{72}$$

The scalar product (71) has been calculated in [33] using the value of the corresponding determinant according to the relation

$$\det\Big(\frac{1-s^{j+k-1}}{1-q^{j+k-1}}\Big)_{1\le j,k\le N}\;=\;\mathcal{V}(\mathbf{q})\mathcal{V}(\mathbf{q}/q)\prod_{k,j=1}^{N}\frac{1-sq^{j-k}}{1-q^{j+k-1}}\,.\tag{73}$$

The relation (73) (would be called as the *Kuperberg's formula*) can be extracted from [45] where the *alternating sign matrices enumeration problem* has been investigated. Generally, s and q are independent parameters in (73). One obtains the determinant (71) at $s=q^{M+1}$. Taking into account (69) and (73), we obtain for (71):

$$\langle\Psi_N(\mathbf{q}^{-\frac12})\,|\,\Psi_N((\mathbf{q}/q)^{\frac12})\rangle\;=\;q^{-\frac{N^2}{2}(N-1)}Z^{\rm cspp}_q(N,N,M)\,.\tag{74}$$

Equation (74) reads that the scalar product of two state-vectors coincides at $q=1$ with $Z^{\rm cspp}_{q=1}(N,N,M)$, i.e., with the number of column strict partitions in $N\times N\times M$ box.

Now lets turn to the expectation value of the ferromagnetic string (30). With regard at (72) and (73), we obtain under the given parametrization:

$$\begin{aligned}\langle\Psi_N(\mathbf{q}^{-\frac12})\mid\bar{\Pi}_n\,|\,\Psi_N((\mathbf{q}/q)^{\frac12})\rangle\;&=\;\sum_{\boldsymbol{\lambda}\subseteq\{(\mathcal{K}/n)^N\}}S_{\boldsymbol{\lambda}}(\mathbf{q})\,S_{\boldsymbol{\lambda}}(\mathbf{q}/q)\;=\\ =\;q^{nN^2}\prod_{k,j=1}^{N}\frac{1-q^{M-n+1+j-k}}{1-q^{j+k-1}}\;&=\;q^{\frac{N^2}{2}(2n+1-N)}Z^{\rm cspp}_q(N,N,M-n)\,.\end{aligned}\tag{75}$$

The average in question is related to the generating function of column strict plane partitions in a box of the smaller size $N\times N\times(M-n)$. The expectation value in left-hand side of (75), being considered at $q\to1$, gives the corresponding number of strict plane partitions:

$$\lim_{q\to1}\langle\Psi_N(\mathbf{q}^{-\frac12})\mid\bar{\Pi}_n\,|\,\Psi_N((\mathbf{q}/q)^{\frac12})\rangle\;=\;Z^{\rm cspp}_1(N,N,M-n)\,.\tag{76}$$

In the same limit, $q\to1$, the corresponding value of the Schur function is given by the ratio of two Vandermonde determinants [40]:

$$S_{\boldsymbol{\lambda}}(\mathbf{1})\;\equiv\;S_{\boldsymbol{\lambda}}(1,1,\ldots,1)\;=\;\mathcal{V}^{-1}(\boldsymbol{\delta})\,\mathcal{V}(\boldsymbol{\lambda}-\boldsymbol{\delta})\,.\tag{77}$$

Right-hand side of (77) coincides with the dimensionality $\mathsf{d}_{\boldsymbol{\lambda}} \equiv \dim \boldsymbol{\pi}_{\boldsymbol{\lambda}}$ of the unitary irreducible representations $\boldsymbol{\pi}_{\boldsymbol{\lambda}}$ with the signature $\boldsymbol{\lambda}$ of the unitary group $\mathcal{U}(N)$. Therefore, $\mathsf{d}_{\boldsymbol{\lambda}} = S_{\boldsymbol{\lambda}}(\mathbf{1})$ [46]. In the limit $q \to 1$, the generating function (46) is specified as follows:

$$\begin{aligned} &\lim_{q\to 1} \mathcal{D}^K_{\beta/2}\left[\langle \Psi_N(\mathbf{q}^{-\frac{1}{2}}) \mid \bar{\Pi}_n\, e^{-\beta\hat{H}}\, \bar{\Pi}_n \mid \Psi_N((\mathbf{q}/q)^{\frac{1}{2}})\rangle\right] = \\ &= \sum_{\boldsymbol{\lambda}^L,\, \boldsymbol{\lambda}^R \subseteq \{(\mathcal{K}/n)^N\}} |P_K(\boldsymbol{\mu}^R \to \boldsymbol{\mu}^L)|\, \mathsf{d}_{\boldsymbol{\lambda}^L}\, \mathsf{d}_{\boldsymbol{\lambda}^R}\,, \end{aligned} \tag{78}$$

where strict and non-strict partitions are related: $\boldsymbol{\mu}^{L,R} = \boldsymbol{\lambda}^{L,R} + \boldsymbol{\delta}$.

The numbers $\mathsf{d}_{\boldsymbol{\lambda}^{L,R}}$ in (78) (dependent "parametrically" on the partitions $\boldsymbol{\lambda}^{L,R}$) are related to appropriate values of the binomial determinants, say, $\mathsf{b}_{\bar{\boldsymbol{\lambda}}}$ (Appendix I). The binomial determinant $\mathsf{b}_{\bar{\boldsymbol{\lambda}}}$ depends on, so-called, *conjugated partition* $\bar{\boldsymbol{\lambda}}$, which is defined by means of the diagram given by transposition (i.e., by reflection with respect of the main diagonal) of the diagram of the non-strict partition $\boldsymbol{\lambda}$ [40]. The conjugated partition $\bar{\boldsymbol{\lambda}}$ is non-strict also, and its parts are: $N \geq \bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{\mathcal{K}-n} \geq 0$ [2]. Strict partition $\bar{\boldsymbol{\mu}} \equiv (\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_{\mathcal{K}-n})$, which corresponds to $\bar{\boldsymbol{\lambda}}$, is defined by the relations: $\bar{\mu}_i = \bar{\lambda}_i + \bar{\delta}_i$, $\bar{\delta}_i = \mathcal{K} - n - i$, $i \in \{1, 2, \ldots, \mathcal{K} - n\}$, i.e., $\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\lambda}} + \bar{\boldsymbol{\delta}}$. In other words, one gets:

$$N + \mathcal{K} - n - 1 = M - n \geq \bar{\mu}_1 > \bar{\mu}_2 > \cdots > \bar{\mu}_{\mathcal{K}-n} \geq 0\,.$$

Strict partition $\boldsymbol{\mu}$ enumerates the spin "down" states placed on sites of the chain of the length $M - n + 1$ (since only the sites from n^{th} to M^{th} are in the play). The other $\mathcal{K} - n = M + 1 - N - n$ sites (labelled by strict partitions, say, $\tilde{\boldsymbol{\mu}} \equiv (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_{\mathcal{K}-n})$) correspond to the spin "down" (hole) states. The parts of the strict partitions $\bar{\boldsymbol{\mu}}$ are in bijective correspondence with the parts of the partitions $\tilde{\boldsymbol{\mu}}$. The corresponding relation reads: $\tilde{\mu}_i = M - \bar{\mu}_{\mathcal{K}-n+1-i}$, $i \in \{1, 2, \ldots, \mathcal{K} - n\}$. In other words,

$$M \geq \tilde{\mu}_1 > \tilde{\mu}_2 > \cdots > \bar{\mu}_{\mathcal{K}-n} \geq n\,.$$

In order to clarify the situation with the strict partitions $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$, it is useful to remind the graphical picture (see Figure 1) which demonstrates the correspondence between strict partitions $\boldsymbol{\mu}$, $\tilde{\boldsymbol{\mu}}$ and non-strict partitions $\boldsymbol{\lambda}$, $\bar{\boldsymbol{\lambda}}$. Namely, to each partition $\boldsymbol{\lambda}$ we associate a set of N-tuples $\mathfrak{G}(\boldsymbol{\lambda})$ as follows [52]:

$$\mathfrak{G}(\boldsymbol{\lambda}) = \left\{\lambda_j - j + \frac{1}{2} \,\middle|\, 1 \leq j \leq N\right\} \subset \mathbb{Z} + \frac{1}{2}\,.$$

On another hand, any non-strict partition $\boldsymbol{\lambda}$ can be represented as a rectangular table (the Young tableau) consisting of N columns so that λ_i, $\forall i$, is the height of ith column ($\lambda_i \leq \mathcal{K} = M - N + 1$). We shift each element of the set $\mathfrak{G}(\boldsymbol{\lambda})$ by $N + \frac{1}{2}$. Then we assign the numbers thus obtained to the projections along the vertical dashed lines onto the horizontal axis. The set of points on the horizontal axis just provides the strict partition $\boldsymbol{\mu}$. For instance, the diagram on Figure 1 is drown for $M = 8$ and $N = 4$ and $n = 0$. We have got respectively: $\boldsymbol{\lambda} = (5, 3, 2, 2)$ and $\boldsymbol{\mu} = (8, 5, 3, 2)$. The corresponding conjugated non-strict and strict partitions are given by $\bar{\boldsymbol{\lambda}} = (4, 4, 2, 1, 1)$ and $\bar{\boldsymbol{\mu}} = (8, 7, 4, 2, 1)$, respectively. The vertical solid lines point out the points labelled by $\tilde{\boldsymbol{\mu}} = (7, 6, 4, 1, 0)$. The relationship between $\bar{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\mu}}$ can be most easily understood graphically (see Figure 1):

[2] Since the non-strict partition $\boldsymbol{\lambda}$ is restricted below: $\mathcal{K} \geq \lambda_1 \geq \lambda_2 \geq \ldots, \lambda_N \geq n$.

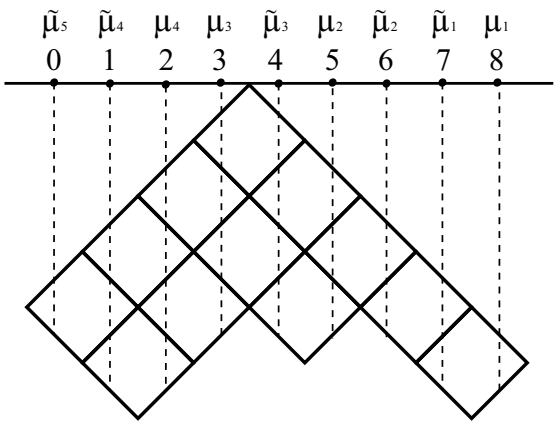

Figure 1: The partitions $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$ for $M = 8$, $N = 4$, and the corresponding Young tableau.

one should transpose the Young tableau, i.e., consider *reflection* of the picture at Figure 1 with respect of the dotted line. Moreover, $n = 2$ implies that 0th and 1st sites are "forbidden", while the sites labelled by $\boldsymbol{\mu}$ are occupied by excitations. Therefore, $\tilde{\mu}_3 = 4$, $\tilde{\mu}_2 = 6$, and $\tilde{\mu}_1 = 7$, just correspond to hole (spin "up") states.

The binomial determinant $\mathsf{b}_{\bar{\boldsymbol{\lambda}}}$ is transformed into another binomial determinant as follows:

$$\begin{aligned} &\begin{pmatrix} N, & N+1, & \ldots, & M-n-1, & M-n \\ \bar{\mu}_{\mathcal{K}-n}, & \bar{\mu}_{\mathcal{K}-n-1}, & \ldots, & \bar{\mu}_2, & \bar{\mu}_1 \end{pmatrix} = \\ &= \begin{pmatrix} N, & N+1, & \ldots, & M-n-1, & M-n \\ \tilde{\mu}_{\mathcal{K}-n}-n, & \tilde{\mu}_{\mathcal{K}-n-1}-n, & \ldots, & \tilde{\mu}_2-n, & \tilde{\mu}_1-n \end{pmatrix}. \end{aligned} \tag{79}$$

We shall not introduce a special notation for the binomial determinant in right-hand side of (79). Both the binomial determinants in (79) can be interpreted in terms of tuples of *nonintersecting* lattice paths. In the case of $\mathsf{b}_{\bar{\boldsymbol{\lambda}}}$, the lattice paths connect the points labelled by the partitions $(N, N+1, \ldots, M-n)$ and $(\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_{\mathcal{K}-n})$ [51]. For instance, the binomial determinant $\mathsf{b}_{\bar{\boldsymbol{\lambda}}} = \mathsf{b}_{(4,4,2,1,1)} \equiv \begin{pmatrix} 4,5,6,7,8 \\ 1,2,4,7,8 \end{pmatrix}$ corresponds to the strict partition $\bar{\mu} = (8,7,4,2,1)$ (see Figure 1). We present ($n = 2$, for simplicity) Figure 2 drawn for the binomial determinant $\mathsf{b}_{\bar{\boldsymbol{\lambda}}} = \mathsf{b}_{(2,1,1)} \equiv \begin{pmatrix} 4,5,6 \\ 1,2,4 \end{pmatrix}$, which corresponds to the strict partition $\tilde{\mu} = (7,6,4)$ (we depict and interpret the non-intersecting lattice paths on Figure 2 closely following the rules of [51]). In our case, the points on the plane are: $A_1 = (0,4)$, $A_2 = (0,5)$, $A_3 = (0,6)$, and $B_1 = (2,2)$, $B_2 = (4,4)$, $B_3 = (5,5)$.

Therefore, right-hand side of (78) is remarkably expressed completely in terms of the numbers of the lattice paths belonging to three tuples. The first tuple contains non-intersecting paths connecting some ordered points labelled by the partitions $(N, N+1, \ldots, M+1-n)$ and the points labelled by the partitions $\tilde{\boldsymbol{\mu}}^L$. The second tuple contains the paths between the spin "down" sites (which are just *complementary* with respect of the corresponding hole positions) labelled by the partitions $\boldsymbol{\mu}^L$ and $\boldsymbol{\mu}^R$. Third tuple includes the paths connecting the sites labelled by $\tilde{\boldsymbol{\mu}}^R$ and the ordered points $(N, N+1, \ldots, M+1-n)$. In other words, the thermal "correlator" of the projectors is also the generating functions of the complicated configuration of the lattice paths. Besides, the numbers $\mathsf{b}_{\bar{\boldsymbol{\lambda}}^L}$ and $\mathsf{b}_{\bar{\boldsymbol{\lambda}}^R}$ themselves can also be interpreted in terms of the boxed plane partitions [51].

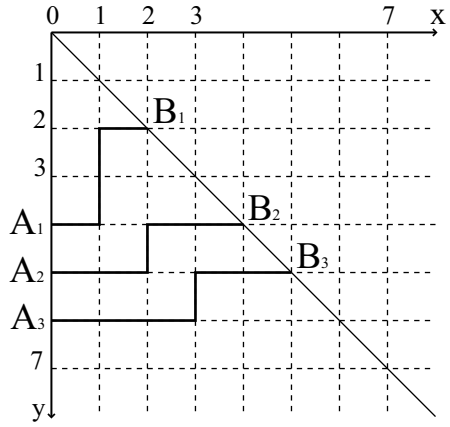

Figure 2: The tuple of non-intersecting paths for the binomial determinant in r.h.s. of (79) corresponding to $(\tilde{\mu}_1 - 2, \tilde{\mu}_2 - 2, \tilde{\mu}_3 - 2) = (5, 4, 2)$.

Now let us turn to the form-factors $\langle \Psi_N(\mathbf{v}^{-2}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}^2) \rangle$ (expressed by (38), (39)) and $\langle \Psi_{N-n}(\mathbf{v}^{-2}) \mid \bar{\mathsf{F}}_n \mid \Psi_N(\mathbf{u}^2) \rangle$ (expressed by (41), (43)) which can also be rewritten at $v_j = q^{-\frac{j}{2}}$ and $u_j = q^{\frac{j-1}{2}}$. For definiteness, let us put $\langle \Psi_N(\mathbf{v}^{-2}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}^2) \rangle$ in the q-parametrization as follows:

$$\begin{aligned} \langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle \; &= \; q^{\frac{n}{2}(N-n)(N-n-1)} \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{K}^{N-n}\}} S_{\hat{\boldsymbol{\lambda}}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q) \; = \\ &= \; \mathcal{V}_N^{-1}(\mathbf{q}) \mathcal{V}_{N-n}^{-1}(\mathbf{q}/q) \, \det \bar{\mathsf{T}} \, , \end{aligned} \tag{80}$$

where

$$\begin{aligned} \bar{\mathsf{T}}_{kj} &= \frac{1 - q^{(M+1)(j+k-1)}}{1 - q^{j+k-1}} \, , \quad 1 \le k \le N - n, \qquad 1 \le j \le N \, , \\ \bar{\mathsf{T}}_{kj} &= q^{j(N-k)} \, , \qquad\qquad N - n + 1 < k \le N, \; 1 \le j \le N \, . \end{aligned} \tag{81}$$

The notations for non-strict partitions $\hat{\boldsymbol{\lambda}}$ and $\boldsymbol{\lambda}$ are taken in the same sense both in (38) and (80). An analogous answer is valid for the form-factor (43). The answer for (80) appears explicitly due to the formula which should be presented as a separate

Proposition: The determinant of the matrix $(\bar{\mathsf{T}})_{1 \le j,k \le N}$ with the entries expressed by (81) is given by the relation:

$$\begin{aligned} \det(\bar{\mathsf{T}})_{1 \le j,k \le N} \; &= \; q^{\frac{n}{2}(N-n)(N-n-1)} \, \mathcal{V}_N(\mathbf{q}) \mathcal{V}_{N-n}(\mathbf{q}/q) \, Z_q(N-n, N, \mathcal{K}) \\ &= \; q^{\frac{n}{2}(N-n)(N-n-1)} \, \mathcal{V}_N(\mathbf{q}) \mathcal{V}_{N-n}(\mathbf{q}/q) \prod_{k=1}^{N} \prod_{j=1}^{N-n} \frac{1 - q^{j+k+\mathcal{K}-1}}{1 - q^{j+k-1}} \, , \end{aligned} \tag{82}$$

where $Z_q(N-n, N, \mathcal{K})$ is the generating function (67) with L and P replaced by $N-n$ and $\mathcal{K}$, respectively.

Proof: The representation (82) taken at $n = 0$ is the same as Eq. (73) being taken at $s = q^{M+1}$. Appendix I contains the proof of (82). The proof in Appendix I, which is based on the Binet–Cauchy formula, differs from the proof of (73) in [45] valid, in principle, for mutually independent s and q.

Taking into account (67), (80) and (82), we obtain:

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle \; = \; q^{\frac{n}{2}(N-n)(N-n-1)} Z_q(N-n, N, \mathcal{K}) \, . \tag{83}$$

Therefore,

$$\lim_{q\to 1}\langle\Psi_N(\mathbf{v})\mid\bar{\mathsf{F}}_n\mid\Psi_{N-n}(\mathbf{u})\rangle\;=\;Z_1(N-n,N,\mathcal{K})\;=\;A(N-n,N,\mathcal{K})\,,\tag{84}$$

i.e., we obtained that $\langle\Psi_N(\mathbf{v})\mid\bar{\mathsf{F}}_n\mid\Psi_{N-n}(\mathbf{u})\rangle$ is the generating function of the MacMahon formula (68) for the number of the plane partitions confined in the box of the size $(N-n)\times N\times(M-N+1)$.

6 Low temperature asymptotics

6.1 The survival probability of ferromagnetic string

Let us go over to the XX-chain long enough and assume that $M\gg 1$, while N is moderate: $1\ll N\ll M$. Now the correlator $F_{k;l}(\beta)$ (50) is approximately given by the modified Bessel function:

$$F_{k;l}(\beta)=I_{k-l}(\beta)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}e^{\beta\cos\phi}e^{i(k-l)\phi}d\phi\,.$$

When β is large (small temperature limit), the behavior of the Bessel function is known for moderate $m\equiv|k-l|$:

$$F_{k;l}(\beta)\simeq\frac{e^{\beta}}{\sqrt{2\pi\beta}}\left(1-\frac{4m^2-1}{8\beta}+\ldots\right),$$

i.e., the power decay is governed by the critical exponent $\xi=-1/2$.

With regard at (53), we put the correlator (16) in the form:

$$\mathcal{T}(\boldsymbol{\theta}^{\rm v},n,\beta)\;=\;\frac{1}{\mathcal{N}^2(\boldsymbol{\theta}^{\rm v})(M+1)^N}\sum_{\{\boldsymbol{\theta}\}}e^{-\beta(E_N(\boldsymbol{\theta})-E_N(\boldsymbol{\theta}^{\rm v}))}\left|\mathcal{V}(e^{i\boldsymbol{\theta}})\,\mathcal{P}(e^{-i\boldsymbol{\theta}},e^{i\boldsymbol{\theta}^{\rm v}})\right|^2,\tag{85}$$

where $\mathcal{N}^2(\boldsymbol{\theta}^{\rm v})$ is given by (26), and $\mathcal{P}(e^{-i\boldsymbol{\theta}},e^{i\boldsymbol{\theta}^{\rm v}})$ is expressed according to (52). Summations are replaced, at large enough M, by the integrations, and we obtain from (85):

$$\begin{aligned}\mathcal{T}(\boldsymbol{\theta}^{\rm v},n,\beta)\;\simeq\;&\frac{1}{\mathcal{N}^2(\boldsymbol{\theta}^{\rm v})N!}\prod_{i=1}^{N}\Big(\int\limits_0^{2\pi}\frac{d\theta_i}{2\pi}\Big)\,e^{\beta\sum\limits_{l=1}^{N}(\cos\theta_l-\cos\theta_l^{\rm v})}\\&\times\left|\mathcal{P}(e^{-i\boldsymbol{\theta}},e^{i\boldsymbol{\theta}^{\rm v}})\right|^2\prod_{1\le k<l\le N}\left|e^{i\theta_k}-e^{i\theta_l}\right|^2,\end{aligned}\tag{86}$$

where the continuous integration variables $\theta_i\in[0,2\pi]$ are due to the change of the discrete variables $k_i\in\mathcal{M}$ as follows: $\theta_{k_i}\mapsto\theta_i$, $\forall i$. Besides, we put approximately:

$$\frac{1}{\mathcal{N}^2(\boldsymbol{\theta}^{\rm v})}\;\approx\;\frac{(2\pi)^{N(N-1)}\Gamma^2(1|N)}{(M+1)^{N^2}}\,,\tag{87}$$

where it is used that

$$\prod_{1\le r<s\le N}|r-s|\;=\;\Gamma(1|N)\,,\qquad\Gamma(1|N)\;\equiv\;\prod_{n=1}^{N}\Gamma(n)\,.$$

Further, we may asymptotically put $\cos\theta_l \approx 1$, $\forall l$, and the integral (86) can be approximated, at large β, as follows:

$$\mathcal{T}(\boldsymbol{\theta}^{\rm v} \approx \mathbf{0}, n, \beta) \simeq \mathcal{P}^2(\mathbf{1},\mathbf{1})\, \frac{(2\pi)^{N(N-2)}\,\Gamma^2(1|N)\,\mathcal{I}}{(M+1)^{N^2}\,N!} \left(\frac{2}{\beta}\right)^{N^2/2}, \tag{88}$$

$$\mathcal{I} \equiv \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} e^{-\sum\limits_{l=1}^{N} x_l^2} \prod_{1\le k<l\le N} \left|x_k - x_l\right|^2 dx_1 dx_2 \ldots dx_N\,, \tag{89}$$

The bold-faced arguments in $\mathcal{P}(\mathbf{1},\mathbf{1})$ imply that $S_{\boldsymbol{\lambda}}(e^{\pm i\boldsymbol{\theta}})$ in (52) is replaced by $S_{\boldsymbol{\lambda}}(1,1,\ldots,1)$ (see (77)) provided the N-tuple of the exponentials $e^{\pm i\boldsymbol{\theta}}$ is substituted by N-tuple of unities.

The integral $\mathcal{I}$ (89) is the *Mehta integral* [47] of the *Gaussian Unitary Ensemble* of random matrices [44], and its value is known:

$$\mathcal{I} \;=\; \frac{\pi^{N/2}\,\Gamma(1|N+1)}{2^{N(N-1)/2}}\,.$$

Then, right-hand side of (88) acquires the form:

$$\mathcal{T}(\boldsymbol{\theta}^{\rm v} \approx \mathbf{0}, n, \beta) \;\simeq\; \left(Z^{\rm cspp}_{q=1}(N,N,M-n)\right)^2 \frac{\mathcal{C}_{N,M}}{\beta^{N^2/2}}\,, \quad \mathcal{C}_{N,M} \;\equiv\; \frac{(2\pi)^{N(N-\frac{3}{2})}\,\Gamma^3(1|N)}{(M+1)^{N^2}}\,. \tag{90}$$

The decay of the survival probability $\mathcal{T}(\boldsymbol{\theta}^{\rm v} \approx \mathbf{0}, n, \beta)$ at large β (low temperature) is governed by the critical exponent $N^2/2$, while its amplitude is proportional to the square of $Z^{\rm cspp}_{q=1}(N,N,M-n)$ (30), i.e., of the number of column strict plane partitions in the corresponding box. The estimate (90) takes equivalent form:

$$\mathcal{T}(\boldsymbol{\theta}^{\rm v} \approx \mathbf{0}, n, \beta) \;\simeq\; \left(Z^{\rm cspp}_{q=1}(N,N,M-n)\right)^2 e^{N^2\log\frac{2\pi}{M+1} - \frac{N^2}{2}\log\beta + 3\boldsymbol{\phi}_N}\,, \tag{91}$$

where

$$\boldsymbol{\phi}_N \;\equiv\; \sum_{k=1}^{N} \log\frac{\Gamma(k)}{(2\pi)^{1/2}}\,. \tag{92}$$

We re-express $Z^{\rm cspp}_1(N,N,M-n)$ in terms of the gamma-functions:

$$Z^{\rm cspp}_1(N,N,M-n) \;\equiv\; \sum_{\boldsymbol{\lambda}\subseteq\{(\mathcal{K}/n)^N\}} \mathsf{d}_{\boldsymbol{\lambda}}\,\mathsf{d}_{\boldsymbol{\lambda}} \tag{93}$$

$$= \prod_{j=1}^{N} \frac{\Gamma(j)\,\Gamma(j+M+1-n)}{\Gamma(j+N)\,\Gamma(j+M+1-n-N)}\,. \tag{94}$$

Let us go further with our estimates. According to Appendix II, the asymptotical representation for $\boldsymbol{\phi}_N$ (92) is valid:

$$\boldsymbol{\phi}_N \;=\; \frac{N^2}{2}\log N \;-\; \frac{3N^2}{4} \;+\; \mathcal{O}(N\log N)\,, \qquad N \gg 1\,. \tag{95}$$

Taking into account the representation (94), we obtain that logarithm of $Z^{\rm cspp}_1(N,N,M-n)$ behaves in leading order as follows (Appendix II):

$$\log Z^{\rm cspp}_1(N,N,M-n) \;\approx\; N^2\log\Big(\mathsf{A}\frac{M+1-n}{N}\Big)\,, \quad M-n \gg N \gg 1\,, \tag{96}$$

where $\mathsf{A} = e^{2(1-\log 2)}$. In other words, Eq. (96) gives us the asymptotical behavior of the number of column strict plane partitions placed into the box of the size $N \times N \times (M-n)$ (very tall box with square bottom). According to (95) and (96), the estimate (91) is specified as follows:

$$\log \mathcal{T}(\boldsymbol{\theta}^{\mathrm{v}} \approx \mathbf{0}, n, \beta) \simeq N^2 \log\Big(\mathsf{C}\frac{(M+1-n)^2}{(M+1)(N\beta)^{1/2}}\Big),$$

where the exponent is taken with leading logarithmic accuracy. The logarithm, $\log \mathcal{T}(\mathbf{0}, n, \beta)$, increases provided both $\frac{M-n}{N}$ and $\frac{N}{\beta}$ increase fast enough.

6.2 The survival probability of domain wall

Let us repeat the same steps as above in order to estimate $\mathcal{F}(\boldsymbol{\theta}^{\mathrm{v}} \approx \mathbf{0}, n, \beta)$ (see (18)). We begin with the consideration of (57) using appropriately (38) and (43) within (59), and we obtain:

$$\langle \Psi_{N-n}(\mathbf{v}) \mid \bar{\mathsf{F}}_n\, e^{-\beta \hat{H}}\, \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u})\rangle = \frac{1}{(M+1)^N} \sum_{M \geq k_1 > k_2 \cdots > k_N \geq 0} e^{-\beta E_N(\boldsymbol{\theta})}$$

$$\times |\mathcal{V}(e^{i\boldsymbol{\theta}})|^2 \sum_{\boldsymbol{\lambda}^L, \boldsymbol{\lambda}^R \subseteq \{\mathcal{K}^{N-n}\}} S_{\boldsymbol{\lambda}^L}(\mathbf{v}^{-2}) S_{\hat{\boldsymbol{\lambda}}^L}(e^{i\boldsymbol{\theta}}) S_{\hat{\boldsymbol{\lambda}}^R}(e^{-i\boldsymbol{\theta}}) S_{\boldsymbol{\lambda}^R}(\mathbf{u}^2)\,. \tag{97}$$

Taking into account (AI.21) and (AI.22), one concludes that right-hand side of (97) is also related to enumeration of certain lattice paths. We use (97) in order to go further:

$$\mathcal{F}(\boldsymbol{\theta}^{\mathrm{v}}, n, \beta) = \frac{1}{\mathcal{N}^2(\boldsymbol{\theta}^{\mathrm{v}})(M+1)^N} \sum_{M \geq k_1 > k_2 \cdots > k_N \geq 0} e^{-\beta(E_N(\boldsymbol{\theta}) - E_N(\boldsymbol{\theta}^{\mathrm{v}}))}$$

$$\times \Big| \mathcal{V}(e^{i\boldsymbol{\theta}}) \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{K}^{N-n}\}} S_{\hat{\boldsymbol{\lambda}}}(e^{-i\boldsymbol{\theta}}) S_{\boldsymbol{\lambda}}(e^{i\boldsymbol{\theta}^{\mathrm{v}}}) \Big|^2 . \tag{98}$$

We repeat the steps leading from from (86) to (91), and the estimate in question takes the form:

$$\mathcal{F}(\boldsymbol{\theta}^{\mathrm{v}} \approx \mathbf{0}, n, \beta) \simeq \big(Z_{q=1}(N-n, N, \mathcal{K})\big)^2\, e^{N^2 \log \frac{2\pi}{M+1} - \frac{N^2}{2}\log\beta + 3\boldsymbol{\phi}_N}\,. \tag{99}$$

We re-express $Z_1(N-n, N, \mathcal{K})$ in terms of the gamma-functions:

$$Z_1(N-n, N, \mathcal{K}) = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{K}^{N-n}\}} \mathsf{d}_{\hat{\boldsymbol{\lambda}}}\, \mathsf{d}_{\boldsymbol{\lambda}}$$

$$= \prod_{j=1}^{N} \frac{\Gamma(j)\,\Gamma(j+M+1-n)}{\Gamma(j+N-n)\,\Gamma(j+M+1-N)}\,. \tag{100}$$

Using Appendix II to estimate right-hand side of (100), we obtain for the logarithm of $Z_1(N-n, N, \mathcal{K})$:

$$\log Z_1(N-n, N, M-N+1) \approx N(N-n)\log\Big(\mathsf{B}\frac{M-n}{2N-n}\Big),$$
$$M, M-n \gg N-n, N \gg 1\,, \tag{101}$$

where B is some constant. In other words, Eq. (101) gives us the asymptotical behavior of the number of all plane partitions placed into the box of the size $(N-n)\times N\times(M-N+1)$ (very tall box with rectangle bottom). According to (95) and (101), the estimate (99) takes the form:

$$\mathcal{F}(\boldsymbol{\theta}^{\rm v}\approx\mathbf{0},n,\beta)\;\simeq\;\exp\Big(N^2\log\frac{N^{3/2}}{M\beta^{1/2}}-2N(N-n)\log\frac{2N-n}{M-n}\Big)\,.$$

It is interesting to note that the behavior of the ratio of $Z_1^{\rm cspp}(N,N,M-n)$ to $Z_1(N-n,N,M-N+1)$ can be estimated. Indeed, according to Eqs. (96) and (101), the corresponding logarithm looks, approximately, as follows:

$$\log\frac{Z_1^{\rm cspp}(N,N,M-n)}{Z_1(N-n,N,M-N+1)}\;=\;N^2\log\Big(\frac{2\mathsf{A}}{\mathsf{B}}\Big)\;+\;nN\log\Big(\frac{\mathsf{B}}{2}\frac{M}{N}\Big)\,. \tag{102}$$

The value of $\log\big(\frac{\mathsf{B}}{2}\frac{M}{N}\big)$ is positive provided M is large enough. Assume that n is also large, $\log\big(\frac{2\mathsf{A}}{\mathsf{B}}\big)\ll n\ll N$. If so, appropriate choice of M (for instance, $\frac{\mathsf{B}}{2}\frac{M}{N}\sim e^N$) can ensure domination of the second term in right-hand side of (102). In other words, the value of $Z_1^{\rm cspp}(N,N,M-n)$ would grow faster than that of $Z_1(N-n,N,M)$. In the case when the first contribution dominates in right-hand side of (102), more detailed investigation is required to determine the sign of $\log\big(\frac{2\mathsf{A}}{\mathsf{B}}\big)$. Therefore, both $Z_1^{\rm cspp}(N,N,M-n)$ and $Z_1(N-n,N,M-N+1)$ can demonstrate various behavior at growing lengths of the sides of the box.

7 Discussion

The XXZ Heisenberg chain has been considered at zero value of the anisotropy parameter. It is crucial that in this limit the Bethe states are expressible in terms of the symmetric Schur functions. The Schur functions have been used to calculate thermal correlation functions of the ferromagnetic string and of the domain wall over the many-particle state-vectors. A relationship between the average value of the projecting operator $\bar{\Pi}_n$ (16) and the generating function of the column strict plane partitions in a box, $Z_q^{\rm cspp}(N,N,P)$ (69), is obtained, as well as between the form-factor of the operator of creation of excitations, $\bar{\mathsf{F}}_n$ (18), and the generating function of arbitrary plane partitions in a box, $Z_q(L,N,P)$ (67). It is shown that the Survival Probability of the Ferromagnetic String turns out to be the generating function related to enumeration both of paths of the random walks of vicious walkers and of certain non-intersecting lattice paths.

The determinant of the Kuperberg type is calculated, and its relationship with the q-binomial determinant is demonstrated. The calculation of the determinant is based on the theory of the symmetric functions.

Asymptotic estimates for the survival probability of the projecting operator and of the domain wall operator are obtained in the limit of small temperature. The critical exponents which govern the decay of the correlation functions are calculated. It is demonstrated that the survival probabilities are asymptotically proportional to the squared numbers of the plane partitions in box. The asymptotical dependence of the survival probabilities on the length of the chain, $M+1$, and on the number of particles, N, is determined provided both M and N are large enough but finite.

Appendix I

The relations (73) and (82) are crucial in the approach developed to the enumeration of the boxed plane partitions. The statement expressed by Eq. (73) has been proved in [45], and thus it would be called as the *Kuperberg's formula*. Reasonings based on the divisibility of the determinant itself have been used in [45]. It is crucil that the Binet-Cauchy formula gives a way to evaluate the right-hand side of (73) straightforwardly although for a special case of s being an integer power of q. The Binet-Cauchy formula is of importance in our calculations, and, moreover, it enables to establish more general relation (82) (Eq. (73) is a particular case of (82)). We provide a proof of (82) related to application of the Binet-Cauchy formula.

Since $M+1$ is the number of sites, while N is the number of excitations, we can safely assume that $M+1 \geq N \geq 1$. So let us fix N and define $\mathcal{K} \equiv M+1-N$, $0 \leq \mathcal{K} \leq M$. Then, we put $s = q^{M+1}$ and re-express (73) as follows:

$$\det\left(\sum_{l=0}^{N+\mathcal{K}-1} q^{l(j+k-1)}\right)_{1\leq j,k\leq N} = \mathcal{V}_N(\mathbf{q})\mathcal{V}_N(\mathbf{q}/q) \prod_{k,j=1}^{N} \frac{1-q^{\mathcal{K}+j+k-1}}{1-q^{j+k-1}}\,, \tag{AI.1}$$

where $\mathcal{K} = 0, 1, \ldots, M$. Left-hand side of (AI.1) is due to the known relation,

$$\sum_{l=0}^{P} Q^l \;=\; \frac{1-Q^{P+1}}{1-Q}\,, \tag{AI.2}$$

while the product $\mathcal{V}_N(\mathbf{q})\mathcal{V}_N(\mathbf{q}/q)$ of the Vandermond determinants is given by (72). Right-hand-side of (AI.1) appears due to the relation (at fixed j):

$$\prod_{k=1}^{N}(1-q^{j-k+M+1}) \;=\; \prod_{l=1}^{N}(1-q^{j+l+M-N})\,. \tag{AI.3}$$

The validity of (AI.1) is most easily seen at $\mathcal{K}=0$. Indeed, left-hand side of (AI.1) at $\mathcal{K}=0$ acquires the form of the determinant of the product of two square matrices, and we obtain:

$$\det\left(\sum_{l=0}^{N-1} q^{jl} q^{l(k-1)}\right)_{1\leq j,k\leq N} = \det\big(q^{j(k-1)}\big)_{1\leq j,k\leq N} \det\big(q^{(j-1)(k-1)}\big)_{1\leq j,k\leq N}\,. \tag{AI.4}$$

It is seen that right-hand side of (AI.4) is the product of two Vandermond determinants, $\mathcal{V}_N(\mathbf{q})\mathcal{V}_N(q^{-1}\mathbf{q})$. It is clear that left-hand side of (AI.1) is expressing the determinant of the matrix being at $\mathcal{K}>0$ a product of two rectangular matrices. The Binet–Cauchy theorem should be applied to the left-hand side of (AI.1) at $\mathcal{K}\neq 0$. However, this requires some preliminary notes concerning the symmetric functions [40].

Before to proceed with the symmetric functions, let us remind certain notations of *q-calculus* [48]. We need the *q-number* $[n]$ being q-analogue of the positive integer $n \in \mathbb{Z}^+$,

$$[n] \;\equiv\; \frac{1-q^n}{1-q}\,, \tag{AI.5}$$

and the *q-factorial* $[n]!$,

$$[n]! \;\equiv\; [1]\,[2]\,\ldots\,[n]\,, \qquad [0]! \;\equiv\; 1\,. \tag{AI.6}$$

The definitions (AI.5) and (AI.6) enable to define the *q-binomial coefficient* $\begin{bmatrix} N \\ r \end{bmatrix}$:

$$\begin{bmatrix} N \\ r \end{bmatrix} \equiv \frac{[N]\,[N-1]\,\ldots\,[N-r+1]}{[r]!} = \frac{[N]!}{[r]!\,[N-r]!}\,. \tag{AI.7}$$

In the limit $q \to 1$, the q-binomial coefficients $\begin{bmatrix} N \\ r \end{bmatrix}$ are transformed into the binomial coefficients $C_N^r \equiv \binom{N}{r}$. Two analogues of the Pascal formula exist for the q-binomial coefficients (AI.7) [48]:

$$\begin{aligned} \begin{bmatrix} N \\ r \end{bmatrix} &= \begin{bmatrix} N-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} N-1 \\ r \end{bmatrix}, \\ \begin{bmatrix} N \\ r \end{bmatrix} &= q^{N-r} \begin{bmatrix} N-1 \\ r-1 \end{bmatrix} + \begin{bmatrix} N-1 \\ r \end{bmatrix}, \end{aligned} \tag{AI.8}$$

where $1 \leq r \leq N-1$. In turn, there exists the following identity for the q-binomial coefficients [48]:

$$\begin{bmatrix} N+N' \\ r \end{bmatrix} = \sum_{j=0}^{\min(r,N)} q^{(N-j)(r-j)} \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} N' \\ r-j \end{bmatrix}. \tag{AI.9}$$

Now let us define the *elementary symmetric functions* $e_r = e_r(\mathbf{x})$ dependent on N variables $\mathbf{x} = (x_1, x_2, \ldots, x_N)$:

$$e_r \equiv \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r}\,. \tag{AI.10}$$

The functions e_r (AI.10) can be obtained as the coefficients in the product:

$$(1+tx_1)(1+tx_2)\ldots(1+tx_N) = 1 + e_1 t + e_2 t^2 + \cdots + e_N t^N\,.$$

Special notations, L_r and R_r, for e_r at $\mathbf{x} = \mathbf{q}$ and $\mathbf{x} = q^{-1}\mathbf{q}$, respectively, are appropriate: $R_r \equiv e_r(q^{-1}\mathbf{q})$ and $L_r \equiv e_r(\mathbf{q})$. The following values are known [40]:

$$R_r = q^{r(r-1)/2} \begin{bmatrix} N \\ r \end{bmatrix}, \qquad L_r = q^{r(r+1)/2} \begin{bmatrix} N \\ r \end{bmatrix}, \tag{AI.11}$$

where the q-binomial coefficients (AI.9) are used.

Let us turn to the definition of the Schur functions (14) which are labelled by non-strict partitions $\boldsymbol{\lambda}$, $\mathcal{K} \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. So-called, *conjugated partitions* $\bar{\boldsymbol{\lambda}}$ can be defined as the diagrams given by transposition (i.e., by reflection with respect of the main diagonal) of the diagrams of the non-strict partitions $\boldsymbol{\lambda}$ [40]. The parts of the non-strict partitions $\bar{\boldsymbol{\lambda}}$ respect the inequality: $N \geq \bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{\mathcal{K}} \geq 0$. With the conjugated diagrams at hands, one can express the Schur functions (14) through the elementary symmetric functions (AI.10) [40]:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}) = \det\big(e_{\bar{\lambda}_i - i + j}(\mathbf{x})\big)_{1 \leq i,j \leq \mathcal{K}}\,. \tag{AI.12}$$

Now let us straightforwardly apply the Binet-Cauchy formula to the left-hand side of (AI.1) (the first line below) and subsequently re-express the answer by means of the Schur functions (14) (the second line below):

$$\begin{aligned}\det\left(\sum_{l=0}^{N+\mathcal{K}-1} q^{l(j+k-1)}\right)_{1\le j,k\le N} &= \sum_{\boldsymbol{\lambda}} \det(q^{j\mu_k})_{1\le j,k\le N}\det(q^{(j-1)\mu_k})_{1\le j,k\le N} \\ &= \mathcal{V}_N(\mathbf{q})\mathcal{V}_N(\mathbf{q}/q)\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{K}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{q})S_{\boldsymbol{\lambda}}(q^{-1}\mathbf{q})\,.\end{aligned} \tag{AI.13}$$

In order to capture the Proposition expressed by (82), let us combine (80) and (81) as follows:

$$\begin{aligned}\det(\bar{\mathsf{T}}) \;=\; & q^{\frac{n}{2}(N-n)(N-n-1)}\mathcal{V}_N(\mathbf{q})\mathcal{V}_{N-n}(\mathbf{q}/q) \\ & \times\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{K}^{N-n}\}} S_{\bar{\boldsymbol{\lambda}}}(\mathbf{q})S_{\boldsymbol{\lambda}}(\mathbf{q}/q)\,,\end{aligned} \tag{AI.14}$$

where

$$\begin{aligned}&\bar{\mathsf{T}}_{kj} = \frac{1-q^{(M+1)(j+k-1)}}{1-q^{j+k-1}}\,, \quad 1\le k\le N-n, \qquad 1\le j\le N\,, \\ &\bar{\mathsf{T}}_{kj} = q^{j(N-k)}\,, \qquad\qquad N-n+1<k\le N,\ 1\le j\le N\,.\end{aligned} \tag{AI.15}$$

In what follows, let us concentrate on the sums of the Schur functions in (AI.13) and (AI.14). Denote them as Σ_N (or, correspondingly, Σ_{N-n}). After the usage of (AI.12), we obtain from (AI.14):

$$\Sigma_{N-n} \;=\; \sum_{\bar{\boldsymbol{\lambda}}\subseteq\{(N-n)^{\mathcal{K}}\}} \det\big(e_{\bar{\lambda}_j-j+k}(\mathbf{q})\big)_{1\le j,k\le\mathcal{K}}\det\big(e_{\bar{\lambda}_k-k+j}(q^{-1}\mathbf{q})\big)_{1\le j,k\le\mathcal{K}}\,, \tag{AI.16}$$

where the summation is now going over all conjugated partitions $\bar{\boldsymbol{\lambda}}$. In more explicit form, (AI.16) takes the form:

$$\begin{aligned}\Sigma_{N-n} \;=\; \sum_{\bar{\boldsymbol{\lambda}}\subseteq\{(N-n)^{\mathcal{K}}\}} & \det\begin{pmatrix} L_{\bar{\lambda}_1} & L_{\bar{\lambda}_1+1} & \dots & L_{\bar{\lambda}_1+\mathcal{K}-1} \\ L_{\bar{\lambda}_2-1} & L_{\bar{\lambda}_2} & \dots & L_{\bar{\lambda}_2+\mathcal{K}-2} \\ \vdots & \vdots & \ddots & \vdots \\ L_{\bar{\lambda}_{\mathcal{K}}-\mathcal{K}+1} & L_{\bar{\lambda}_{\mathcal{K}}-\mathcal{K}+2} & \dots & L_{\bar{\lambda}_{\mathcal{K}}}\end{pmatrix} \\ & \times\det\begin{pmatrix} R_{\bar{\lambda}_1} & R_{\bar{\lambda}_1-1} & \dots & R_{\bar{\lambda}_1-\mathcal{K}+1} \\ R_{\bar{\lambda}_2+1} & R_{\bar{\lambda}_2} & \dots & R_{\bar{\lambda}_2-\mathcal{K}+2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\bar{\lambda}_{\mathcal{K}}+\mathcal{K}-1} & R_{\bar{\lambda}_{\mathcal{K}}+\mathcal{K}-2} & \dots & R_{\bar{\lambda}_{\mathcal{K}}}\end{pmatrix},\end{aligned} \tag{AI.17}$$

where the definitions (AI.11) are taken into account. Besides, $R_0 = L_0 = 1$ (by definition) while both R_r and L_r are zero at $r > N$. We again use the Binet-Cauchy formula to replace the sum $\sum_{\bar{\boldsymbol{\lambda}}}$ in the right-hand side of (AI.17) by the determinant of the symmetric matrix

of the size $\mathcal{K}\times\mathcal{K}$ as follows:

$$\Sigma_{N-n} = \\ = \det\begin{pmatrix} \sum_{i=0}^{N-n} L_iR_i & \sum_{i=0}^{N-n} L_{i+1}R_i & \sum_{i=0}^{N-n} L_{i+2}R_i & \dots & \sum_{i=0}^{N-n} L_{i+\mathcal{K}-1}R_i \\ \sum_{i=0}^{N-n} L_iR_{i+1} & \sum_{i=0}^{N-n} L_iR_i & \sum_{i=0}^{N-n} L_{i+1}R_i & \dots & \sum_{i=0}^{N-n} L_{i+\mathcal{K}-2}R_i \\ \sum_{i=0}^{N-n} L_iR_{i+2} & \sum_{i=0}^{N-n} L_iR_{i+1} & \sum_{i=0}^{N-n} L_iR_i & \dots & \sum_{i=0}^{N-n} L_{i+\mathcal{K}-3}R_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{N-n} L_iR_{i+\mathcal{K}-1} & \sum_{i=0}^{N-n} L_iR_{i+\mathcal{K}-2} & \sum_{i=0}^{N-n} L_iR_{i+\mathcal{K}-3} & \dots & \sum_{i=0}^{N-n} L_iR_i \end{pmatrix}. \tag{AI.18}$$

The entries in (AI.18) are given by the sums which can be determined by means of (AI.9) and (AI.11) as follows ($0\le s\le\mathcal{K}-1$):

$$\begin{aligned} \sum_{i=0}^{N-n} L_{i+s}R_i &= q^{\frac{s}{2}(s+1)}\begin{bmatrix}2N-n\\ N-s\end{bmatrix}, \\ \sum_{i=0}^{N-n} L_iR_{i+s} &= q^{\frac{s}{2}(s-1)}\begin{bmatrix}2N-n\\ N+s\end{bmatrix}. \end{aligned} \tag{AI.19}$$

With the sums (AI.19) at hands, we obtain the determinant of the matrix in (AI.18) with the entries given by the q-binomial coefficients:

$$\Sigma_{N-n} = \\ = \det\begin{pmatrix} C^{N}_{q,2N-n} & q\,C^{N-1}_{q,2N-n} & q^3\,C^{N-2}_{q,2N-n} & \dots & q^{\frac{\mathcal{K}}{2}(\mathcal{K}-1)}\,C^{N-\mathcal{K}+1}_{q,2N-n} \\ C^{N+1}_{q,2N-n} & C^{N}_{q,2N-n} & q\,C^{N-1}_{q,2N-n} & \dots & q^{\frac{\mathcal{K}-1}{2}(\mathcal{K}-2)}\,C^{N-\mathcal{K}+2}_{q,2N-n} \\ q\,C^{N+2}_{q,2N-n} & C^{N+1}_{q,2N-n} & C^{N}_{q,2N-n} & \dots & q^{\frac{\mathcal{K}-2}{2}(\mathcal{K}-3)}\,C^{N-\mathcal{K}+3}_{q,2N-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q^{\frac{\mathcal{K}-1}{2}(\mathcal{K}-2)}\,C^{N+\mathcal{K}-1}_{q,2N-n} & q^{\frac{\mathcal{K}-2}{2}(\mathcal{K}-3)}\,C^{N+\mathcal{K}-2}_{q,2N-n} & q^{\frac{\mathcal{K}-3}{2}(\mathcal{K}-4)}\,C^{N+\mathcal{K}-3}_{q,2N-n} & \dots & C^{N}_{q,2N-n} \end{pmatrix}. \tag{AI.20}$$

We used $C^{N}_{q,N'}$ in (AI.20) as the short-hand notation for the q-binomial coefficient $\begin{bmatrix}N'\\ N\end{bmatrix}$. In the limit $q\to 1$, the q-binomial coefficient $C^{N}_{q,N'}$ tends to the usual binomial coefficient $C^{N}_{N'}$.

Let us calculate the determinant (AI.20) using the properties of the q-binomial coefficients (AI.8). As a first step, we combine the rows in (AI.20) as follows: the $\mathcal{K}^{\text{th}}$ is multiplied by q^{N+1} and the $(\mathcal{K}-1)^{\text{th}}$ is added to it, the $(\mathcal{K}-1)^{\text{th}}$ is multiplied by q^{N+1} and the $(\mathcal{K}-2)^{\text{th}}$ is added to it, ..., the 2nd is multiplied by q^{N+1} and the 1st is added to it. The second step concerns the rows of matrix thus obtained: the $\mathcal{K}^{\text{th}}$ is multiplied by q^{N+2} and the $(\mathcal{K}-1)^{\text{th}}$ is added to it, the $(\mathcal{K}-1)^{\text{th}}$ is multiplied by q^{N+2} and the $(\mathcal{K}-2)^{\text{th}}$ is added to it, ..., the 3rd is multiplied by q^{N+2} and the 2nd is added to it. After $\mathcal{K}-1$ steps, we introduce the factor

$$q^{-(N+1)(\mathcal{K}-1)-(N+2)(\mathcal{K}-2)-\ldots-(N+\mathcal{K}-1)\times 1},$$

and obtain the answer which we put in the following form:

$$\begin{aligned} \Sigma_{N-n} &= q^{\frac{N}{2}(1-\mathcal{K})\mathcal{K}}\times \\ &\times\begin{pmatrix} 2N-n, & 2N-n+1, & \dots, & 2N-n+\mathcal{K}-2, & 2N-n+\mathcal{K}-1 \\ N-n, & N-n+1, & \dots, & N-n+\mathcal{K}-2, & N-n+\mathcal{K}-1 \end{pmatrix}_q. \end{aligned} \tag{AI.21}$$

We introduced in (AI.21) the notation for, so-called, *q-binomial determinant*, [49, 50]. It is appropriate to recall that the *binomial determinant* has been defined in [51] as follows:

$$\left(\begin{array}{llll} 2N-n, & 2N-n+1, & \ldots, & 2N-n+\mathcal{K}-2, \quad 2N-n+\mathcal{K}-1 \\ N-n, & N-n+1, & \ldots, & N-n+\mathcal{K}-2, \quad N-n+\mathcal{K}-1 \end{array} \right) \equiv$$

$$\equiv \det \left(\begin{array}{cccccc} C_{2N-n}^{N-n} & C_{2N-n}^{N-n+1} & C_{2N-n}^{N-n+2} & \ldots & C_{2N-n}^{N-n+\mathcal{K}-2} & C_{2N-n}^{N-n+\mathcal{K}-1} \\ C_{2N-n+1}^{N-n} & C_{2N-n+1}^{N-n+1} & C_{2N-n+1}^{N-n+2} & \ldots & C_{2N-n+1}^{N-n+\mathcal{K}-2} & C_{2N-n+1}^{N-n+\mathcal{K}-1} \\ C_{2N-n+2}^{N-n} & C_{2N-n+2}^{N-n+1} & C_{2N-n+2}^{N-n+2} & \ldots & C_{2N-n+2}^{N-n+\mathcal{K}-2} & C_{2N-n+2}^{N-n+\mathcal{K}-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{2N-n+\mathcal{K}-2}^{N-n} & C_{2N-n+\mathcal{K}-2}^{N-n+1} & C_{2N-n+\mathcal{K}-2}^{N-n+2} & \ldots & C_{2N-n+\mathcal{K}-2}^{N-n+\mathcal{K}-2} & C_{2N-n+\mathcal{K}-2}^{N-n+\mathcal{K}-1} \\ C_{2N-n+\mathcal{K}-1}^{N-n} & C_{2N-n+\mathcal{K}-1}^{N-n+1} & C_{2N-n+\mathcal{K}-1}^{N-n+2} & \ldots & C_{2N-n+\mathcal{K}-1}^{N-n+\mathcal{K}-2} & C_{2N-n+\mathcal{K}-1}^{N-n+\mathcal{K}-1} \end{array} \right). \tag{AI.22}$$

The binomial determinant (AI.22) is positive, and it gives the number of certain paths on a two-dimensional integer lattice. Moreover, the binomial determinants are related to the shifted Young tableaux, as well as enjoy many other interesting properties [51]. The q-binomial determinant introduced in (AI.21) corresponds to the case when all the entries in (AI.22) are replaced by the q-binomial coefficients: $C_{N'}^{N} \to C_{q,N'}^{N}$.

Evaluation of the q-binomial determinant in (AI.21) leads to the answer:

$$\begin{aligned} \Sigma_{N-n} \; &= \; q^{\frac{N}{2}(1-\mathcal{K})\mathcal{K}} \prod_{j=1}^{N-n} \prod_{k=1}^{\mathcal{K}} \frac{[N+j+k-1]}{[j+k-1]} \times \\ &\times \det \left(\begin{array}{cccccc} 1 & C_{q,N}^{1} & C_{q,N}^{2} & \ldots & C_{q,N}^{\mathcal{K}-2} & C_{q,N}^{\mathcal{K}-1} \\ 1 & C_{q,N+1}^{1} & C_{q,N+1}^{2} & \ldots & C_{q,N+1}^{\mathcal{K}-2} & C_{q,N+1}^{\mathcal{K}-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & C_{q,N+\mathcal{K}-2}^{1} & C_{q,N+\mathcal{K}-2}^{2} & \ldots & C_{q,N+\mathcal{K}-2}^{\mathcal{K}-2} & C_{q,N+\mathcal{K}-2}^{\mathcal{K}-1} \\ 1 & C_{q,N+\mathcal{K}-1}^{1} & C_{q,N+\mathcal{K}-1}^{2} & \ldots & C_{q,N+\mathcal{K}-1}^{\mathcal{K}-2} & C_{q,N+\mathcal{K}-1}^{\mathcal{K}-1} \end{array} \right). \end{aligned} \tag{AI.22}$$

After a transformation, the determinant in (AI.22) acquires the form:

$$q^{\frac{N}{2}(\mathcal{K}-1)\mathcal{K}} \times \det \left(\begin{array}{cccccc} 1 & C_{q,N}^{1} & C_{q,N}^{2} & \ldots & C_{q,N}^{\mathcal{K}-2} & C_{q,N}^{\mathcal{K}-1} \\ 0 & 1 & q^{-1}C_{q,N}^{1} & \ldots & q^{1\cdot(3-\mathcal{K})}C_{q,N}^{\mathcal{K}-3} & q^{1\cdot(2-\mathcal{K})}C_{q,N}^{\mathcal{K}-2} \\ 0 & 0 & 1 & \ldots & q^{2(4-\mathcal{K})}C_{q,N}^{\mathcal{K}-4} & q^{2\cdot(3-\mathcal{K})}C_{q,N}^{\mathcal{K}-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & q^{(\mathcal{K}-2)\cdot(-1)}C_{q,N}^{1} \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{array} \right). \tag{AI.23}$$

In other words, this determinant is equal to $q^{\frac{N}{2}(\mathcal{K}-1)\mathcal{K}}$. We combine (AI.22) and (AI.23) and obtain the expression for the determinant Σ_{N-n} (AI.18). The double product in (AI.22) is expressed as follows:

$$\prod_{j=1}^{N-n} \prod_{k=1}^{\mathcal{K}} \frac{[N+j+k-1]}{[j+k-1]} \; = \; \prod_{j=1}^{N-n} \prod_{k=1}^{\mathcal{K}} \frac{1-q^{N+j+k-1}}{1-q^{j+k-1}} \; = \; Z_q(N-n, \mathcal{K}, N)\,, \tag{AI.24}$$

where $Z_q(N-n, \mathcal{K}, N)$ corresponds to the definition of the generating function (67). Furthermore, the double product in left-hand side of (AI.24) can equivalently be re-

arranged as follows:

$$\prod_{j=1}^{N-n}\prod_{k=1}^{N}\frac{[\mathcal{K}+j+k-1]}{[j+k-1]} \;=\; \prod_{j=1}^{N-n}\prod_{k=1}^{N}\frac{1-q^{\mathcal{K}+j+k-1}}{1-q^{j+k-1}} \;=\; Z_q(N-n,N,\mathcal{K})\,. \tag{AI.25}$$

It is clear, with regard at the definition (67), that the representations (AI.24) and (AI.25) are indeed coinciding. Thus, we obtain for Σ_{N-n} (AI.18):

$$\Sigma_{N-n} \;=\; Z_q(N-n,N,\mathcal{K})\,. \tag{AI.26}$$

Substituting Σ_{N-n} (AI.26) into (AI.14) one just obtains the statement expressed by (82). As a particular case, it is seen from (AI.26) that substituting Σ_N into (AI.13) one obtains (AI.1). Therefore, Eqs. (82) and (AI.1) are justified. □

The relation (77) states that $S_{\boldsymbol{\lambda}}(\mathbf{1})$ coincides with the dimensionality $\mathsf{d}_{\boldsymbol{\lambda}}$ of the irreducible representation of $\mathcal{U}(N)$. It is interesting to have a look at (AI.12) at $\mathbf{x}\to\mathbf{1}$. It is appropriate to think of this limit in the sense of the $q\to 1$ limit of the q-parametrized relations (AI.11). The distinction between L_r and R_r disappears at $q=1$, and we obtain:

$$\mathsf{d}_{\boldsymbol{\lambda}} \;=\; \det\bigl(C_N^{\bar\mu_i+j-\mathcal{K}}\bigr)_{1\le i,j\le\mathcal{K}}\,, \tag{AI.27}$$

where the strict partition $\bar{\boldsymbol{\mu}}$ corresponds to the conjugated partition $\bar{\boldsymbol{\lambda}}$: $\bar\mu_j=\bar\lambda_j+\mathcal{K}-j$, $j\in\{1,2,\ldots,\mathcal{K}\}$. Equation (AI.21) is valid since $\mathcal{K}\ge l(\bar{\boldsymbol{\lambda}})$ holds ($l(\bar{\boldsymbol{\lambda}})$ is the number of parts of the partition $\bar{\boldsymbol{\lambda}}$). Using (AI.8) at $q=1$,

$$C_{N+1}^{\bar\mu_l+1} \;=\; C_N^{\bar\mu_l} \;+\; C_N^{\bar\mu_l+1}\,,$$

we can re-express right-hand side of (AI.27) in the form of the binomial determinant as follows:

$$\mathsf{d}_{\boldsymbol{\lambda}} \;=\; \begin{pmatrix} N, & N+1, & \ldots, & N+\mathcal{K}-2, & N+\mathcal{K}-1 \\ \bar\mu_{\mathcal{K}}, & \bar\mu_{\mathcal{K}-1}, & \ldots, & \bar\mu_2, & \bar\mu_1 \end{pmatrix} \;\equiv\; \mathsf{b}_{\bar{\boldsymbol{\lambda}}}\,. \tag{AI.28}$$

The binomial determinant $\mathsf{b}_{\bar{\boldsymbol{\lambda}}}$ (AI.28) can be interpreted in terms of tuples of the lattice paths connecting the points labelled by the partitions $(N+\mathcal{K}-1,N+\mathcal{K}-2,\ldots,N)$ and $(\bar\mu_1,\bar\mu_2,\ldots,\bar\mu_{\mathcal{K}})$ [51]. The relation (AI.28) just provides the combinatorial interpretation of $\mathsf{d}_{\boldsymbol{\lambda}}$.

Appendix II

We use the Binet representation for $\log\Gamma(z)$ [53]:

$$\log\Gamma(z) \;=\; \bigl(z-\frac{1}{2}\bigr)\log z - z + \frac{1}{2}\log 2\pi + \int\limits_0^\infty e^{-tz}B(t)dt\,, \qquad \Re z>0\,, \tag{AII.1}$$

where

$$B(t) \;\equiv\; \frac{1}{t}\Bigl(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\Bigr)\,.$$

The function $B(t)$ is expanded for sufficiently small t as follows:

$$B(t) \;=\; B_2\Big(\frac{1}{2} - \frac{t^2}{5!} + \frac{t^4}{7!} + \dots\Big)\,,$$

where B_2 is the Bernoulli number, $B_2 = \frac{1}{6}$. The representation (AII.1) enables at $N \gg 1$ the validity of the estimate:

$$\boldsymbol{\phi}_N \simeq \sum_{k=1}^{N}\big(k - \frac{1}{2}\big)\log k \;-\; \frac{N(N+1)}{2} \;+\; \sum_{k=1}^{N}\int\limits_0^1 B(t)e^{-tk}dt \;+\; \int\limits_1^\infty \frac{B(t)}{e^t - 1}dt\,. \qquad \text{(AII.2)}$$

In turn, Eq. (AII.2) leads to (95) provided the Euler–Maclaurin formula is used.

Moreover, we introduce the notations:

$$\boldsymbol{\phi}'_N \;\equiv\; \sum_{k=1}^{N}\log\frac{\Gamma(k+N)}{(2\pi)^{1/2}}\,,$$

$$\boldsymbol{\phi}''_N \;\equiv\; \sum_{k=1}^{N}\log\frac{\Gamma(k+L)}{(2\pi)^{1/2}}\,,$$

$$\boldsymbol{\phi}'''_N \;\equiv\; \sum_{k=1}^{N}\log\frac{\Gamma(k+N+L)}{(2\pi)^{1/2}}\,.$$

In analogous way as above, usage of (AII.1) gives us the estimate for $\boldsymbol{\phi}''_N$ at $N, L \gg 1$:

$$\boldsymbol{\phi}''_N \;=\; \frac{(L+N)^2}{2}\log(L+N) - \frac{L^2}{2}\log L - \frac{3N}{2}\big(L+\frac{N}{2}\big) + \mathcal{O}\big((N+L)\log(N+L)\big)\,. \quad \text{(AII.3)}$$

As a particular case, we obtain from (AII.3) the estimate for $\boldsymbol{\phi}'_N$:

$$\boldsymbol{\phi}'_N \;=\; \frac{3N^2}{2}\log N \;+\; N^2\big(2\log 2 - \frac{9}{4}\big) \;+\; \mathcal{O}(N\log N)\,, \qquad N \gg 1\,. \qquad \text{(AII.4)}$$

Bearing in mind estimation of (96), it is appropriate to use (AII.3) in order to obtain the difference $\boldsymbol{\phi}'''_N - \boldsymbol{\phi}''_N$ in leading order (with respect of N^2) as follows:

$$\boldsymbol{\phi}'''_N - \boldsymbol{\phi}''_N \;\approx\; N^2\log(2N+L) \;+\; \frac{N^2}{2}\,, \qquad N, L \gg 1\,. \qquad \text{(AII.5)}$$

Then, Eq. (96) is obtained from (95), (AII.4), and (AII.5) taken at $L = M + 1 - n - N$. In order to obtain (101), we use (95) and (AII.3), where we put respectively $L = N - n$, $L = M + 1 - n$, or $L = M + 1 - N$.

References

[1] L. D. Faddeev, L. A. Takhtajan, *Quantum inverse scattering method and the XYZ Heisenberg model*, Usp. Mat. Nauk, **34** (1979), No. 5(209), 13–63. [Translation: L. A. Takhtadzhyan, L. D. Faddeev, Russ. Math. Surveys, **34**, 11–68 (1979).]

[2] P. P. Kulish, E. K. Sklyanin, *Quantum spectral transform method. Recent developments*, Lecture Notes in Phys., **151**, Springer, Berlin, etc., 1982, pp. 61–119.

[3] N. M. Bogoliubov, A. G. Izergin, V. E. Korepin, *Correlation Functions of Integrable Systems and Quantum Inverse Scattering Method*, Nauka, Moscow, 1992. [In Russian]

[4] V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, Cambridge, 1993.

[5] V. E. Korepin, *Calculation of norms of Bethe wave functions*, Comm. Math. Phys. **86** (1982), No. 3, 391–418.

[6] A. G. Izergin, V. E. Korepin, *Correlation functions for the Heisenberg XXZ-antiferromagnet*, Comm. Math. Phys. **99** (1985), No. 2, 271–302.

[7] F. H. L. Eßler, H. Frahm, A. G. Izergin, V. E. Korepin, *Determinant representation for correlation functions of spin-1/2 XXX and XXZ Heisenberg magnets*, Comm. Math. Phys., **174** (1995), No. 1, 191–214.

[8] N. Kitanine, J. M. Maillet, N. Slavnov, V. Terras, *Correlation functions of the XXZ spin-$\frac{1}{2}$ Heisenberg chain at the free fermion point from their multiple integral representations*, Nucl. Phys. B **642** (2002), No. 3, 433–455.

[9] N. Kitanine, J. M. Maillet, N. A. Slavnov, V. Terras, *On the algebraic Bethe Ansatz approach to the correlation functions of the XXZ spin-1/2 Heisenberg chain*, arXiv:hep-th/0505006

[10] N. A. Slavnov, *The algebraic Bethe ansatz and quantum integrable systems*, Russian Math. Surveys, **62** (2007), No. 4, 727–766.

[11] E. Lieb, T. Schultz, D. Mattis, *Two soluble models of an antiferromagnetic chain*, Ann. Phys. (NY), **16** (1961), No. 3, 407–466.

[12] Th. Niemeijer, *Some exact calculations on a chain of spin $\frac{1}{2}$. I, II*, Physica, **36** (1967), No. 3, 377–419; **39** (1968), No. 3, 313–326.

[13] F. Colomo, A. G. Izergin, V. E. Korepin, V. Tognetti, *Correlators in the Heisenberg XXO chain as Fredholm determinants*, Phys. Lett. A, **169** (1992), No.4, 243–247.

[14] F. Colomo, A. G. Izergin, V. E. Korepin, V. Tognetti, *Temperature correlation functions in the XXO Heisenberg chain*, Theor. Math. Phys., **94** (1993), No.1, 11–38.

[15] M. E. Fisher, *Walks, walls, wetting, and melting*, J. Statist. Phys., **34** (1984), No. 5–6, 667–729.

[16] P. J. Forrester, *Exact solution of the lock step model of vicious walkers*, J. Phys. A.: Math. Gen., **23** (1990), No. 7, 1259–1273.

[17] T. Nagao, P. J. Forrester, *Vicious random walkers and a discretization of Gaussian random matrix ensembles*, Nucl. Phys. B, **620** (2002), No. 3, 551–565.

[18] A. J. Guttmann, A. L. Owczarek, X. G. Viennot, *Vicious walkers and Young tableaux I: without walls*, J. Phys. A: Math. Gen., **31** (1998), No. 40, 8123–8135.

[19] C. Krattenthaler, A. J. Guttmann, X. G. Viennot, *Vicious walkers, friendly walkers and Young tableaux: II. With a wall*, J. Phys. A: Math. Gen., **33** (2000), No. 48, 8835–8866.

[20] C. Krattenthaler, A. J. Guttmann, X. G. Viennot, *Vicious walkers, friendly walkers, and Young tableaux. III. Between two walls*, J. Statist. Phys., **110** (2003), No. 3–6, 1069–1086.

[21] M. Katori, H. Tanemura, *Scaling limit of vicious walks and two-matrix model*, Phys. Rev. E, **66** (2002), No. 1, 011105 [12 pages].

[22] M. Katori, H. Tanemura, T. Nagao, N. Komatsuda, *Vicious walks with a wall, noncolliding meanders, and chiral and Bogoliubov–de Gennes random matrices*, Phys. Rev. E, **68** (2003), No. 2, 021112 [16 pages].

[23] M. Katori, H. Tanemura, *Nonintersecting paths, noncolliding diffusion processes and representation theory*, In: Combinatorial Methods in Representation Theory and their Applications. RIMS Kokyuroku, **1438** (2005), 83–102.

[24] P. Bak, C. Tang, K. Wiesenfeld, *Self-organized criticality*, Phys. Rev. A, **38** (1998), No. 1, 364–374.

[25] D. Huse, M. Fisher, *Commensurate melting, domain walls, and dislocations*, Phys. Rev. B, **29** (1984), No. 1, 239–270.

[26] J. W. Essam, A. J. Guttmann, *Vicious walkers and directed polymer networks in general dimensions*, Phys. Rev. E, **52** (1995), No. 6, 5849–5862.

[27] S. Yu. Grigoriev, V. B. Priezzhev, *Random walk of annihilating particles on the ring*, Teoret. Mat. Fiz., **146** (2006), No. 3, 488–498. [In Russian]

[28] J. W. van de Leur, A. Yu. Orlov, *Random turn walk on a half line with creation of particles at the origin.* arXiv:0801.0066

[29] N. M. Bogoliubov, *XX Heisenberg chain and random walks*, Zap. Nauchn. Sem. POMI **325** (2005), 13–27; English transl., J. Math. Sci. **138** (2006), No. 3, 5636–5643.

[30] N. M. Bogoliubov, *The integrable models for the vicious and friendly walkers*, Zap. Nauchn. Sem. POMI **335** (2006), 59–74; English transl., J. Math. Sci., **143** (2007), No. 1, 2729–2737.

[31] N. M. Bogoliubov, C. Malyshev, *A path integration approach to the correlators of XY Heisenberg magnet and random walks*, In: Proceedings of the 9th Intern. Conf. "Path Integrals: New Trends and Perspectives" (Dresden, Germany, September 23–28, 2007). Eds., W. Janke, A. Pelster (World Sci., Singapore, 2008), pp. 508–513. arXiv:0810.4816

[32] N. M. Bogoliubov, C. Malyshev, *The correlation functions of the XX Heisenberg magnet and random walks of vicious walkers*, Theor. Math. Phys., **159** (2009), No. 2, 563–574. arXiv:0903.3227

[33] N. M. Bogoliubov, C. Malyshev, *The correlation functions of the XXZ Heisenberg chain in the case of zero or infinite anisotropy, and random walks of vicious walkers*, St. Petersburg Math. J., **22** (2011), No. 3, 359-377. arXiv:0912.1138

[34] N. M. Bogoliubov, C. Malyshev, *Ising limit of a Heisenberg XXZ magnet and some temperature correlation functions*, Theor. Math. Phys., **169** (2011), No. 2, 1517-1529. arXiv:1106.5455

[35] N. M. Bogoliubov, *Form-factors, plane partitions and random walks*, Zap. Nauchn. Sem. POMI **360** (2008), 5–29; English transl., J. Math. Sci., **158** (2009), No. 6, 771–786.

[36] S. Sachdev, *Quantum Phase Transitions*, Cambridge University Press, Cambridge, 1999.

[37] F. Colomo, A. G. Izergin, V. Tognetti, *Correlation functions in the XXO Heisenberg chain and their relations with spectral shapes*, J. Phys. A: Math. Gen., **30** (1997), No. 2, 361–370.

[38] V. Korepin, J. Terilla, *Thermodynamic interpretation of quantum error correcting criterion*, Quantum Information Processing, **1** (2002), No. 4, 225–242.

[39] B.-Q. Jin, V. E. Korepin, *Entanglement, Toeplitz determinants and Fisher-Hartwig conjecture*, J. Stat. Phys., **116** (2004), No. 1-4, 79–95.

[40] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, Oxford, 1995.

[41] F. R. Gantmakher, *Theory of Matrices*, Nauka, Moscow, 1988. [In Russian]

[42] M. Shiroishi, M. Takahashi, Y. Nishiyama, *Emptiness Formation Probability for the One-Dimensional Isotropic XY Model* , J. Phys. Soc. Jap., **70** (2001), 3535–3543.

[43] A. G. Abanov, F. Franchini, *Emptiness Formation Probability for the Anisotropic XY Spin Chain in a Magnetic Field*, Phys. Lett. A, **316**, No. 5 (2003), 342–349.

[44] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach* (AMS, Providence, 1999).

[45] G. Kuperberg, *Another proof of the alternative-sign matrix conjecture*, Int. Math. Res. Notices, **1996** (1996), 139–150.

[46] D. P. Zhelobenko, A. I. Stern, *Representations of the Lie Groups*, Nauka, Moscow, 1983. [In Russian]

[47] M. L. Mehta, *Random Matrices*, Academic Press, London, 1991.

[48] A. Klimyk, K. Schmudgen, *Quantum Groups and their Representations*, Springer, Berlin, 1997.

[49] A. M. Ostrowski, *On some determinants with combinatorial numbers*, J. reine angew. Math. **216** (1964), 25–30.

[50] L. Carlitz, *Some determinants of q-binomial coefficients*, J. reine angew. Math. **226** (1967), 216–220.

[51] I. Gessel, G. Viennot, *Binomial determinants, paths, and hook length formulae*, Advances in Mathematics, **58** (1985), 300–321.

[52] A. Okounkov, *Infinite wedge and random partitions*, Selecta Mathematica, New Series, **7** (2001), No. 1, 57–81.

[53] W. Magnus, F. Oberhettinger, R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer, Berlin, 1966.