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# INNER MAPPINGS AND HYPERBOLIC GRADIENTS

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Abstract

Let $\varphi$ be a holomorphic mapping between complex unit balls. We obtain quantitative versions of the following heuristic principle: if the hyperbolic gradient of $\varphi$ does not grow sufficiently rapidly, then $\varphi$ is far from being inner.

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## 1. Introduction

Let $B_n$ denote the unit ball of $\mathbb{C}^n$, $n \geq 1$. Let $\sigma_n$ denote Lebesgue measure on the unit sphere $\partial B_n$; $\sigma_n(\partial B_n) = 1$. Given a holomorphic mapping $\varphi : B_n \to B_m$, $m, n \in \mathbb{N}$, the radial limit $\varphi^*(\zeta)$ is defined at $\sigma_n$-almost every point of $\partial B_n$. If $|\varphi^*| = 1$ $\sigma_n$-a.e., then $\varphi$ is called *inner*.

1.1. **Holomorphic self-mappings of the unit disc.** The starting point for this paper is the following result for $n = m = 1$:

**Theorem 1.1** ([12, Theorem 1.1], [1, Theorem 5.1])**.**

(i) *Assume that $\omega$ is an increasing function on $[0, 1]$, $\omega(0) = 0$, and*

$$\int_0 \frac{\omega^2(t)}{t}\, dt < \infty.$$

*Let $\varphi : B_1 \to B_1$ be a holomorphic function such that*

$$\frac{|\varphi'(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2} \leq \omega(1 - |z|) \quad \textit{for all } z \in B_1. \tag{1.1}$$

*Then $\varphi$ is not inner.*

(ii) *Assume that $\omega$ is an increasing function on $[0, 1]$, $\omega(0) = 0$, $\omega(t)/t^{1-\varepsilon}$ decreases for some $\varepsilon > 0$, and*

$$\int_0 \frac{\omega^2(t)}{t}\, dt = \infty.$$

*Then there exists an inner function $\varphi : B_1 \to B_1$ such that*

$$\frac{|\varphi'(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2} \leq \omega(1 - |z|) \quad \textit{for all } z \in B_1.$$

The above result is sharp, in a sense. However, by [7, Theorem 1.9(a)], the following improvement of Theorem 1.1(i) holds.

**Theorem 1.2** (see [7, p. 687])**.** *If $\omega$ and $\varphi$ satisfy the assumptions of Theorem 1.1(i), then $|\varphi^*| < 1$ $\sigma$-a.e.*

The quotient

$$\frac{|\varphi'(z)|}{1 - |\varphi(z)|^2}$$

is often called the hyperbolic derivative of $\varphi$. So, Theorems 1.1(i) and 1.2 are precise formulations of the following heuristic principle:

*If the hyperbolic derivative of $\varphi$ does not grow sufficiently rapidly, then $\varphi$ is far from being inner.*

Also, different motivations for Theorems 1.1 and 1.2 are given in [12, 1, 7].

In this paper, we improve Theorem 1.2.

**Theorem 1.3.** *If $\omega$ and $\varphi$ satisfy the assumptions of Theorem 1.1(i), then*

$$\int_{\partial B_1} \left( \frac{1}{1-|\varphi^*(\zeta)|^2} \right)^K d\sigma_1(\zeta) < \infty \tag{1.2}$$

*for all $K > 0$.*

It is interesting to remark that, by [8, Corollary 3.25], condition (1.2) holds for all $K > 0$ if and only if the composition operator $C_\varphi : f \mapsto f \circ \varphi$ is compact on the Hardy–Orlicz space $H^{\Psi_2}(B_1)$, where $\Psi_2(x) = e^{x^2} - 1$.

Theorem 1.3 suggests that the conclusion of Theorem 1.1(i) remains true when the hyperbolic factor $(1 - |\varphi(z)|^2)^{-1}$ in estimate (1.1) is replaced by a smaller function of $(1 - |\varphi(z)|^2)$. A quantitative version of this observation is formulated as Corollary 5.2.

1.2. **Holomorphic mappings between complex unit balls.** In fact, Theorem 1.3 extends to the holomorphic mappings $\varphi : B_n \to B_m$ with arbitrary $n, m \in \mathbb{N}$. So, the main result of the present paper is the following theorem:

**Theorem 1.4.** *Assume that $\omega$ is an increasing function on $[0, 1]$, $\omega(0) = 0$, and*

$$\int_0 \frac{\omega^2(t)}{t}\, dt < \infty.$$

*Let $n, m \in \mathbb{N}$ and let $\varphi : B_n \to B_m$ be a holomorphic mapping such that*

$$\frac{|\nabla\varphi(z)|(1-|z|^2)}{1-|\varphi(z)|^2} \le \omega(1-|z|), \quad z \in B_n, \tag{1.3}$$

*where $|\nabla\varphi(z)|^2 = \sum_{j=1}^n \left|\frac{\partial\varphi}{\partial z_j}(z)\right|^2$. Then*

$$\int_{\partial B_n} \left( \frac{1}{1-|\varphi^*(\zeta)|^2} \right)^K d\sigma_n(\zeta) < \infty$$

*for all $K > 0$.*

**Organization of the paper.** The main idea behind the proof of Theorem 1.4 is to consider the composition operator $C_\varphi$ from the Bloch space $\mathcal{B}(B_m)$ into a space $X$ that is sufficiently close to $H^\infty(B_n)$. As a model, we consider the case $X = \mathrm{BMOA}(B_n)$ in Section 2. Theorem 1.4 is proved in Section 3. Non-hyperbolic generalizations of Theorems 1.4 and 1.1(i) are obtained in Sections 4 and 5, respectively.

## 2. A model: Bloch-to-BMOA compositions

2.1. **The Bloch space.** Let $H(B_m)$ denote the space of holomorphic functions on $B_m$. The Bloch space $\mathcal{B}(B_m)$ consists of those functions $f \in H(B_m)$ for which

$$\|f\|_{\mathcal{B}(B_m)} = |f(0)| + \sup_{w \in B_m} |\mathcal{R}f(w)|(1-|w|^2) < \infty,$$

where

$$\mathcal{R}f(w) = \sum_{j=1}^{m} w_j \frac{\partial f}{\partial w_j}(w), \quad w \in B_m,$$

is the radial derivative of $f$. Let

$$\nabla f(w) = \left( \frac{\partial f}{\partial w_1}(w), \dots, \frac{\partial f}{\partial w_m}(w) \right), \quad w \in B_m,$$

denote the complex gradient. Observe that

$$|f(0)| + \sup_{w \in B_m} |\nabla f(w)|(1 - |w|^2)$$

is an equivalent norm on $\mathcal{B}(B_m)$; see, for example, [13, Theorem 3.4].

2.2. **Carleson measures and** BMOA**.** For $\zeta \in \partial B_n$ and $r > 0$, put

$$B(\zeta, r) = \{z \in B_n : \ |1 - \langle z, \zeta \rangle|^{\frac{1}{2}} < r\};$$
$$Q(\zeta, r) = \{\xi \in \partial B_n : \ |1 - \langle \xi, \zeta \rangle|^{\frac{1}{2}} < r\}.$$

A positive measure $\mu$ on $B_n$ is called a Carleson measure if there exists a constant $C > 0$ such that $\mu(B(\zeta, r)) \le C r^{2n}$ for all $\zeta \in \partial B_n$.

Let $\nu_n$ denote Lebesgue measure on $B_n$; $\nu_n(B_n) = 1$. Consider a function $f \in H(B_n)$. By [13, Theorem 5.19], $f \in \mathrm{BMOA}(B_n)$ if and only if

$$|\mathcal{R}f(z)|^2 (1 - |z|^2) \, d\nu_n(z)$$

is a Carleson measure.

2.3. **Bloch-to-BMOA composition operators.** Given a holomorphic mapping $\varphi = (\varphi_1, \dots, \varphi_m) : B_n \to B_m$, put $\mathcal{R}\varphi = (\mathcal{R}\varphi_1, \dots, \mathcal{R}\varphi_m)$, where $\mathcal{R}\varphi_j : B_n \to \mathbb{C}$ is the radial derivative of $\varphi_j$, $j = 1, \dots, m$. Consider the function

$$\Omega_{\varphi, \mathcal{R}}(t) = \max_{|z| = 1 - t} \frac{|\mathcal{R}\varphi(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2}, \quad 0 < t \le 1.$$

We need a result about the composition operator $C_\varphi : f \mapsto f \circ \varphi$.

**Proposition 2.1** ([5, Proposition 4.1])**.** *Let $\varphi : B_n \to B_m$ be a holomorphic mapping. Assume that the composition operator $C_\varphi$ maps $\mathcal{B}(B_m)$ to* BMOA$(B_n)$*. Then there exist constants $\varepsilon = \varepsilon(n, m, \|C_\varphi\|_{\mathcal{B} \to \mathrm{BMOA}}) > 0$ and $C = C(n) > 0$ such that*

$$\int_{\partial B_n} \exp \sqrt{\varepsilon \log \frac{1}{1 - |\varphi^*(\zeta)|^2}} \, d\sigma_n(\zeta) \le C.$$

The following assertion is a model for Theorem 1.4.

**Corollary 2.2.** *Let $n, m \in \mathbb{N}$ and let $\varphi : B_n \to B_m$ be a holomorphic mapping. Assume that*

$$\int_0 \frac{\Omega^2_{\varphi,\mathcal{R}}(t)}{t}\, dt < \infty. \tag{2.1}$$

*Then there exists a constant $\varepsilon = \varepsilon(n, m, \Omega_{\varphi,\mathcal{R}}) > 0$ such that*

$$\int_{\partial B_n} \exp \sqrt{\varepsilon \log \frac{1}{1 - |\varphi^*(\zeta)|^2}}\, d\sigma_n(\zeta) < \infty.$$

*Proof.* Let $f \in \mathcal{B}(B_m)$, that is,

$$|\nabla f(w)|(1 - |w|^2) \le C, \quad w \in B_m.$$

We have

$$\mathcal{R}(f \circ \varphi)(z) = \sum_{k=1}^{m} \frac{\partial f}{\partial w_k}(\varphi(z)) \cdot \mathcal{R}\varphi_k(z), \quad z \in B_n.$$

Therefore,

$$|\mathcal{R}(f \circ \varphi)(z)| \le |\nabla f(\varphi(z))||\mathcal{R}\varphi(z)| \le C \frac{|\mathcal{R}\varphi(z)|}{1 - |\varphi(z)|^2}, \quad z \in B_n.$$

For $\zeta \in \partial B_n$, put $\Pi(\zeta, r) = \{\rho\xi : \ 1 - r < \rho < 1 \text{ and } \xi \in Q(\zeta, r)\}$. Then $B(\zeta, r) \subset \Pi(\zeta, Cr)$ for an absolute constant $C > 0$. Also, we have $\sigma_n(Q(\zeta, r)) \le Cr^{2n}$. Therefore, (2.1) guarantees that

$$|\mathcal{R}(f \circ \varphi)(z)|^2 (1 - |z|^2)\, d\nu_n(z)$$

is a Carleson measure. Thus, $f \circ \varphi \in \mathrm{BMOA}(B_n)$. It remains to apply Proposition 2.1. □

**Corollary 2.3** (cf. Theorem 1.1)**.** *Let $m \in \mathbb{N}$.*

(i) *Assume that* (2.1) *holds for a holomorphic mapping $\varphi : B_1 \to B_m$. Then $\varphi$ is not inner.*

(ii) *Assume that $\omega$ is an increasing function on $[0, 1]$, $\omega(0) = 0$, $\omega(t)/t^{1-\varepsilon}$ decreases for some $\varepsilon > 0$, and*

$$\int_0 \frac{\omega^2(t)}{t}\, dt = \infty.$$

*Then there exists an inner mapping $\varphi : B_1 \to B_m$ such that $\Omega_{\varphi,\mathcal{R}}(t) \le \omega(t)$ for all $t \in (0, 1]$.*

*Proof.* Part (i) follows from Corollary 2.2. To prove part (ii), it suffices to consider the mapping $(\varphi, 0, \dots, 0) : B_1 \to B_m$, where the inner function $\varphi : B_1 \to B_1$ is provided by Theorem 1.1(ii). □

2.4. **Mappings into the unit disc.** Consider a holomorphic mapping $\varphi : B_n \to B_1$. If (2.1) holds, then $\varphi$ is not inner by Corollary 2.2. However, for $n \geq 2$, the inner mappings $\varphi : B_n \to B_1$ are known to be very rare and singular objects. So, the same conclusion remains true when $\Omega_{\varphi,\mathcal{R}}$ is replaced by its non-hyperbolic analog. Namely, put

$$\omega_{\varphi,\mathcal{R}}(t) = \max_{|z|=1-t} |\mathcal{R}\varphi(z)|(1-|z|^2), \quad 0 < t \leq 1.$$

**Proposition 2.4.** *Let $\varphi : B_n \to B_1$ be a holomorphic mapping. Assume that $n \geq 2$ and*

$$\int_0 \frac{\omega^2_{\varphi,\mathcal{R}}(t)}{t}\, dt < \infty. \tag{2.2}$$

*Then $\varphi$ is not inner.*

*Proof.* Recall that a finite positive measure $\mu$ on $B_n$ is called a vanishing Carleson measure if

$$\lim_{r\to 0+} \frac{\mu(B(\zeta, r))}{r^{2n}} = 0$$

uniformly on $\zeta \in \partial B_n$. We argue as in the proof of Corollary 2.2. So, (2.2) guarantees that

$$|\mathcal{R}\varphi(z)|^2(1-|z|^2)\, d\nu_n(z)$$

is a vanishing Carleson measure. Thus, $\varphi \in \mathrm{VMOA}(B_n)$ by [13, Theorem 5.19]. Recall that $n \geq 2$, hence, $\varphi$ is not inner by [10, Corollary 3.3]. □

By Theorem 1.1(ii), condition (2.1) is sharp for the class of the inner mappings $\varphi : B_1 \to B_1$. Clearly, there are inner mappings $\varphi : B_1 \to B_1$ with property (2.2). In fact, if $n = 1$ and (2.2) holds, then $\varphi$ is inner if and only if $\varphi$ is a finite Blaschke product (see, for example, [1, Remark after Theorem 5.1]).

The author is not aware whether conditions (2.1) and (2.2) are sharp for the class of the inner mappings $\varphi : B_n \to B_1$ when $n \geq 2$. Note that $\Omega_{\varphi,\mathcal{R}}$ is bounded by the Schwarz–Pick lemma. Clearly, we have $\omega_{\varphi,\mathcal{R}} \leq \Omega_{\varphi,\mathcal{R}}$. Finally, by [3, Corollary 3.1], there exist inner mappings $\varphi : B_n \to B_1$ such that $\Omega_{\varphi,\mathcal{R}}(t) \to 0$ as $t \to 0+$.

## 3. Proof of Theorem 1.4

We need two auxiliary lemmas.

### 3.1. A reverse estimate for Bloch functions.

**Lemma 3.1** ([5, Lemma 3.2])**.** *Let $m \in \mathbb{N}$ and let $0 < p < \infty$. Then there exist constants $J = J(m) \in \mathbb{N}$, $\tau_{m,p} > 0$ and there exist functions $F_{j,x} \in \mathcal{B}(B_m)$, $1 \leq j \leq J$, $0 \leq x \leq 1$, such that $\|F_{j,x}\|_{\mathcal{B}(B_m)} \leq 1$ and*

$$\sum_{j=1}^{J} \int_0^1 |F_{j,x}(w)|^p\, dx \geq \tau_{m,p} \left(\log \frac{1}{1-|w|^2}\right)^{\frac{p}{2}}$$

*for all $w \in B_m$.*

It is worth to mention that $J(1) = 1$ in the above lemma.

3.2. **Exponential integrals.** In this subsection, we use dyadic martingales on arcs $[\zeta, \xi) \subset \partial B_1$. Given an arc $I_0 \subset \partial B_1$, let $\mathcal{F}_k$ be the $\sigma$-algebra generated by the dyadic subarcs $I_k \subset I_0$, $\sigma_1(I_k) = 2^{-k}\sigma_1(I_0)$. For a real dyadic martingale $M = (M_k, \mathcal{F}_k)$, the square function $SM$ is defined by the identity

$$SM = \sum_{k=1}^{\infty} (\Delta M_k)^2,$$

where $\Delta M_k = M_k - M_{k-1}$. We need the following result:

**Theorem 3.2** (see [2])**.** *Let $(M_k, \mathcal{F}_k)$ be a real dyadic martingale on an arc $I_0 \subset \partial B_1$. Assume that $D > 0$ and*

$$\|SM\|_\infty < \frac{1}{4D}.$$

*Then the limit $M_\infty(\zeta) = \lim_{k\to\infty} M_k(\zeta)$ exists at $\sigma_1$-almost every point $\zeta \in I_0$ and*

$$\frac{1}{\sigma_1(I_0)} \int_{I_0} \exp\left(D(M_\infty - M_0)^2(\zeta)\right) d\sigma_1(\zeta) \le C < \infty,$$

*where $C > 0$ is an absolute constant.*

The proof of the following lemma is based on certain ideas from [6].

**Lemma 3.3.** *Let $g \in H(B_1)$. Assume that*

$$|g'(z)|(1 - |z|^2) \le \omega(1 - |z|) \quad \text{for all } z \in B_1,$$

*where $\omega$ is an increasing function on $[0, 1]$, $\omega(0) = 0$, and*

$$\int_0 \frac{\omega^2(t)}{t}\, dt < \infty.$$

*Let $A > 0$. Then there exists a constant $C = C(A, \omega, |g(0)|) < \infty$ such that*

$$\int_{\partial B_1} \exp\left(A|g(r\zeta)|^2\right) d\sigma_1(\zeta) \le C$$

*for all $0 \le r < 1$.*

*Proof.* We have

$$\int_0^1 |g'(r\zeta)|^2(1 - r^2)\, dr \le \int_0^1 \frac{\omega^2(1 - r)}{1 - r^2}\, dr < \infty$$

for all $\zeta \in \partial B_1$. Thus,

$$\int_{\partial B_1} \left( \int_0^1 |g'(r\zeta)|^2(1 - r^2)\, dr \right)^{\frac{p}{2}} d\sigma_1(\zeta) \le C(\omega) < \infty$$

for all $p < \infty$. Hence, $\|g\|_{H^p(B_1)} \le C(p, \omega, |g(0)|)$, where $H^p(B_1)$ denotes the Hardy space (see, for example, [14]). In particular, $\|g\|_{H^1(B_1)} \le C(\omega, |g(0)|)$. So, the boundary values $g^*(\zeta)$ are defined $\sigma_1$-a.e., $g^* \in L^1(\partial B_1)$, and $g = P[g^*]$, where $P[g^*]$ is the Poisson integral of $g^*$.

Now, consider the function $u = \operatorname{Re} g : B_1 \to \mathbb{R}$. Note that

$$|\nabla u(z)|(1-|z|^2) \le \omega(1-|z|).$$

Recall that $\omega$ is increasing on $[0,1]$. So, by Green's formula, there exists an absolute constant $C > 0$ such that

$$\left| \int_{I_+} u^*(\zeta)\, d\sigma_1(\zeta) - \int_{I_-} u^*(\zeta)\, d\sigma_1(\zeta) \right| \le C\sigma_1(I_\pm)\omega(\sigma_1(I_\pm)) \tag{3.1}$$

for any adjacent arcs $I_+, I_- \subset \partial B_1$ such that $\sigma_1(I_+) = \sigma_1(I_-)$ (see, for example, the proof of the implication (b)$\Rightarrow$(a) from [6, Theorem 1.1]).

Let $I_0 \subset \partial B_1$ be an arc. Consider the dyadic martingale $M$ generated on $I_0$ by the function $u^*(\zeta)$. Namely, put

$$M_k|_{I_k} = \frac{1}{\sigma_1(I_k)} \int_{I_k} u^*(\zeta)\, d\sigma_1(\zeta),$$

where $I_k$ is a dyadic arc of rank $k$, $k \ge 0$. By Lebesgue's differentiation theorem, the limit $M_\infty(\zeta) = \lim_{k\to\infty} M_k(\zeta)$ exists at $\sigma_1$-almost every point $\zeta \in I_0$; moreover, $M_\infty = u^*$ $\sigma_1$-a.e.

Estimate (3.1) guarantees that

$$SM(\zeta) = \sum_{k=1}^{\infty} (M_k(\zeta) - M_{k-1}(\zeta))^2 \le C \sum_{k=1}^{\infty} \omega^2(2^{-k}\sigma_1(I_0)) \quad \text{for all } \zeta \in I_0.$$

Since $\omega$ is increasing, we have

$$\sum_{k=1}^{\infty} \omega^2(2^{-k}\sigma_1(I_0)) \le C \int_0^{\sigma_1(I_0)} \frac{\omega^2(t)}{t}\, dt.$$

Hence, there exists $n_0 \in \mathbb{N}$ so large that

$$\|SM\|_\infty < \frac{1}{16A}$$

for all $I_0$ with $\sigma_1(I_0) \le \frac{1}{n_0}$. Fix such a number $n_0 = n_0(A, \omega)$.

Now, let $I_0 \subset \partial B_1$ be an arbitrary arc such that $\sigma_1(I_0) = \frac{1}{n_0}$. By Theorem 3.2, we have

$$\frac{1}{\sigma_1(I_0)} \int_{I_0} \exp\left(4A(\operatorname{Re} g^*(\zeta) - M_0)^2\right)\, d\sigma_1(\zeta) \le C. \tag{3.2}$$

Also, note that

$$\begin{aligned} M_0 = \frac{1}{\sigma_1(I_0)} \int_{I_0} \operatorname{Re} g^*(\zeta)\, d\sigma_1(\zeta) &\le \frac{\|g\|_{H^1(B_1)}}{\sigma_1(I_0)} \\ &\le n_0 C(\omega, |g(0)|) \le C(A, \omega, |g(0)|). \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we obtain

$$\frac{1}{\sigma_1(I_0)} \int_{I_0} \exp\left(2A(\operatorname{Re} g^*(\zeta))^2\right)\, d\sigma_1(\zeta) \le C(A, \omega, |g(0)|).$$

Analogously, we have

$$\frac{1}{\sigma_1(I_0)}\int_{I_0}\exp\left(2A(\operatorname{Im} g^*(\zeta))^2\right)\,d\sigma_1(\zeta)\le C(A,\omega,|g(0)|).$$

Note that $\exp(A(a^2+b^2))\le\exp(2Aa^2)+\exp(2Ab^2)$ for all $a,b\in\mathbb{R}$. Therefore,

$$n_0\int_{I_0}\exp\left(A|g^*(\zeta)|^2\right)\,d\sigma_1(\zeta)\le C(A,\omega,|g(0)|).$$

Thus,

$$\int_{\partial B_1}\exp\left(A|g^*(\zeta)|^2\right)\,d\sigma_1(\zeta)\le C(A,\omega,|g(0)|).$$

Finally, recall that

$$g(r\zeta)=\int_{\partial B_1}g^*(\xi)\,d\mu_{r\zeta}(\xi),\quad 0\le r<1,\ \zeta\in\partial B_1,$$

where

$$\mu_{r\zeta}(\xi)=\frac{1-r^2}{|r\zeta-\xi|^2}\,d\sigma_1(\xi),\quad \xi\in\partial B_1,$$

is the Poisson kernel for the point $r\zeta\in B_1$. So, by Jensen's inequality,

$$\begin{aligned}\int_{\partial B_1}\exp(A|g(r\zeta)|^2)\,d\sigma_1(\zeta)&\le\int_{\partial B_1}\exp\left(A\left(\int_{\partial B_1}|g^*(\xi)|\,d\mu_{r\zeta}(\xi)\right)^2\right)\,d\sigma_1(\zeta)\\&\le\int_{\partial B_1}\int_{\partial B_1}\exp\left(A|g^*(\xi)|^2\right)\,d\mu_{r\zeta}(\xi)\,d\sigma_1(\zeta)\\&=\int_{\partial B_1}\exp\left(A|g^*(\zeta)|^2\right)\,d\sigma_1(\zeta)\\&\le C(A,\omega,|g(0)|)\end{aligned}$$

for all $0\le r<1$. □

3.3. **Proof of Theorem 1.4.** If $\varphi:B_n\to B_m$ is a holomorphic mapping, then

$$|\mathcal{R}\varphi(z)|\le|\nabla\varphi(z)|,\quad z\in B_n.$$

So, the following assertion implies Theorem 1.4.

**Theorem 3.4.** *Let $n,m\in\mathbb{N}$ and let $\varphi:B_n\to B_m$ be a holomorphic mapping such that*

$$\frac{|\mathcal{R}\varphi(z)|(1-|z|^2)}{1-|\varphi(z)|^2}\le\omega(1-|z|),\quad z\in B_n,\tag{3.4}$$

*where $\omega$ satisfies the assumptions of Lemma 3.3. Then*

$$\int_{\partial B_n}\left(\frac{1}{1-|\varphi^*(\zeta)|^2}\right)^K\,d\sigma_n(\zeta)<\infty$$

*for all $K>0$.*

*Proof.* Let the constant $\tau = \tau_{m,2} > 0$ and the functions $F_{j,x} \in \mathcal{B}(B_m)$, $1 \le j \le J$, $0 \le x \le 1$, be those provided by Lemma 3.1 with $q = 2$. Note that

$$|\nabla F_{j,x}(w)|(1-|w|^2) \le C, \quad w \in B_m,\ 1 \le j \le J,\ 0 \le x \le 1. \tag{3.5}$$

Fix a point $\zeta \in \partial B_n$. Consider the slice-mapping $\varphi_\zeta(\lambda) = \varphi(\lambda\zeta)$, $\lambda \in B_1$. Observe that

$$(F_{j,x} \circ \varphi_\zeta)'(\lambda) = \sum_{k=1}^{m} \frac{\partial F_{j,x}}{\partial w_k}(\varphi_\zeta(\lambda)) \cdot (\varphi_\zeta)'_k(\lambda).$$

We have $\lambda(\varphi_\zeta)'_k(\lambda) = (\mathcal{R}\varphi_k)(\lambda\zeta)$, hence,

$$\begin{aligned}
|\lambda||(F_{j,x} \circ \varphi_\zeta)'(\lambda)|(1-|\lambda|^2) &\le |\nabla F_{j,x}(\varphi_\zeta(\lambda))||\mathcal{R}\varphi(\lambda\zeta)|(1-|\lambda|^2) \\
&\le C\frac{|\mathcal{R}\varphi(\lambda\zeta)|(1-|\lambda\zeta|^2)}{1-|\varphi(\lambda\zeta)|^2} \\
&\le C\omega(1-|\lambda|), \quad \lambda \in B_1,
\end{aligned}$$

by (3.5) and (3.4). So, Lemma 3.3 is applicable to the function $F_{j,x} \circ \varphi_\zeta \in H(B_1)$. Recall that $\|F_{j,x}\|_{\mathcal{B}(B_m)} \le 1$, thus, $|(F_{j,x} \circ \varphi_\zeta)(0)|$ is estimated by a constant $C = C(\varphi(0))$. Therefore, Lemma 3.3 guarantees that

$$C = C(A, \omega, \varphi(0)) \ge \int_{\partial B_1} \exp\left(A|F_{j,x} \circ \varphi_\zeta(r\xi))|^2\right) d\sigma_1(\xi)$$

for all $1 \le j \le J$, $0 \le x \le 1$, $0 \le r < 1$. So, we have

$$C \ge \frac{1}{J}\sum_{j=1}^{J}\int_0^1 \int_{\partial B_1} \exp\left(A|F_{j,x} \circ \varphi_\zeta(r\xi))|^2\right) d\sigma_1(\xi)\, dx$$

for all $0 \le x \le 1$, $0 \le r < 1$. Now, Fubini's theorem, Jensen's inequality and Lemma 3.1 with $q = 2$ guarantee that

$$\begin{aligned}
C &\ge \int_{\partial B_1} \frac{1}{J}\sum_{j=1}^{J}\int_0^1 \exp\left(A|F_{j,x} \circ \varphi_\zeta(r\xi))|^2\right) dx\, d\sigma_1(\xi) \\
&\ge \int_{\partial B_1} \exp\left(\frac{A}{J}\sum_{j=1}^{J}\int_0^1 A|F_{j,x} \circ \varphi_\zeta(r\xi))|^2\, dx\right) d\sigma_1(\xi) \\
&\ge \int_{\partial B_1} \exp\left(\frac{A\tau}{J}\log\frac{1}{1-|\varphi_\zeta(r\xi)|^2}\right) d\sigma_1(\xi) \\
&= \int_{\partial B_1} \left(\frac{1}{1-|\varphi_\zeta(r\xi)|^2}\right)^{A\tau/J} d\sigma_1(\xi)
\end{aligned}$$

for all $0 \le r < 1$. Note that the above constant $C = C(A, \omega, \varphi(0))$ does not depend on $\zeta \in \partial B_n$. So, putting $A = KJ/\tau$ and integrating by slices, we obtain

$$C \ge \sup_{0 \le r < 1} \int_{\partial B_n} \left( \frac{1}{1 - |\varphi(r\zeta)|^2} \right)^K d\sigma_n(\zeta).$$

By Fatou's lemma, the proof of Theorem 3.4 is complete. □

## 4. Theorem 1.4 and growth spaces

The following result shows that the conclusion of Theorem 1.1(ii) remains true when $(1 - |\varphi(z)|^2)^{-1}$ is replaced by an arbitrarily large function of $(1 - |\varphi(z)|^2)$.

**Theorem 4.1** ([1, Theorem 5.3])**.** *Assume that $\omega$ is an increasing function on $[0, 1]$, $\omega(0) = 0$, $\omega(t)/t^{1-\varepsilon}$ decreases for some $\varepsilon > 0$, and*

$$\int_0 \frac{\omega^2(t)}{t}\, dt = \infty.$$

*Let $v : (0, 1] \to (0, +\infty)$ be a continuous function, $v(0+) = 0$. Then there exists an inner function $\varphi : B_1 \to B_1$ such that*

$$\frac{|\varphi'(z)|(1 - |z|^2)}{v(1 - |\varphi(z)|^2)} \le \omega(1 - |z|) \quad \textit{for all } z \in B_1.$$

So, it is natural to look for stronger conclusions when the hyperbolic gradient

$$\frac{|\nabla \varphi(z)|}{1 - |\varphi(z)|^2}$$

in estimate (1.3) is replaced, for example, by

$$\frac{|\nabla \varphi(z)|}{(1 - |\varphi(z)|^2)^\gamma}$$

for some $\gamma > 1$.

By analogy with Theorem 1.4, the above problem is closely related with properties of the composition operator $C_\varphi : f \mapsto f \circ \varphi$, $f \in \mathcal{A}^{-\beta}(B_m)$, $\beta > 0$. By definition, the growth space $f \in \mathcal{A}^{-\beta}(B_m)$ consists of those $f \in H(B_m)$ for which

$$\|f\|_{\mathcal{A}^{-\beta}(B_m)} = \sup_{z \in B_m} |f(z)|(1 - |z|^2)^\beta < \infty.$$

In a sense, the Bloch space $\mathcal{B}(B_m)$ is an extension of the scale $\{\mathcal{A}^{-\beta}(B_m)\}_{\beta > 0}$ to the end-point $\beta = 0$.

It is well known that $\|f\|_{\mathcal{A}^{-\beta}(B_m)}$ is equivalent to the following Bloch-type norm:

$$\|f\|_{\mathcal{B}^{\beta+1}(B_m)} = |f(0)| + \sup_{z \in B_m} |\nabla f(z)|(1 - |z|)^{\beta+1}.$$

The following lemma is a particular case of Lemma 1.2 from [4].

**Lemma 4.2.** *Given $\beta > 0$ and $m \in \mathbb{N}$, there exists $F_j \in \mathcal{A}^{-\beta}(B_m)$, $1 \le j \le J = J(m)$, such that*

$$\sum_{j=1}^{J} |F_j(z)| \ge \frac{1}{(1-|z|)^{\beta}}, \quad z \in B_m. \tag{4.1}$$

The role of the above lemma will be similar to that of Lemma 3.1.

**Theorem 4.3.** *Assume that $\omega$ is an increasing function on $[0,1]$, $\omega(0) = 0$, and*

$$\int_0 \frac{\omega^2(t)}{t}\, dt < \infty.$$

*Let $\alpha > 1$ and let $\varphi : B_n \to B_m$ be a holomorphic mapping such that*

$$\frac{|\mathcal{R}\varphi(z)|(1-|z|^2)}{(1-|\varphi(z)|^2)^{\alpha}} \le \omega(1-|z|), \quad z \in B_n.$$

*Then*

$$\int_{\partial B_n} \exp\left[\frac{K}{(1-|\varphi^*(\zeta)|^2)^{\alpha-1}}\right] d\sigma_n(\zeta) < \infty$$

*for all $K > 0$.*

*Proof.* Let the functions $F_j \in \mathcal{A}^{1-\alpha}(B_m)$, $1 \le j \le J$, be those provided by Lemma 4.2 with $\beta = \alpha - 1 > 0$. Note that

$$|\nabla F_j(w)|(1-|w|^2)^{\alpha} \le C, \quad w \in B_m,\ 1 \le j \le J. \tag{4.2}$$

Fix a point $\zeta \in \partial B_n$. We argue as in the proof of Theorem 3.4. So, given $A > 0$, we have

$$C = C(A, \omega, \varphi(0)) \ge \int_{\partial B_1} \exp\left(A|F_j \circ \varphi_\zeta(r\xi))|^2\right) d\sigma_1(\xi)$$

for all $1 \le j \le J$, $0 \le r < 1$ by (4.1), (4.2) and Lemma 3.3. Therefore,

$$C \ge \frac{1}{J}\sum_{j=1}^{J} \int_{\partial B_1} \exp\left(A|F_j \circ \varphi_\zeta(r\xi))|^2\right) d\sigma_1(\xi)$$

for all $0 \le r < 1$. Applying Jensen's inequality and Lemma 4.2, we have

$$\begin{aligned} C &\ge \int_{\partial B_1} \frac{1}{J}\sum_{j=1}^{J} \exp\left(A|F_j \circ \varphi_\zeta(r\xi))|^2\right) d\sigma_1(\xi) \\ &\ge \int_{\partial B_1} \exp\left(\frac{A}{J}\sum_{j=1}^{J} A|F_j \circ \varphi_\zeta(r\xi))|^2\right) d\sigma_1(\xi) \\ &= \int_{\partial B_1} \exp\left[\frac{A}{J(1-|\varphi_\zeta(r\xi)|^2)^{\alpha-1}}\right] d\sigma_1(\xi) \end{aligned}$$

for all $0 \le r < 1$. To finish the proof, put $A = KJ$, integrate by slices and apply Fatou's lemma. $\square$

## 5. Theorem 1.1(i) and inner functions

As mentioned in the introduction, it is natural to expect that the conclusion of Theorem 1.1(i) remains true when the hyperbolic factor $(1-|\varphi(z)|^2)^{-1}$ is replaced by a smaller function of $(1-|\varphi(z)|^2)$. To quantify this observation, assume that $\Omega, \omega : (0,1] \to (0,+\infty)$ are bounded measurable functions. By definition, $\Omega$ is called *admissible* if the implication

$$\int_0 \frac{\omega^2(t)}{t}\, dt < \infty \quad \Rightarrow \quad \varphi \text{ is not inner}$$

holds for all holomorphic functions $\varphi : B_1 \to B_1$ such that

$$\frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}\Omega(1-|\varphi(z)|^2) \le \omega(1-|z|^2) \quad \text{for all } z \in B_1. \tag{5.1}$$

For example, up to additional assumptions about $\omega$, Theorem 1.1(i) says that $\Omega \equiv 1$ is admissible.

To characterize the admissible functions, we use the following result, which is essentially obtained in [12].

**Theorem 5.1** (cf. [12, Theorem 1.1])**.** *Let $\Omega : (0,1] \to (0,+\infty)$ be a measurable function such that*

$$\int_0 \frac{\Omega^2(t)}{t}\, dt = \infty. \tag{5.2}$$

*If $\varphi : B_1 \to B_1$ is an inner function, then*

$$\int_{B_1} |\varphi'(z)|^2 \frac{\Omega^2(1-|\varphi(z)|^2)}{(1-|\varphi(z)|^2)^2}(1-|z|^2)\, d\nu_1(z) = \infty. \tag{5.3}$$

*Proof.* Recall that the Nevanlinna counting function is defined by the identity

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}(w)} \log\frac{1}{|z|}, \quad w \in B_1 \setminus \{\varphi(0)\}.$$

We have

$$\int_{B_1} |\varphi'(z)|^2 \frac{\Omega^2(1-|\varphi(z)|^2)}{(1-|\varphi(z)|^2)^2} \log\frac{1}{|z|}\, d\nu_1(z) = \int_{B_1} \frac{\Omega^2(1-|w|^2)}{(1-|w|^2)^2} N_\varphi(w)\, d\nu_1(w) \tag{5.4}$$

by a change of variable formula; see [11, p. 398].

Since $\varphi$ is inner, we have

$$N_\varphi(w) = \log\left|\frac{1-\overline{w}\varphi(0)}{w-\varphi(0)}\right|$$

for all $w \in B_1 \setminus \{\varphi(0)\}$ outside a set of logarithmic capacity zero, hence, for all $w \in B_1 \setminus E$ with $\nu_1(E) = 0$ (see [9]). Therefore,

$$CN_\varphi(w) \ge 1-|w|^2$$

for all $w \in B_1 \setminus E$. So, the right-hand side of (5.4) is infinite by (5.2). Hence, (5.3) holds. □

**Corollary 5.2.** *Let $\Omega : (0,1] \to (0,+\infty)$ be a bounded measurable function. Then $\Omega$ is admissible if and only if*

$$\int_0 \frac{\Omega^2(t)}{t}\,dt = \infty. \tag{5.5}$$

*Proof.* Let (5.5) hold. Assume that $\varphi : B_1 \to B_1$ is holomorphic, (5.1) holds, and

$$\int_0 \frac{\omega^2(t)}{t}\,dt < \infty.$$

Then

$$\int_{B_1} |\varphi'(z)|^2 \frac{\Omega^2(1-|\varphi(z)|^2)}{(1-|\varphi(z)|^2)^2}(1-|z|^2)\,d\nu_1(z) \le \int_{B_1} \frac{\omega^2(1-|z|^2)}{1-|z|^2}\,d\nu_1(z) < \infty.$$

Hence, by (5.5), Theorem 5.1 guarantees that $\varphi$ is not inner. In other words, $\Omega$ is admissible.

To prove the converse implication, assume that

$$\int_0 \frac{\Omega^2(t)}{t}\,dt < \infty.$$

Then (5.1) holds with $\omega = \Omega$ for the inner function $\varphi(z) = z$. So, $\Omega$ is not admissible. □

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