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Exceptional collections on varieties of Borel subgroups of groups of rank 2

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ABSTRACT:

In the present notes we construct new examples of exceptional collections of line bundles on a variety of Borel subgroups of a simple linear algebraic group G of rank 2 over an algebraically closed field of characteristic zero.

Key words: exceptional collection, complete flag variety, Steinberg basis

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1. Introduction

Let $D^b(X)$ be the derived category of coherent sheaves on a smooth projective variety X over a field k. We say that an object E of $D^b(X)$ is *exceptional* [6, Def. 1.1] if $Ext^*(E,E) = k$. Let W be a finite set and let $\mathcal{P}$ be a partial order on W. An ordered set (with respect to $\mathcal{P}$) of exceptional objects $\{E_w\}_{w\in W}$ in $D^b(X)$ is called a *$\mathcal{P}$-exceptional collection*, if

$$Ext^*(E_w, E_{w'}) = 0 \text{ for all } w < w' \text{ with respect to } \mathcal{P}.$$

If $\mathcal{P}$ is a total order, then a $\mathcal{P}$-exceptional collection is called simply an *exceptional collection* [6, Def. 1.2]. Finally, a $\mathcal{P}$-exceptional collection $\{E_w\}_{w\in W}$ is called *full* if it generates the category $D^b(X)$, i.e. the smallest triangulated category containing $\{E_w\}_{w\in W}$ is $D^b(X)$ itself. Note that if there is a full exceptional collection of vector bundles $\{\mathcal{E}_w\}_{w\in W}$, then the classes $\{[\mathcal{E}_w]\}_{w\in W}$ generate $K_0(X)$.

The following conjecture is well-known:

Conjecture. Let X be a projective homogeneous variety of a split semisimple linear algebraic group G over a field k. Then there exists a full exceptional collection of vector bundles in $D^b(X)$.

The first example of a full exceptional collection of bundles for $\mathbb{P}^n$ was provided by Beilinson [2]. The next step was done by Kapranov in [7] where he constructed full exceptional collections for flag varieties of type A and quadrics. Later on an essential progress (see [8, §1.1] for a historical overview) have been achieved for varieties of other Dynkin types culminating in the recent work by Kuznetsov and Polishuk [8], where exceptional collections were constructed for all homogeneous varieties of classical types.

In the present notes we provide new examples (see Corollary 3.3 and Examples (2), (3), (4), (5), (6)) of ($\mathcal{P}$-)exceptional collections on varieties of Borel subgroups of groups G of rank 2, where $\mathcal{P}$ is a left weak Bruhat order on the Weyl group of G. Observe that in contrast to all known examples our collections consist of line bundles and, moreover, form $\mathbb{Z}$-bases of Grothendieck's K_0. To construct such a collection we introduce a purely combinatorial algorithm based on a different description of the Steinberg basis [10] obtained in [1]. We expect that this algorithm can be applied to construct exceptional collections on varieties of higher ranks.

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2. Preliminaries

In the present section we recall several basic facts concerning root systems, weights, associated line bundles and the Grothendieck K_0 (see [3], [4], [5], [9]).

2.1. Let G be a simple simply connected linear algebraic group of rank n over an algebraically closed field of characteristic 0. We fix a split maximal torus T and a Borel subgroup B such that $T \subset B \subset G$.

Let Λ be the weight lattice of the root system Φ of G. Observe that Λ is the group of characters of T. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots and let $\{\omega_1, \dots, \omega_n\}$ be the respective set of fundamental weights (a basis of Λ), i.e. $\alpha_i^\vee(\omega_j) = \delta_{ij}$. Let Φ^+ denote the set of all positive roots and let Λ^+ denote the cone of dominant weights.

2.2. Consider the integral group ring $\mathbb{Z}[\Lambda]$; its elements are finite linear combinations $\sum_i a_i e^{\lambda_i}$, $\lambda_i \in \Lambda$, $a_i \in \mathbb{Z}$. Observe that $\mathbb{Z}[\Lambda]$ can be identified with the representation ring of T. Let $\mathfrak{B}$ denote the variety of Borel subgroups of G, i.e. the variety of Borel subgroups conjugate to B.

Consider the characteristic map for K_0 (see [4, §2.8])

$$\mathfrak{c}\colon \mathbb{Z}[\Lambda] \to K_0(\mathfrak{B})$$

defined by sending e^λ to the class of the associated line bundle $\mathcal{L}(\lambda)$ over $\mathfrak{B}$. It is a surjective ring homomorphism with kernel generated by augmented invariants. More precisely, if $\mathbb{Z}[\Lambda]^W$ denotes the subring of W-invariant elements and $\epsilon\colon \mathbb{Z}[\Lambda] \to \mathbb{Z},\ e^\lambda \mapsto 1$ is the augmentation map, then $\ker \mathfrak{c}$ is generated by elements $x \in \mathbb{Z}[\Lambda]^W$ such that $\epsilon(x) = 0$.

2.3. The Weyl group W acts linearly on Λ via simple reflections s_i as

$$s_i(\lambda) = \lambda - \alpha_i^\vee(\lambda)\alpha_i, \quad \lambda \in \Lambda.$$

Let ρ denote the half-sum of all positive roots. Observe that ρ is also the sum of the fundamental weights. We define another action of the Weyl group on Λ as

$$w * \lambda = w(\lambda + \rho) - \rho, \quad \lambda \in \Lambda.$$

Observe that this is a group action although it is not linear.

A weight $\lambda \in \Lambda$ is called singular if it lies on the wall of a Weyl chamber, i.e. there exists $\alpha \in \Phi^+$ such that $(\lambda, \alpha) = 0$ or, equivalently, if $w(\lambda) = \lambda$ for some non-identity w. A weight $\lambda + \rho$ is singular $\iff$ $w(\lambda + \rho) = \lambda + \rho$ $\iff$ $w * \lambda = \lambda$ for some non-identity w $\iff$ there is no $w \in W$ such that $w * \lambda \in \Lambda^+$.

A weight is called regular if it is not a singular weight. Observe that ρ is a minimal regular dominant weight. A weight $\lambda + \rho$ is regular $\iff$ there is a unique $w \in W$ such that $w * \lambda \in \Lambda^+$.

2.4. We recall the following fundamental fact known as the Borel-Weil-Bott theorem (see [5, p.392]): If $\lambda + \rho$ is singular, then

$$H^*(\mathfrak{B}; \mathcal{L}(\lambda)) = 0.$$

If $\lambda + \rho$ is regular, then

$$H^i(\mathfrak{B}; \mathcal{L}(\lambda)) = \begin{cases} 0 & l(w) \neq i \\ V_\lambda^* & l(w) = i. \end{cases}$$

where w is the unique element of W with $w * \lambda \in \Lambda^+$ and V_λ^* is the dual of the irreducible representation of G with highest weight λ.

2.5. Following [1] for each $w \in W$ consider the cones Λ^+ and $w^{-1}\Lambda^+$. Let H_α denote the hyperplane orthogonal to a positive root $\alpha \in \Phi^+$. We say that H_α separates Λ^+ and $w^{-1}\Lambda^+$ if

$$\Lambda^+ \subset \{\lambda \in \Lambda \mid (\lambda, \alpha) \geq 0\} \text{ and } w^{-1}\Lambda^+ \subset \{\lambda \in \Lambda \mid (\lambda, \alpha) \leq 0\}.$$

or, equivalently, if $(w^{-1}\rho, \alpha) < 0$. Let H_w denote the union of all such hyperplanes, i.e.

$$H_w = \bigcup_{(w^{-1}\rho, \alpha) < 0} H_\alpha.$$

Consider a subset $A_w = w^{-1}\Lambda^+ \setminus H_w$ consisting of weights $\lambda \in w^{-1}\Lambda^+$ separated from Λ^+ by the same set of hyperplanes as $w^{-1}\rho$. By [1, Lem. 6] there is a unique element $\lambda_w \in A_w$ such that for each $\mu \in A_w$ we have $\mu - \lambda_w \in w^{-1}\Lambda^+$. In fact, the set A_w can be viewed as a cone $w^{-1}\Lambda^+$ shifted to the vertex λ_w.

2.6. **Example.** In particular, for the identity $1 \in W$ we have $\lambda_1 = 0$. Let $w = s_j$ be a simple reflection, then

$$w^{-1}\Lambda^+ = \mathbb{N}_0 s_j(\omega_j) \oplus \bigoplus_{i \neq j} \mathbb{N}_0 \omega_i = \mathbb{N}_0(\omega_j - \alpha_j) \oplus \bigoplus_{i \neq j} \mathbb{N}_0 \omega_i$$

and $A_w = (\omega_j - \alpha_j) + w^{-1}\Lambda^+$. Hence, in this case we have $\lambda_w = \omega_j - \alpha_j$.

For the longest element $w_0 \in W$ we have $\lambda_{w_0} = -\rho$.

2.7. By [1, Thm. 2] the integral group ring $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with the basis $\{e^{\lambda_w}\}_{w\in W}$. As there is an isomorphism

$$\mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z} = \mathbb{Z}[\Lambda]/\ker \mathfrak{c} \simeq K_0(\mathfrak{B})$$

classes of the associated line bundles $\{\mathfrak{c}(e^{\lambda_w}) = [\mathcal{L}(\lambda_w)]\}_{w\in W}$ form a $\mathbb{Z}$-basis of $K_0(\mathfrak{B})$.

3. Exceptional collections

Consider the variety $\mathfrak{B}$ of Borel subgroups. We have $Hom(\mathcal{L},\mathcal{L}) = k$ for any line bundle over $\mathfrak{B}$. Let $\mathcal{L}(\lambda)$ be the line bundle over $\mathfrak{B}$ associated to the weight λ. By definition

$$\mathcal{L}(\lambda) \text{ is exceptional if and only if } Ext^i(\mathcal{L}(\lambda),\mathcal{L}(\lambda)) = 0 \text{ for all } i \neq 0.$$

For any two line bundles we have

$$Ext^i(\mathcal{L}(\lambda),\mathcal{L}(\lambda')) = H^i(\mathfrak{B},\mathcal{L}(\lambda)^* \otimes \mathcal{L}(\lambda')) = H^i(\mathfrak{B},\mathcal{L}(\lambda'-\lambda)).$$

Therefore, by the Borel-Weil-Bott theorem any line bundle $\mathcal{L}(\lambda)$ over $\mathfrak{B}$ is exceptional (since 0 is dominant).

3.1. **Definition.** A collection of weights $\{\lambda_w\}_{w\in W}$ is called *$\mathcal{P}$-exceptional* if the respective collection of line bundles $\{\mathcal{L}(\lambda_w)\}_{w\in W}$ is $\mathcal{P}$-exceptional and it forms a basis of $K_0(\mathfrak{B})$. Observe that by the Borel-Weil-Bott theorem the first condition is equivalent to

$$\lambda_{w'} - \lambda_w + \rho \text{ is singular for all } w <_{\mathcal{P}} w', \tag{1}$$

i.e. $\lambda_{w'} - \lambda_w + \rho$ lies on the wall of a Weyl chamber.

Now we are ready to state the main result of this paper

3.2. **Theorem.** *Let $\mathfrak{B}$ be the variety of Borel subgroups of a group G of rank 2. Then there exists a $\mathcal{P}$-exceptional collection of weights $\{\lambda_w\}_{w\in W}$, where $\mathcal{P}$ is the left weak Bruhat order on the Weyl group W. Moreover, if G is not of type G_2, then there exists an exceptional collection of weights $\{\lambda_w\}_{w\in W}$ for some total order on W.*

3.3. **Corollary.** *Let $\mathfrak{B}$ be the variety of Borel subgroups of a group G of rank 2. Then there exists a $\mathcal{P}$-exceptional collection of line bundles on $\mathfrak{B}$ which forms a $\mathbb{Z}$-basis of $K_0(\mathfrak{B})$, where $\mathcal{P}$ is the left weak Bruhat order on W. Moreover, if $\mathfrak{B}$ is not of type G_2, then there exists an exceptional collection of line bundles on $\mathfrak{B}$ that forms a $\mathbb{Z}$-basis of $K_0(\mathfrak{B})$.*

The proof of the theorem consists of two steps:

First, we find a maximal $\mathcal{P}$-exceptional subcollection of weights among the weights λ_w constructed in 2.5. This is done by direct computations using (1). Then we modify remaining weights to fit in the collection, i.e. to satisfy (1), and to remain a basis. To guarantee the second we modify the weights using the following fact:

3.4. Let $\mathcal{B}$ be a basis of $\mathbb{Z}[\Lambda]$ over $\mathbb{Z}[\Lambda]^W$ and let $e^\lambda \in \mathcal{B}$ be such that for some W-invariant set $A = \{\lambda_1, \lambda_2, ..., \lambda_k\}$ we have $e^{\lambda+\lambda_i} \in \mathcal{B}$ for all $i < k$ and $e^{\lambda+\lambda_k} \notin \mathcal{B}$. Then the following set forms a basis

$$\mathcal{B} \cup \{e^{\lambda+\lambda_k}\} \setminus \{e^{\lambda+\lambda_1}\}.$$

Indeed, there is a decomposition

$$e^{\lambda+\lambda_k} = (e^{\lambda_1} + e^{\lambda_2} + \cdots + e^{\lambda_k})e^\lambda - e^{\lambda+\lambda_1} - e^{\lambda+\lambda_2} - \cdots - e^{\lambda+\lambda_{k-1}}$$

with the coefficients from $\mathbb{Z}[\Lambda]^W$ and an invertible coefficient at $e^{\lambda+\lambda_1}$.

For instance, for B_2 the rule says that if we have a square (shifted orbit of a fundamental weight) where the center and three vertices are the basis weights, then replacing one of these basis weigths by the missing vertex gives a basis. See the B_2-case for a detailed picture. In case of G_2 we use a hexagon (shifted orbit of a fundamental weight) instead of the square, where the center and all but one vertex are the basis weights.

Proof of the theorem. We use the following notation: a product of simple reflections $w = s_{i_1}s_{i_2}\cdot\ldots\cdot s_{i_k}$ is denoted by $[i_1, i_2, \ldots, i_k]$; the identity is denoted by $[]$. Given a presentation $\lambda = a_1\omega_1 + \ldots + a_n\omega_n$ in terms of fundamental weights we denote λ by $(a_1, \ldots, a_n)$. We denote by $\mathcal{P}$ the left weak Bruhat order on the Weyl group W.

A_2**-case:** The Weyl group W consists of the following elements:

$$W = \{[], [1], [2], [2,1], [1,2], [1,2,1]\}.$$

The respective basis weights $\{\lambda_w\}_{w\in W}$ of 2.5 are given by (here the i-th weight is indexed by the i-th element of the Weyl group):

$$\{(0,0), (-1,1), (1,-1), (-1,0), (0,-1), (-1,1)\}. \tag{2}$$

Direct computations using (1) show that $\{\lambda_w\}_{w\in W}$ is a $\mathcal{P}$-exceptional collection.

Using 3.4 we can modify the weights $\{\lambda_w\}$ to obtain the following exceptional collection of weights (there are no Ext's from left to the right):

$$\{(0,0), (-1,0).(-2,0), (1,-1), (0,-1), (-1,-1)\}. \tag{3}$$

B_2 **or** C_2**-case:** The Weyl group W consists of the following elements:

$$W = \{[], [1], [2], [2,1], [1,2], [1,2,1], [2,1,2], [1,2,1,2]\}.$$

The respective basis weights $\{\lambda_w\}_{w\in W}$ of 2.5 are given by:

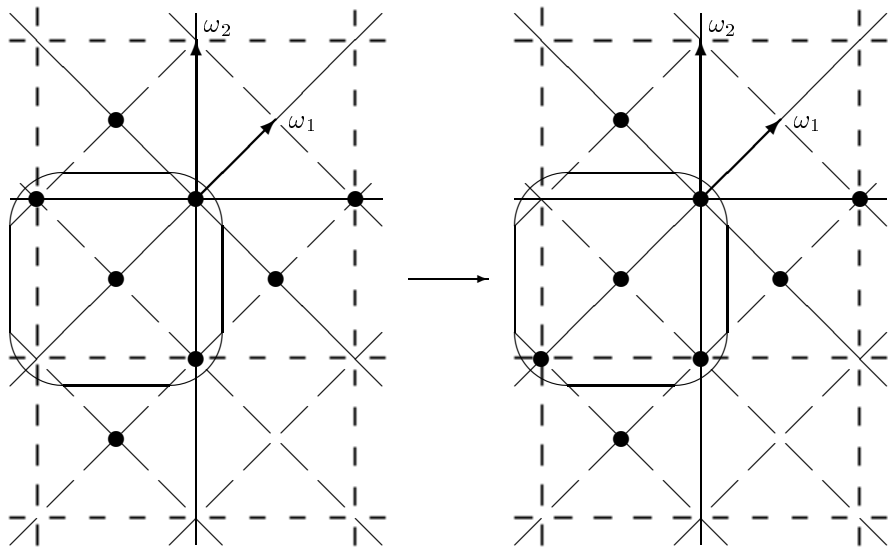
$$\{(0,0), (-1,1), (2,-1), (-2,1), (1,-1), (-1,0), (0,-1), (-1,-1)\}$$

(here $\omega_1 = (e_1 + e_2)/2$ and $\omega_2 = e_2$). Direct computations show that $\{\lambda_w\}_{w\in W}$ is a $\mathcal{P}$-exceptional collection except of the weight $\lambda_{[2,1]} = (-2,1)$: indeed, the property (1) fails only for the weights 0 and $(-2,1)$, i.e. $\lambda_{[2,1]} + \rho$ is regular or, equivalently, $[1] * \lambda_{[2,1]} = 0$ is dominant.

We modify the basis weights using 3.4: An element $e^{(-2,0)}$ has the following representation with respect to the initial basis:

$$e^{(-2,0)} = (e^{(1,0)} + e^{(1,-1)} + e^{(-1,1)} + e^{(-1,0)})e^{(-1,0)} - e^{(0,0)} - e^{(-2,1)} - e^{(0,-1)}$$

where $(e^{(1,0)} + e^{(1,-1)} + e^{(-1,1)} + e^{(-1,0)}) \in \mathbb{Z}[\Lambda]^W$. Hence, we can substitute $e^{(-2,1)}$ by $e^{(-2,0)}$. The following picture illustrates our arguments (thick points represent the basis, solid lines are the walls of Weyl chambers):



Finally, after reindexing we obtain a $\mathcal{P}$-exceptional collection

$$\{(0,0),(-1,1),(2,-1),(-1,0),(1,-1),(-2,0),(0,-1),(-1,-1)\}. \tag{4}$$

Repeating 3.4 we obtain the following exceptional collection of weights (no Ext's from left to the right)

$$\{(1,0),(0,0),(-1,0),(-2,0),(2,-1),(1,-1),(0,-1),(-1,-1)\}. \tag{5}$$

G_2**-case:** The Weyl group W consists of the following 12 elements:

$$W = \{[],[1],[2],[2,1],[1,2],[1,2,1],[2,1,2],[2,1,2,1],[1,2,1,2],$$
$$[1,2,1,2,1],[2,1,2,1,2],[1,2,1,2,1,2]\}.$$

The respective basis weights $\{\lambda_w\}_{w\in W}$ of 2.5 are given by:

$$\{(0,0),(-1,1),(3,-1),(-3,2),(2,-1),(-2,1),(3,-2),$$
$$(-3,1),(1,-1),(-1,0),(0,-1),(-1,-1)\}.$$

Using 3.4 we obtain the following $\mathcal{P}$-exceptional collection

$$\{(0,0),(-1,1),(3,-1),(-3,1),(2,-1),(-1,0),(1,-1),$$
$$(-2,0),(0,-1),(-3,0),(2,-2),(-1,-1)\}. \tag{6}$$

Indeed, the difference between the initial basis and the modified one consists in the substitution of $e^{(-3,2)}, e^{(-2,1)}, e^{(3,-2)}$ by $e^{(-3,0)}, e^{(-2,0)}, e^{(2,-2)}$. We proceed in several steps using the same reasoning as in the B_2-case. Denote

$$A = e^{(1,0)} + e^{(2,-1)} + e^{(1,-1)} + e^{(-1,0)} + e^{(-2,1)} + e^{(-1,1)},$$
$$B = e^{(0,1)} + e^{(3,-1)} + e^{(3,-2)} + e^{(0,-1)} + e^{(-3,1)} + e^{(-3,2)},$$

the sums of the elements corresponding to the orbits of the fundamental weights. Note that $A, B \in \mathbb{Z}[\Lambda]^W$. We have the following decompositions with respect to the initial basis (coefficients belong to $\mathbb{Z}[\Lambda]^W$):

$$e^{(2,-2)} = Ae^{(1,-1)} - e^{(0,0)} - e^{(2,-1)} - e^{(3,-2)} - e^{(0,-1)} - e^{(-1,0)},$$

$$e^{(-4,2)} = Ae^{(-2,1)} - e^{(0,0)} - e^{(-1,0)} - e^{(-3,1)} - e^{(-3,2)} - e^{(-1,1)}.$$

Hence we can substitute $e^{(2,-2)}$ for $e^{(3,-2)}$ and $e^{(-4,2)}$ for $e^{(-3,2)}$. Using the decomposition

$$e^{(-2,0)} = Ae^{(-1,0)} - e^{(0,0)} - e^{(1,-1)} - e^{(-1,0)} - e^{(-3,1)} - e^{(-2,1)}$$

we substitute $e^{(-2,0)}$ for $e^{(-2,1)}$. Then, using

$$e^{(-4,1)} = Be^{(-1,0)} - e^{(-4,2)} - e^{(-1,1)} - e^{(2,-1)} - e^{(2,-2)} - e^{(-1,-1)}$$

substitute $e^{(-4,1)}$ for $e^{(-4,2)}$. At last, using

$$e^{(-3,0)} = Ae^{(-2,0)} - e^{(-1,0)} - e^{(0,-1)} - e^{(-1,-1)} - e^{(-4,1)} - e^{(-3,1)}$$

we substitute $e^{(-3,0)}$ for $e^{(-4,1)}$, obtaining the required basis.

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