ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

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РЕДКОЛЛЕГИЯ

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Free boundary problems of magnetohydrodynamics in multi-connected domains

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ABSTRACT:

We prove local in time solvability of a free boundary problem for a viscous incompressible electrically conducting capillary liquid. Concerning initial data (velocity vector field and magnetic field) we assume that they satisfy only natural compatibility conditions. The solution is obtained in anisotropic Sobolev spaces.

Key words and phrases: Free boundary problems, magnetohydrodynamics, Sobolev spaces.

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1. Formulation of the problem.

The subject of the present paper is the evolution free boundary problem for a viscous incompressible electrically conducting capillary liquid occupying a bounded variable domain Ω_{1t} with a free boundary Γ_t that is supposed to be a connected compact surface. The liquid is subjected to the external mass forces $\boldsymbol{f}(x,t)$, capillary forces on Γ_t and forces arising due to the presence of the magnetic field. Moreover, in a fixed domain Ω_3 an electric current of density $\boldsymbol{j}(x,t)$ is defined. This vector field should have zero normal component on $\partial\Omega_3$:

$$\boldsymbol{j}\cdot\boldsymbol{n}=0 \tag{1.1}$$

and vanish outside Ω_3.

Both Ω_{1t} and Ω_3 are surrounded by a vacuum region Ω_{2t}. The domain $\Omega=\overline{\Omega}_{1t}\cup\overline{\Omega}_3\cup\Omega_{2t}$ is bounded by a compact perfectly conducting surface S. If is assumed that the domains Ω_{1t} and Ω_3 are disjoint, as well as the surfaces S and Γ_t.

The case $S=\emptyset$, $\Omega=\mathbb{R}^3$ is not excluded.

The state of the medium in Ω is characterized by the following functions:

$\boldsymbol{v}(x,t)$: the velocity vector field of the liquid,
$p(x,t)$: the pressure function,
$\boldsymbol{H}(x,t)$, $x\in\Omega$: the magnetic intensity,
$\boldsymbol{B}(x,t)$: the magnetic induction,
$\boldsymbol{E}(x,t)$, $x\in\Omega$: the electric intensity.

The vector fields $\boldsymbol{H}(x,t)$ and $\boldsymbol{B}(x,t)$ are connected by the relation

$$\boldsymbol{B}(x,t)=\mu\boldsymbol{H}(x,t)$$

where μ is a piece-wise constant function of magnetic permeability taking positive values μ_1 in Ω_1, μ_2 in Ω_2, μ_3 in Ω_3. The domain Ω_3 may consist of several components with different values of μ_3.

The motion of the liquid is governed by the Navier-Stokes equations

$$\begin{cases}\boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}-\nabla\cdot T(\boldsymbol{v},p)-\nabla\cdot T_M(\boldsymbol{H})=\boldsymbol{f}(x,t),\\ \nabla\cdot\boldsymbol{v}(x,t)=0,\quad x\in\Omega_{1t},\ \ t>0,\end{cases} \tag{1.2}$$

where

$T(\boldsymbol{v},p)=-pI+\nu S(\boldsymbol{v})$ is the viscous stress tensor,
$S(\boldsymbol{v})=\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T$is the doubled rate-of-strain tensor,
$T_M(\boldsymbol{H})=\mu(\boldsymbol{H}\otimes\boldsymbol{H}-\frac{1}{2}|\boldsymbol{H}|^2I)$ is the magnetic stress tensor.

Electric and magnetic fields satisfy the system of the Maxwell equations with the omitted displacement current

$$\begin{cases}\mu\boldsymbol{H}_t=-rot\boldsymbol{E},\quad \nabla\cdot\boldsymbol{H}=0,\qquad x\in\Omega_{1t}\cup\Omega_{2t}\cup\Omega_3,\\ rot\boldsymbol{H}=\alpha(\boldsymbol{E}+\mu(\boldsymbol{v}\times\boldsymbol{H})),\quad x\in\Omega_{1t},\ \ t>0,\\ rot\boldsymbol{H}=\alpha\boldsymbol{E}+\boldsymbol{j}(x,t),\quad x\in\Omega_3,\\ rot\boldsymbol{H}=0,\quad \nabla\cdot\boldsymbol{H}=0,\quad \nabla\cdot\boldsymbol{E}=0,\quad x\in\Omega_{2t}.\end{cases} \tag{1.3}$$

where α is a piece-wise constant function of conductivity, positive for $x\in\Omega_{1t}\cup\Omega_3$ and equal to zero in Ω_{2t}. The last equation for $\boldsymbol{E}$ follows from the fact that the vacuum region Ω_{2t} can not contain electric charges.

Let us pass to the boundary conditions on the exterior boundary S and on the interfaces Γ_t and S_3 separating media with different physical properties. On fixed surfaces $S_3=\partial\Omega_3$ and S

standard boundary and jump conditions are prescribed (see [1,2]):

$$\begin{cases} \boldsymbol{H}\cdot\boldsymbol{n}=0, \quad \boldsymbol{E}_\tau=0, \quad x\in S, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{H}_\tau]=0, \quad [\boldsymbol{E}_\tau]=0, \quad x\in S_3, \end{cases} \tag{1.4}$$

and on the free surface Γ_t we have

$$\begin{cases} \big(T(\boldsymbol{v},p)+[T_M(\boldsymbol{H})]\big)\boldsymbol{n}=\sigma\boldsymbol{n}H, \quad V_n=\boldsymbol{v}\cdot\boldsymbol{n}, \\ [\mu\boldsymbol{H}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{H}_\tau]=0, \end{cases} \tag{1.5}$$

where H is the doubled mean curvature of Γ_t,
$\boldsymbol{H}_\tau=\boldsymbol{H}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{H})$, $\boldsymbol{E}_\tau=\boldsymbol{E}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{E})$ are tangential components of $\boldsymbol{H}$ and $\boldsymbol{E}$,
$[\boldsymbol{F}]$ is a jump of the vector field $\boldsymbol{F}(x)$, $x\in\Omega_{1t}\cup\Omega_{2t}\cup\Omega_3$, on Γ_t and S_3,
V_n is the velocity of evolution of Γ_t in the direction of the exterior normal $\boldsymbol{n}$.

It remains to write the jump condition for $\boldsymbol{E}_\tau$ on Γ_t. Usually the jump conditions are deduced from the Maxwell equations, assuming that they are satisfied in the sense of the distribution theory. For instance, from

$$\int_K \boldsymbol{B}\cdot\nabla\varphi(x)dx=0$$

where $\varphi\in C_0^\infty(K)$, $K\subset\Omega$, $K\cup\Gamma_t\neq\emptyset$ it follows that

$$-\int_K \nabla\cdot\boldsymbol{B}\varphi dx+\int_{\Gamma_t\cap K}[\boldsymbol{B}\cdot\boldsymbol{n}]\varphi dS=0,$$

and this implies

$$\nabla\cdot\boldsymbol{B}=0, \quad x\in K\setminus\Gamma_t, \quad [\boldsymbol{B}\cdot\boldsymbol{n}]=0, \quad x\in\Gamma_t\cap K. \tag{1.6}$$

In the same way the jump condition $[\boldsymbol{H}_\tau]|_{\Gamma_t}=0$ is deduced. Moreover, if the equation $\boldsymbol{B}_t=-rot\boldsymbol{E}$ is satisfied in a generalized sense, i.e.,

$$\int_{t_0}^{t_0+\tau}\int_K(-\boldsymbol{B}\cdot\boldsymbol{\varphi}_t+\boldsymbol{E}\cdot rot\boldsymbol{\varphi})dxdt=0,$$

then on the manifold $\mathfrak{G}_\tau=\{x\in\Gamma_t,\ t\in(t_0,t_0+\tau)\}\subset\mathbb{R}^4$ the relation

$$\mathfrak{n}_t[\boldsymbol{B}_\tau]+[\mathfrak{n}_x\times\boldsymbol{E}]=0, \tag{1.7}$$

holds, where $\mathfrak{n}_x=(\mathfrak{n}_1,\mathfrak{n}_2,\mathfrak{n}_3)$ and $\mathfrak{n}_t$ are components of the normal vector $\mathfrak{n}$ to $\mathfrak{G}_\tau$ in $\mathbb{R}^4$. In particular, for fixed interfaces, when $\mathfrak{n}_t=0$, the condition (1.7) takes a standard form $[\mathfrak{n}_x\times\boldsymbol{E}]=0$, i.e., $[\boldsymbol{E}_\tau]=0$.

The following proposition is important for the construction of the solution of (1.2)-(1.5), (1.7).

Proposition 1. *If the relations (1.6) and (1.7) are satisfied, then*

$$[\boldsymbol{n}\cdot rot\boldsymbol{E}]=-[\boldsymbol{n}\cdot\boldsymbol{B}_t], \quad x\in\Gamma_t. \tag{1.8}$$

Proof. Suppose the surface $\mathfrak{G}_\tau\subset K\times(t_0,t_0+\tau)\subset\mathbb{R}^4$ is given by the equation $\xi_3=z(\xi_1,\xi_2,t)$, where (ξ_1,ξ_2,ξ_3) are Cartesian coordinates in $\mathbb{R}^3$. The functions given on $\mathfrak{G}_\tau$ can be considered as functions of ξ_1,ξ_2,t. The normal $\mathfrak{n}$ to $\mathfrak{G}_\tau$ is parallel to the vector $(\widetilde{\boldsymbol{n}},z_t)$, where $\widetilde{\boldsymbol{n}}=(z_{\xi_1},z_{\xi_2},-1)$.

It is easily verified that

$$\begin{aligned} &-[\widetilde{\boldsymbol{n}}\cdot rot\boldsymbol{E}]=[z_{\xi_1}(E_{2,3}-E_{3,2})+z_{\xi_2}(E_{3,1}-E_{1,3})-(E_{1,2}-E_{2,1})] \\ &=\big[\frac{\partial}{\partial\xi_1}(z_{\xi_2}E_3+E_2)+\frac{\partial}{\partial\xi_2}(-E_1-z_{\xi_1}E_3)\big]=\frac{\partial}{\partial\xi_1}[\widetilde{\boldsymbol{n}}\times\boldsymbol{E}]_1+\frac{\partial}{\partial\xi_2}[\widetilde{\boldsymbol{n}}\times\boldsymbol{E}]_2, \end{aligned} \tag{1.9}$$

where $E_{i,j}$ is the partial derivative of E_i with respect to ξ_j and

$$\frac{\partial \boldsymbol{E}}{\partial \xi_\alpha} = \boldsymbol{E}_{,\alpha} + z_{\xi_\alpha}\boldsymbol{E}_{,3}, \quad \alpha = 1, 2,$$

are derivatives calculated taking into account the dependence of $\boldsymbol{E}$ on ξ_3. By (1.7), the equation (1.9) is equivalent to

$$\begin{aligned} -[\widetilde{\boldsymbol{n}} \cdot \boldsymbol{rot}\boldsymbol{E}] &= -\frac{\partial}{\partial \xi_1} z_t[B_1] - \frac{\partial}{\partial \xi_2} z_t[B_2] \\ &= -[z_{\xi_1,t}B_1 + z_t B_{1,\xi_1} + z_{\xi_1} z_t B_{1,3} + z_{\xi_2,t}B_2 + z_t B_{2,\xi_2} + z_{\xi_2} z_t B_{2,3}]. \end{aligned} \tag{1.10}$$

Now we differentiate $[\widetilde{\boldsymbol{n}} \cdot \boldsymbol{B}] = 0$ with respect to t, which leads to

$$[z_{\xi_1,t}B_1 + z_{\xi_1}B_{1,t} + z_t z_{\xi_1}B_{1,3} + z_{\xi_2,t}B_2 + z_{\xi_2}B_{2,t} + z_{\xi_2} z_t B_{2,3} - B_{3,t} - z_t B_{3,t}] = 0. \tag{1.11}$$

When we add (1.11) to (1.10) and take the equation $\nabla \cdot \boldsymbol{B} = 0$ into account, we obtain

$$-[\widetilde{\boldsymbol{n}} \cdot \boldsymbol{rot}\boldsymbol{E}] = [z_{\xi_1}B_{1,t} + z_{\xi_2}B_{2,t} - B_{3,t}] = [\widetilde{\boldsymbol{n}} \cdot \boldsymbol{B}_t],$$

q.e.d.

The proof of the Proposition is due to Dr. N.Filonov.

Let us go back to the formulation of the problem. At the initial moment $t = 0$ the configuration of the liquid and the values of $\boldsymbol{v}(x, 0)$ and $\boldsymbol{H}(x, 0)$ are prescribed:

$$\boldsymbol{v}(x, 0) = \boldsymbol{v}_0(x), \quad x \in \Omega_{10}, \quad \boldsymbol{H}(x, 0) = \boldsymbol{H}_0(x), \quad x \in \Omega_{10} \cup \Omega_{20} \cup \Omega_3, \tag{1.12}$$

where Ω_{10}, Ω_{20} are given domains. Finally, we need some normalization conditions for $\boldsymbol{E}(x, t)$, $x \in \Omega_2$. Indeed, together with $\boldsymbol{E}$ the vector field $\boldsymbol{E} + \boldsymbol{E}'$ also satisfies (1.3),(1.4), if $\boldsymbol{E}' = 0$ in $\Omega_{1t} \cup \Omega_3$ and $\boldsymbol{E}'$ is so called Dirichlet vector field in Ω_{2t}, satisfying the conditions

$$rot\boldsymbol{E}'(x, t) = 0, \quad \nabla \cdot \boldsymbol{E}'(x, t) = 0, \quad x \in \Omega_{2t}, \quad \boldsymbol{E}'_\tau = 0, \quad x \in \partial\Omega_{2t},$$

in other words, $\boldsymbol{E}'(x, t) = \sum_{j=1}^{b_2} C_j(t)\chi_j(x, t)$,

$$\nabla^2\chi_j(x, t) = 0, \quad x \in \Omega_{2t}, \quad \chi_j(x, t)\Big|_{x \in S_k} = \delta_{jk}, \quad \chi_j(x, t)|_{x \in \Gamma_t} = 0,$$

where S_k, $k = 1, ..., b_2$ are all the connected components of $\partial\Omega_{2t}$, except Γ_t and b_2 is the second Betti number of Ω_{2t}. The normalization conditions can be taken in the form

$$\int_{S_k} \boldsymbol{E} \cdot \boldsymbol{n} dS = 0. \tag{1.13}$$

Since the matrix with the elements $\int_{S_i} \frac{\partial \chi_j}{\partial n} dS$, $i, j = 1, ..., b_2$ is not degenerate, the equations (1.13) define $C_j(t)$ (and the vector field $\boldsymbol{E}(x, t)$, $x \in \Omega_{2t}$) in a unique way.

The aim of the paper is to prove local (in time) unique solvability of the problem (1.2)-(1.5), (1.7), (1.12), (1.13) under the assumption that the initial data satisfy only natural compatibility conditions

$$\begin{aligned} &\nabla \cdot \boldsymbol{v}_0(x) = 0, \quad x \in \Omega_{10}, \quad [(S(\boldsymbol{v}_0)\boldsymbol{n})_\tau] = 0, \quad x \in \Gamma_0, \\ &\nabla \cdot \boldsymbol{H}_0(x) = 0, \quad x \in \Omega_{10} \cup \Omega_{20} \cup \Omega_{30}, \quad rot\boldsymbol{H}_0(x) = 0, \quad x \in \Omega_{20}, \\ &[\boldsymbol{H}_{0\tau}] = 0, \quad [\mu\boldsymbol{H}_0 \cdot \boldsymbol{n}_0] = 0, \quad x \in \Gamma_0, \\ &[\boldsymbol{H}_{0\tau}] = 0, \quad [\mu\boldsymbol{H}_0 \cdot \boldsymbol{n}] = 0, \quad x \in S_3, \\ &\boldsymbol{H}_0(x) \cdot \boldsymbol{n} = 0, \quad x \in S, \end{aligned} \tag{1.14}$$

where $\boldsymbol{n}_0$ is the exterior normal to Γ_0.

The precise formulation of the result is given in Theorem 2.1.

2. Transformation of the problem and formulation of the main result.

It is customary to write free boundary problems as nonlinear problems in fixed domains. We introduce now the corresponding coordinate transformation. We assume that Γ_0 is located in the neighborhood of a smooth connected surface $\mathcal{G}$ of arbitrary topological type, and can be regarded as a normal perturbation of $\mathcal{G}$:

$$\Gamma_0 = \{x = y + \boldsymbol{N}(y)\rho_0(y), \quad y \in \mathcal{G}\},$$

where ρ_0 is a given small function and $\boldsymbol{N}(y)$ is the exterior normal to $\mathcal{G}$. Moreover, we assume that also for $t > 0$

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho(y,t), \quad y \in \mathcal{G}\}, \tag{2.1}$$

with an unknown function $\rho(y,t)$ such that $\rho(y,0) = \rho_0(y)$. We extend $\boldsymbol{N}(y)$ and $\rho(y,t)$ from $\mathcal{G}$ into Ω in such a way that the extension $\boldsymbol{N}^*$ of $\boldsymbol{N}$ is a smooth non-zero regular function in Ω and ρ^* vanishes in Ω_3 and near S and satisfies the inequalities (4.11) (hence ρ^* is small for small ρ). We denote by $\mathcal{F}_1$ the domain bounded by $\mathcal{G}$ and we set $\mathcal{F}_2 = \Omega \setminus (\mathcal{F}_1 \cup \overline{\Omega}_3)$, $\Omega_3 \equiv \mathcal{F}_3$ (for uniformity of notation).

The transformation

$$x \equiv e_\rho(y,t) = y + \boldsymbol{N}^*(y)\rho^*(y,t), \quad y \in \mathcal{G} \tag{2.2}$$

maps $\mathcal{F}_i$ on Ω_{it}, $i = 1,2$, and leaves Ω_3 invariant. We denote by $\mathcal{L} = \mathcal{L}(y,\rho^*) = (l_{ij})_{i,j=1,2,3}$ the Jacobi matrix of the transformation (1.6) and we set $L = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1} = (\widehat{L}_{ij})_{i,j=1,2,3}$; $\widehat{\mathcal{L}}$ is the co-factors matrix of $\mathcal{L}$. We note that $L(y,\rho^*) = 1$ and $\mathcal{L} = \widehat{\mathcal{L}} = I$ for $y \in \mathcal{F}_3$ and $y \in S$.

We set

$$\boldsymbol{u}(y,t) = \boldsymbol{v}(e_\rho,t) \quad q(y,t) = p(e_\rho,t),$$

introduce new unknown vector fields

$$\boldsymbol{h} = \widehat{\mathcal{L}}\boldsymbol{H}(e_\rho,t), \quad \boldsymbol{e} = \widehat{\mathcal{L}}\boldsymbol{E}(e_\rho,t), \tag{2.3}$$

and make use of the formulas

$$\boldsymbol{n}(e_\rho) = \frac{\widehat{\mathcal{L}}^T(y,\rho)\boldsymbol{N}(y)}{|\widehat{\mathcal{L}}^T(y,\rho)\boldsymbol{N}(y)|}, \quad y \in \mathcal{G}, \tag{2.4}$$

$$\begin{aligned} &\nabla\cdot\boldsymbol{h}(y,t) = \widehat{\mathcal{L}}^T\nabla_y\cdot\boldsymbol{H}(e_\rho,t) = L(y,\rho^*)\nabla_x\cdot\boldsymbol{H}(x,t)|_{x=e_\rho(y,t)},\\ &\nabla\cdot\boldsymbol{e}(y,t) = L(y,\rho^*)\nabla_x\cdot\boldsymbol{E}(x,t)|_{x=e_\rho(y,t)},\\ &rot_x\boldsymbol{H}(x,t) = \frac{\mathcal{L}}{L}rot_y\mathcal{P}\boldsymbol{h}(y,t), \quad rot_x\boldsymbol{E}(x,t) = \frac{\mathcal{L}}{L}rot_y\mathcal{P}\boldsymbol{e}(y,t), \end{aligned} \tag{2.5}$$

where $x = e_\rho(y,t)$ and

$$\mathcal{P}(y,\rho^*) = \frac{\mathcal{L}^T}{L}\mathcal{L}. \tag{2.6}$$

Repeating the calculations from [3], Sec.1, we show that the transformation (2.1) converts (1.2)-(1.5) in

$$\begin{cases} \boldsymbol{u}_t - \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u} + (\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\\ -\widetilde{\nabla}\cdot\widetilde{T}(\boldsymbol{u},q) - \widetilde{\nabla}\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h}) = \boldsymbol{f}(e_\rho,t),\\ \nabla\cdot\widehat{\mathcal{L}}\boldsymbol{u} = 0, \quad y \in \mathcal{F}_1, \quad t > 0, \end{cases} \tag{2.7}$$

$$
(2.8)\qquad \begin{cases} \mu(\boldsymbol{h}_t - \frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h} - \rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h}) = -rot\mathcal{P}\boldsymbol{e}, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\\ \mathcal{P}rot\mathcal{P}\boldsymbol{h} = \alpha(\mathcal{P}\boldsymbol{e} + \mu(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y\in\mathcal{F}_1,\\ rot\mathcal{P}\boldsymbol{h} = 0, \quad \nabla\cdot\boldsymbol{h} = 0, \quad \nabla\cdot\boldsymbol{e} = 0, \quad x\in\mathcal{F}_2,\\ rot\boldsymbol{h} = \alpha\boldsymbol{e} + \boldsymbol{j}(y,t), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y\in\mathcal{F}_3, \end{cases}
$$

$$
(2.9)\qquad \begin{cases} \boldsymbol{h}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{e}_\tau = 0, \quad y\in S,\\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad [\boldsymbol{e}_\tau] = 0, \quad y\in S_3, \end{cases}
$$

$$
(2.10)\qquad \begin{cases} [\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h} - \frac{\widetilde{\mathcal{L}}\widetilde{\mathcal{L}}^T\boldsymbol{N}}{|\widetilde{\mathcal{L}}^T\boldsymbol{N}|^2}(\boldsymbol{h}\cdot\boldsymbol{N})] = 0,\\ \widetilde{T}(\boldsymbol{u},q)\boldsymbol{n}(e_\rho) + [T(\frac{\mathcal{L}}{L}\boldsymbol{h})\boldsymbol{n}(e_\rho)] = \sigma H\boldsymbol{n}, \quad \rho_t = \frac{\boldsymbol{u}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}{\Lambda(y,\rho)}, \quad y\in\mathcal{G}, \end{cases}
$$

where $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla_y$ is the transformed gradient ∇_x ("T" means transposition, $\mathcal{L}^{-T} = (\mathcal{L}^{-1})^T$);
$\widetilde{S}(\boldsymbol{u}) = \widetilde{\nabla}\boldsymbol{u} + (\widetilde{\nabla}\boldsymbol{u})^T$ is the transformed rate-of-strain tensor,
$\widetilde{T}(\boldsymbol{u},q) = -qI + \nu\widetilde{S}(\boldsymbol{u})$ is the transformed stress tensor,
$\Lambda(y,\rho) = \boldsymbol{N}(y)\cdot\widehat{\mathcal{L}}(y,\rho)\boldsymbol{N}(y) = 1 - \rho\mathcal{H}(y) + \rho^2\mathcal{K}(y)$
and $\mathcal{H}$, $\mathcal{K}$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively.

Now we turn to the boundary condition (1.7). We compute the components $\mathfrak{n}_t$ and $\mathfrak{n}_x$ of the normal to the surface $\{x\in\Gamma_t, t\in(0,T)\}$ in $\mathbb{R}^4$. We consider the four-dimensional transformation $(x,t') = \mathfrak{e}_\rho(y,t)$ defined by

$$
(2.11)\qquad x = e_\rho(y,t), \quad t' = t
$$

and we make use of the four-dimensional analog of the formula (2.4), i.e.,

$$
\mathfrak{n}(e_\rho,t) = \frac{\widehat{\mathfrak{L}}^T\boldsymbol{N}_0}{|\widehat{\mathfrak{L}}^T\boldsymbol{N}_0|},
$$

where $\boldsymbol{N}_0(y) = (N_1(y), N_2(y), N_3(y), 0)$ is the normal to $\mathcal{G}\times(0,T)$ and $\widehat{\mathfrak{L}}$ is the co-factors matrix corresponding to the transformation (2.11). The Jacobi matrix of this transformation is given by

$$
\mathfrak{L} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & N_1^*(y)\rho_t^*(y,t)\\ l_{21} & l_{22} & l_{23} & N_2^*(y)\rho_t^*(y,t)\\ l_{31} & l_{32} & l_{33} & N_3^*(y)\rho_t^*(y,t)\\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

hence

$$
\widehat{\mathfrak{L}} = \begin{pmatrix} \widehat{L}_{11} & \widehat{L}_{12} & \widehat{L}_{13} & -(\widehat{\mathcal{L}}\boldsymbol{N})_1\rho_t(y,t)\\ \widehat{L}_{21} & \widehat{L}_{22} & \widehat{L}_{23} & -(\widehat{\mathcal{L}}\boldsymbol{N})_2\rho_t(y,t)\\ \widehat{L}_{31} & \widehat{L}_{32} & \widehat{L}_{33} & -(\widehat{\mathcal{L}}\boldsymbol{N})_3\rho_t(y,t)\\ 0 & 0 & 0 & L \end{pmatrix}
$$

for $y\in\mathcal{G}$ and

$$
\widehat{\mathfrak{L}}^T\boldsymbol{N}_0 = (\widehat{\mathcal{L}}^T\boldsymbol{N}, -\Lambda\rho_t).
$$

It follows that (1.7) is equivalent to

$$
(2.12)\qquad -\Lambda\rho_t[\mu\boldsymbol{H}] + [\widehat{\mathcal{L}}^T\boldsymbol{N}\times\boldsymbol{E}] = 0, \quad x = e_\rho(y,t)
$$

or, in view of the algebraic identity $A\boldsymbol{f}\times A\boldsymbol{g} = \widehat{A}^T(\boldsymbol{f}\times\boldsymbol{g})$ and the kinematic boundary condition, to

$$-(\boldsymbol{u}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N})[\mu\boldsymbol{h}] + L[\boldsymbol{N}\times\mathcal{P}\boldsymbol{e}] = 0. \tag{2.13}$$

Finally, we have the initial and normalization conditions

$$\begin{cases} \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y) = \boldsymbol{v}_0(e_{\rho_0}), \quad y\in\mathcal{F}_1, \quad \rho(y,0) = \rho_0(y), \quad y\in\mathcal{G}, \\ \boldsymbol{h}_0(y,0) = \boldsymbol{h}_0(y) = \widehat{\mathcal{L}}(y,\rho_0^*)\boldsymbol{H}(e_{\rho_0}), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \\ \displaystyle\int_{S_k}\boldsymbol{e}(y,t)\cdot\boldsymbol{n}(y)dS = 0, \quad k=1,...,h_2(\mathcal{F}_2). \end{cases} \tag{2.14}$$

We find the solution of the problem (2.7)-(2.10), (2.13), (2.14) in anisotropic Sobolev-Slobodetskii spaces. We recall the definition of the corresponding norms. Let Ω be a domain in $\mathbb{R}^n$. The (isotropic) Sobolev space $W_2^l(\Omega)$ with $l>0$ is the space of functions $u(x)$, $x\in\Omega$, with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{0\le|j|\le l}\|D^ju\|^2_{L_2(\Omega)} \equiv \sum_{0\le|j|\le l}\int_\Omega|D^ju(x)|^2dx,$$

if $l=[l]$, i.e. l is an integral number, and

$$\|u\|^2_{W_2^l(\Omega)} = \|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|j|=[l]}\int_\Omega\int_\Omega|D^ju(x)-D^ju(y)|^2\frac{dxdy}{|x-y|^{n+2\lambda}},$$

if $l=[l]+\lambda$, $\lambda\in(0,1)$. As usual, D^ju denotes a (generalized) partial derivative $\frac{\partial^{|j|}u}{\partial x_1^{j_1}...\partial x_n^{j_n}}$ where $j=(j_1,j_2,...j_n)$ and $|j|=j_1+...+j_n$. The anisotropic space $W_2^{l,l/2}(Q_T)$, $Q_T=\Omega\times(0,T)$, can be defined as

$$L_2((0,T),W_2^l(\Omega))\cap W_2^{l/2}((0,T),L_2(\Omega))$$

and supplied with the norm

$$\|u\|^2_{W_2^{l,l/2}(Q_T)} = \int_0^T\|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_\Omega\|u(x,\cdot)\|^2_{W_2^{l/2}(0,T)}dx. \tag{2.15}$$

There exist many other equivalent norms in $W_2^{l,l/2}(Q_T)$; some of them will be used below. Sobolev spaces of functions given on smooth surfaces, in particular, on $\mathcal{G}$ and on $G_T=\mathcal{G}\times(0,T)$, are introduced in a standard way, with the help of local maps and partition of unity. We also find it convenient to introduce the spaces $W_2^{l,0}(Q_T) = L_2(0,T;W_2^l(\Omega))$ and $W_2^{0,l/2}(Q_T) = W_2^{l/2}(0,T;L_2(\Omega))$; the squares of norms in these spaces coincide, respectively, with the first and the second integral in (2.15).

In order to obtain uniform estimates of the solutions of (1.8) for small T, we introduce in $W_2^{l,l/2}(Q_T)$ equivalent norms defined by

$$\|u\|_{\widehat{W}_2^{l,l/2}(Q_T)} = \|u\|_{W_2^{l,l/2}(Q_T)},$$

if $l/2$ is an integer or $l/2=[l/2]+\lambda$, $\lambda\in(1/2,1)$,

$$\|u\|^2_{\widehat{W}_2^{l,l/2}(Q_T)} = \|u\|^2_{W_2^{l,l/2}(Q_T)} + \frac{1}{T^{2\lambda}}\|\frac{\partial^{[l/2]}}{\partial t^{[l/2]}}u\|^2_{L_2(Q_T)},$$

if $\lambda\in(0,1/2)$, and

$$\|u\|^2_{H^{l,l/2}(O_T)} = \|u\|^2_{\widehat{W}_2^{l,l/2}(O_T)} + \sum_{0\le j<(l-1)/2}\sup_{t<T}\|D_t^ju(\cdot,t)\|^2_{W_2^{l-1-2j}(\Omega)}.$$

Similar norms can be introduced on the manifold $G_T = \mathcal{G}\times(0,T)$. The advantages furnished by working with H-norms are discussed in [4, Propositions 1.1 and 1.2].

We prove the following theorem.

Theorem 1. *Let $\boldsymbol{u}_0 \in W_2^{l+1}(\mathcal{F}_1)$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$, $\boldsymbol{h}_0 \in W_2^{l'+1}(\mathcal{F}_i)$, $i=1,2,3$, with $1/2 < l' < l < 1$, and let the compatibility conditions*

$$
\begin{aligned}
&\widetilde{\nabla}\cdot\boldsymbol{u}_0 = 0, \quad y\in\mathcal{F}_1, \quad \widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n}_0(e_{\rho_0}) - \boldsymbol{n}_0(\boldsymbol{n}_0\cdot\widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n}_0) = 0, \quad y\in\mathcal{G},\\
&\nabla\cdot\boldsymbol{h}_0 = 0, \quad y\in\mathcal{F}_i, \quad i=1,2,3,\\
&rot\mathcal{P}(y,\rho_0)\boldsymbol{h}_0 = 0, \quad y\in\mathcal{F}_2,\\
&[\mu\boldsymbol{h}_0\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_{0\tau}] = \Big(\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2} - \boldsymbol{N}\Big)[\boldsymbol{h}_0\cdot\boldsymbol{N}], \quad y\in\mathcal{G},\\
&[\mu\boldsymbol{h}_0\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_{0\tau}] = 0, \quad y\in S_3,\\
&\boldsymbol{h}_0\cdot\boldsymbol{n}|_S = 0,
\end{aligned}
\tag{2.16}
$$

and the smallness condition

$$
\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \le \epsilon \ll 1 \tag{2.17}
$$

be satisfied. Assume also that

$$
\begin{aligned}
&\boldsymbol{f}\in W_2^{l/2}(0,T;W_2^2(\mathbb{R}^3))\cap W_2^{l+2,0}(\mathbb{R}^3\times(0,T_0)),\\
&\boldsymbol{j}\in W_2^{l'/2}(0,T;W_2^1(\mathcal{F}_3))\cap W_2^{l'+1,0}(\mathcal{F}_3\times(0,T_0))
\end{aligned}
\tag{2.18}
$$

and that the condition (1.1) holds. Then the problem (2.7)-(2.11) has a unique solution defined in a certain (small) time interval $(0,T)$ with the following regularity properties:

$$
\begin{aligned}
&\boldsymbol{u}\in W_2^{2+l.1+l/2}(Q_T^1), \quad \nabla q\in W_2^{l,l/2}(Q_T^1),\\
&q\in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G})),\\
&\rho\in W_2^{l+5/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G})),\\
&\rho_t\in W_2^{l+3/2,l/2+3/4}(G_T), \quad \boldsymbol{h}^{(i)}\in W_2^{l'+2,l'/2+1}(Q_T^i),\\
&\boldsymbol{e}^{(i)}\in W_2^{l'+1,0}(Q_T^i)\cap W_2^{l'/2}(0,T;W_2^1(\mathcal{F}_i)),
\end{aligned}
$$

where $Q_T^i = \mathcal{F}_i\times(0,T)$, $G_T = \mathcal{G}\times(0,T)$, $\boldsymbol{h}^{(i)} = \boldsymbol{h}|_{Q_T^i}$, $\boldsymbol{e}^{(i)} = \boldsymbol{e}|_{Q_T^i}$, $i=1,2,3$. The solution satisfies the inequality

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{H^{l+2,l/2+1}(Q_T^1)} + \|\nabla q\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} + \|q\|_{W_2^{l+1/2,0}(G_T)} + \|q\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}_T))}\\
&+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{\widehat{W}_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}_T))} + \|\rho_t\|_{H^{l+3/2,l/2+3/4}(G_T))}\\
&+ \sup_{t<T}\|\rho(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})} + \sum_{i=1}^3 \|\boldsymbol{h}^{(i)}\|_{H^{l'+2,l'/2+1}(Q_T^i)}\\
&+ \sum_{i=1}^3 \big(\|\boldsymbol{e}^{(i)}\|_{W_2^{l'+1,0}(Q_T^i)} + \|\boldsymbol{e}^{(i)}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^1(\mathcal{G}))}\big)\\
&\le c\Big(\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} + \|\mathcal{H}\|_{W_2^{l'+1/2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\\
&+ \sum_{i=1}^3 \|\boldsymbol{h}_0\|_{W_2^{l'+1}(\Omega_{i0})} + \|\boldsymbol{j}\|_{W_2^{l'+1,0}(Q_T^3)} + \|\boldsymbol{j}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^1(\mathcal{F}_3))}\Big).
\end{aligned}
\tag{2.19}
$$

For the problem (1.2)-(1.5), (1.7), (1.12), (1.13) this means that it is solvable in the time interval $(0,T)$ and

$$\begin{aligned}
&\boldsymbol{v}\circ e_\rho \in W_2^{2+l.1+l/2}(Q_T^1), \quad \nabla p\circ e_\rho \in W_2^{l,l/2}(Q_T^1),\\
&p\circ e_\rho \in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G})),\\
&\rho \in W_2^{l+5/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G})),\\
&\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T), \quad \boldsymbol{H}^{(i)}\circ e_\rho \in W_2^{l'+2,l'/2+1}(Q_T^i),\\
&\boldsymbol{E}^{(i)}\circ e_\rho \in W_2^{l'+1,0}(Q_T^i)\cap W_2^{l'/2}(0,T;W_2^1(\mathcal{F}_i)), \quad i=1,2,3.
\end{aligned}$$

Theorem 1 is proved in the paper [3] under the following extra assumptions: $\boldsymbol{f}=0$, $\boldsymbol{j}=0$, $\Omega_3=\emptyset$, Ω_1 and Ω are simply connected. In this case the equations $rot\boldsymbol{H}=0$, $\nabla\cdot\boldsymbol{H}=0$, $x\in\Omega_{2t}$ imply

$$\boldsymbol{H}^{(2)}=\nabla\varphi, \tag{2.20}$$

where φ is a solution of the Neumann problem

$$\nabla^2\varphi(x,t)=0, \quad x\in\Omega_{2t}, \quad \mu_2\frac{\partial\varphi}{\partial n}=\mu_1\boldsymbol{H}^{(1)}\cdot\boldsymbol{n}, \quad x\in\Gamma_t, \quad \frac{\partial\varphi}{\partial n}=0, \quad x\in S \tag{2.21}$$

and $\boldsymbol{H}^{(i)}(x,t)=\boldsymbol{H}(x,t)|_{x\in\Omega_{it}}$. Hence $\boldsymbol{H}^{(2)}$ is completely determined by $\boldsymbol{H}^{(1)}\cdot\boldsymbol{n}|_{x\in\Gamma_t}$, which permits to exclude $\boldsymbol{E}$ and to work only with the function $\boldsymbol{H}^{(1)}$ satisfying the relations

$$\begin{cases}
\mu_1\boldsymbol{H}_t+\alpha^{-1}rotrot\boldsymbol{H}-\mu_1 rot(\boldsymbol{v}\times\boldsymbol{H})=0,\\
\nabla\cdot\boldsymbol{H}(x,t)=0, \quad x\in\Omega_{1t},\\
\mu_1\boldsymbol{H}^{(1)}\cdot\boldsymbol{n}=\mu_2\dfrac{\partial\varphi}{\partial n}, \quad \boldsymbol{H}_\tau=\nabla_\tau\varphi(x,t), \quad x\in\Gamma_t,\\
\boldsymbol{H}(x,0)=\boldsymbol{H}_0(x), \quad x\in\Omega_{10}.
\end{cases}$$

In the general case (2.20) is not true, nevertheless, one can separate the determination of $\boldsymbol{u},\rho,\boldsymbol{h}$ from that of $\boldsymbol{e}$. Let $\boldsymbol{\psi}\in W_2^{1,0}(Q_T^i)$, $i=1,2,3$ be a test vector field satisfying the conditions

$$\nabla\cdot\boldsymbol{\psi}(y,t)=0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad rot\boldsymbol{\psi}(y,t)=0, \quad y\in\mathcal{F}_2, \tag{2.22}$$

$$\begin{aligned}
&[\mu\boldsymbol{\psi}\cdot\boldsymbol{N}]=0, \quad [\boldsymbol{\psi}_\tau]=0, \quad y\in\mathcal{G},\\
&[\mu\boldsymbol{\psi}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{\psi}_\tau]=0, \quad y\in S_3,\\
&\boldsymbol{\psi}\cdot\boldsymbol{n}=0, \quad y\in S.
\end{aligned} \tag{2.23}$$

Since, in view of (2.13),

$$\int_\Omega rot\mathcal{P}\boldsymbol{e}(y,t)\cdot\boldsymbol{\psi}(y,t)dy=\int_\Omega \mathcal{P}\boldsymbol{e}\cdot rot\boldsymbol{\psi}dy+\int_{\mathcal{G}}\frac{\boldsymbol{u}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}{L}[\mu\boldsymbol{h}]\cdot\boldsymbol{\psi}dS,$$

it is easy to obtain from the equations (2.8) the integral identity

$$\begin{aligned}
&\int_0^T\int_\Omega \mu(\boldsymbol{h}_t-\boldsymbol{\Phi}(\boldsymbol{h},\rho))\cdot\boldsymbol{\psi}(y,t)dydt+\int_0^T\int_{\mathcal{F}_1\cup\mathcal{F}_3}\alpha^{-1}\mathcal{P}rot\mathcal{P}\boldsymbol{h}\cdot rot\boldsymbol{\psi}(y,t)dydt\\
&-\mu_1\int_0^T\int_{\mathcal{F}_1}(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})\cdot rot\boldsymbol{\psi}(y,t)dydt+\int_{\mathcal{G}}\boldsymbol{\Psi}(\boldsymbol{u},\boldsymbol{h},\rho)\cdot\boldsymbol{\psi}dS\\
&=\int_0^T\int_{\mathcal{F}_3}\alpha^{-1}\boldsymbol{j}(y,t)\cdot rot\boldsymbol{\psi}(y,t)dydt,
\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}\mathbf{\Phi}(\boldsymbol{h},\rho) &= \frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h} + \rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h}\\ \mathbf{\Psi}(\boldsymbol{u},\boldsymbol{h},\rho) &= \frac{\boldsymbol{u}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}{L}[\mu\boldsymbol{h}].\end{aligned} \tag{2.25}$$

This identity, together with initial and boundary conditions for $\boldsymbol{h}$ and the equation $rot\mathcal{P}\boldsymbol{h}=0$, constitutes a nonlinear problem studied in Sec.4.

3. Linear problems

The proof of the solvability of the problem (1.7) is based on the analysis of the following non-homogeneous linear problems:

1. Find $(\boldsymbol{v},p,\rho)$ such that

$$\begin{cases}\boldsymbol{v}_t-\nu\nabla^2\boldsymbol{v}+\boldsymbol{m}(y,t)\rho^*+\nabla p=\boldsymbol{f}(y,t),\\ \nabla\cdot\boldsymbol{v}=f(y,t)=\nabla\cdot\boldsymbol{F}(y,t),\quad y\in\mathcal{F}_1,\quad t>0,\\ T(\boldsymbol{v},p)\boldsymbol{N}(y)+\sigma\boldsymbol{N}(y)\mathfrak{B}\rho=\boldsymbol{d}(y,t),\\ \rho_t+\boldsymbol{V}(x)\cdot\nabla_\tau\rho-\boldsymbol{v}\cdot\boldsymbol{N}(y)=g(y,t),\quad y\in\mathcal{G},\\ \boldsymbol{v}(y,0)=\boldsymbol{v}_0(y),\quad y\in\mathcal{F}_1,\quad \rho(y,0)=\rho_0(y),\quad y\in\mathcal{G},\end{cases} \tag{3.1}$$

where $\mathfrak{B}\rho=-\Delta_{\mathcal{G}}\rho-b(y)\rho$, $b=(\mathcal{H}^2-2\mathcal{K})$, $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$,

2. Find the vector field $\boldsymbol{H}(y,t)$, satisfying the equations

$$\begin{aligned}&rot\boldsymbol{H}(y,t)=rot\boldsymbol{\ell}(y,t),\quad y\in\mathcal{F}_2,\quad \boldsymbol{H}(y,0)=\boldsymbol{H}_0(y),\quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\\ &[\mu\boldsymbol{H}\cdot\boldsymbol{N}]=0,\quad [\boldsymbol{H}_\tau]=\boldsymbol{a},\quad y\in\mathcal{G},\quad [\mu\boldsymbol{H}\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{H}_\tau]=0,\quad y\in S_3,\quad \boldsymbol{H}\cdot\boldsymbol{n}=0,\quad y\in S.\end{aligned} \tag{3.2}$$

and the integral identity

$$\begin{aligned}&\int_0^T\int_\Omega\mu\boldsymbol{H}_t\cdot\boldsymbol{\psi}(y,t)dydt+\int_{\mathcal{F}_1\cup\mathcal{F}_3}\alpha^{-1}rot\boldsymbol{H}\cdot rot\boldsymbol{\psi}(y,t)dydt\\ &=\int_0^T\int_{\mathcal{F}_1\cup\mathcal{F}_3}\boldsymbol{G}_1(y,t)\cdot rot\boldsymbol{\psi}(y,t)dydt+\int_0^T\int_\Omega\mu\boldsymbol{G}_2(y,t)\cdot\boldsymbol{\psi}(y,t)dydt,\end{aligned} \tag{3.3}$$

with the same kind of the test function $\boldsymbol{\psi}(y,t)$ as in (2.24).

In addition, we need to consider the auxiliary problem

$$\begin{cases}rot\boldsymbol{h}(y)=\boldsymbol{k}(y),\quad \nabla\cdot\boldsymbol{h}=0,\quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}]=0,\quad [\boldsymbol{h}_\tau]=\boldsymbol{a},\quad y\in\mathcal{G},\\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{h}_\tau]=0,\quad y\in S_3,\\ \boldsymbol{h}\cdot\boldsymbol{n}(y)=0,\quad y\in S.\end{cases} \tag{3.4}$$

We start with the existence theorem for the problem (3.1).

Theorem 2. *Assume that $l\in(1/2,1)$ and that the data of the problem (3.1) possess the following regularity properties: $\boldsymbol{f}\in W_2^{l,l/2}(Q_T^1)$, $f\in W_2^{l+1,0}(Q_T^1)$, $\boldsymbol{F}_t\in W_2^{0,l/2}(Q_T^1)$, $\boldsymbol{d}\cdot\boldsymbol{N}\in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))$, $\boldsymbol{d}-\boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})\in W_2^{l+1/2,l/2+1/4}(G_T)$, $g\in W_2^{l+3/2,l/2+3/4}(G_T)$, $\boldsymbol{v}_0\in W_2^{l+1}(\mathcal{F}_1)$, $\rho_0\in W_2^{l+2}(\mathcal{G})$, where $T<\infty$, $Q_T^1=\mathcal{F}_1\times(0,T)$,*

$G_T = \mathcal{G} \times (0,T)$. *Moreover, let* $\boldsymbol{m} \in W_2^{l,l/2}(Q_T^1)$, $\boldsymbol{V} \in W_2^{l+3/2}(\mathcal{G})$ *and let the compatibility conditions*

$$\begin{aligned} &\nabla \cdot \boldsymbol{v}_0(x) = f(x,0), \quad x \in \mathcal{F}_1, \\ &\nu \Pi_{\mathcal{G}} S(\boldsymbol{v}_0)\boldsymbol{N} = \Pi_{\mathcal{G}} \boldsymbol{d}(x,0), \quad x \in \mathcal{G}, \end{aligned} \tag{3.5}$$

be satisfied, where

$$\Pi_{\mathcal{G}} \boldsymbol{d} = \boldsymbol{d} - \boldsymbol{N}(\boldsymbol{N} \cdot \boldsymbol{d})$$

is the tangential component of $\boldsymbol{d}$ *on* $\mathcal{G}$. *Then the problem* (3.1) *has a unique solution* $\boldsymbol{v}, p, \rho$ *such that*

$$\begin{aligned} &\boldsymbol{v} \in W_2^{l+2,l/2+1}(Q_T^1), \quad \nabla p \in W_2^{l,l/2}(Q_T^1), \quad p \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G})), \\ &\rho \in W_2^{l+5/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G})), \quad \rho_t \in W_2^{l+3/2,l/2+3/4}(G_T), \end{aligned}$$

and the solution satisfies the inequality

$$\begin{aligned} &\|\boldsymbol{v}\|_{H^{l+2,l/2+1}(Q_T^1)} + \|\nabla p\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + \|p\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}_T))} \\ &+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{\widehat{W}_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}_T))} + \|\rho_t\|_{H^{l+3/2,l/2+3/4}(G_T))} \\ &\le c\Big(\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} + \|f\|_{W_2^{l+1,0}(Q_T^1)} + \|\boldsymbol{F}_t\|_{\widehat{W}_2^{0,l/2}(Q_T^1)} \\ &+ \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{H^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \\ &+ \|g\|_{H^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big). \end{aligned} \tag{3.6}$$

This theorem is proved in [5] in the case $\boldsymbol{m} = 0$. The term $\boldsymbol{m}\rho^*$ in (3.1) is weak and can be estimated by the interpolation inequality

$$\begin{aligned} &\|m\rho^*\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} \le c\eta(\|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{\widehat{W}_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}_T))} \\ &+ \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T))}) + c(\eta)\|\rho\|_{L_2(G_T)}, \end{aligned}$$

with arbitrarily small η, so the result of [5] extends to the case $\boldsymbol{m} \neq 0$ in a standard way.

Before discussing the problems (3.2), (3.3) and (3.4), we recall some basic results concerning the space $L_2(\mathcal{D})$ of square summable vector fields given in the domain $\mathcal{D} \subset \mathbb{R}^3$ with a smooth connected boundary $\partial\mathcal{D}$. We introduce the finite dimensional spaces of the Neumann and Dirichlet vector fields

$$\begin{aligned} U_n(\mathcal{D}) &= \{\boldsymbol{u} \in W_2(\mathcal{D}): \quad \nabla \cdot \boldsymbol{u} = 0, \quad rot\boldsymbol{u} = 0, \quad \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\mathcal{D}} = 0\}, \\ U_d(\mathcal{D}) &= \{\boldsymbol{v} \in W_2(\mathcal{D}): \quad \nabla \cdot \boldsymbol{v} = 0, \quad rot\boldsymbol{v} = 0, \quad \boldsymbol{v}_\tau|_{\partial\mathcal{D}} = 0\}. \end{aligned}$$

The dimensions of these spaces are equal to the first and the second Betti numbers of $\mathcal{D}$, $b_1(\mathcal{D})$ and $b_2(\mathcal{D})$, respectively. Following [6,7], we recall the structure of the Neumann vector fields. In the case $b_1(\mathcal{D}) > 0$ there exist $b_1(\mathcal{D})$ closed contours in $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$ generating the first homology group of $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$, end every contour Λ generates the Neumann vector field of the form

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$$

with $\boldsymbol{u}_1$ given through the Biot-Savart law

$$\boldsymbol{u}_1(x) = \int_\Lambda \frac{x-y}{|x-y|^3} \times d\boldsymbol{l}_y$$

and $\boldsymbol{u}_2 = \nabla\varphi$, where φ is a solution of the problem

$$\triangle\varphi = 0, \quad \text{in } D, \quad \frac{\partial\varphi}{\partial n}\Big|_{\partial D} = -\boldsymbol{u}_1 \cdot \boldsymbol{n}\Big|_{\partial D}.$$

As for $b_2(\mathcal{D})$, it is equal to the number of connected components Γ_j of the boundary of $\mathcal{D}$ minus one (denoted by Γ_0), and the basis in $U_d(\mathcal{D})$ consists of vector fields $\boldsymbol{v}_k(x) = \nabla\Phi_k(x)$, where Φ_k is a solution of the Dirichlet problem

$$\Delta^2\Phi_k = 0, \quad \Phi_k|_{\Gamma_j} = \delta_{kj}, \quad k,j = 1,...,b_2(\mathcal{D}), \quad \Phi_k|_{\Gamma_0} = 0.$$

We also recall the Weyl decomposition formula for the space $L_2(\mathcal{D})$ [8]:

$$L_2(\mathcal{D}) = \overset{\circ}{G}(\mathcal{D}) \oplus U(\mathcal{D}) \oplus \overset{\circ}{J}(\mathcal{D}) = G(\mathcal{D}) \oplus \overset{\circ}{J}(\mathcal{D}) = \overset{\circ}{G}(\mathcal{D}) \oplus J(\mathcal{D}), \tag{3.7}$$

where

$$\begin{aligned}
\overset{\circ}{G}(\mathcal{D}) &= \{\boldsymbol{u} \in L_2(\mathcal{D}): \quad \boldsymbol{u} = \nabla\varphi, \quad \varphi \in W_2^1(\mathcal{D}), \quad \varphi|_{\partial\mathcal{D}} = 0\},\\
U(\mathcal{D}) &= \{\boldsymbol{u} \in L_2(\mathcal{D}): \quad \boldsymbol{u} = \nabla\psi, \quad \psi \in W_2^1(\mathcal{D}), \quad \nabla^2\psi = 0\},\\
\overset{\circ}{J}(\mathcal{D}) &= \{\boldsymbol{u} \in L_2(\mathcal{D}): \quad \nabla\cdot\boldsymbol{u} = 0, \quad \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\mathcal{D}} = 0\};
\end{aligned}$$

$G(\mathcal{D}) = \overset{\circ}{G}(\mathcal{D}) \oplus U(\mathcal{D})$ is the space of the potential vector fields and $J(\mathcal{D}) = \overset{\circ}{J}(\mathcal{D}) \oplus U(\mathcal{D})$ is the space of divergence free vector fields.

It is clear that $U_n(\mathcal{D}) \subset \overset{\circ}{J}(\mathcal{D})$ and $U_d(\mathcal{D}) \subset U(\mathcal{D})$.

Following [7], we introduce in the domain $\Omega = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ the $b_1(\Omega)$-dimensional space of the modified Neumann vector fields $\widetilde{U}_n(\Omega)$ whose elements $\widetilde{\boldsymbol{u}}_k(y)$ satisfy the relations

$$\begin{aligned}
&rot\widetilde{\boldsymbol{u}}_q(x) = 0, \quad \nabla\cdot\widetilde{\boldsymbol{u}}_q(x) = 0, \quad x \in \mathcal{F}_i, \quad i = 1,2,3,\\
&[\mu\widetilde{\boldsymbol{u}}_q\cdot\boldsymbol{N}] = 0, \quad [\widetilde{\boldsymbol{u}}_{q\tau}] = 0, \quad x \in \mathcal{G},\\
&[\mu\widetilde{\boldsymbol{u}}_q\cdot\boldsymbol{n}] = 0, \quad [\widetilde{\boldsymbol{u}}_{q\tau}] = 0, \quad x \in S_3,\\
&\widetilde{\boldsymbol{u}}_q(x)\cdot\boldsymbol{n}(x) = 0, \quad x \in S.
\end{aligned}$$

It is clear that $\widetilde{\boldsymbol{u}}_q(x) = \boldsymbol{u}_q(x) + \nabla\omega_q(x)$,

$$\begin{aligned}
&\nabla^2\omega_q(x) = 0, \quad x \in \mathcal{F}_i, \quad i = 1,2,3,\\
&[\mu\frac{\partial\omega_q}{\partial N}] = -[\mu]\boldsymbol{u}_q\cdot\boldsymbol{N}, \quad [\omega_q] = 0, \quad x \in \mathcal{G},\\
&[\mu\frac{\partial\omega_q}{\partial n}] = -[\mu]\boldsymbol{u}_q\cdot\boldsymbol{n}, \quad [\omega_q] = 0, \quad x \in S_3,\\
&\frac{\partial\omega_q}{\partial n} = 0, \quad x \in S.
\end{aligned}$$

We denote by $\mathcal{H}^l(\Omega)$ the space of the vector fields $\boldsymbol{\psi} \in W_2^l(\mathcal{F}_i)$, $i = 1,2,3$, satisfying (2.22), (2.23). For $l < 1/2$ the boundary conditions (2.23) are replaced with

$$[\mu\boldsymbol{\psi}\cdot\boldsymbol{N}] = 0, \quad y \in \mathcal{G}, \quad [\mu\boldsymbol{\psi}\cdot\boldsymbol{n}] = 0, \quad y \in S_3, \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = 0, \quad y \in S, \tag{3.8}$$

because $\boldsymbol{\psi}_\tau$ has no sense in this case. By $\mathcal{H}^0(\Omega)$ we mean the closure of $\mathcal{H}^l(\Omega)$ in $L_2(\Omega)$. If $\boldsymbol{\psi} \in \mathcal{H}^0(\Omega)$, then $\boldsymbol{\psi}^{(i)} \in J(\mathcal{F}_i)$, $i = 1.3.$ and $\boldsymbol{\psi}^{(2)} = \nabla\varphi(y) + \sum_{i=1}^{b_1(\mathcal{F}_2)} c_j\boldsymbol{w}_j(y)$, where $\boldsymbol{w}_j \in U_n(\mathcal{F}_2)$ and φ is the solution of the Neumann problem

$$\begin{aligned}
&\nabla^2\varphi(y) = 0, \quad y \in \mathcal{F}_2, \quad \mu_2\frac{\partial\varphi}{\partial N} = \mu_1\boldsymbol{\psi}^{(1)}\cdot\boldsymbol{N}, \quad y \in \mathcal{G},\\
&\mu_2\frac{\partial\varphi}{\partial n} = \mu_3\boldsymbol{\psi}^{(3)}\cdot\boldsymbol{n}, \quad y \in \mathcal{S}_3, \quad \frac{\partial\varphi}{\partial n} = 0 \quad y \in S.
\end{aligned} \tag{3.9}$$

Along with (3.7), we shall use in Ω the decomposition

$$L_{2,\mu}(\Omega) = G_\mu(\Omega) \oplus \overset{\circ}{J}_\mu(\Omega), \tag{3.10}$$

where $L_{2,\mu}(\Omega)$ is the space $L_2(\Omega)$ supplied with the scalar product

$$(\boldsymbol{u},\boldsymbol{v})_\mu = \int_\Omega \mu \boldsymbol{u}\cdot\boldsymbol{v}dy, \tag{3.11}$$

$\overset{\circ}{J}_\mu(\Omega)$ is the space of divergence free vector fields satisfying the conditions (2.23), and $G_\mu(\Omega)$ is the space of the elements of the form $\nabla\varphi$, where $\varphi\in W_2^1(\Omega)$ is the solution of the problem

$$\begin{aligned}&\nabla^2\varphi = \nabla\cdot\boldsymbol{u}(y), \quad [\varphi]|_{\mathcal{G}\cup S_3}=0, \quad [\mu\frac{\partial\varphi}{\partial N}]\big|_{\mathcal{G}} = [\mu\boldsymbol{u}\cdot\boldsymbol{N}]|_{\mathcal{G}},\\ &[\mu\frac{\partial\varphi}{\partial n}]\big|_{S_3} = [\mu\boldsymbol{u}\cdot\boldsymbol{n}]|_{S_3}, \quad \frac{\partial\varphi}{\partial n}\big|_S = \boldsymbol{u}\cdot\boldsymbol{n}|_S,\end{aligned} \tag{3.12}$$

or, in a weak form,

$$\int_\Omega \mu\nabla\varphi\cdot\nabla\eta dy = \int_\Omega \mu\boldsymbol{u}\cdot\nabla\eta dy, \quad \forall\eta\in W_2^1(\Omega).$$

The projection operator P_μ on $\overset{\circ}{J}_\mu(\Omega)$ is continuous not only in $L_2(\Omega)$, but also in $\cup_{i=1}^3 W_2^r(\mathcal{F}_i)$, which is a consequence of the estimates of the solution of (3.12).

It is clear that $\mathcal{H}^l(\Omega)\subset\overset{\circ}{J}_\mu(\Omega)$ and $\widetilde{U}_n(\Omega)\subset\overset{\circ}{J}_\mu(\Omega)$. We denote by $P_{\mathcal{H}}\boldsymbol{f}$, $\boldsymbol{f}\in\overset{\circ}{J}_\mu(\Omega)$, the projection of $\boldsymbol{f}$ on $\mathcal{H}^0$ and by $P_{\mathcal{H}_\perp}\boldsymbol{f}$ the projection of $\boldsymbol{f}$ on the subspace $\mathcal{H}^0_\perp\subset\mathcal{H}^0$ whose elements are orthogonal to $\widetilde{U}_n(\Omega)$:

$$\int_\Omega \mu\boldsymbol{\psi}\cdot\widetilde{\boldsymbol{u}}_q dy = 0, \quad q=1,...,b_1(\Omega). \tag{3.13}$$

It is easily verified that

$$P_{\mathcal{H}}\boldsymbol{f}(y) = \begin{cases}\boldsymbol{f}^{(i)}(y), \quad y\in\mathcal{F}_i, \quad i=1,3,\\ \nabla\varphi(y) + \sum\limits_{j=1}^{b_1(\mathcal{F}_2)} a_j\boldsymbol{w}_j(y), \quad y\in\mathcal{F}_2,\end{cases}$$

where $\varphi(y)$ is a solution of the Neumann problem

$$\begin{cases}\nabla^2\varphi(y)=0, \quad y\in\mathcal{F}_2,\\ \frac{\partial\varphi}{\partial n}\big|_S = 0, \quad \frac{\partial\varphi}{\partial N} = \boldsymbol{f}^{(2)}(y)\cdot\boldsymbol{N}, \quad y\in\mathcal{G}\cup S_3,\end{cases}$$

$\boldsymbol{w}_j(y)$ are the Neumann vector fields in $\mathcal{F}_2$ orthonormalized in $L_2(\mathcal{F}_2)$ and

$$a_k = \int_{\mathcal{F}_2} \boldsymbol{f}^{(2)}(y)\cdot\boldsymbol{w}_k(y)dy, \quad k=1,...,b_1(\mathcal{F}_2).$$

As P_μ, the operators $P_{\mathcal{H}}$ and $P_{\mathcal{H}_\perp}$ are continuous not only in $L_2(\Omega)$, but also in $\cup_{i=1}^3 W_2^r(\mathcal{F}_i)$.

Now we turn to the problem (3.4).

Theorem 3. *If $\boldsymbol{k}\in W_2^{l+1}(\mathcal{F}_i)$, $i=1,2,3$ and the conditions*

$$\begin{aligned}&\nabla\cdot\boldsymbol{k}(y,t)=0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\\ &\boldsymbol{k}\cdot\boldsymbol{n}|_S=0, \quad [\boldsymbol{k}\cdot\boldsymbol{n}]_{S_3}=0, \quad [\boldsymbol{k}\cdot\boldsymbol{N}] = \boldsymbol{N}\cdot rot\boldsymbol{a}, \quad y\in\mathcal{G}\end{aligned}$$

hold, then the problem (3.3) has a unique solution $\boldsymbol{h}\in W_2^{l+1}(\mathcal{F}_i)$, $i=1,2,3$, orthogonal to the space $\widetilde{U}_n$:

$$\int_\Omega \mu\boldsymbol{h}\cdot\widetilde{\boldsymbol{u}}_q dx = 0, \quad q=1,...,b_1(\Omega). \tag{3.14}$$

The solution satisfies the inequality

$$\sum_{i=1}^{3}\|\boldsymbol{h}\|_{W_2^{l+2}(\mathcal{F}_i)}\le c(\sum_{i=1}^{3}\|\boldsymbol{k}\|_{W_2^{l+1}(\mathcal{F}_i)}+\|\boldsymbol{a}\|_{W_2^{l+3/2}(\mathcal{G})}). \tag{3.15}$$

Moreover, if

$$\boldsymbol{k}=rot\boldsymbol{K}(y),\quad [\boldsymbol{K}_\tau]|_{\mathcal{G}\cup S_3}=0,\quad \boldsymbol{K}_\tau|_S=0,\quad \boldsymbol{a}=[\boldsymbol{A}]|_{\mathcal{G}} \tag{3.16}$$

with $\boldsymbol{A}$ *given in* $\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3$ *and satisfying the conditions*

$$\boldsymbol{A}^{(3)}=0,\quad \boldsymbol{A}^{(1)}\cdot\boldsymbol{N}|_{\mathcal{G}}=\boldsymbol{A}^{(2)}\cdot\boldsymbol{N}|_{\mathcal{G}}=0,$$

then

$$\|\boldsymbol{h}\|_{L_2(\Omega)}\le c(\|\boldsymbol{K}\|_{L_2(\Omega)}+\|\boldsymbol{A}\|_{L_2(\Omega)}+\|\boldsymbol{a}\|_{W_2^{-1/2}(\mathcal{G})}). \tag{3.17}$$

Proof. Following [3], we construct the solution in the form

$$\boldsymbol{h}(y)=\boldsymbol{a}^*+\nabla\boldsymbol{\psi}+\boldsymbol{\xi}+\sum_{j=1}^{b_1(\Omega)}c_j\widetilde{\boldsymbol{u}}_j, \tag{3.18}$$

where $\boldsymbol{a}^*$ is the extension of $\boldsymbol{a}$ into Ω such that $\boldsymbol{a}^*(y)=0$ for $y\in\mathcal{F}_2\cup\mathcal{F}_3$ and

$$\|\boldsymbol{a}^*\|_{W_2^{l+2}(\mathcal{F}_1)}\le c\|\boldsymbol{a}\|_{W_2^{l+3/2}(\mathcal{G})} \tag{3.19}$$

The functions $\psi(x)$ and $\boldsymbol{\xi}$ we define as solutions to the problems

$$\begin{cases}\nabla^2\psi=-\nabla\cdot\boldsymbol{a}^*(x),\quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\\ [\psi]=0,\quad [\mu\dfrac{\partial\psi}{\partial N}]=0,\quad x\in\mathcal{G},\quad [\psi]=0,\quad [\mu\dfrac{\partial\psi}{\partial n}]=0,\quad x\in S_3,\quad \dfrac{\partial\psi}{\partial N}\Big|_S=0;\end{cases} \tag{3.20}$$

$$\begin{cases}rot\boldsymbol{\xi}=\boldsymbol{j}(x)-rot\boldsymbol{a}^*(x),\quad \nabla\cdot\boldsymbol{\xi}=0,\quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\\ [\mu\boldsymbol{\xi}\cdot\boldsymbol{N}]=0,\quad [\boldsymbol{\xi}_\tau]=0,\quad x\in\mathcal{G},\quad [\mu\boldsymbol{\xi}\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{\xi}_\tau]=0,\quad x\in S_3,\quad \boldsymbol{\xi}\cdot\boldsymbol{n}|_S=0.\end{cases} \tag{3.21}$$

The solution of (3.21) has the form $\boldsymbol{\xi}(x)=\boldsymbol{\xi}_1+\nabla\omega$ with

$$\boldsymbol{\xi}_1(x)=\frac{1}{4\pi}rot\int_\Omega\frac{\boldsymbol{k}(y)-rot\boldsymbol{a}^*(y)}{|x-y|}dy,$$

$$\begin{cases}\nabla^2\omega(x)=0,\quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\quad [\omega]=0,\quad [\mu\dfrac{\partial\omega}{\partial N}]=-[\mu]\boldsymbol{\xi}_1\cdot\boldsymbol{N},\quad x\in\mathcal{G},\\ [\omega]=0,\quad [\mu\dfrac{\partial\omega}{\partial n}]=-[\mu]\boldsymbol{\xi}_1\cdot\boldsymbol{n},\quad x\in S_3,\quad \dfrac{\partial\omega}{\partial n}=-\boldsymbol{\xi}_1\cdot\boldsymbol{n},\quad y\in S.\end{cases} \tag{3.22}$$

The constants c_j are found from the orthogonality conditions (3.14):

$$0=\int_\Omega\mu(\boldsymbol{a}^*+\nabla\psi+\boldsymbol{\xi})\cdot\widetilde{\boldsymbol{u}}_k dx+\sum_{j=1}^{b_1(\Omega)}c_j\int_\Omega\mu\widetilde{\boldsymbol{u}}_j\cdot\widetilde{\boldsymbol{u}}_k dx, \tag{3.23}$$

$k=1,...,b_1(\Omega)$. It is clear that

$$\sum_{i=1}^{b_1}|c_i|\le c\|\boldsymbol{a}^*+\nabla\psi+\boldsymbol{\xi}\|_{L_2(\Omega)},$$

which implies

$$\sum_{i=1}^{3}\|\sum_{j=1}^{b_1}c_j\widetilde{\boldsymbol{u}}_j\|_{W_2^{l+2}(\mathcal{F}_i)}\le c\|\boldsymbol{a}^*+\nabla\psi+\boldsymbol{\xi}\|_{L_2(\Omega)}. \tag{3.24}$$

Making use of well known estimates of the Newtonian potential and of solutions of the elliptic boundary value problems (3.20) and (3.22), we obtain

$$\sum_{i=1}^{3}\|\nabla\psi\|_{W_2^{l+2}(\mathcal{F}_i)}\le c\|\boldsymbol{a}^*\|_{W_2^{l+2}(\mathcal{F}_1)}\le c\|\boldsymbol{a}\|_{W_2^{l+3/2}(\mathcal{G})},$$

$$\sum_{i=1}^{3}\|\boldsymbol{\xi}\|_{W_2^{l+1}(\mathcal{F}_i)}\le c\sum_{i=1}^{3}\|\boldsymbol{k}-rot\boldsymbol{a}^*\|_{W_2^{l+1}(\mathcal{F}_i)}\le c(\sum_{i=1}^{3}\|\boldsymbol{k}\|_{W_2^{l+1}(\mathcal{F}_i)}+\|\boldsymbol{a}\|_{W_2^{l+3/2}(\mathcal{G})}).$$

Together with (3.24), these inequalities yield (3.15) The uniqueness follows from the fact that the solution of a homogeneous problem (3.4) belongs to $U_n(\Omega)$.

Now we prove (3.17). We use the following representation formula for $\boldsymbol{h}$:

$$\boldsymbol{h}=\boldsymbol{A}+\nabla\Psi+\boldsymbol{X}+\sum_{j=1}^{b_1(\Omega)}c_j\widetilde{\boldsymbol{u}}_j,$$

where Ψ and $\boldsymbol{X}$ are solutions to the problems

$$\begin{cases}\nabla^2\Psi(x)=-\nabla\cdot\boldsymbol{A}(x), & x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad \dfrac{\partial\Psi}{\partial N}\Big|_S=0,\\ [\Psi]=0, \quad [\mu\dfrac{\partial\Psi}{\partial N}]=0, & x\in\mathcal{G}, \quad [\Psi]=0, \quad [\mu\dfrac{\partial\Psi}{\partial n}]=0, \quad x\in S_3,\end{cases} \tag{3.25}$$

$$\begin{cases}rot\boldsymbol{X}=\boldsymbol{j}(x)-rot\boldsymbol{A}(x), \quad \nabla\cdot\boldsymbol{X}=0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad \boldsymbol{X}\cdot\boldsymbol{n}|_S=0,\\ [\mu\boldsymbol{X}\cdot\boldsymbol{N}]=0, \quad [\boldsymbol{X}_\tau]=0, \quad x\in\mathcal{G}, \quad [\mu\boldsymbol{X}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{X}_\tau]=0, \quad x\in S_3.\end{cases}$$

Since $\boldsymbol{A}|_S=0$, $[\mu\boldsymbol{A}\cdot\boldsymbol{N}]|_{\mathcal{G}}=0$, $[\mu\boldsymbol{A}\cdot\boldsymbol{n}]|_{S_3}=0$, the problem (3.25) is solvable, and

$$\|\nabla\Psi\|_{L_2(\Omega)}\le c\|\boldsymbol{A}\|_{L_2(\Omega)}. \tag{3.26}$$

As for $\boldsymbol{X}$, we represent it as the sum

$$\boldsymbol{X}(x)=\boldsymbol{X}_1(x)+\nabla U(x),$$

$$\boldsymbol{X}_1(x)=\frac{1}{4\pi}rot\int_\Omega\frac{rot(\boldsymbol{K}(y)-\boldsymbol{A}(y))}{|x-y|}dy,$$

$$\begin{cases}\nabla^2U(x)=0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad [U(x)]=0, \quad [\mu\dfrac{\partial U}{\partial N}]=-[\mu]\boldsymbol{X}_1\cdot\boldsymbol{N}, \quad x\in\mathcal{G},\\ [U(x)]=0, \quad [\mu\dfrac{\partial U}{\partial n}]=-[\mu]\boldsymbol{X}_1\cdot\boldsymbol{n}, \quad x\in S_3, \quad \dfrac{\partial U}{\partial n}|_S=-\boldsymbol{X}_1\cdot\boldsymbol{n}.\end{cases}$$

Since $[\boldsymbol{N}\times(\boldsymbol{K}-\boldsymbol{A})]|_{\mathcal{G}}=-\boldsymbol{N}\times\boldsymbol{a}$, $[\boldsymbol{n}\times(\boldsymbol{K}-\boldsymbol{A})]|_{S_3}=0$,, the vector field

$$\boldsymbol{X}_1(x)=\frac{1}{4\pi}rot\Big(\int_\Omega\nabla\frac{1}{|x-y|}\cdot(\boldsymbol{K}(y)-\boldsymbol{A}(y))dy-\int_{\mathcal{G}}\frac{\boldsymbol{N}(y)\times\boldsymbol{a}(y)}{|x-y|}dS\Big)$$

satisfies the inequality

$$\|\boldsymbol{X}_1\|_{L_2(\Omega)}\le c(\|\boldsymbol{K}\|_{L_2(\Omega)}+\|\boldsymbol{A}\|_{L_2(\Omega)}+\|\boldsymbol{a}\|_{W_2^{-1/2}(\mathcal{G})}), \tag{3.27}$$

that is obtained as in [3] (see (2.24)), as well as the estimate

$$\|\nabla U\|_{L_2(\Omega)}\le c\|\boldsymbol{X}_1\|_{L_2(\Omega)}, \tag{3.28}$$

[3, (2.26). Putting the inequalities (3.26)-(3.28) together, we obtain (3.17). The theorem is proved.

Corollary *Assume that* $\boldsymbol{k}$, $\boldsymbol{K}$, $\boldsymbol{a}$, $\boldsymbol{A}$ *in (3.3), (3.16) depend on* $t\in(0,T)$, $\boldsymbol{k}\in W_2^{l+1,0}(Q_T^j)$, $\boldsymbol{k}(\cdot,t)\in W_2^l(\mathcal{F}_j)$, $j=1,2,3$, $\boldsymbol{a}\in W_2^{l+3/2,0}(G_T)$, $\boldsymbol{a}(\cdot,t)\in W_2^{l+1/2}(\mathcal{G})$ $\forall t<T$, $\boldsymbol{K}_t\in W_2^{0,l/2}(Q_T)$, $\boldsymbol{a}\in W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))$, $\boldsymbol{A}_t^{(i)}\in W_2^{0,l/2}(Q_T^i)$, $i=1,2$. *Then*

$$
\begin{aligned}
&\sum_{j=1}^3\|\boldsymbol{h}^{(j)}\|_{H^{l+2,l/2+1}(Q_T^j)}\\
&\le c\Big(\sum_{i=1}^3(\|\boldsymbol{k}^{(i)}\|_{W_2^{l+1,0}(Q_T^i)}+\sup_{t<T}\|\boldsymbol{k}^{(i)}(\cdot,t)\|_{W_2^l(\mathcal{F}_i)}+\|\boldsymbol{K}_t^{(i)}\|_{\widehat{W}_2^{0,l/2}(Q_T^i)})\\
&+\|\boldsymbol{a}\|_{W_2^{l+3/2,0}(G_T)}+\sup_{t<T}\|\boldsymbol{a}\|_{W_2^{l+1/2}(\mathcal{G})}+\|\boldsymbol{a}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&+\|\boldsymbol{A}_t\|_{\widehat{W}_2^{0,l/2}(Q_T^1\cup Q_T^3)}\Big).
\end{aligned}\tag{3.29}
$$

Estimate (3.29) follows from (3.15) and from (3.17) applied to the time derivative $\boldsymbol{h}_t$and to the finite difference of the time derivative $\boldsymbol{h}_t(x,t+s)-\boldsymbol{h}_t(x,t)$ (cf. the analogous corollary in [3]).

Let us pass to the problem 2. (3.2), (3.3), (2.23).

Theorem 4. *Assume that* $\boldsymbol{G}_1\in W_2^{l+1,0}(Q_T^i)\cap W_2^{l/2}(0,T;W_2^1(\mathcal{F}_i))$, $i=1,3$, $\boldsymbol{G}_2\in W_2^{l,l/2}(Q_T^i)$, $\boldsymbol{H}_0\in W_2^{l+1}(\mathcal{F}_i)$, $i=1,2,3$, $\boldsymbol{\ell}\in W_2^{l+1,0}(Q_T^2)\cap W_2^{l/2}(0,T;W_2^1(\mathcal{F}_2))$, $\boldsymbol{\ell}(y,t)|_{y\in S_3\cup S}=0$, $\boldsymbol{a}\in W_2^{l+3/2,0}(G_T)$, $\boldsymbol{a}(\cdot,t)\in W_2^{l+1/2}(\mathcal{G})$, $\forall t\in(0,T)$, $\boldsymbol{a}_t\in W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))$, $\boldsymbol{a}=[\boldsymbol{A}]$, $\boldsymbol{A}_t\in W_2^{0,l/2}(Q_T)$ *and that the compatibility conditions*

$$
\begin{aligned}
&\nabla\cdot\boldsymbol{H}_0(y)=0,\quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\quad rot\boldsymbol{H}_0(y)=rot\boldsymbol{\ell}(y,0),\quad y\in\mathcal{F}_2,\\
&[\mu\boldsymbol{H}_0\cdot\boldsymbol{N}]=0,\quad [\boldsymbol{H}_{0\tau}]=\Big(\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2}\boldsymbol{N}\Big|_{\rho=\rho_0}-\boldsymbol{N}\Big)[\boldsymbol{H}_0\cdot\boldsymbol{N}],\\
&[\mu\boldsymbol{H}_0\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{H}_{0\tau}]=0,\quad y\in S_3,\quad \boldsymbol{H}_0\cdot\boldsymbol{n}=0,\quad y\in S.
\end{aligned}\tag{3.30}
$$

are satisfied. Then the problem 2. has a unique solution $\boldsymbol{H}\in W_2^{l+2,l/2+1}(Q_T^i)$, $i=1,2,3$, *and*

$$
\begin{aligned}
&\sum_{i=1}^3\|\boldsymbol{H}\|_{H^{l+2,l/2+1}(Q_T^i)}\\
&\le c(T)(\sum_{j=1,3}(\|\boldsymbol{G}_1\|_{W_2^{l+1,0}(Q_T^j)}+\|\boldsymbol{G}_1\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_j))})+\sum_{i=1}^3\|\boldsymbol{G}_2\|_{\widehat{W}_2^{l,l/2}(Q_T^i)}\\
&+\|\boldsymbol{\ell}\|_{W_2^{l+2,0}(Q_T^2)}+\sup_{t<T}\|\boldsymbol{\ell}(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})}+\|\boldsymbol{\ell}_t\|_{\widehat{W}_2^{0,l/2}(Q_T^2)}+\|\boldsymbol{a}_t\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&+\|\boldsymbol{a}\|_{W_2^{l+3/2}(G_T)}+sup_{t<T}\|\boldsymbol{a}(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})}+\|\boldsymbol{A}_t\|_{\widehat{W}_2^{0,l/2}(Q_T)}+\sum_{i=1}^3\|\boldsymbol{H}_0\|_{W_2^{l+1}(\mathcal{F}_i)}).
\end{aligned}\tag{3.31}
$$

The solution is unique in the class of vector fields with the above-mentioned regularity properties that are orthogonal to $\widetilde{U}_n(\Omega)$:

$$
\int_\Omega\mu\boldsymbol{H}\cdot\widetilde{\boldsymbol{u}}_k(y)dy=0,\quad k=1,...,b_1(\Omega).
$$

Proof. *Step 1. Reduction to the case* $\boldsymbol{\ell}=0$.

We extend $\boldsymbol{\ell}$ from $\mathcal{F}_2$ in $\mathcal{F}_1$ in such a way that

$$\|\boldsymbol{\ell}\|_{W_2^{l+2}(\mathcal{F}_1\cup\mathcal{F}_2)} \le c\|\boldsymbol{\ell}\|_{W_2^{l+2}(\mathcal{F}_2)},$$
$$\|\boldsymbol{\ell}\|_{L_2(\mathcal{F}_1\cup\mathcal{F}_2)} \le c\|\boldsymbol{\ell}\|_{L_2(\mathcal{F}_2)},$$

then

$$\begin{aligned}&\|\boldsymbol{\ell}\|_{W_2^{l+2,0}(Q_T^1\cup Q_T^2)} + \|\boldsymbol{\ell}\|_{\widehat{W}_2^{0,1+l/2}(Q_T^1\cup Q_T^2)}\\ &\le c(\|\boldsymbol{\ell}\|_{W_2^{l+2,0}(Q_T^2)} + \|\boldsymbol{\ell}\|_{\widehat{W}_2^{0,1+l/2}(Q_T^2)}).\end{aligned} \tag{3.32}$$

For $y\in\mathcal{F}_3$ we set $\boldsymbol{\ell}(y,t)=0$. We define $\boldsymbol{h}_1$ as a solution of the problem (3.4) with $\boldsymbol{k}=rot\boldsymbol{\ell}$; it satisfies the inequality (3.29) with $\boldsymbol{K}=\boldsymbol{\ell}$, $\boldsymbol{k}=rot\boldsymbol{\ell}$. For $\boldsymbol{h}=\boldsymbol{H}-\boldsymbol{h}_1$ we obtain the problem 2. with $\boldsymbol{\ell}=0$, $\boldsymbol{a}=0$ and with $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ replaced by $\boldsymbol{G}_1'=\boldsymbol{G}_1-\alpha^{-1}rot\boldsymbol{h}_1$, $\boldsymbol{G}_2'=\boldsymbol{G}_2-\boldsymbol{h}_{1t}$.

Step 2. Proof of the solvability of the problem 2. with $\ell=0$, $\boldsymbol{a}=0$.
This problem can be written in the form

$$\begin{cases}\displaystyle\int_0^T\int_\Omega \mu\boldsymbol{h}_t\cdot\boldsymbol{\psi}(y,t)dydt+\int_{\mathcal{F}_1\cup\mathcal{F}_3}\alpha^{-1}rot\boldsymbol{h}\cdot rot\boldsymbol{\psi}(y,t)dydt\\ \displaystyle=\int_0^T\int_{\mathcal{F}_1\cup\mathcal{F}_3}\boldsymbol{g}_1(y,t)\cdot rot\boldsymbol{\psi}(y,t)dydt+\int_0^T\int_\Omega\mu\boldsymbol{g}_2(y,t)\cdot\boldsymbol{\psi}(y,t)dydt,\\ \nabla\cdot\boldsymbol{h}(y,t)=0,\quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\quad rot\boldsymbol{h}(y,t)=0,\quad y\in\mathcal{F}_2,\\ [\mu\boldsymbol{h}\cdot\boldsymbol{N}]=0,\quad [\boldsymbol{h}_\tau]=0,\quad y\in\mathcal{G},\\ [\mu\boldsymbol{h}\cdot\boldsymbol{n}]=0,\quad [\boldsymbol{h}_\tau]=0,\quad y\in S_3,\quad \boldsymbol{h}\cdot\boldsymbol{n}=0,\quad y\in S,\\ \boldsymbol{h}(y,0)=\boldsymbol{H}_0(y)-\boldsymbol{h}_1(y,0)\equiv\boldsymbol{h}_0(y),\end{cases} \tag{3.33}$$

where

$$\boldsymbol{g}_1=P_J\boldsymbol{G}_1',\quad \boldsymbol{g}_2=P_{\mathcal{H}}P_\mu\boldsymbol{G}_2',$$

P_J is the orthogonal projection on the space $\overset{\circ}{J}(\mathcal{F}_1\cup\mathcal{F}_3)$.

Now we decompose $\boldsymbol{g}_2$ and $\boldsymbol{h}$ in the sum of linear combinations of the modified Neumann vector fields $\widetilde{\boldsymbol{u}}_q(x)$ in Ω and of the vector fields that are orthogonal to $\widetilde{U}_n(\Omega)$. We set

$$\boldsymbol{h}=\boldsymbol{h}'+\boldsymbol{h}^{''},\quad \boldsymbol{g}_2=\boldsymbol{g}_2'+\boldsymbol{g}^{''},$$

where

$$\boldsymbol{h}'=P_{\mathcal{H}_\perp}\boldsymbol{h},\quad \boldsymbol{g}_2'=P_{\mathcal{H}_\perp}\boldsymbol{g}_2,\quad \boldsymbol{h}^{''}(y,t)=\sum_{j=1}^{b_1(\Omega)}k_j(t)\widetilde{\boldsymbol{u}}_j(y),\quad \boldsymbol{g}_2^{''}(y,t)=\sum_{j=1}^{b_1(\Omega)}\gamma_j(t)\widetilde{\boldsymbol{u}}_j(y).$$

In the same way we decompose the test function $\boldsymbol{\psi}$ and the initial datum $\boldsymbol{h}_0$. Then (3.33) is decomposed in two problems. Setting $\boldsymbol{\psi}=\boldsymbol{\psi}^{''}$, we reduce (3.33) to

$$\boldsymbol{h}_t^{''}=\boldsymbol{g}_2^{''},\quad \boldsymbol{h}^{''}(y,0)=\boldsymbol{h}_0^{''}(y), \tag{3.34}$$

and for $\boldsymbol{h}'$ we obtain the problem

$$
(3.35)\qquad \begin{cases} \int_0^T \int_\Omega \mu \boldsymbol{h}'_t \cdot \boldsymbol{\psi}'(y,t)dydt + \int_{\mathcal{F}_1\cup\mathcal{F}_3} \alpha^{-1} rot\boldsymbol{h}' \cdot rot\boldsymbol{\psi}'(y,t)dydt \\ = \int_0^T \int_{\mathcal{F}_1\cup\mathcal{F}_3} \boldsymbol{g}_1(y,t)\cdot rot\boldsymbol{\psi}'(y,t)dydt + \int_0^T \int_\Omega \mu \boldsymbol{g}'_2(y,t)\cdot \boldsymbol{\psi}'(y,t)dydt, \\ \nabla\cdot \boldsymbol{h}'(y,t) = 0, \quad y\in \mathcal{F}_1 \cup \in \mathcal{F}_2 \cup \in \mathcal{F}_3, \quad rot\boldsymbol{h}'(y,t) = 0, \quad \in \mathcal{F}_2, \\ [\mu \boldsymbol{h}'\cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{h}'_\tau] = 0, \quad y\in \mathcal{G}, \\ [\mu \boldsymbol{h}'\cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{h}'_\tau] = 0, \quad y\in S_3, \quad \boldsymbol{h}'\cdot \boldsymbol{n} = 0, \quad y\in S, \\ \boldsymbol{h}'(y,0) = \boldsymbol{h}'_0(y), \end{cases}
$$

Clearly, (3.34) is easily solved. We have

$$
(3.36)\qquad \boldsymbol{h}''(y,t) = \boldsymbol{h}''_0(y) + \int_0^t \boldsymbol{g}''_2(y,\tau)d\tau,
$$

which is equivalent to

$$
k_j(t) = k_j(0) + \int_0^t \gamma_j(\tau)d\tau, \quad j = 1,...,b_1(\Omega).
$$

As for (3.35), this problem is studied in [7]. Indeed, let E be the extension operator defined on the space $\overset{\circ}{J}(\mathcal{F}_1\cup\mathcal{F}_3)\cap W_2^1(\mathcal{F}_1\cup\mathcal{F}_3)$ such that $\mathrm{E}\boldsymbol{\psi}\in W_2^1(\Omega)$ and $(\mathrm{E}\boldsymbol{\psi})_\tau = 0$ on S. Then

$$
\int_{\mathcal{F}_1\cup\mathcal{F}_3} \boldsymbol{g}_1\cdot rot\boldsymbol{\psi}dy = \int_\Omega \mathrm{E}\boldsymbol{g}_1\cdot rot\boldsymbol{\psi}dy = \int_\Omega rot\mathrm{E}\boldsymbol{g}_1\cdot\boldsymbol{\psi}dy = \int_\Omega \mu P_{\mathcal{H}_\perp}\mu^{-1}rot\mathrm{E}\boldsymbol{g}_1\cdot\boldsymbol{\psi}dy,
$$

$$
\int_{\mathcal{F}_1\cup\mathcal{F}_3} \alpha^{-1}rot\boldsymbol{h}'\cdot rot\boldsymbol{\psi}'(y,t)dy = \int_\Omega \mu P_{\mathcal{H}_\perp}(\mu\alpha)^{-1}rot(\mathrm{E}rot\boldsymbol{h})\cdot rot\boldsymbol{\psi}dy,
$$

consequently (3.35) can be written as the Cauchy problem

$$
(3.37)\qquad \boldsymbol{h}'_t + \boldsymbol{\mathcal{A}}\boldsymbol{h}' = \boldsymbol{g}'_2 + \boldsymbol{g}_3 \equiv \boldsymbol{g}_4, \quad \boldsymbol{h}'|_{t=0} = \boldsymbol{h}'_0,
$$

where

$$
(3.38)\qquad \boldsymbol{g}_3 = P_{\mathcal{H}_\perp}\mu^{-1}rot\mathrm{E}\boldsymbol{g}_1,
$$

and

$$
(3.39)\qquad \boldsymbol{\mathcal{A}}\boldsymbol{h}' = P_{\mathcal{H}_\perp}\mu^{-1}rot\mathrm{E}\alpha^{-1}rot\boldsymbol{h}'
$$

The operator $\mathcal{A}$ is a positive operator defined on the space of vector fields $\boldsymbol{h}\in W_2^2(\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3)\cup\mathcal{H}^2(\Omega)$ (see details in [7]).

Now we pass to the proof of (3.31). It is clear that

$$
(3.40)\qquad \begin{aligned} &\sum_{i=1}^3 \|\boldsymbol{h}''\|_{H^{l+2,l/2+1}(Q_T^i)} \\ &\le c(T)(\|\boldsymbol{k}\|_{\widehat{W}_2^{1+l/2}(0,T)} + |\boldsymbol{k}_0| + \|\boldsymbol{\gamma}\|_{\widehat{W}_2^{l/2}(0,T))}) \\ &\le c(T)\sum_{i=1}^3 (\|\boldsymbol{h}_0\|_{W_2^{l+1}(\mathcal{F}_i)} + \|\boldsymbol{g}_2\|_{\widehat{W}_2^{l,l/2}(Q_T^i)}), \end{aligned}
$$

because

$$\int_\Omega \mu \boldsymbol{h}_0(y)\cdot \widetilde{\boldsymbol{u}}_k(y)dy = \sum_{q=1}^{b_1(\Omega)} k_q(0)\int_\Omega \mu \widetilde{\boldsymbol{u}}_k(y)\cdot \widetilde{\boldsymbol{u}}_q(y)dy,$$

$$\int_\Omega \mu \boldsymbol{g}_2(y,t)\cdot \widetilde{\boldsymbol{u}}_k(y)dy = \sum_{q=1}^{b_1(\Omega)} \gamma_q(t)\int_\Omega \mu \widetilde{\boldsymbol{u}}_k(y)\cdot \widetilde{\boldsymbol{u}}_q(y)dy,$$

Next, we estimate the functions $\boldsymbol{g}_2'$ and $\boldsymbol{g}_3$ in (3.37) making use of the boundedness of the projection operators P_J, P_μ, $P_{\mathcal{H}}$, $P_{\mathcal{H}_\perp}$:

$$\|P_J \boldsymbol{G}_1'\|_{W_2^r(\mathcal{F}_i)} \le c\|\boldsymbol{G}_1'\|_{W_2^r(\mathcal{F}_i)}, \quad i=1,3, \quad r=1,\ r=l+1,$$

$$\|P_\mu \boldsymbol{G}_2'\|_{L_2(\Omega)} \le c\|\boldsymbol{G}_2'\|_{L_2(\Omega)}, \quad \sum_{i=1}^3 \|P_\mu \boldsymbol{G}_2'\|_{W_2^l(\mathcal{F}_i)} \le c\|\boldsymbol{G}_2'\|_{W_2^l(\mathcal{F}_i)};$$

moreover we require that

$$\|\mathrm{E}\boldsymbol{f}\|_{W_2^r(\Omega)} \le c\sum_{i=1,3}\|\boldsymbol{f}\|_{W_2^r(\mathcal{F}_i)}, \quad r=1,l+1.$$

This implies

$$\begin{aligned}
&\sum_{i=1}^3 \|\boldsymbol{g}_3\|_{\widehat{W}_2^{l,l/2}(Q_T^i)} \le c\sum_{i=1,3}(\|\boldsymbol{g}_1\|_{W_2^{l+1,0}(Q_T^i)} + \|\boldsymbol{g}_1\|_{\widehat{W}_2^{l/2}(0,T:W_2^1(\mathcal{F}_i))})\\
&\le c\sum_{i=1,3}(\|\boldsymbol{G}_1'\|_{W_2^{l+1,0}(Q_T^i)} + \|\boldsymbol{G}_1'\|_{\widehat{W}_2^{l/2}(0,T:W_2^1(\mathcal{F}_i))}), \qquad (3.41)\\
&\sum_{i=1,3}\|\boldsymbol{g}_2'\|_{\widehat{W}_2^{l,l/2}(Q_T^i)} \le c\sum_{i=1,3}\|\boldsymbol{G}_2'\|_{\widehat{W}_2^{l,l/2}(Q_T^i)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{i=1}^3 \|\boldsymbol{g}_4\|_{\widehat{W}_2^{l,l/2}(Q_T^i)} \le c\Big(\sum_{i=1}^3 \|\boldsymbol{G}_2'\|_{\widehat{W}_2^{l,l/2}(Q_T^i)}\\
&+\sum_{i=1,3}(\|\boldsymbol{G}_1'\|_{W_2^{l+1,0}(Q_T^i)} + \|\boldsymbol{G}_1'\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_i))})\Big). \qquad (3.42)
\end{aligned}$$

Now we make use of the following result.

Theorem 5. *For arbitrary* $\boldsymbol{g}_4 \in \cap_{i=1}^3 W_2^{l,l/2}(Q_T^i)\cap L_2(0,T;\mathcal{H}_\perp^l(\Omega))$ *and* $\boldsymbol{h}_0' \in \mathcal{H}_\perp^{l+1}(\Omega)$ *the problem (3.37) has a unique solution*

$$\boldsymbol{h}' \in \cap_{i=1}^3 W_2^{2+l,1+l/2}(Q_T^i)\cap L_2(0,T;\mathcal{H}_\perp^{1+2}(\Omega))$$

and

$$\sum_{i=1}^3 \|\boldsymbol{h}'\|_{H^{2+l,1+l/2}(Q_T^i)} \le c\sum_{i=1}^3(\|\boldsymbol{g}_4\|_{\widehat{W}_2^{l,l/2}(Q_T^i)} + \|\boldsymbol{h}_0'\|_{W_2^{l+1}(\mathcal{F}_i)}). \qquad (3.43)$$

This result in obtained in [7] for $l=0$, $\mathcal{F}_3=\emptyset$, and in [3] it is extended to $l\in(1/2,1)$ in the case of simply connected $\mathcal{F}_1$ and Ω, when $\widetilde{U}_n(\Omega)=\emptyset$. The general case is considered in a similar manner.

The inequality (3.31) for the solution of the problem 2. is a consequence of (3.29), (3.40)-(3.43). The uniqueness follows from the uniqueness of the solution of (3.37). Theorem 4 is proved.

4. Nonlinear problem

In this section we consider the main nonlinear problem

$$
\begin{cases}
\boldsymbol{u}_t - \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u} + (\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \\
-\widetilde{\nabla}\cdot\widetilde{T}(\boldsymbol{u},q) - \widetilde{\nabla}\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h}) = \boldsymbol{f}(e_\rho,t), \\
\nabla\cdot\widehat{\mathcal{L}}\boldsymbol{u} = 0, \quad y\in\mathcal{F}_1, \quad t>0, \\
\widetilde{T}(\boldsymbol{u},q)\boldsymbol{n}(e_\rho) + [T_M(\frac{\mathcal{L}}{L}\boldsymbol{h})\boldsymbol{n}] = \sigma H(e_\rho))\boldsymbol{n}, \quad \rho_t = \dfrac{\boldsymbol{u}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}{\Lambda(y,\rho)}, \quad y\in\mathcal{G}, \\
\displaystyle\int_0^T\int_\Omega \mu(\boldsymbol{h}_t - \boldsymbol{\Phi}(\boldsymbol{h},\rho))\cdot\boldsymbol{\psi}(y,t)dydt + \int_0^T\int_{\mathcal{F}_1\cup\mathcal{F}_3}\alpha^{-1}\mathcal{P}rot\mathcal{P}\boldsymbol{h}\cdot rot\boldsymbol{\psi}(y,t)dydt \\
\displaystyle+\mu_1\int_0^T\int_{\mathcal{F}_1}(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})\cdot rot\boldsymbol{\psi}(y,t)dydt + \int_{\mathcal{G}}\boldsymbol{\Psi}(\boldsymbol{u},\boldsymbol{h},\rho)\cdot\boldsymbol{\psi}dS \\
\displaystyle= \int_0^T\int_{\mathcal{F}_3}\alpha^{-1}\boldsymbol{j}(y,t)\cdot rot\boldsymbol{\psi}(y,t)dydt, \\
\nabla\cdot\boldsymbol{h}(y,t) = 0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \quad rot\mathcal{P}\boldsymbol{h}(y,t) = 0, \quad y\in\mathcal{F}_2, \\
[\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \big(\dfrac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2} - \boldsymbol{N}\big)[\boldsymbol{h}\cdot\boldsymbol{N}], \quad y\in\mathcal{G}, \\
[\mu\boldsymbol{h}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_\tau] = 0, \quad y\in S_3, \quad \boldsymbol{h}\cdot\boldsymbol{n} = 0, \quad y\in S, \\
\boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y\in\mathcal{F}_1, \quad \boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,
\end{cases}
\tag{4.1}
$$

where $\boldsymbol{\psi}$ is an arbitrary test function satisfying the conditions (2.22), (2.23).

As in [3], we transform the problem (4.1) by separating the linear and nonlinear parts with respect to $\boldsymbol{u}, q, \rho, \boldsymbol{h}$ in all the equations and separating the tangential and normal components in the equation $\widetilde{T}(\boldsymbol{u},q)\boldsymbol{n}(e_\rho) = \sigma H(e_\rho)\boldsymbol{n}$. We also make use of the formulas

$$
\begin{aligned}
&\boldsymbol{f}(e_\rho,t) = \boldsymbol{f}(y,t) + \int_0^1 \frac{d}{ds}\boldsymbol{f}(e_{s\rho},t)ds = \boldsymbol{f}(y,t) + \frac{d}{ds}f(e_{s\rho},t)\Big|_{s=0} \\
&+\int_0^1(1-s)\frac{d^2}{ds^2}\boldsymbol{f}(e_{s\rho},t)ds = \boldsymbol{f}(y,t) + \rho^*(y,t)(\boldsymbol{N}^*\cdot\nabla)\boldsymbol{f}(y,t) \\
&+\int_0^1(1-s)\frac{d^2}{ds^2}\boldsymbol{f}(e_{s\rho},t)ds, \\
&H(e_\rho) = -\nabla_x\cdot\boldsymbol{n}(x)|_{x=e_\rho(y,t)} = -\mathcal{L}^{-T}(y,\rho)\nabla\cdot\frac{\mathcal{L}^T(y,\rho)\boldsymbol{N}(y)}{|\mathcal{L}^T(y,\rho)\boldsymbol{N}(y)|} \\
&= \mathcal{H}(y) - \mathfrak{B}\rho - \int_0^1(1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla\cdot\frac{\mathcal{L}^T(y,s\rho)\boldsymbol{N}(y)}{|\mathcal{L}^T(y,s\rho)\boldsymbol{N}(y)|},
\end{aligned}
\tag{4.2}
$$

where $\mathfrak{B}\rho = -\Delta_{\mathcal{G}}\rho - (\mathcal{H}^2 - 2\mathcal{K})\rho$ is the first variation of $-(H(e_\rho) - \mathcal{H}(y))$ with respect to ρ. (Without restriction of generality, we may assume that $\boldsymbol{f}(x,t)$ is extended into the whole space $\mathbb{R}^3$). Since $\boldsymbol{\Psi}\cdot\boldsymbol{\psi} = \boldsymbol{\Psi}_\tau\cdot\boldsymbol{\psi}_\tau = (\boldsymbol{N}\times\boldsymbol{\Psi})\cdot(\boldsymbol{N}\times\boldsymbol{\psi})$, the surface integral in (4.1) can be written in the form

$$
\int_{\mathcal{G}}\boldsymbol{\Psi}\cdot\boldsymbol{\psi}(y,t)dS = -\int_{\mathcal{F}_1} rot(\boldsymbol{N}^*\times\boldsymbol{\Psi}^*)\cdot\boldsymbol{\psi}(y,t)dy + \int_{\mathcal{F}_1}(\boldsymbol{N}^*\times\boldsymbol{\Psi}^*)\cdot rot\boldsymbol{\psi}(y,t)dy,
$$

where $\boldsymbol{N}^*$ is the extension of $\boldsymbol{N}$ in Ω mentioned in Sec.2 and

$$\boldsymbol{\Psi}^* = \frac{\boldsymbol{u}\cdot\widehat{\mathcal{L}}^T(y,\rho^*)\boldsymbol{N}^*(y)}{L(y,\rho^*)}[\mu](\boldsymbol{h}-\boldsymbol{N}^*(\boldsymbol{N}^*\cdot\boldsymbol{h})), \quad y\in\mathcal{F}_1. \tag{4.3}$$

Finally, we introduce the vector field $\boldsymbol{V}\in W_2^{l+3/2}(\mathcal{G})$, satisfying the condition

$$\|\boldsymbol{V}-\boldsymbol{u}_0\|_{W_2^{l+1/2}(\mathcal{G})}\le\delta\ll 1. \tag{4.4}$$

It is easily seen that (4.1) is equivalent to

$$\begin{cases}
\boldsymbol{u}(y,t)-\nu\nabla^2\boldsymbol{u}+\nabla q-\rho^*(y)(\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{f}(y,t)=\boldsymbol{f}(y,t)+\boldsymbol{l}_1(\boldsymbol{u},q,\rho),\\
\nabla\cdot\boldsymbol{u}=l_2(\boldsymbol{u},\rho), \quad y\in\mathcal{F}_1,\\
\Pi_{\mathcal{G}}S(\boldsymbol{u})\boldsymbol{N}(y)=\boldsymbol{l}_3(\boldsymbol{u},\rho),\\
-q+\nu\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}+\sigma\mathfrak{B}\rho=l_4(\boldsymbol{u},\rho)+l_5(\rho)+\sigma\mathcal{H}(y),\\
\rho_t+\boldsymbol{V}(y)\cdot\nabla_\tau\rho-\boldsymbol{u}\cdot\boldsymbol{N}=l_6(\boldsymbol{u},\rho),\\
\displaystyle\int_0^T\int_\Omega\mu\boldsymbol{h}_t\cdot\boldsymbol{\psi}(y,t)dydt+\int_0^T\int_{\mathcal{F}_1\cup\mathcal{F}_3}\alpha^{-1}rot\boldsymbol{h}\cdot rot\boldsymbol{\psi}(y,t)dydt=\\
\displaystyle=\int_0^T\int_{\mathcal{F}_1}\boldsymbol{l}_7(\boldsymbol{u},\boldsymbol{h},\rho)\cdot rot\boldsymbol{\psi}(y,t)dydt+\int_\Omega\mu\boldsymbol{\Phi}(\boldsymbol{h},\rho)\cdot\boldsymbol{\psi}(y,t)dydt\\
\displaystyle+\int_0^T\int_{\mathcal{F}_1}rot(\boldsymbol{N}^*\times\boldsymbol{\Psi}^*)\cdot\boldsymbol{\psi}(y,t)dydt-\int_0^T\int_{\mathcal{F}_1}(\boldsymbol{N}^*\times\boldsymbol{\Psi}^*)\cdot rot\boldsymbol{\psi}(y,t)dy\\
\displaystyle+\int_0^T\int_{\mathcal{F}_3}\alpha^{-1}\boldsymbol{j}(y,t)\cdot rot\boldsymbol{\psi}(y,t)dydt,\\
\nabla\cdot\boldsymbol{h}(y,t)=0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,\\
rot\boldsymbol{h}=rot\boldsymbol{l}_8(\boldsymbol{h},\rho), \quad y\in\mathcal{F}_2,\\
[\mu\boldsymbol{h}\cdot\boldsymbol{N}]=0, \quad [\boldsymbol{h}_\tau]=\boldsymbol{l}_9(\boldsymbol{h},\rho), \quad y\in\mathcal{G},\\
[\mu\boldsymbol{h}\cdot\boldsymbol{n}]=0, \quad [\boldsymbol{h}_\tau]=0, \quad y\in S_3, \quad \boldsymbol{h}\cdot\boldsymbol{n}=0, \quad y\in S,\\
\boldsymbol{u}(y,0)=\boldsymbol{u}_0(y), \quad y\in\mathcal{F}_1, \quad \boldsymbol{h}(y,0)=\boldsymbol{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,
\end{cases} \tag{4.5}$$

where

$$
(4.6)\qquad \begin{cases}
\boldsymbol{l}_1(\boldsymbol{u},q,\rho)=\nu(\widetilde{\nabla}^2-\nabla^2)\boldsymbol{u}+(\nabla-\widetilde{\nabla})q+\rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u}\\
-(\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\widetilde{\nabla}\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h})+\int_0^1(1-s)\frac{d^2}{ds^2}\boldsymbol{f}(e_{s\rho},t)ds,\\
l_2(\boldsymbol{u},\rho)=(I-\widehat{\mathcal{L}}^T)\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{L}(\boldsymbol{u},\rho),\\
\boldsymbol{L}(\boldsymbol{u},\rho)=(I-\widehat{\mathcal{L}})\boldsymbol{u},\quad y\in\mathcal{F}_1,\\
\boldsymbol{l}_3(\boldsymbol{u},\rho)=\Pi_{\mathcal{G}}(\Pi_{\mathcal{G}}S(\boldsymbol{u})\boldsymbol{N})(y)-\Pi\widetilde{S}(\boldsymbol{u})\boldsymbol{n}(e_\rho(y)),\\
l_4(\boldsymbol{u},\boldsymbol{h},\rho)=\nu(\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}-\boldsymbol{n}\cdot\widetilde{S}(\boldsymbol{u})\boldsymbol{n})-[\boldsymbol{n}(e_\rho)\cdot T_M(\frac{\mathcal{L}}{L}\boldsymbol{h})\boldsymbol{n}],\\
l_5(\rho)=-\sigma\int_0^1(1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla\cdot\frac{\mathcal{L}^T(y,s\rho)\boldsymbol{N}}{|\mathcal{L}^T(y,s\rho)\boldsymbol{N}|}ds,\\
l_6(\boldsymbol{u},\boldsymbol{h},\rho)=(\frac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{\Lambda(y,\rho)}+\nabla_\tau\rho-\boldsymbol{N})\cdot\boldsymbol{u}+(\boldsymbol{V}-\boldsymbol{u})\cdot\nabla_\tau\rho,\quad y\in\mathcal{G},\\
\boldsymbol{l}_7(\boldsymbol{h},\rho)=\alpha^{-1}(rot\boldsymbol{h}-\mathcal{P}rot\mathcal{P}\boldsymbol{h})+\mu_1(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h}),\quad y\in\mathcal{F}_1,\\
\boldsymbol{l}_8(\boldsymbol{h},\rho)=(I-\mathcal{P})\boldsymbol{h},\quad y\in\mathcal{F}_2,\\
\boldsymbol{l}_9(\boldsymbol{h},\rho)=(\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|^2}-\boldsymbol{N})[\boldsymbol{h}\cdot\boldsymbol{N}]=[\boldsymbol{A}(\boldsymbol{h},\rho)],\quad y\in\mathcal{G},\\
\boldsymbol{A}^{(i)}(\boldsymbol{h},\rho)=(\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\boldsymbol{N}^*}{|\widehat{\mathcal{L}}^T\boldsymbol{N}^*|^2}-\boldsymbol{N}^*)(\boldsymbol{h}^{(i)}\cdot\boldsymbol{N}^*),\quad y\in\mathcal{F}_i,\quad i=1,2,\\
\Pi\boldsymbol{f}=\boldsymbol{f}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{f}),\quad \Pi_{\mathcal{G}}\boldsymbol{g}=\boldsymbol{g}-\boldsymbol{N}(\boldsymbol{g}\cdot\boldsymbol{N}).
\end{cases}
$$

We note that the vector field

$$
\boldsymbol{\Phi}(\boldsymbol{h},\rho)=\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\boldsymbol{h}+\rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\frac{1}{L}\mathcal{L}\boldsymbol{h}=\boldsymbol{h}_t(y,t)-\widehat{\mathcal{L}}(\boldsymbol{H}_t(x,t)|_{x=e_\rho})
$$

is divergence free and $\boldsymbol{\Phi}=0$ in $\mathcal{F}_3$ and in the neighborhood of $S\cup S_3$.

The solvability of the problem (4.4) can be proved by successive approximations, according to a usual scheme

$$
(4.7)\qquad \begin{cases}
\boldsymbol{u}_{m+1,t}(y,t) - \nu\nabla^2\boldsymbol{u}_{m+1} + \nabla q_{m+1} - \rho^*_{m+1}(y)(\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{f}(y,t) \\
= \boldsymbol{f}(y,t) + \boldsymbol{l}_1(\boldsymbol{u}_m, q_m, \rho_m), \\
\nabla\cdot\boldsymbol{u}_{m+1} = l_2(\boldsymbol{u}_m,\rho_m), \quad y\in\mathcal{F}_1, \\
\Pi_{\mathcal{G}} S(\boldsymbol{u}_{m+1})\boldsymbol{N}(y) = \boldsymbol{l}_3(\boldsymbol{u}_m,\rho_m), \\
-q_{m+1} + \nu\boldsymbol{N}\cdot S(\boldsymbol{u}_{m+1})\boldsymbol{N} + \sigma\mathfrak{B}\rho_{m+1} = l_4(\boldsymbol{u}_m,\rho_m) + l_5(\rho_m) + \sigma\mathcal{H}(y), \\
\rho_{m+1,t} + \boldsymbol{V}(y)\cdot\nabla_\tau\rho_{m+1} - \boldsymbol{u}_{m+1}\cdot\boldsymbol{N} = l_6(\boldsymbol{u}_m,\rho_m), \\
\int_0^T\int_\Omega \mu\boldsymbol{h}_{m+1,t}\cdot\boldsymbol{\psi}(y,t)dydt + \int_0^T\int_{\mathcal{F}_1\cup\mathcal{F}_3}\alpha^{-1}rot\boldsymbol{h}_{m+1}\cdot rot\boldsymbol{\psi}(y,t)dydt \\
= \int_0^T\int_{\mathcal{F}_1}\boldsymbol{l}_7(\boldsymbol{u}_m,\boldsymbol{h}_m,\rho_m)\cdot rot\boldsymbol{\psi}(y,t)dydt + \int_\Omega\mu\boldsymbol{\Phi}(\boldsymbol{h}_m,\rho_m)\cdot\boldsymbol{\psi}(y,t)dydt \\
+\int_0^T\int_{\mathcal{F}_1} rot(\boldsymbol{N}^*\times\boldsymbol{\Psi}^*_m)\cdot\boldsymbol{\psi}(y,t)dydt - \int_0^T\int_{\mathcal{F}_1}(\boldsymbol{N}^*\times\boldsymbol{\Psi}^*_m)\cdot rot\boldsymbol{\psi}(y,t)dydt \\
+\int_0^T\int_{\mathcal{F}_3}\alpha^{-1}\boldsymbol{j}(y,t)\cdot rot\boldsymbol{\psi}(y,t)dydt, \\
\nabla\cdot\boldsymbol{h}_{m+1}(y,t) = 0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \\
rot\boldsymbol{h}_{m+1} = rot\boldsymbol{l}_8(\boldsymbol{h}_m,\rho_m), \quad y\in\mathcal{F}_2, \\
[\mu\boldsymbol{h}_{m+1}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_{m+1,\tau}] = \boldsymbol{l}_9(\boldsymbol{h}_m,\rho_m), \quad y\in\mathcal{G}, \\
[\mu\boldsymbol{h}_{m+1}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{h}_{m+1,\tau}] = 0, \quad y\in S_3, \quad \boldsymbol{h}_{m+1}\cdot\boldsymbol{n} = 0, \quad y\in S, \\
\boldsymbol{u}_{m+1}(y,0) = \boldsymbol{u}_0(y), \quad y\in\mathcal{F}_1, \quad \boldsymbol{h}_{m+1}(y,0) = \boldsymbol{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,
\end{cases}
$$

where $m = 1, ...,$ and $\boldsymbol{\Psi}^*_m$ is defined by (4.3) with $\boldsymbol{u}_m, \boldsymbol{h}_m, \rho_m$ instead of $\boldsymbol{u}, \boldsymbol{h}, \rho$.

The first approximation, $(\boldsymbol{u}_1, q_1, \rho_1, \boldsymbol{h}_1)$, is defined for $t\in(0,\infty)$ in the following way: $q_1 = 0$, $\boldsymbol{u}_1$, ρ_1 and $\boldsymbol{h}_1$ satisfy the initial conditions

$$
\begin{aligned}
&\boldsymbol{u}_1(y,0) = \boldsymbol{u}_0(y), \quad y\in\mathcal{F}_1, \quad \rho_1(y,0) = \rho_0(y), \quad y\in\mathcal{G}, \\
&\boldsymbol{h}_1(y,0) = \boldsymbol{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3,
\end{aligned}
$$

the equation $\nabla\cdot\boldsymbol{h}_1(y,t) = 0$, $y\in\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3$ and the inequalities

$$
(4.8)\qquad \begin{aligned}
&\|\boldsymbol{u}_1\|_{H^{l+2,l/2+1}(Q^1_\infty)} \le c\|\boldsymbol{u}_1\|_{W_2^{l+2,l/2+1}(Q^1_\infty)} \le c\|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)}, \\
&\|\rho_1\|_{W_2^{l+5/2,0}(G_\infty)} + \|\rho_{1,t}\|_{W_2^{l+3/2,l/2+3/4}(G_\infty)} \le c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}, \\
&\sum_{i=1}^3\|\boldsymbol{h}_1\|_{W_2^{l+2,l/2+1}(Q^i_\infty)} \le c\sum_{i=1}^3\|\boldsymbol{h}_0\|_{W_2^{l+1}(\mathcal{F}_i)}.
\end{aligned}
$$

The proof of the solvability of (4.7) is based on the estimates of non-linear terms and of the solutions of the linear problems 1. and 2. studied in Sec. 2. Let

(4.9)

$$
\begin{aligned}
X_m(T) = {} & \|\boldsymbol{u}_m\|_{H^{l+2,l/2+1}(Q^1_T)} + \|\nabla q_m\|_{\widehat{W}_2^{l,l/2}(Q^1_T)} + \|q_m\|_{W_2^{l+1/2,0}(G_T)} + \|q_m\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \\
& + \|\rho_m\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_m\|_{\widehat{W}_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))} + \|\rho_{m,t}\|_{H^{l+3/2,l/2+3/4}(G_T)} \\
& + \sup_{t<T}\|\rho_m(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})} + \sum_{i=1}^3\|\boldsymbol{h}^{(i)}_m\|_{H^{l'+2,l'/2+1}(Q^i_T)}
\end{aligned}
$$

and

$$
\begin{aligned}
Z_m(T) &= \|\boldsymbol{l}_1(\boldsymbol{u}_m, q_m, \boldsymbol{h}_m, \rho)\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} + \|l_2(\boldsymbol{u}_m, \rho_m)\|_{W_2^{l+1,0}(Q_T^1)} + \sup_{t<T} \|l_2(\boldsymbol{u}_m, \rho_m)\|_{W_2^l(\mathcal{F}_1)} \\
&+ \|\boldsymbol{L}_t(\boldsymbol{u}_m, \rho_m)\|_{\widehat{W}_2^{0,l/2}(Q_T^1)} + \|\boldsymbol{l}_3(\boldsymbol{u}_m, \rho_m)\|_{H^{l+1/2,l/2+1/4}(G_T)} + \|l_4(\boldsymbol{u}_m, \boldsymbol{h}_m, \rho_m)\|_{W_2^{l+1/2,0}(G_T)} \\
&+ \|l_5(\rho_m)\|_{W_2^{l+1/2,0}(G_T)} + \|l_4(\boldsymbol{u}_m, \boldsymbol{h}_m, \rho_m)\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} + \|l_5(\rho_m)\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \\
&+ \|l_6(\boldsymbol{u}_m, \rho_m)\|_{H^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{l}_7\|_{W_2^{l'+1,0}(Q_T^1)} + \|\boldsymbol{l}_7\|_{\widehat{W}_2^{l'/2}(0,T;W_2^1(\mathcal{F}_1))} \\
&+ \|\boldsymbol{l}_8\|_{H^{l'+2,l'/2+1}(Q_T^2)} + \|\boldsymbol{l}_9(\boldsymbol{h}_m, \rho_m)\|_{H^{l'+3/2,l'/2+3/4}(G_T)} + \|\boldsymbol{A}_t(\boldsymbol{h}_m, \rho_m)\|_{\widehat{W}_2^{0,l'/2}(Q_T)} \\
&+ \|\boldsymbol{l}_{9,t}(\boldsymbol{h}_m, \rho_m)\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))} + \sum_{i=1}^{3} \|\Phi(\boldsymbol{h}_m, \rho_m)\|_{\widehat{W}_2^{l',l'/2}(Q_T^i)} \\
&+ \|\boldsymbol{\Psi}^*(\boldsymbol{u}_m, \boldsymbol{h}_m, \rho_m)\|_{W_2^{l'+1,0}(Q_T^1)} + \|\boldsymbol{\Psi}^*(\boldsymbol{u}_m, \boldsymbol{h}_m, \rho_m)\|_{\widehat{W}_2^{l'/2}(0,T;W_2^1(\mathcal{F}_1))}.
\end{aligned}
$$

The following proposition is an analog of Theorem 7 in [3].

Proposition 2. *Assume that (4.4) holds and*

$$
\sup_{t<T} \|\rho_m(\cdot,t)\|_{W_2^{l+3/2}(\mathcal{G})} \le \delta \ll 1, \tag{4.10}
$$

moreover, let the extension ρ^* of ρ (see (2.2)) possess the properties

$$
\frac{\partial \rho^*}{\partial N}\Big|_{\mathcal{G}} = 0,
$$

$\rho^*(y,t) = 0$, *when $y \in \mathcal{F}_3$ or y belongs to a neighborhood of S, and*

$$
\begin{aligned}
&\|\rho^*(\cdot,t)\|_{W_2^{r+1/2}(\Omega)} \le c\|\rho\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+5/2], \\
&\|\rho_t^*(\cdot,t)\|_{W_2^{r+1/2}(\Omega)} \le c\|\rho_t\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+3/2], \quad l \in (1/2, 1).
\end{aligned} \tag{4.11}
$$

Then

$$
Z_m \le \delta_1 \sum_{j=1}^{3} X_m^j, \tag{4.12}
$$

where δ_1 is a small number dependent on δ and T.

The non-linear terms (4.6) are identical or very similar to the nonlinear terms (1.10), (1.11) in [3], and they are estimated in the same manner (cf. [3, Sec.4, 4, Sec.3]). We point out that the estimate of $\boldsymbol{l}_9$ are made here slightly better than in [3], namely, it holds

$$
\begin{aligned}
&\|\boldsymbol{a}_t(\boldsymbol{h}_m, \rho_m)\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))} \\
&\le c\delta \|\boldsymbol{h}_{m,t} \cdot \boldsymbol{N}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))} + cT^{(1-l')/2} \|\boldsymbol{h}^{(i)}\|_{H^{2+l',1+l'/2}(Q_T^i)} \|\rho\|_{H^{3/2+l,3/4+l/2}(G_T)} \\
&\le c\delta \|\boldsymbol{h}_{m,t}\|_{\widehat{W}_2^{0,l'/2}(Q_T)} + cT^{(1-l')/2} \|\boldsymbol{h}^{(i)}\|_{H^{2+l',1+l'/2}(Q_T^i)} \|\rho\|_{H^{3/2+l,3/4+l/2}(G_T)},
\end{aligned}
$$

because $\boldsymbol{h}_m$, $m \ge 1$, are divergence free. Therefore the norm

$$
\sum_{i=1}^{2} \|\boldsymbol{h}_{m,t}^{(i)}\|_{W_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}
$$

is not included in $X_m(T)$ (although it can be estimated, too).

Making use of the inequalities (3.6) and (3.31) applied to the problem (4.7), we obtain

$$
X_{m+1} \le c_1 \delta_1 \sum_{j=1}^{3} X_m^j + c_2 N,
$$

where $\delta_1 \ll 1$,

$$N = \|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\mathcal{H}\|_{W_2^{l+1/2}(\mathcal{G})}$$
$$+ \sum_{i=1}^{3} \|\boldsymbol{h}_0\|_{W_2^{l'+1}(\mathcal{F}_i)} + \|\boldsymbol{j}\|_{W_2^{l+1,0}(Q_T^3)} + \|\boldsymbol{j}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^1(\mathcal{F}_3))} + \|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T^1)}$$

and the constants c_1 and c_2 are independent of T.

It follows that in the case of small δ_1 the estimate

$$X_{m+1} \le 2c_2 N \tag{4.13}$$

holds and, in addition,

$$\begin{aligned} \|\rho_{m+1}(\cdot,t)\|_{W_2^{l+3/2}(\mathcal{G})} &\le \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + \int_0^t \|\rho_{m,\tau}(\cdot,\tau)\|_{W_2^{l+3/2}(\mathcal{G})} d\tau \\ &\le \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + 2c_2 N\sqrt{T} \le \epsilon + 2c_2\sqrt{T}N \le \delta. \end{aligned} \tag{4.14}$$

Thus, we have proved that (4.13) is satisfied for all $m \ge 1$. The convergence of the sequences $\boldsymbol{u}_m, q_m, \boldsymbol{h}_m, \rho_m$ follows from the estimates of the differences $\boldsymbol{u}_{m+1}-\boldsymbol{u}_m$, $q_{m+1}-q_m$, $\boldsymbol{h}_{m+1}-\boldsymbol{h}_m$, $\rho_{m+1}-\rho_m$ (cf. [3], Theorem 8). Making m tend to infinity, we obtain the inequality

$$X(T) \le 2c_2 N \tag{4.15}$$

for the solution of the problem (4.7) constructed above (here $X(T)$ is defined by (4.9) with $\boldsymbol{u}, q, \rho, \boldsymbol{h}$ instead of $\boldsymbol{u}_m, q_m, \rho_m, \boldsymbol{h}_m$).

The uniqueness of the solution obtained in this way follows from the same kind of estimates for the differences $\boldsymbol{u}-\boldsymbol{u}'$, $q-q'$, $\rho-\rho'$, $\boldsymbol{h}-\boldsymbol{h}'$ of two possible solutions of (4.7).

We conclude the proof of Theorem 1 by construction of the vector field $\boldsymbol{e}$ assuming that the solution of (4.1) is already obtained. For this we need to solve the problem

$$\begin{cases} rot\boldsymbol{\mathcal{E}}(y) = \mu\boldsymbol{\xi}(y), \quad \nabla\cdot\mathcal{P}^{-1}\boldsymbol{\mathcal{E}} = 0, \quad y \in \mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \\ [\mu\mathcal{P}^{-1}\boldsymbol{\mathcal{E}}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{\mathcal{E}}_\tau] = \boldsymbol{a}(y), \quad y \in \mathcal{G}, \\ [\mu\mathcal{P}^{-1}\boldsymbol{\mathcal{E}}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{\mathcal{E}}_\tau] = 0, \quad y \in S_3, \quad \boldsymbol{\mathcal{E}}_\tau = 0, \quad y \in S. \end{cases} \tag{4.16}$$

Theorem 6. *Assume that $\boldsymbol{\xi} \in W_2^l(\mathcal{F}_i)$, $i = 1,2,3$, $\boldsymbol{a} \in W_2^{l+3/2}(\mathcal{G})$ and that the compatibility and orthogonality conditions*

$$\begin{aligned} &\nabla\cdot\boldsymbol{\xi}(y) = 0, \quad y \in \mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3, \\ &[\mu\boldsymbol{\xi}\cdot\boldsymbol{N}] = \boldsymbol{N}\cdot rot\boldsymbol{a}, \quad y \in \mathcal{G}, \quad [\mu\boldsymbol{\xi}\cdot\boldsymbol{n}] = 0, \quad y \in S_3, \quad \boldsymbol{\xi}\cdot\boldsymbol{n} = 0, \quad y \in S \end{aligned} \tag{4.17}$$

$$\int_\Omega \mu\boldsymbol{\xi}(y)\cdot\widetilde{\boldsymbol{u}}_q(y)dy = \int_{\mathcal{G}} (\boldsymbol{N}\times\boldsymbol{a})\cdot\widetilde{\boldsymbol{u}}_q dS, \quad \widetilde{\boldsymbol{u}}_q \in \widetilde{U}_n(\Omega), \quad q = 1,...,b_1(\Omega) \tag{4.18}$$

are satisfied. Then the problem (4.16) has a unique solution $\boldsymbol{\mathcal{E}} \in W_2^{l+1}(\mathcal{F}_i)$, $i = 1,2,3$ orthogonal to the $b_2(\Omega)$-dimensional space $\widetilde{U}_d$ of vector fields $\boldsymbol{v}(y($ such that

$$\begin{aligned} &rot\boldsymbol{v} = 0, \quad \nabla\cdot\mathcal{P}^{-1}\boldsymbol{v}(y) = 0, \\ &[\boldsymbol{v}_\tau] = 0, \quad y \in S_3\cup\mathcal{G}, \quad [\mu\mathcal{P}^{-1}\boldsymbol{v}\cdot\boldsymbol{N}]|_{\mathcal{G}} = 0, \quad [\mu\mathcal{P}^{-1}\boldsymbol{v}\cdot\boldsymbol{n}]|_{S_3} = 0, \quad \boldsymbol{v}_\tau|_S = 0. \end{aligned} \tag{4.19}$$

The solution satisfies the inequality

$$\sum_{i=1}^{3} \|\boldsymbol{\mathcal{E}}\|_{W_2^{r+1}(\mathcal{F}_i)} \le c\Big(\sum_{i=1}^{3} \|\boldsymbol{\xi}\|_{W_2^r(\mathcal{F}_i)} + \|\boldsymbol{a}\|_{W_2^{r+1/2}(\mathcal{G})}\Big) \tag{4.20}$$

with $r = l, r = 0$.

Proof. We seek the solution in the form

$$\boldsymbol{\mathcal{E}} = \boldsymbol{a}^* + \boldsymbol{\mathcal{E}}_1 + \nabla Z(y) + \sum_{j=1}^{b_2(\Omega)} d_j \boldsymbol{v}_j(y), \tag{4.21}$$

where $\boldsymbol{a}^*$ is the extension of $\boldsymbol{a}$, as in Theorem 3,

$$\boldsymbol{\mathcal{E}}_1(y) = \frac{1}{4\pi} rot \int_\Omega \frac{(\mu\boldsymbol{\xi}(z) - rot\boldsymbol{a}^*(z))dz}{|y-z|},$$

$$\begin{cases} \nabla \cdot \mathcal{P}^{-1}\nabla Z = -\nabla\mathcal{P}^{-1}(\boldsymbol{a}^* + \boldsymbol{\mathcal{E}}_1(y)), \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3, \\ Z(y) = -g(y), \quad y \in S, \quad [Z(y)] = 0, \quad y \in \mathcal{G} \cup S_3, \\ [\mu\mathcal{P}^{-1}\nabla Z \cdot \boldsymbol{N}] = -[\mu\mathcal{P}^{-1}(\boldsymbol{a}^* + \boldsymbol{\mathcal{E}}_1) \cdot \boldsymbol{N}], \quad y \in \mathcal{G}, \\ [\mu\mathcal{P}^{-1}\nabla Z \cdot \boldsymbol{n}] = -[\mu\mathcal{P}^{-1}(\boldsymbol{a}^* + \boldsymbol{\mathcal{E}}_1) \cdot \boldsymbol{n}], \quad y \in S_3. \end{cases} \tag{4.22}$$

It is easily seen that $rot\boldsymbol{\mathcal{E}}_1 = \mu\boldsymbol{\xi} - rot\boldsymbol{a}^*$. The function g is defined as follows. Since $(\mu\boldsymbol{\xi} - rot\boldsymbol{a}^*) \cdot \boldsymbol{n}|_S = 0$, we have $\int_\Sigma rot\boldsymbol{\mathcal{E}}_1 \cdot \boldsymbol{n} dS = 0$ for arbitrary $\Sigma \subset S$, and this relation holds also for $\Sigma = \Sigma_k$, $k = 1, ..., b_1(\Omega)$, where Σ_k are cuts of Ω that make Ω simply connected (this follows from the orthogonality conditions (4.18), i.e., from

$$\int_\Omega (\mu\boldsymbol{\xi} - rot\boldsymbol{a}^*) \cdot \boldsymbol{u}_q dy = 0,$$

see [6]). Hence by the Stokes formula, the relation

$$\int_\gamma \boldsymbol{\mathcal{E}}_1 \cdot d\boldsymbol{l} = 0$$

holds for arbitrary closed contour $\gamma \subset S$, which implies $\boldsymbol{\mathcal{E}}_1|_S = \nabla_\tau g(y)$ with a certain single-valued $g(y)$. Now the relations (4.16) are easily verified.

Making use of the estimates of the volume potential and of the solution of elliptic problem for Z (4.22), we estimate $\mathcal{E}_1$ and ∇Z, after which the term $\sum_{j=1}^{b_2(\Omega)} d_j \boldsymbol{v}_j(y)$ is estimated with the help of the condition of orthogonality of $\boldsymbol{\mathcal{E}}$ to $\boldsymbol{v}_j$ (cf. the proof of (3.15)). In this way we obtain (4.20). The uniqueness of the solution in the class indicated in the statement of the theorem is obvious. The theorem is proved.

Now we pass to the construction of $\boldsymbol{e}$. We want to solve the problem (4.16) with $\boldsymbol{\xi} = -\boldsymbol{h}_t + \Phi$ and $\boldsymbol{a} = -\boldsymbol{N} \times \boldsymbol{\Psi}$; in this case the condition $[\boldsymbol{\mathcal{E}}_\tau] = \boldsymbol{a}$ on $\mathcal{G}$ is equivalent to

$$[\boldsymbol{N} \times \boldsymbol{\mathcal{E}}] = \boldsymbol{\Psi}. \tag{4.23}$$

Let us verify the assumptions (4.17), (4.18) of Theorem 6. From Proposition 1 the following statements follows: arbitrary $\boldsymbol{b}$ and $\boldsymbol{\mathcal{E}}$ given on $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\mathcal{G}$, respectively, and such that

$$\nabla \cdot \boldsymbol{b} = 0, \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad [\boldsymbol{b} \cdot \boldsymbol{N}] = 0, \quad y \in \mathcal{G},$$

$$[\boldsymbol{N} \times \boldsymbol{\mathcal{E}}] = \frac{\boldsymbol{u} \cdot \widehat{\mathcal{L}}^T \boldsymbol{N}}{L}[\boldsymbol{b}], \quad y \in \mathcal{G},$$

satisfy

$$[\boldsymbol{N} \cdot rot\boldsymbol{\mathcal{E}}] = -[\boldsymbol{b}_t - \frac{1}{L}\widehat{\mathcal{L}}^T \mathcal{L}\boldsymbol{b} - \rho_t \widehat{\mathcal{L}}(\mathcal{L}^{-1}(\boldsymbol{N} \cdot \nabla)\frac{1}{L}\mathcal{L}\boldsymbol{b} \cdot \boldsymbol{N}].$$

For $\boldsymbol{b} = \mu \boldsymbol{h}$, this coincides with the main compatibility condition in (4.17). As for (4.18), this condition can be verified by setting $\boldsymbol{\psi} = \widetilde{\boldsymbol{u}}_q$ in the integral identity in (4.1). Hence the problem (4.16) with the above-mentioned data is solvable, and the solution satisfies the identity

$$\begin{aligned} &\int_0^T \int_{\mathcal{F}_1 \cup \mathcal{F}_3} (-\mathcal{E} + \alpha^{-1} rot \boldsymbol{h}) \cdot rot \boldsymbol{\psi}(y,t) dy dt = \\ &= \mu_1 \int_0^T \int_{\mathcal{F}_1} \boldsymbol{l}_7(\boldsymbol{u}, \boldsymbol{h}, \rho) \cdot rot \boldsymbol{\psi}(y,t) dy dt + \int_0^T \int_{\mathcal{F}_3} \alpha^{-1} \boldsymbol{j}(y,t) \cdot rot \boldsymbol{\psi}(y,t) dy dt. \end{aligned} \tag{4.24}$$

By Theorem 3, $rot \boldsymbol{\psi}$ runs over the whole $\overset{\circ}{J}(\mathcal{F}_1 \cup \mathcal{F}_3)$, when $\boldsymbol{\psi}$ runs over $\mathcal{H}^1(\Omega)$. It follows that

$$\begin{aligned} &-\mathcal{E}^{(1)} + \alpha_1^{-1} \mathcal{P} rot \mathcal{P} \boldsymbol{h}^{(1)} - \mu_1 (\mathcal{L}^{-1} \boldsymbol{u} \times \boldsymbol{h}^{(1)}) = \nabla Y^{(1)}, \\ &-\mathcal{E}^{(3)} + \alpha^{-1} rot \boldsymbol{h}^{(3)} - \alpha^{-1} \boldsymbol{j}(y,t) = \nabla Y^{(3)}. \end{aligned} \tag{4.25}$$

We set

$$\mathcal{P} \boldsymbol{e} = \mathcal{E} + \nabla Y, \quad y \in \mathcal{F}_1 \cup \mathcal{F}_3$$

and

$$\mathcal{P} \boldsymbol{e} = \mathcal{E} + \nabla Y^{(2)} + \sum_{j=1}^{b_2(\Omega)} C_j(t) \nabla \omega_j(y), \quad y \in \mathcal{F}_2,$$

where $Y^{(2)}$ is a solution of the problem

$$\nabla \cdot \mathcal{P}^{-1} \nabla Y^{(2)} = 0, \quad y \in \mathcal{F}_2, \quad Y^{(2)}|_{y \in \mathcal{G}} = Y^{(1)}, \quad Y^{(2)}|_{y \in S_3} = Y^{(3)}, \tag{4.26}$$

and ω_j are solutions of

$$\nabla \cdot \mathcal{P}^{-1} \nabla \omega_j = 0, \quad y \in \mathcal{F}_2, \quad \omega_j(y)|_{y \in S_k} = \delta_{jk}, \quad j,k = 1, ..., b_2(\mathcal{F}_2), \quad \omega_j|_{y \in \mathcal{G}} = 0.$$

Then the equations

$$\nabla \cdot \boldsymbol{e}^{(2)}(y,t) = 0, \quad y \in \mathcal{F}_2, \quad [(\mathcal{P} \boldsymbol{e})_\tau] = 0, \quad y \in \mathcal{G} \cup S_3$$

are satisfied.

We pass to the estimates of $\boldsymbol{e}$. By (4.20),

$$\sum_{i=1}^3 \|\mathcal{E}\|_{W_2^{l+1,0}(Q_T^i)} \le c\Big(\sum_{i=1}^3 \|\boldsymbol{h}_t - \Phi\|_{W_2^{l,0}(Q_T^i)} + \|\boldsymbol{\Psi}\|_{W_2^{l+1/2,0}(G_T)} \Big). \tag{4.27}$$

Applying the same inequality with $l = 0$ to $\mathcal{E}$ and to the finite difference $\Delta_t(h)\mathcal{E} = \mathcal{E}(y, t+h) - \mathcal{E}(y,t)$, we easily obtain

$$\sum_{i=1}^3 \|\mathcal{E}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_i))} \le c\Big(\sum_{i=1}^3 \|\boldsymbol{h}_t - \Phi\|_{\widehat{W}_2^{0,l/2}(Q_T^i)} + \|\Psi\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \Big). \tag{4.28}$$

Hence

$$\begin{aligned} &\sum_{i=1}^3 (\|\mathcal{E}\|_{W_2^{l+1,0}(Q_T^i)} + \|\mathcal{E}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_i))}) \\ &\le c \sum_{i=1}^3 \|\boldsymbol{h}_t - \Phi\|_{\widehat{W}_2^{l,l/2}(Q_T^i)} + \|\boldsymbol{\Psi}\|_{W_2^{l+1/2,0}(G_T)} + \|\Psi\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \Big). \end{aligned} \tag{4.29}$$

Further, from (4.25) it follows that

$$
\begin{aligned}
&\|\nabla Y^{(1)}\|_{W_2^{l+1,0}(Q_T^1)} \le \|\mathcal{E}\|_{W_2^{l+1,0}(Q_T^1)} \\
&+ \|\alpha_1^{-1}\mathcal{P}rot\mathcal{P}\boldsymbol{h} - \mu_1(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})\|_{W_2^{l+1,0}(Q_T^1)}, \\
&\|\nabla Y^{(1)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_1))} \le \|\mathcal{E}\|_{\widehat{W}_2^{l/2}(0,T;W_2^(\mathcal{F}_1))} \\
&+ \|\alpha_1^{-1}\mathcal{P}rot\mathcal{P}\boldsymbol{h} - \mu_1(\mathcal{L}^{-1}\boldsymbol{u}\times\boldsymbol{h})\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_1))}, \\
&\|\nabla Y^{(3)}\|_{W_2^{l+1,0}(Q_T^1)} \le \|\mathcal{E}\|_{W_2^{l+1,0}(Q_T^3)} + \|\alpha^{-1}(rot\boldsymbol{h} - \boldsymbol{j})\|_{W_2^{l+1,0}(Q_T^1)}, \\
&\|\nabla Y^{(3)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_3))} \le \|\mathcal{E}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_3))} + \|\alpha^{-1}(rot\boldsymbol{h} - \boldsymbol{j})\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_3))},
\end{aligned}
\tag{4.30}
$$

and, since $Y^{(2)}$ is a solution of (4.22), we have

$$
\begin{aligned}
&\|\nabla Y^{(2)}\|_{W_2^{l+1,0}(Q_T^2)} + \|\nabla Y^{(2)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_2))} \\
&\le c(\|\nabla Y^{(1)}\|_{W_2^{l+1,0}(Q_T^1)} + \|\nabla Y^{(1)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_1))} \\
&+ \|\nabla Y^{(3)}\|_{W_2^{l+1,0}(Q_T^3)} + \|\nabla Y^{(3)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_3))}).
\end{aligned}
\tag{4.31}
$$

The functions $C_j(t)$ are found from the condition (1.13), since the matrix with the elements $\int_{S_k}\mathcal{P}^{-1}\nabla\omega_j\cdot\boldsymbol{n}dS$ is non-degenerate. Indeed, otherwise there would exist the constants a_j such that

$$
\int_{S_k}\mathcal{P}^{-1}\nabla A(y)\cdot\boldsymbol{n}dS = 0, \quad k = 1,...,b_2(\Omega)
$$

where $A(y) = \sum_{j=1}^{b_2(\Omega)} a_j\omega_j$. This would imply

$$
\begin{aligned}
0 &= \int_\Omega A\nabla\cdot\mathcal{P}^{-1}\nabla A dy = \int_{\partial\mathcal{F}_2}\mathcal{P}^{-1}\nabla A(y)\cdot\boldsymbol{n}A(y)dS - \int_{\mathcal{F}_2}\mathcal{P}^{-1}\nabla A(y)\cdot\nabla A(y)dy \\
&= -\int_{\mathcal{F}_2}\mathcal{P}^{-1}\nabla A(y)\cdot\nabla A(y)dy,
\end{aligned}
$$

because $A|_{S_k} = const$, $k = 0,...,b_2(\mathcal{F}_2)$. Hence $\nabla A = 0$ and $a_j = 0$.

By (1.13),

$$
\begin{aligned}
0 &= \int_{S_k}\boldsymbol{e}\cdot\boldsymbol{n}dS = \int_{S_k}\mathcal{P}\boldsymbol{e}\cdot\boldsymbol{n}dS = \int_{S_k}(\mathcal{E} + \nabla Y^{(2)})\cdot\boldsymbol{n}dS \\
&+ \sum_{j=1}^{b_2(\Omega)} C_j(t)\int_{S_k}\mathcal{P}\nabla\omega_j\cdot\boldsymbol{n}dS,
\end{aligned}
\tag{4.32}
$$

from which it follows that

$$
\begin{aligned}
&\|\sum_{j=1}^{b_2} C_j\mathcal{P}\omega_j\|_{W_2^{l+1,0}(Q_T^2)} + \|\sum_{j=1}^{b_2} C_j\mathcal{P}\omega_j\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_2))} \le c(\|\nabla Y^{(2)}\|_{W_2^{l+1,0}(Q_T^2)} \\
&+ \|\nabla Y^{(2)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_2))} + \|\mathcal{E}\|_{W_2^{l+1,0}(Q_T^2)} + \|\mathcal{E}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_2))}).
\end{aligned}
\tag{4.33}
$$

Finally,

$$\sum_{i=1}^{3}\|\boldsymbol{e}\|_{W_2^{l+1,0}(Q_T^i)}+\|\boldsymbol{e}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_i))}\le c\sum_{i=1}^{3}(\|\boldsymbol{\mathcal{E}}\|_{W_2^{l+1,0}(Q_T^i)}$$
$$+\|\boldsymbol{\mathcal{E}}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_i))}+\|\nabla Y^{(i)}\|_{W_2^{l+1,0}(Q_T^i)}+\|\nabla Y^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_i))} \tag{4.34}$$
$$+\|\sum_{j=1}^{b_2}C_j\mathcal{P}\omega_j\|_{W_2^{l+1,0}(Q_T^2)}+\|\sum_{j=1}^{b_2}C_j\mathcal{P}\omega_j\|_{\widehat{W}_2^{l/2}(0,T;W_2^1(\mathcal{F}_2))}).$$

Inequality (2.16) for $\boldsymbol{e}$ is a consequence of (4.27)-(4.34), (4.12), (4.13). This completes the proof of Theorem 1.

The solution of the problem (1.2)-(1.5), (1.7), (1.12), (1.13) is defined by

$$\boldsymbol{v}(x,t)=\boldsymbol{u}(e_\rho^{-1}(x,t),t),\quad p(x,t)=q(e_\rho^{-1}(x,t),t),$$
$$\boldsymbol{H}(x,t)=\frac{\mathcal{L}(y,\rho^*)}{L(y,\rho^*)}\boldsymbol{h}(x,t)\big|_{y=e_\rho^{-1}(x,t)},\quad \boldsymbol{E}(x,t)=\frac{\mathcal{L}(y,\rho^*)}{L(y,\rho^*)}\boldsymbol{e}(y,t)\big|_{y=e_\rho^{-1}(x,t)},$$

the free surface Γ_t is given by (2.1).

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