ПРЕПРИНТЫ ПОМИ РАН

Учредитель: Федеральное государственное бюджетное учреждение науки
Санкт-Петербургское отделение Математического института
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Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16 октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых коммуникаций

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

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http://www. pdmi.ras.ru /preprint/

Заведующая информационно-издательским сектором Симонова В.Н

PDMI Preprint 18/2012

Exact constants in Poincaré type inequalities for functions with zero mean boundary traces

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Abstract

In the paper, we investigate Poincaré type inequalities for the functions having zero mean value on the whole boundary of a Lipschitz domain or on a measurable part of the boundary. We derive exact and easily computable constants for some basic domains (rectangles, cubes, and right triangles). In the last section, we derive an a estimate of the difference between the exact solutions of two boundary value problems. Constants in Poincaré type inequalities enter these estimates, which provide guaranteed a posteriori error control.

1. Introduction

Let Ω be a connected bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. The classical Poincaré inequality reads

$$\|w\|_{2,\Omega} \leq C_P(\Omega)\|\nabla w\|_{2,\Omega}, \qquad \forall w \in \widetilde{H}^1(\Omega), \tag{1.1}$$

where

$$\widetilde{H}^1(\Omega) := \left\{w \in H^1(\Omega) \mid \{\!\!\{\, w \,\}\!\!\}_\Omega = 0\right\}.$$

Here and later on $\{\!\!\{\, g \,\}\!\!\}_\omega$ denotes the mean value of g on the set ω.

It was shown by Steklov [10] that the constant in (1.1) is equal to $\lambda^{-\frac{1}{2}}$, where λ is the smallest positive eigenvalue of the problem

$$\begin{aligned} -\Delta u =& \lambda u \quad \text{in} \quad \Omega; \\ \partial_{\mathbf{n}} u =& 0 \quad\ \ \text{on} \quad \partial\Omega, \end{aligned} \tag{1.2}$$

It has been shown (see [7]) that for convex domains in $\mathbb{R}^d$ an upper bound of the Poincaré constant is expressed throughout the diameter of Ω, namely,

$$C_P(\Omega) \leq \frac{\mathrm{diam}\Omega}{\pi}. \tag{1.3}$$

Other results related to constants in Poincaré type inequalities can be found in [2, 3, 4, 5, 6] and some other publications cited therein.

In this paper, we consider estimates similar to (1.1), for the functions having zero mean on a certain part of the boundary (or on the whole boundary). They are as follows:

$$\|w\|_{2,\Omega} \leq C_1(\Omega,\Gamma)\|\nabla w\|_{2,\Omega}, \qquad \forall w \in H^1(\Omega,\Gamma), \tag{1.4}$$

$$\|w\|_{2,\Gamma} \leq C_2(\Omega,\Gamma)\|\nabla w\|_{2,\Omega}, \qquad \forall w \in H^1(\Omega,\Gamma), \tag{1.5}$$

where Γ is a measurable part of $\partial\Omega$ (we assume that $(d-1)$-measure of Γ is positive),

$$H^1(\Omega,\Gamma) = \left\{w \in H^1(\Omega) \mid \{\!\!\{\, w \,\}\!\!\}_\Gamma = 0\right\}.$$

Since the quantity $\|w\|_\ell := \|\nabla w\|_{2,\Omega} + |\int_\Gamma w\,ds|$ is a norm equivalent to the original norm of $H^1(\Omega)$, existence of the constants $C_1(\Omega,\Gamma)$ and $C_2(\Omega,\Gamma)$ is easy to prove.

In this paper, we find sharp values of the constants in Poincaré type inequalities for rectangular domains and also for some classes of triangles. Our analysis is based on the fact (obtained by standard variational arguments) that the extremal function in (1.4) is an eigenfunction $u \in H^1(\Omega,\Gamma)$ of the boundary value problem

$$\begin{aligned} &-\Delta u = \lambda u \qquad \text{in} \quad \Omega; \\ &\partial_{\mathbf{n}} u = \mu \equiv \frac{\lambda}{|\Gamma|}\int_\Omega u\,dx \quad \text{on} \quad \Gamma; \qquad \partial_{\mathbf{n}} u = 0 \quad \text{on} \quad \partial\Omega\setminus\Gamma, \end{aligned} \tag{1.6}$$

which corresponds to the least eigenvalue $\lambda > 0$.

Analogously, the extremal function in (1.5) is an eigenfunction $u \in H^1(\Omega,\Gamma)$ of the boundary value problem

$$\begin{aligned} &\Delta u = 0 \qquad \text{in} \quad \Omega; \\ &\partial_{\mathbf{n}} u = \lambda u \quad \text{on} \quad \Gamma; \qquad \partial_{\mathbf{n}} u = 0 \quad \text{on} \quad \partial\Omega\setminus\Gamma, \end{aligned} \tag{1.7}$$

which corresponds to the least positive eigenvalue.

In both cases the sharp constant in (1.4), (1.5) is equal to $\lambda^{-\frac{1}{2}}$. It is easy to show that the eigenfunctions of the problems (1.6) and (1.7) form complete orthogonal systems in $L_2(\Omega)$ and in $L_2(\Gamma)$, respectively. Thus, the analysis is reduced to finding the corresponding minimal positive eigenvalues.

In short, the outline of the paper is as follows. Section 2 is concerned with exact constants for rectangular domains in $\mathbb{R}^2$. In Section 3, we find the constants for right triangles and in Section 4 for a parallelepiped. Section 5 is intended to present an example, which shows that the estimates can be used in quantitative analysis of differential equations. In this example, we consider two elliptic boundary value problems with different boundary conditions and source terms. The second problem is viewed as a certain simplification of the first one. This means that if the functions presenting source terms and Dirichlet or Neumann boundary conditions have complicated nonlinear behavior in some sets, then they are replaced by simple (e.g., constant) functions. We show that if Ω can be decomposed into a collection of simple subdomains (for which the constants C_P, C_1 and C_2 are known), then an easily computable bound of the difference between two exact solutions can be deduced. We outline that the computation of this bound does not require solving a boundary value problem and needs only integration of known functions. In particular, this estimate can be used to find a suitable initial mesh in finite element, finite difference, or discontinuous Galerkin methods. Our analysis is performed with the example of a simple linear elliptic equation. However, by similar arguments one can obtain similar estimates for other differential equations associated with the pair of conjugate operators grad and $-$div. Other applications of (1.4) and (1.5) are related to a posteriori error estimation methods for partial differential equations, where computable bounds between exact solutions and approximations often involve constants in Poincaré type inequalities (see [8]).

2. Exact constants for rectangles

In this section, we assume that Ω is a rectangle with lengthes of sides h_1 and h_2. We find exact values of the constants in (1.4) and (1.5) for the following two cases: Γ coincides with one side of Ω and Γ coincides with the whole $\partial\Omega$.

2.1. **Case 1: Γ coincides with one side of the rectangle.** In this case it is convenient to select the coordinate system such that (see Fig. 1) $\Omega = (0, h_1) \times (0, h_2)$. Without a loss of generality, we assume that

$$\Gamma = \{x_1 = 0, x_2 \in [0, h_2]\}.$$

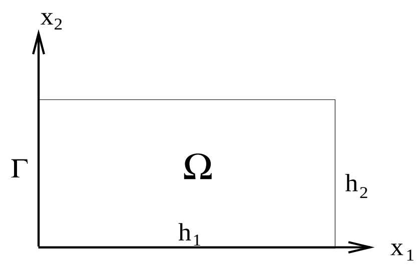

Figure 1

Theorem 2.1. *Sharp constants in* (1.4) *and* (1.5) *are equal to* $\frac{1}{\pi}\max\{2h_1; h_2\}$ *and* $\left(\frac{\pi}{h_2}\tanh(\frac{\pi h_1}{h_2})\right)^{-\frac{1}{2}}$*, respectively.*

Proof. Separating variables we obtain that the eigenfunctions of the problem (1.6)

$$u_{km}(x) = \cos\left(\frac{\pi m}{h_1}x_1\right)\cos\left(\frac{\pi k}{h_2}x_2\right), \quad m = 0,1,2,\dots;\ k = 1,2,\dots;$$
$$u_{0m}(x) = \sin\left(\frac{\pi(m+\frac{1}{2})}{h_1}x_1\right), \qquad m = 0,1,2,\dots.$$

They form a complete orthogonal system in $L_2(\Omega)$. Therefore, the least eigenvalue of the problem (1.6) is $\min\{\lambda_{00};\lambda_{10}\} = \min\{\left(\frac{\pi}{2h_1}\right)^2; \left(\frac{\pi}{h_2}\right)^2\}$, and the first statement follows.

Consider another inequality. Similarly, we find that the eigenfunctions of (1.7) are

$$u_k(x) = \cos\left(\frac{\pi k}{h_2}x_2\right)\cosh\left(\frac{\pi k}{h_2}(x_1 - h_1)\right), \qquad k = 0,1,2,\dots.$$

They form a complete orthogonal system in $L_2(\Gamma)$. Therefore, the corresponding least eigenvalue of the problem (1.7) is $\lambda_1 = \frac{\pi}{h_2}\tanh(\frac{\pi h_1}{h_2})$. $\square$

Remark 2.1. *It is convenient to present the constant* C_2 *in terms of parameters* h *and* κ*, which characterize the size and the shape of* Ω*, respectively. We set* $h_1 = \kappa h$ *and* $h_2 = h$*. Then,* $C_2 = C_*(\kappa)\sqrt{h}$*, where* $C_*(\kappa) = \frac{1}{\sqrt{\pi \tanh \pi\kappa}}$ *(see Fig. 2).*

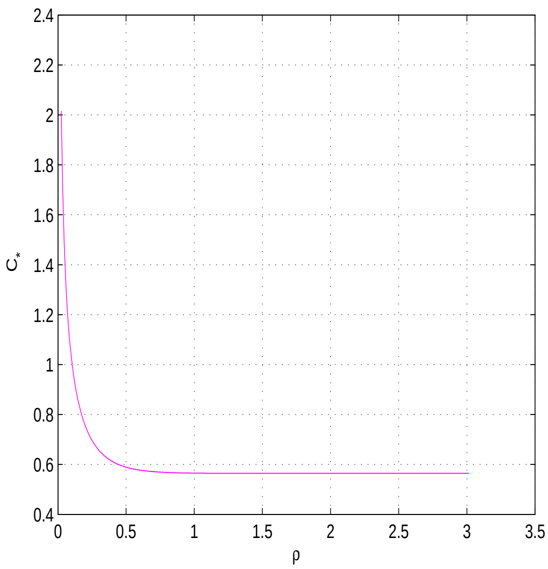

FIGURE 2. The graph of $C_*(\kappa)$.

2.2. **Case 2:** $\Gamma = \partial\Omega$**.** In this case, the problem is symmetric with respect to two axes. Therefore, it is convenient to select the coordinate system such that $\Omega = (-\frac{h_1}{2}, \frac{h_1}{2}) \times (-\frac{h_2}{2}, \frac{h_2}{2})$ (see Fig. 3). Due to the biaxial symmetry all the eigenfunctions of (1.6) and (1.7) are either even or odd with respect to the axes x_1 and x_2.

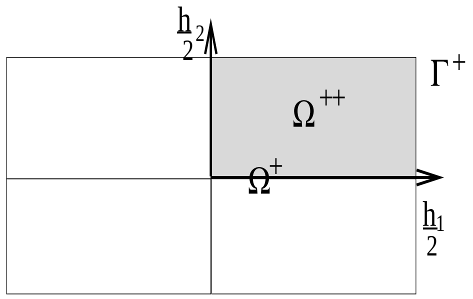

FIGURE 3

Theorem 2.2. *The sharp constant in* (1.4) *is equal to* $\frac{1}{\pi}\max\{h_1; h_2\}$.

Proof. First, we consider the eigenfunctions of (1.6), which are odd with respect to x_1. In this case $\mu = 0$, and we arrive at the following problem:

$$
\begin{aligned}
-\Delta u =& \lambda u \quad \text{in} \quad \Omega^+ := (0, \frac{h_1}{2}) \times (-\frac{h_2}{2}, \frac{h_2}{2}), \\
u =& 0 \quad \text{on} \quad \{x_1 = 0\} \cap \Omega, \qquad \partial_{\mathbf{n}} u = 0 \quad \text{on} \quad \partial\Omega^+ \setminus \{x_1 = 0\}.
\end{aligned} \tag{2.1}
$$

It is easy to see that the functions

$$
\begin{aligned}
u^{(1)}_{km}(x) &= \sin\Big(\frac{\pi(2k+1)}{h_1} x_1\Big) \cos\Big(\frac{2\pi m}{h_2} x_2\Big), \qquad k, m = 0, 1, \ldots; \\
u^{(2)}_{km}(x) &= \sin\Big(\frac{\pi(2k+1)}{h_1} x_1\Big) \sin\Big(\frac{\pi(2m+1)}{h_2} x_2\Big), \quad k, m = 0, 1, \ldots,
\end{aligned}
$$

are eigenfunctions to the problem (2.1). They form a system of orthogonal functions, which is complete in $L_2(\Omega^+)$. Therefore, the least eigenvalue of the problem (2.1) is $\lambda^{(1)}_{00} = \big(\frac{\pi}{h_1}\big)^2$.

Eigenfunctions of the problem (1.6), which are odd with respect to x_2 can be constructed quite similarly and we find that the corresponding least eigenvalue is $\big(\frac{\pi}{h_2}\big)^2$.

It remains to consider eigenfunctions even with respect to both variables. They belong to the space $H^1(\Omega^{++}, \Gamma^+)$, where $\Omega^{++} := (0, \frac{h_1}{2}) \times (0, \frac{h_2}{2})$, and $\Gamma^+ = \Gamma \cap \partial\Omega^{++}$ (see Fig. 3). In this case, we need to solve the problem

$$
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in} \quad \Omega^{++}; \\
\partial_{\mathbf{n}} u &= \mu \quad \text{on} \quad \Gamma^+; \qquad \partial_{\mathbf{n}} u = 0 \quad \text{on} \quad \partial\Omega^{++} \setminus \Gamma^+.
\end{aligned} \tag{2.2}
$$

Moreover, the eigenvalues λ^e_k of the problem (2.2) (enumerated in the increasing order and repeated according to their multiplicity) are critical values of the Rayleigh quotient

$$Q[v] \equiv \frac{\|\nabla v\|^2_{2,\Omega^{++}}}{\|v\|^2_{2,\Omega^{++}}} \tag{2.3}$$

over the space $H^1(\Omega^{++},\Gamma^+)$.

Consider now the functional Q on the whole space $H^1(\Omega^{++})$. By the variational principle (see, e.g., [1], (1.15)), its critical values $\widetilde{\lambda}^e_k$ enumerated in the increasing order and repeated according to their multiplicity satisfy the relation[1] $\widetilde{\lambda}^e_k \le \lambda^e_k \le \widetilde{\lambda}^e_{k+1}$. Therefore, if there exists an eigenvalue of the problem (2.2) in the interval $(\widetilde{\lambda}^e_0, \widetilde{\lambda}^e_1)$ then it is necessarily λ^e_0.

Note that $\widetilde{\lambda}^e_k$ are eigenvalues of the conventional Neumann problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega^{++}; \qquad \partial_{\mathbf{n}} u = 0 \quad \text{on} \quad \partial\Omega^{++},$$

and thus, $\widetilde{\lambda}^e_0 = 0$, $\widetilde{\lambda}^e_1 = \min\{\big(\frac{2\pi}{h_1}\big)^2; \big(\frac{2\pi}{h_2}\big)^2\}$.

Now we observe that the equation

$$\frac{h_1}{2}\cot\Big(\frac{\omega h_2}{2}\Big) + \frac{h_2}{2}\cot\Big(\frac{\omega h_1}{2}\Big) + \frac{2}{\omega} = 0 \tag{2.4}$$

has a unique solution ω_0 in the interval $(0, \min\{\frac{2\pi}{h_1}; \frac{2\pi}{h_2}\})$ since the function in the left-hand side of (2.4) decreases from $+\infty$ to $-\infty$ on this interval. Direct calculation shows that the function

$$v_0(x) = \frac{\cos(\omega_0 x_1)}{\sin(\frac{\omega_0 h_1}{2})} + \frac{\cos(\omega_0 x_2)}{\sin(\frac{\omega_0 h_2}{2})}$$

solves the problem (2.2) with $\lambda = \omega_0^2$. We note that (2.4) is just the condition $\int_{\Gamma_+} v_0 \, ds = 0$.

Thus, we conclude that $\lambda^e_0 = \omega_0^2$. However, it is easy to see that

$$\omega_0 > \min\{\frac{\pi}{h_1}; \frac{\pi}{h_2}\}.$$

Therefore, the least eigenvalue of the problem (1.6) is $\min\{\big(\frac{\pi}{h_1}\big)^2; \big(\frac{\pi}{h_2}\big)^2\}$, and the statement follows. $\square$

Theorem 2.3. *The sharp constant in* (1.5) *equals* $\big(\frac{2z_0}{\sqrt{h_1h_2}}\tanh(\frac{z_0}{\alpha_0})\big)^{-\frac{1}{2}}$, *where* $z_0 = z_0(\alpha)$ *is a unique root of the equation*

$$\tanh\Big(\frac{z}{\alpha}\Big)\tan(z\alpha) = 1, \tag{2.5}$$

such that $z_0\alpha < \frac{\pi}{2}$, *while* $\alpha_0 = \sqrt{\frac{\max\{h_1;h_2\}}{\min\{h_1;h_2\}}}$.

Proof. First, we consider the eigenfunctions of (1.7), which are even with respect to both variables. They belong to the space $H^1(\Omega^{++},\Gamma^+)$ and solve the following problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad \Omega^{++}; \\ \partial_{\mathbf{n}} u &= \lambda u \quad \text{on} \quad \Gamma^+; \qquad \partial_{\mathbf{n}} u = 0 \quad \text{on} \quad \partial\Omega^{++}\setminus\Gamma^+. \end{aligned} \tag{2.6}$$

[1]Note that $H^1(\Omega^{++},\Gamma^+)$ has codimension 1 in $H^1(\Omega^{++})$.

Moreover, the eigenvalues Λ_k^e of the problem (2.6) complemented by zero, enumerated in the increasing order and repeated according to their multiplicity are critical values of the Rayleigh quotient

$$\mathcal{Q}^+[v] \equiv \frac{\|\nabla v\|^2_{2,\Omega^{++}}}{\|v\|^2_{2,\Gamma^+}}$$

over the space $H^1(\Omega^{++})$. Consider another Rayleigh quotient

$$\widetilde{\mathcal{Q}}[v] \equiv \frac{\|\nabla v\|^2_{2,\Omega^{++}}}{\|v\|^2_{2,\partial\Omega^{++}}}$$

on the same space. Since $\widetilde{\mathcal{Q}}[v] \le \mathcal{Q}^+[v]$, by the variational principle its critical values $\widetilde{\Lambda}_k^e$, which are also enumerated in the increasing order and repeated according to their multiplicity, satisfy the relation $\widetilde{\Lambda}_k^e \le \Lambda_k^e$. However, by homogeneity argument $\widetilde{\Lambda}_k^e = 2\lambda_k$. Therefore, an eigenfunction of (1.7), which is even with respect to both variables cannot correspond to the least eigenvalue[2].

Further, we consider the eigenfunctions odd with respect to x_1. They lead to the following problem in Ω^+:

$$\begin{aligned} \Delta u =&0 \quad \text{in} \quad \Omega^+; \qquad u = 0 \quad \text{on} \quad \{x_1 = 0\}; \\ \partial_{\mathbf{n}} u =&\lambda u \quad \text{on} \quad \partial\Omega^+ \setminus \{x_1 = 0\}. \end{aligned} \tag{2.7}$$

We claim that the eigenfunction of (2.7) corresponding to the least eigenvalue should preserve its sign in Ω^+. Indeed, the function

$$v(x_1, x_2) = |u(|x_1|, x_2)| \cdot \operatorname{sign}(x_1)$$

belongs to $H^1(\Omega, \Gamma)$ and provides the same value λ of the Rayleigh quotient

$$\mathcal{Q}[v] \equiv \frac{\|\nabla v\|^2_{2,\Omega}}{\|v\|^2_{2,\Gamma}} \tag{2.8}$$

as u. If λ minimizes $\mathcal{Q}$ on $H^1(\Omega, \Gamma)$ then v must be a solution of (1.7), which is possible only if v is positive in Ω^+, and the claim follows. Moreover, since eigenfunctions of (2.7) are orthogonal in $L_2(\partial\Omega^+ \setminus \{x_1 = 0\})$, an eigenfunction positive in Ω^+ should correspond to the least eigenvalue.

Now we observe that the equation

$$\tan\left(\frac{\omega h_1}{2}\right) \tanh\left(\frac{\omega h_2}{2}\right) = 1, \tag{2.9}$$

---

[2]We can suggest another proof of this fact, which is interesting by itself. Let u be a solution of (2.6). We claim that at least one of sets $\varpi_\pm = \Omega^{++} \cap \{u \gtrless 0\}$ has a connected component which touches Γ^+ but does not touch the coordinate axes. Indeed, consider a connected component of ϖ_+ touching Γ^+ (in view of the condition $\int_{\Gamma^+} u = 0$, such a component exists). If this component touches both axes then any connected component of ϖ_- touching Γ^+ is separated either from $\{x_1 = 0\}$ or from $\{x_2 = 0\}$. To be definite, let ϖ be a connected component of ϖ_- which touches Γ^+ but does not touch $\{x_1 = 0\}$. Then the function

$$v(x_1, x_2) = u(|x_1|, |x_2|) \cdot \chi_\varpi(|x_1|, |x_2|) \cdot \operatorname{sign}(x_1)$$

belongs to $H^1(\Omega, \Gamma)$ and provides the same value Λ of the Rayleigh quotient (2.8) as u. If Λ minimizes $\mathcal{Q}$ on $H^1(\Omega, \Gamma)$ then v should be a solution of (1.7) which is impossible. Unfortunately, this argument is purely 2-dimensional.

obviously has a unique solution ω_1 in the interval $(0, \frac{2\pi}{h_1})$. Direct calculation shows that the function

$$v_1(x) = \sin(\omega_1 x_1)\cosh(\omega_1 x_2)$$

is positive in Ω^+ and solves the problem (2.7) with $\lambda = \omega_1 \tanh\left(\frac{\omega_1 h_2}{2}\right)$ (the equation (2.9) is just the equality of quotient $\partial_{\mathbf{n}} u/u$ on sides of rectangle). Substituting $z_0 = \frac{\omega_1}{2}\sqrt{h_1 h_2}$ we conclude that the least eigenvalue of (2.7) is equal to $\frac{2z_0}{\sqrt{h_1 h_2}} \tanh(\frac{z_0}{\underline{\alpha}})$ where z_0 is root of (2.5) with $\underline{\alpha} = \sqrt{\frac{h_1}{h_2}}$.

In a similar way, considering eigenfunctions of the problem (1.7) odd with respect to x_2 we obtain the least eigenvalue $\frac{2z_0}{\sqrt{h_1 h_2}} \tanh(\frac{z_0}{\overline{\alpha}})$ where z_0 is root of (2.5) with $\overline{\alpha} = \sqrt{\frac{h_2}{h_1}}$.

To complete the proof it suffices to show that the function $f(\alpha) = z_0 \tanh(\frac{z_0}{\alpha})$ decreases on $(0, +\infty)$. We claim that, in fact, $\alpha f(\alpha)$ is a decreasing function. Indeed, differentiation of (2.5) after some transformations yields

$$\frac{d}{d\alpha}\big(\alpha f(\alpha)\big) = \frac{2z_0(1 - \tanh^4(\frac{z_0}{\alpha}))}{1 + \alpha^2 - \tanh^2(\frac{z_0}{\alpha})(1 - \alpha^2)} \cdot \left[\frac{\tanh(\frac{z_0}{\alpha})}{1 + \tanh^2(\frac{z_0}{\alpha})} - z_0\alpha\right].$$

The fraction here is obviously positive. Further, (2.5) implies $z_0\alpha > \frac{\pi}{4}$. Thus,

$$\frac{\tanh(\frac{z_0}{\alpha})}{1 + \tanh^2(\frac{z_0}{\alpha})} - z_0\alpha < \frac{1}{2} - \frac{\pi}{4} < 0,$$

and the claim follows. $\square$

3. Exact constants for an isosceles right triangle

In this section, we assume that Ω is an isosceles right triangle. We find exact values of the constants in (1.4) and (1.5) for the following three cases: Γ is a leg; Γ coincides with two legs; Γ is the hypotenuse.

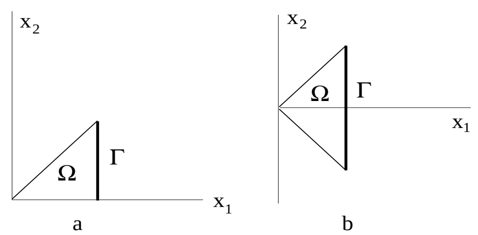

Figure 4

3.1. **Case 1: Γ is a leg.** In this case it is convenient to select the coordinate system such that (see Fig. 4a) $\Omega = \{0 < x_2 < x_1 < h\}$ and $\Gamma = \{x_1 = h, x_2 \in [0, h]\}$.

Theorem 3.1. *The exact constant in* (1.4) *is equal to* $\widetilde{z}_0^{-1} h$, *where* $\widetilde{z}_0$ *is a unique root of the equation*

$$z\cot(z) + 1 = 0 \tag{3.1}$$

in the interval $(0, \pi)$.

Proof. As in the proof of Theorem 2.2, the eigenvalues $\lambda_k^\vartriangle$ of the problem (1.6) (which are enumerated in the increasing order and repeated according to their multiplicity) are critical values of the Rayleigh quotient $Q[v] \equiv \frac{\|\nabla v\|_{2,\Omega}^2}{\|v\|_{2,\Omega}^2}$ over the space $H^1(\Omega, \Gamma)$.

Consider the functional Q on the whole space $H^1(\Omega)$. In accordance with the variational principle, the corresponding critical values $\widetilde{\lambda}_k^\vartriangle$ (enumerated in the increasing order and repeated according to their multiplicity) satisfy the relation $\widetilde{\lambda}_k^\vartriangle \le \lambda_k^\vartriangle \le \widetilde{\lambda}_{k+1}^\vartriangle$. Therefore, if the interval $(\widetilde{\lambda}_0^\vartriangle, \widetilde{\lambda}_1^\vartriangle)$ contains an eigenvalue of the problem (1.6), then it is necessarily $\lambda_0^\vartriangle$.

Note that $\widetilde{\lambda}_k^\vartriangle$ are eigenvalues of the conventional Neumann problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega; \qquad \partial_{\mathbf{n}} u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.2}$$

By even reflection with respect to the line $\{x_1 = x_2\}$ we conclude that any eigenfunction of (3.2) is an eigenfunction of the Neumann problem in the square $(0, h) \times (0, h)$. In particular, $\widetilde{\lambda}_0^\vartriangle = 0$ corresponds to the eigenfunction $\widetilde{u}_0 \equiv 1$, and $\widetilde{\lambda}_1^\vartriangle = \left(\frac{\pi}{h}\right)^2$ corresponds to the eigenfunction $\widetilde{u}_1(x) = \cos(\frac{\pi x_1}{h}) + \cos(\frac{\pi x_2}{h})$.

Now we observe that the equation (3.1) obviously has a unique solution in the interval $(0, \pi)$. Direct calculation shows that the function

$$\widetilde{v}_0(x) = \cos\left(\frac{\widetilde{z}_0 x_1}{h}\right) + \cos\left(\frac{\widetilde{z}_0 x_2}{h}\right)$$

solves the problem (1.6) with $\lambda = (\frac{\widetilde{z}_0}{h})^2$ (the equation (3.1) is just the condition $\int_\Gamma \widetilde{v}_0 \, ds = 0$). Thus, we conclude that $\lambda_0^\vartriangle = (\frac{\widetilde{z}_0}{h})^2$, and the statement follows. $\square$

Remark 3.1. *Approximate value of the root in* (3.1) *is* 2.02876. *Thus, the constant in Theorem* 3.1 *is approximately* $0.4929h$.

Theorem 3.2. *The sharp constant in* (1.5) *is equal to* $\left(\frac{\widehat{z}_0}{h}\tanh(\widehat{z}_0)\right)^{-\frac{1}{2}}$ *where* $\widehat{z}_0$ *is a unique root of the equation*

$$\tan(z) + \tanh(z) = 0 \tag{3.3}$$

in the interval $(0, \pi)$.

Proof. Using the monotone rearrangement (see, e.g., [5]) with respect to x_2 we can suppose that the minimizer v of the Rayleigh quotient $\mathcal{Q}[v]$ over the space $H^1(\Omega, \Gamma)$ is monotone decreasing in x_2. Therefore, $v|_\Gamma$ has exactly one change of sign. Moreover, since any other eigenfunction of the problem (1.7) is orthogonal to 1 and to v in $L_2(\Gamma)$, an eigenfunction u such that $u|_\Gamma$ has exactly one change of sign should coincide with v up to a constant multiplier.

Now we observe that the equation (3.3) obviously has a unique solution in the interval $(0, \pi)$. Direct calculation shows that the function

$$\widetilde{v}_1(x) = \cos\left(\frac{\widehat{z}_0 x_1}{h}\right)\cosh\left(\frac{\widehat{z}_0 x_2}{h}\right) + \cosh\left(\frac{\widehat{z}_0 x_1}{h}\right)\cos\left(\frac{\widehat{z}_0 x_2}{h}\right)$$

solves the problem (1.7) with $\lambda = \frac{\widehat{z}_0}{h}\tanh(\widehat{z}_0)$ (the equation (3.3) is just the condition $\int_\Gamma \widetilde{v}_1 \, ds = 0$). Since $\frac{\pi}{2} < \widehat{z}_0 < \pi$, $\widetilde{v}_1|_\Gamma$ is monotone decreasing in x_2. Therefore, $\widetilde{v}_1|_\Gamma$ has exactly one change of sign, and the statement follows. $\square$

Remark 3.2. *Approximate value of the root in* (3.3) *is* 2.3650. *Thus, the constant in Theorem 3.2 is approximately* $0.6560h^{\frac{1}{2}}$.

3.2. **Case 2: Γ coincides with two legs.** In this case we again assume that (see Fig. 4a) $\Omega = \{0 < x_2 < x_1 < h\}$ and $\Gamma = \{x_1 = h, x_2 \in [0, h]\} \cup \{x_2 = 0, x_1 \in [0, h]\}$.

Theorem 3.3. *The sharp constant in* (1.4) *is equal to* $\frac{h}{\pi}$.

Proof. By even reflection with respect to the line $\{x_1 = x_2\}$ we conclude that any eigenfunction of (1.6) is an eigenfunction of the same problem in the square $\Omega' = (0, h) \times (0, h)$ with $\Gamma = \partial\Omega'$. This problem is solved in Theorem 2.2, and the least positive eigenvalue equals $(\frac{\pi}{h})^2$. The dimension of corresponding eigenspace equals 2 and contains the function $\cos(\frac{\pi}{h}x_1) + \cos(\frac{\pi}{h}x_2)$ which solves the original problem in the triangle. $\square$

Theorem 3.4. *The sharp constant in* (1.5) *equals* $\big(\frac{2z_0}{h}\tanh(z_0)\big)^{-\frac{1}{2}}$, *where* z_0 *is a unique root of the equation* (2.5) *with* $\alpha = 1$ *such that* $z < \frac{\pi}{2}$.

Proof. We again use even reflection with respect to the line $\{x_1 = x_2\}$ and reduce our problem to the problem in the square $(0, h) \times (0, h)$. This problem is solved in Theorem 2.3, and the least positive eigenvalue equals $\frac{2z_0}{h}\tanh(z_0)$. The dimension of corresponding eigenspace equals 2 and contains the function

$$\sin\Big(z_0\Big(\frac{2x_1}{h} - 1\Big)\Big)\cosh\Big(z_0\Big(\frac{2x_2}{h} - 1\Big)\Big) + \cosh\Big(z_0\Big(\frac{2x_1}{h} - 1\Big)\Big)\sin\Big(z_0\Big(\frac{2x_2}{h} - 1\Big)\Big)$$

which solves the original problem in the triangle. $\square$

Remark 3.3. *Approximate value of the root in* (2.5) *with* $\alpha = 1$ *is* 0.93755. *Thus, the constant in Theorem* 3.2 *is approximately* $0.8523h^{\frac{1}{2}}$.

3.3. **Case 3: Γ is the hypotenuse.** In this case it is convenient to select the coordinate system such that (see Fig. 4b) $\Omega = \{0 < |x_2| < x_1 < h\}$ and $\Gamma = \{x_1 = h, x_2 \in [-h, h]\}$.

Theorem 3.5. *The sharp constant in* (1.4) *is equal to* $\widetilde{z}_0^{-1}h$, *where* $\widetilde{z}_0$ *is defined in Theorem* 3.1.

Proof. First, we consider the eigenfunctions of (1.6), which are odd with respect to x_2. In this case $\mu = 0$, and we arrive at the following problem in $\widetilde{\Omega}^+ = \{0 < x_2 < x_1 < h\}$:

$$\begin{aligned} -\Delta u =& \lambda u \quad \text{in} \quad \widetilde{\Omega}^+; \qquad u = 0 \quad \text{on} \quad \{x_2 = 0\}; \\ \partial_{\mathbf{n}} u =& 0 \qquad \text{on} \quad \partial\widetilde{\Omega}^+ \setminus \{x_2 = 0\}. \end{aligned} \tag{3.4}$$

Similarly to Theorems 3.3 and 3.4, we use even reflection with respect to the line $\{x_1 = x_2\}$ and reduce (3.4) to the problem in the square $(0, h) \times (0, h)$. Thus, we conclude that the least eigenvalue of the problem (3.4) is equal to $\frac{1}{2}(\frac{\pi}{h})^2$ and corresponds to the eigenfunction

$$\widehat{u}_0(x) = \sin\Big(\frac{\pi x_1}{2h}\Big)\sin\Big(\frac{\pi x_2}{2h}\Big).$$

Next, we consider the eigenfunctions, which are even with respect to x_2. Then we arrive at the problem (1.6) in $\widetilde{\Omega}^+$ which is solved in Theorem 3.1.

To complete the proof we compare $\frac{1}{2}(\frac{\pi}{h})^2$ and $(\frac{\widetilde{z}_0}{h})^2$. It is easy to check that $\frac{\pi}{\sqrt{2}} \cdot \cot(\frac{\pi}{\sqrt{2}}) < -1$. Since $t \mapsto t \cdot \cot(t)$ is a decreasing function on $(0, \pi)$, this means $\frac{\pi}{\sqrt{2}} > \widetilde{z}_0$, and the statement follows. $\square$

Theorem 3.6. *The exact constant in* (1.5) *is equal to* $h^{\frac{1}{2}}$.

Proof. First, we consider eigenfunctions of the problem (1.7) even with respect to x_2. Then we arrive at the problem (1.7) in $\widetilde{\Omega}^+$ which is solved in Theorem 3.2.

Further, let us consider the eigenfunctions, which are odd with respect to x_1. We arrive at the following problem in $\widetilde{\Omega}^+$:

$$
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad \widetilde{\Omega}^+; & \partial_{\mathbf{n}} u &= 0 \quad \text{on} \quad \{x_1 = x_2\}; \\
u &= 0 \quad \text{on} \quad \{x_2 = 0\}; & \partial_{\mathbf{n}} u &= \lambda u \quad \text{on} \quad \{x_1 = h\}.
\end{aligned} \tag{3.5}
$$

Direct calculation shows that the function $x_1 x_2$ is positive in $\widetilde{\Omega}^+$ and solves the problem (3.5) with $\lambda = \frac{1}{h}$. Similarly to the problem (2.7), it should correspond to the least eigenvalue.

To complete the proof we compare $\frac{\widehat{z}}{h}\tanh(\widehat{z}_0)$ and $\frac{1}{h}$. Since $\widehat{z}_0 > \frac{\pi}{2}$, we have $\widehat{z}_0 \tanh(\widehat{z}_0) > \frac{\pi}{2}\tanh(\frac{\pi}{2}) > 1$, and the statement follows. $\square$

4. Constants in three dimensional case

Theorems 2.1–2.3 can be extended to functions of three variables. The corresponding proofs are quite similar. Therefore, we present them in a concise form paying major attention to 3D specifics.

Theorem 4.1. *Let $\Omega = (0, h_1) \times (0, h_2) \times (0, h_3)$ and $\Gamma = \partial\Omega \cap \{x_1 = 0\}$. Then the sharp constants in* (1.4) *and* (1.5) *are equal to $\frac{1}{\pi}\max\{2h_1; h_2; h_3\}$ and $\big(\frac{\pi}{\max\{h_2;h_3\}}\tanh(\frac{\pi h_1}{\max\{h_2;h_3\}})\big)^{-\frac{1}{2}}$, respectively.*

We omit the proof, which is quite similar to the proof of Theorem 2.1.

Remark 4.1. *Let $h = \max\{h_2; h_3\}$ and $h_1 = \kappa h$. Then $C_1(\Omega, \Gamma) = \frac{h}{\pi}\max\{1, 2\kappa\}$* and *$C_2(\Omega, \Gamma) = C_*(\kappa)\sqrt{h}$, where $C_*(\kappa) = \frac{1}{\sqrt{\pi \tanh \pi\kappa}}$ (see Fig. 2).*

Theorem 4.2. *Let $\Omega = (-\frac{h_1}{2}, \frac{h_1}{2}) \times (-\frac{h_2}{2}, \frac{h_2}{2}) \times (-\frac{h_3}{2}, \frac{h_3}{2})$ and $\Gamma = \partial\Omega$. Then the exact constant in* (1.4) *is equal to $\frac{1}{\pi}\max\{h_1; h_2; h_3\}$.*

Proof. The proof is similar to the proof of Theorem 2.2. Instead of (2.4) we obtain the equation

$$
\frac{h_1 h_2}{2}\cot\left(\frac{\omega h_3}{2}\right) + \frac{h_1 h_3}{2}\cot\left(\frac{\omega h_2}{2}\right) + \frac{h_2 h_3}{2}\cot\left(\frac{\omega h_1}{2}\right) + \frac{2}{\omega}(h_1 + h_2 + h_3) = 0.
$$

Its unique solution in the interval $(0, \frac{2\pi}{h})$ (here $h = \max\{h_1; h_2; h_3\}$) is greater than $\frac{\pi}{h}$, and the statement follows. $\square$

Theorem 4.3. *Let Ω and Γ be as in Theorem* 4.2. *Assume (for the sake of definiteness only) that $h_1 \le h_2 \le h_3$. Then the exact constant in* (1.5) *is equal to $\big(\frac{2z_1}{h_1}\tanh(z_1)\big)^{-\frac{1}{2}}$, where (z_1, z_2) is a unique solution of the system*

$$
\begin{aligned}
\frac{z_1}{h_1}\tanh(z_1) &= \frac{z_2}{h_2}\tanh(z_2) = \\
&= \frac{z_1}{h_1}\sqrt{1 + \frac{\tanh^2(z_1)}{\tanh^2(z_2)}} \cdot \cot\left(\frac{z_1 h_3}{h_1}\sqrt{1 + \frac{\tanh^2(z_1)}{\tanh^2(z_2)}}\right),
\end{aligned} \tag{4.1}
$$

such that $\frac{z_1 h_3}{h_1}\sqrt{1 + \frac{\tanh^2(z_1)}{\tanh^2(z_2)}} < \frac{\pi}{2}$.

Proof. Similarly to the proof of Theorem 2.3, we conclude that the eigenfunction of (1.7) corresponding to the least eigenvalue must be odd with respect to one of the coordinate axes (for instance, with respect to x_3). This gives the following analog of (2.7) in the domain $\widehat{\Omega}^+ = (-\frac{h_1}{2}, \frac{h_1}{2}) \times (-\frac{h_2}{2}, \frac{h_2}{2}) \times (0, \frac{h_3}{2})$:

$$\begin{aligned} \Delta u =&0 \quad \text{in} \quad \widehat{\Omega}^+; \qquad u = 0 \quad \text{on} \quad \{x_3 = 0\}; \\ \partial_{\mathbf{n}} u =& \Lambda u \quad \text{on} \quad \partial\widehat{\Omega}^+ \setminus \{x_3 = 0\}. \end{aligned} \tag{4.2}$$

Repeating the proof of Theorem 2.3, we find that the eigenfunction of (4.2), which is positive in $\widehat{\Omega}^+$, corresponds to the least eigenvalue.

It is easy to see that the equation

$$\mu \tanh\left(\frac{\mu h_1}{2}\right) = \nu \tanh\left(\frac{\nu h_2}{2}\right) \tag{4.3a}$$

defines an increasing function $\nu = \nu(\mu)$ on $\mathbb{R}_+$, and the equation

$$\mu \tanh\left(\frac{\mu h_1}{2}\right) = \sqrt{\mu^2 + \nu^2(\mu)} \cdot \cot\left(\sqrt{\mu^2 + \nu^2(\mu)}\ \frac{h_3}{2}\right) \tag{4.3b}$$

has a unique solution μ_0 such that $\sqrt{\mu_0^2 + \nu^2(\mu_0)}\ \frac{h_3}{2} < \frac{\pi}{2}$. Direct calculation shows that the function

$$U_1(x) = \cosh(\mu_0 x_1) \cosh(\nu(\mu_0) x_2) \sin\left(\sqrt{\mu_0^2 + \nu^2(\mu_0)}\, x_3\right)$$

is positive in $\widehat{\Omega}^+$ and solves the problem (4.2) with $\Lambda = \mu_0 \tanh\left(\frac{\mu_0 h_1}{2}\right)$ (note that (4.3) reflects the boundary conditions on the sides of parallelepiped). Substituting $z_1 = \frac{\mu_0 h_1}{2}$, $z_2 = \frac{\nu(\mu_0) h_2}{2}$, we conclude after some manipulations that the least eigenvalue of (4.2) equals $\frac{2z_1}{h_1} \tanh(z_1)$ where (z_1, z_2) is solution of (4.1).

The eigenfunctions odd with respect to other coordinates can be constructed quite analogously. However, some additional calculations show that U_1 is the best eigenfunction provided that h_3 is the longest edge of Ω. Thus, the statement follows. □

5. An application of the estimates

In this section, we discuss the meaning of the above derived estimates for quantitative analysis of solutions to partial differential equations with the paradigm of a linear elliptic equation. However, similar analysis can be performed for other differential equations associated with the pair of conjugate operators grad and $-$div.

Assume that the boundary $\partial\Omega$ consists of two measurable nonintersecting parts Γ^D and Γ^N associated with Dirichlet and Neumann boundary conditions, respectively. Consider the following elliptic boundary value **problem** $\mathcal{P}$:

$$\operatorname{div}(A\nabla u) + f = 0 \qquad \text{in} \quad \Omega, \tag{5.1}$$

$$u = u_0 \qquad \text{on} \quad \Gamma^D, \tag{5.2}$$

$$A\nabla u \cdot n = F \qquad \text{on} \quad \Gamma^N. \tag{5.3}$$

Here the dot stands for the scalar product of vectors,

$$f \in L^2(\Omega), \quad F \in L^2(\Gamma^N), \quad \text{and} \quad u_0 \in H^1(\Omega). \tag{5.4}$$

We assume that the matrix A is symmetric, bounded, and satisfies the uniform ellipticity condition

$$A\xi \cdot \xi \geq c|\xi|^2, \quad c > 0,$$

Standard (generalized) solution to the problem $\mathcal{P}$ is a function $u \in H^1(\Omega)$ such that

$$u - u_0 \in V_0 := \{w \in H^1(\Omega) : w|_{\Gamma^D} = 0\}$$

and

$$\int\limits_{\Omega} A\nabla u \cdot \nabla w \, dx = \int\limits_{\Omega} fw \, dx + \int\limits_{\Gamma_N} Fw \, ds, \qquad \forall w \in V_0. \tag{5.5}$$

Well known results in the elliptic theory guarantee the existence and uniqueness of the solution u.

In practice, finding a solution u is replaced by finding a sequence of approximate solutions u_n converging to u. Usually, u_n is constructed as a Galerkin approximation associated with a certain finite dimensional space V_n If the functions f, F, and u_0 are complicated (e.g., rapidly change or oscillate in some parts of Ω, Γ^N, and Γ^D), then finding u_n may be a difficult problem. For example, if approximations are constructed with the help of simple (e.g., piecewise affine) functions and the boundary conditions are defined by complicated nonlinear functions, then the boundary condition on Γ^D and Γ^N cannot be exactly satisfied. Similar difficulty arises if a curvilinear boundary is approximated by piecewise affine functions. Numerical computation of the integrals involving f and F (which is necessary in all variational–difference numerical schemes) leads to errors generated by the fact that on mesh cells the source terms are usually simplified (e.g., replaced by mean values). All these errors have a common source: they arise because in reality the construction of u_n is based on a different problem $\widehat{P}$. Approximation and integration errors induce additional errors in discrete solutions, which are usually estimated only in an asymptotic sense. However, in quantitative analysis we need concrete values of them. Indeed, if such an error is smaller than the desired accuracy level (which in the majority of cases is known a priori), then we can ignore inconveniences in boundary conditions and inaccuracy in local representations of source terms. On the other hand, if it is essentially larger, then the mesh and integration methods are invalid for our purposes (i.e., this is a signal to use a finer mesh and/or more accurate integration methods). Thus, a guaranteed and easily computable estimate of the error can help to select suitable meshes, approximations of source terms, and quadrature formulas without directly solving a boundary value problem.

Below we show how the required estimate can be deduced with the help of the Poincaré type inequalities considered in previous sections. We note that estimates of errors caused by simplification of the coefficients entering A has been recently derived in [9], so that estimation of summed effect can be done by combining these estimates.

Let us assume that Ω is split into a set $\mathcal{O}$ of "simple" nonoverlapping subdomains Ω_i (e.g., they can be cells of a certain mesh). Each Ω_i belongs to one of the following three subsets:

$$\begin{aligned}
\mathcal{O}^D &:= \{\Omega_i \subset \Omega \mid \partial\Omega_i \cap \Gamma^D := \Gamma_i^D \neq \emptyset\},\\
\mathcal{O}^N &:= \{\Omega_i \subset \Omega \mid \partial\Omega_i \cap \Gamma^N := \Gamma_i^N \neq \emptyset\},\\
\mathcal{O}^I &:= \mathcal{O} \setminus (\mathcal{O}^D \cup \mathcal{O}^N).
\end{aligned}$$

In other words, $\mathcal{O}^I$ contain interior subdomains, $\mathcal{O}^D$ contain subdomains associated with Γ^D, and elements of $\mathcal{O}^N$ are the subdomains associated with Γ^N. Then, $\overline{\Omega} = \overline{\Omega}^D \cup \overline{\Omega}^I \cup \overline{\Omega}^N$, where Ω^D, Ω^N, and Ω^I consist of Ω_i from $\mathcal{O}^D$, $\mathcal{O}^N$, and $\mathcal{O}^I$, respectively (see Fig. 5).

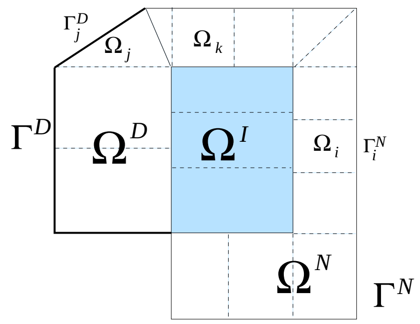

FIGURE 5

Now, instead of $\mathcal{P}$ we consider a modified (simplified) **problem** $\widehat{\mathcal{P}}$:

$$\operatorname{div}(\widehat{A}\nabla\widehat{u}) + \widehat{f} = 0 \quad \text{in} \quad \Omega, \tag{5.6}$$

$$\widehat{u} = \widehat{u}_0 \quad \text{on} \quad \Gamma^D, \tag{5.7}$$

$$A\nabla\widehat{u} \cdot n = \widehat{F} \quad \text{on} \quad \Gamma^N. \tag{5.8}$$

Our goal is to deduce an estimate of the difference between the exact solutions of these two problems. For this purpose, we define the energy norm

$$|\!|\!| u - \widehat{u} |\!|\!|^2 := \int_\Omega A\nabla(u-\widehat{u}) \cdot \nabla(u-\widehat{u})\, dx$$

and note that, by the ellipticity condition,

$$|\!|\!| u - \widehat{u} |\!|\!| \geq \sqrt{c} \cdot \|\nabla(u-\widehat{u})\|_{2,\Omega}. \tag{5.9}$$

Now we introduce the quantities

$$D_1 := \Big(\sum_{\Omega_i \in \mathcal{O}} \mathbb{C}_i^2 \|f - \widehat{f}\|^2_{2,\Omega_i} \Big)^{1/2}, \tag{5.10}$$

$$D_2 := \Big(\sum_{\Omega_i \in \mathcal{O}^N} C_2(\Omega_i, \Gamma_i^N)^2 \|F - \widehat{F}\|^2_{2,\Gamma_i^N} \Big)^{1/2}, \tag{5.11}$$

where

$$\mathbb{C}_i = \begin{cases} C_P(\Omega_i) & \text{if} \quad \Omega_i \in \mathcal{O}^I \cup \mathcal{O}^N, \\ C_1(\Omega_i, \Gamma_i^D) & \text{if} \quad \Omega_i \in \mathcal{O}^D. \end{cases} \tag{5.12}$$

These quantities are easily computable provided that the constants C_P, C_1, and C_2 associated with the corresponding subdomains are known. Indeed, if f, $\widehat{f}$, F, and $\widehat{F}$ are defined, then finding the quantities is reduced to integration of known functions and *does not require solving a boundary value problem.* The theorem below shows that a guaranteed and directly computable bound of the difference between two exact solutions can be expressed throughout D_1, D_2 and other easily computable quantities.

Theorem 5.1. *Let u and $\widehat{u}$ be the solutions of* (5.1)–(5.3) *and* (5.6)–(5.8), *respectively. Suppose also that the right-hand sides of* (5.6) *and* (5.8) *satisfy the conditions*

$$\{\!\!\{ f - \widehat{f} \}\!\!\}_{\Omega_i} = 0 \qquad \forall\, \Omega_i \in \mathcal{O}^I \cup \mathcal{O}^N, \tag{5.13}$$

$$\{\!\!\{ F - \widehat{F} \}\!\!\}_{\Gamma_i^N} = 0 \qquad \forall\, \Omega_i \in \mathcal{O}^N. \tag{5.14}$$

Then

$$|\!|\!| u - \widehat{u} |\!|\!| \leq \rho_1 + \sqrt{\rho_2 + \rho_1^2}, \tag{5.15}$$

where

$$2\rho_1 = \frac{D_1 + D_2}{\sqrt{c}} + |\!|\!| \phi |\!|\!|; \qquad \rho_2 = \mathcal{I}_0 + \mathcal{I}_1(\phi) + \mathcal{I}_2(\phi), \tag{5.16}$$

$$\mathcal{I}_0 = \sum_{\Omega_i \in \mathcal{O}^D} \{\!\!\{ u_0 - \widehat{u}_0 \}\!\!\}_{\Gamma_i^D} \int\limits_{\Omega_i} (f - \widehat{f})\, dx, \tag{5.17}$$

$$\mathcal{I}_1(\phi) = \int\limits_{\Omega} (f - \widehat{f})\phi\, dx, \qquad \mathcal{I}_2(\phi) = \int\limits_{\Gamma^N} (F - \widehat{F})\, \phi\, ds, \tag{5.18}$$

and ϕ is an arbitrary function in $H^1(\Omega)$ such that $\phi = u_0 - \widehat{u}_0$ on Γ^D.

Proof. We use (5.5), (5.6)–(5.8) and obtain

$$\int\limits_{\Omega} A(\nabla u - \nabla \widehat{u}) \cdot \nabla w\, dx = \int\limits_{\Omega} (f - \widehat{f}) w\, dx + \int\limits_{\Gamma^N} (F - \widehat{F})\, w\, ds \quad \forall w \in V_0. \tag{5.19}$$

Since $w = u - \widehat{u} - \phi \in V_0$ we can use it as a trial function. Then

$$\int\limits_{\Omega} A\nabla(u - \widehat{u}) \cdot \nabla w\, dx = |\!|\!| u - \widehat{u} |\!|\!|^2 + \int\limits_{\Omega} A\nabla(u - \widehat{u}) \cdot \nabla \phi\, dx \geq \tag{5.20}$$
$$\geq |\!|\!| u - \widehat{u} |\!|\!|^2 - |\!|\!| u - \widehat{u} |\!|\!|\, |\!|\!| \phi |\!|\!|.$$

Consider the first term in the right-hand side of (5.19):

$$\int\limits_{\Omega} (f - \widehat{f}) w\, dx \tag{5.21}$$
$$= \sum_{\Omega_i \in \mathcal{O}^I \cup \mathcal{O}^N} \int\limits_{\Omega_i} (f - \widehat{f})(u - \widehat{u}) dx + \sum_{\Omega_i \in \mathcal{O}^D} \int\limits_{\Omega_i} (f - \widehat{f})(u - \widehat{u}) dx + \mathcal{I}_1(\phi).$$

The terms of the first sum in (5.21) are estimated using (5.13) and (1.1):

$$\int\limits_{\Omega_i} (f - \widehat{f})(u - \widehat{u}) dx = \int\limits_{\Omega_i} (f - \widehat{f})(u - \widehat{u} - \{\!\!\{ u - \widehat{u} \}\!\!\}_{\Omega_i})\, dx$$
$$\leq C_P(\Omega_i) \| f - \widehat{f} \|_{2,\Omega_i} \| \nabla(u - \widehat{u}) \|_{2,\Omega_i}, \qquad \Omega_i \in \mathcal{O}^I \cup \mathcal{O}^N.$$

The terms of the second sum are estimated with the help of (1.4) as follows:

$$\int_{\Omega_i} (f-\widehat{f})(u-\widehat{u})\,dx = \int_{\Omega_i} (f-\widehat{f})(u-\widehat{u}-\{\!\!\{\, u-\widehat{u}\,\}\!\!\}_{\Gamma_i^D})\,dx$$

$$+\int_{\Omega_i} (f-\widehat{f})\{\!\!\{\, u-\widehat{u}\,\}\!\!\}_{\Gamma_i^D}\,dx \le C_1(\Omega_i,\Gamma_i^D)\|f-\widehat{f}\|_{2,\Omega_i}\|\nabla(u-\widehat{u})\|_{2,\Omega_i}$$

$$+\{\!\!\{\, u-\widehat{u}\,\}\!\!\}_{\Gamma_i^D}\int_{\Omega_i}(f-\widehat{f})dx, \qquad \Omega_i\in\mathcal{O}^D.$$

Summing up these estimates and using (5.9) and (5.17) we obtain

$$\int_{\Omega}(f-\widehat{f})w\;dx \le \sum_{\Omega_i\in\mathcal{O}} \mathbb{C}_i\|f-\widehat{f}\|_{2,\Omega_i}\|\nabla(u-\widehat{u})\|_{2,\Omega_i}+\mathcal{I}_0+\mathcal{I}_1(\phi)$$
$$\le \frac{D_1}{\sqrt{c}}|\!|\!|u-\widehat{u}|\!|\!|+\mathcal{I}_0+\mathcal{I}_1(\phi). \tag{5.22}$$

In a similar way, using (5.14) and (1.5) we deduce

$$\int_{\Gamma_i^N}(F-\widehat{F})(u-\widehat{u})\,ds \le C_2(\Omega,\Gamma_i^N)\|F-\widehat{F}\|_{2,\Gamma_i^N}\|\nabla(u-\widehat{u})\|_{2,\Omega_i}, \tag{5.23}$$

and, therefore, by (5.9) and (5.18),

$$\int_{\Gamma^N}(F-\widehat{F})w\;ds \le \frac{D_2}{\sqrt{c}}|\!|\!|u-\widehat{u}|\!|\!|+\mathcal{I}_2(\phi). \tag{5.24}$$

Now (5.19), (5.21), (5.22), and (5.24) imply the estimate

$$|\!|\!|u-\widehat{u}|\!|\!|^2 \le 2\rho_1|\!|\!|u-\widehat{u}|\!|\!|+\rho_2, \tag{5.25}$$

where the quantities ρ_1 and ρ_2 are defined by (5.16).

The quadratic inequality (5.25) easily implies (5.15). □

Theorem 5.1 presents the most general form of the estimate. If $\widehat{u}_0=u_0$, then this estimate can be significantly simplified. Indeed, in this case one can choose $\phi\equiv 0$, and (5.15) is reduced to

$$|\!|\!|u-\widehat{u}|\!|\!| \le \frac{D_1+D_2}{\sqrt{c}}. \tag{5.26}$$

Moreover, in this case we can replace $C_1(\Omega_i,\Gamma_i^D)$ in (5.12) by a smaller constant $C_F(\Omega_i,\Gamma_i^D)$ such that

$$\|w\|_{2,\Omega}\le C_F(\Omega_i,\Gamma_i^D)\|\nabla w\|_{2,\Omega}, \qquad \forall w\in H^1(\Omega_i)\;:\;w|_{\Gamma_i^D}=0.$$

For simple domains such as rectangles or isoceles right triangles this constant is well-known.

6. Acknowledgments

This work was in part supported by RFBR grants 11-01-00825 (the first author) and 11-01-00531-a (the second author).

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