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ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

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LINEAR PROBLEMS ARISING IN THE STUDY OF GLOBAL SOLVABILITY OF A FREE BOUNDARY PROBLEM IN MAGNETOHYDRODYNAMICS

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ABSTRACT:

We consider linear problems arising in linearization of a free boundary problem in magnetohydrodynamics. Linear problem for velocity vector field has a slightly different form as usual as a consequence of the non-zero velocity of motion of the barycenter. Weighted estimates with the weight e^{at}, $a > 0$ for solutions of these problems in Sobolev norms are proved. On this base we intend to prove the global solvability for the free boundary problem in magnetohydrodynamics under the assumptions that the initial data are small and the initial position of the free boundary is close to the sphere. Here we formulate the theorem for the nonlinear problem, the paper with the detailed proof is in preparation.

Key words: magnetohydrodynamics, linear problem, weighted estimates, Sobolev norms, free boundary

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1 Introduction

We consider the free boundary problem governing the motion of a finite isolated mass of a viscous incompressible electrically conducting capillary liquid. It is assumed that the liquid is contained in a bounded variable domain Ω_{1t} whose boundary consists of two disjoint components: the free boundary Γ_t and a fixed surface Σ that ia also a boundary of a fixed domain D. The domain $\bar{D} \cup \Omega_{1t}$ is surrounded by a bounded vacuum region Ω_{2t}; we set $\Omega = \Omega_{1t} \cup \Gamma_t \cup \Omega_{2t}$; Ω is bounded by Σ and the exterior surface S. It is assumed that both S and Σ are perfect conductors, $\Gamma_t \cap S = \varnothing$, $\Gamma_t \cap \Sigma = \varnothing$. The problem consists of determination of Ω_{it} together with the functions $\boldsymbol{v}(x,t)$, $p(x,t)$ defined for $x \in \Omega_{1t}$ and $\boldsymbol{H}(x,t)$, $x \in \Omega_{1t} \cup \Omega_{2t}$ and satisfying the following system of equations, initial and boundary conditions:

$$
\begin{aligned}
&\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla\cdot T(\boldsymbol{v},p) - \nabla\cdot T_M(\boldsymbol{H}) = 0, \qquad \nabla\cdot\boldsymbol{v}(x,t) = 0, \quad x \in \Omega_{1t},\\
&\mu_1 \boldsymbol{H}_t + \alpha^{-1} rot rot \boldsymbol{H} - \mu_1 rot(\boldsymbol{v}\times\boldsymbol{H}) = 0, \qquad \nabla\cdot\boldsymbol{H}(x,t) = 0, \quad x \in \Omega_{1t}, \quad t > 0,\\
&rot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H}(x,t) = 0, \quad x \in \Omega_{2t},\\
&\boldsymbol{v}(x,t) = 0, \quad x \in \Sigma,\\
&T(\boldsymbol{v},p) + [T_M(\boldsymbol{H})])\boldsymbol{n} = \sigma\boldsymbol{n}\mathcal{H}, \quad \boldsymbol{V}_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x \in \Gamma_t,\\
&[\mu\boldsymbol{H}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad x \in \Gamma_t,\\
&\boldsymbol{H}(x,t)\cdot\boldsymbol{n}(x) = 0, \quad x \in S,\\
&\boldsymbol{H}(x,t)\cdot\boldsymbol{n}(x) = 0, \quad rot_\tau\boldsymbol{H} = 0, \quad x \in \Sigma,\\
&\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_{10}, \qquad \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x), \quad x \in \Omega_{10} \cup \Omega_{20}.
\end{aligned} \tag{1.1}
$$

We have used the following notation:

ν, α, σ are positive constants (the kinematic viscosity, conductivity, coefficient of the surface tension),

$\mathcal{H}$ is the doubled mean curvature of Γ_t,

$T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$ is the stress tensor,

$S(\boldsymbol{v}) = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)_{i,j=1,2,3}$ is the doubled rate-of-strain tensor,

$\boldsymbol{V}_n$ is the velocity of evolution of the surface Γ_t in the direction of the exterior normal $\boldsymbol{n}$ to Γ_t,

μ is a piecewise constant function (equals to $\mu_i > 0$ in Ω_{it}) - magnetic permeability,

$T_M(\boldsymbol{H}) = \mu(\boldsymbol{H}\otimes\boldsymbol{H} - \frac{1}{2}I|\boldsymbol{H}|^2)$ is the magnetic stress tensor.

Ω_{10} is a given initial configuration of the liquid,

$\partial\Omega_{10} = \Sigma \cup \Gamma_0$,

$[u] = u^{(1)} - u^{(2)}$ - jump of $u(x)$ on Γ_t, $u^{(i)} = u|_{x\in\overline{\Omega}_{it}}$.

Similar problem but without a rigid domain D is studied in the paper [1]. It is proved that for arbitrary $\Omega_{10} \subset \Omega$ and arbitrary initial data $\boldsymbol{v}_0(x)$, $\boldsymbol{H}_0(x)$ given in Ω_{10} and $\Omega_{10} \cup \Omega_{20}$, respectively, and satisfying natural compatibility conditions, this problem has a unique solution defined on a certain (small) time interval.

Our goal is to show that the problem (1.1) is uniquely solvable in an infinite time interval $t > 0$, under the additional assumptions that the initial data are small and the surface Γ_0 is close to a sphere. We also intend to prove that the velocity, pressure and the magnetic field tend to zero exponentially and Γ_t tends to a sphere when $t \to \infty$.

Now we make some auxiliary constructions. Let $\Omega_t = \Omega_{1t} \cup \bar{D}$ and let $|\Omega_t| = |\Omega_0| = \frac{4}{3}\pi R_0^3$. When we imagine that D is also filled with the liquid of the same density 1, then the barycenter of Ω_t is located at the point

$$\xi(t) = \frac{1}{|\Omega_0|} \int_{\Omega_t} x dx.$$

We assume that $\xi(0) = 0$. The velocity of the motion of the barycenter is

$$\xi'(t) = \frac{1}{|\Omega_0|} \frac{d}{dt} \int_{\Omega_t} x dx = \frac{1}{|\Omega_0|} \int_{\Omega_{1t}} \boldsymbol{v}(x,t) dx,$$

if $\boldsymbol{v}(x,t) = 0$ for $x \in D$.

Let

$$\Gamma_0 = \{x = y + \boldsymbol{N}(y)\rho_0(y), \quad y \in S_{R_0}, \}$$

and

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho(y,t) + \xi(t), \quad y \in S_{R_0}, \}$$

where $\boldsymbol{N}(y) = \frac{y}{|y|}$ is the exterior normal to $S_{R_0} = \{|y| = R_0\}$. In order to write the problem (1.1) in the fixed domain we construct the mapping of $\Omega = \Omega_{1t} \cup \Gamma_t \cup \Omega_{2t}$ on $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$, where $\mathcal{F}_1$ is the domain bounded by Σ and S_{R_0} and $\mathcal{F}_2 = \Omega \setminus \overline{\mathcal{F}}_1$; $\partial\mathcal{F}_2 = S \cup S_{R_0}$.

We take a smooth cut-off function $\chi(y)$ equal to 1, when y belongs to the layer $R_0 - d_0 \leq |y| \leq R_0 + d_0$ (we assume that this layer is contained in Ω), and we extend $\boldsymbol{N}(y)$ and $\rho(y,t)$ from S_{R_0} into Ω, as in [1], in particular, we assume that the extension $\rho^*(y,t) = 0$ near S and Σ. In our problem it is possible to set $N^* = \frac{y}{|y|}$. Now we define the mapping

$$x = y + \boldsymbol{N}^*(y)\rho^*(y,t) + \chi(y)\xi(t) \equiv e_{\rho,\xi}, \quad y \in \Omega. \tag{1.2}$$

When ρ and $\xi(t)$ are sufficiently small (which is certainly the case for $t \leq t_0$), then (1.2) establishes one-to-one correspondence between $\mathcal{F}_i$ and Ω_{it}, $i = 1, 2$. We denote by $\mathcal{L}(y, \rho^*, \xi)$ the Jacobi matrix of the transformation (1.2) and we set $L = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$. We note that

$$\rho_t(y,t) = n_r(\boldsymbol{v} \cdot \boldsymbol{n} - \xi'_t(t) \cdot \boldsymbol{n})|_{x=e_{\rho,\xi}},$$

where n_r is the radial component of the normal to Γ_t, i.e., $n_r = \frac{R(y,t)}{\sqrt{R^2 + (\nabla_\omega R)^2}}$, $\quad R = R_0 + \rho$, ∇_ω being the gradient on the unit sphere $|y| = 1$, i.e., the gradient with respect to the angular variables.

As in [1], the transformation (1.2) converts (1.1) in a nonlinear problem in the fixed domain $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$,. We separate linear and nonlinear parts in this problem, then it can be written in the following form:

$$
\begin{aligned}
&\boldsymbol{u}_t(y,t) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q, \boldsymbol{h}, \rho),\\
&\nabla\cdot\boldsymbol{u} = l_2(\boldsymbol{u},\rho), \quad y\in\mathcal{F}_1,\ t>0,\\
&\Pi_0 S(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u},\rho),\\
&-q + \nu\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}(y) + \sigma B_0\rho = l_4(\boldsymbol{u},\boldsymbol{h},\rho) + l_5(\rho),\\
&\rho_t - \boldsymbol{u}\cdot\boldsymbol{N}(y) + \frac{1}{|\Omega_0|}\int_{\mathcal{F}_1}\boldsymbol{u}dy\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u},\rho), \quad y\in S_{R_0},\\
&\mu_1\boldsymbol{h}_t + \alpha^{-1} rot rot\boldsymbol{h} = \boldsymbol{l}_7(\boldsymbol{h},\boldsymbol{u},\rho),\\
&\nabla\cdot\boldsymbol{h} = 0, \quad y\in\mathcal{F}_1,\\
&rot\boldsymbol{h} = rot\boldsymbol{l}_8(\boldsymbol{h},\rho), \quad \nabla\cdot\boldsymbol{h} = 0, \quad y\in\mathcal{F}_2,\\
&[\mu\boldsymbol{h}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \boldsymbol{l}_9(\boldsymbol{h},\rho), \quad y\in S_{R_0},\\
&\boldsymbol{h}(y,t)\cdot\boldsymbol{n}(y) = 0, \quad y\in S\cup\Sigma, \quad rot_\tau\boldsymbol{h} = 0, \quad y\in\Sigma,\\
&\boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y\in\mathcal{F}_1, \quad \boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2,\\
&\rho(y,0) = \rho_0(y), \quad y\in S_{R_0},
\end{aligned}
\tag{1.3}
$$

where $\boldsymbol{u}(y,t) = \boldsymbol{v}\circ e_{\rho,\xi}$, $q(y,t) = p\circ e_{\rho,\xi}$, $\boldsymbol{h}(y,t) = \widehat{\mathcal{L}}(y,\rho^*,\xi)(\boldsymbol{H}\circ e_{\rho,\xi})$, l_j are the nonlinear terms.

$$
\begin{aligned}
&\Pi\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{f}), \quad \Pi_0\boldsymbol{g} = \boldsymbol{g} - \boldsymbol{N}(\boldsymbol{g}\cdot\boldsymbol{N}),\\
&B_0\rho = -\frac{1}{R_0^2}(\Delta_{S_1}\rho + 2\rho),
\end{aligned}
\tag{1.4}
$$

where Δ_{S_1} is the Laplacean on the unit sphere S_1. The expression $\sigma B_0\rho$ is the first variation of $\sigma(H(x) + \frac{2}{R_0})$ with respect to ρ, and $l_5(\rho)$ is a nonlinear remainder. The conditions $|\Omega_0| = \frac{4}{3}\pi R_0^3$ and $\int_{\Omega_{10}} x_i dx = 0$ imply

$$
\int_{S_1}((R_0+\rho_0)^3 - R_0^3)dS = 0, \quad \int_{S_1} y_i((R_0+\rho_0)^4 - R_0^4)dS = 0, \quad i = 1,2,3,
$$

i.e.,

$$
\begin{aligned}
&\int_{S_1}\rho_0(y)dS = -\frac{1}{R_0}\int_{S_1}\rho^2(y)dS - \frac{1}{3R_0^2}\int_{S_1}\rho^3(y)dS,\\
&\int_{S_1} y_i\rho_0(y)dS = -\frac{1}{R_0}\int_{S_1} y_i\rho^2(y)dS - \frac{3}{2R_0^2}\int_{S_1} y_i\rho^3(y)dS - \frac{1}{4R_0^3}\int_{S_1} y_i\rho^4(y)dS.
\end{aligned}
\tag{1.5}
$$

2 Linear problems

Omitting the nonlinear terms in (1.3), we arrive at a linear problem, which is separated in two independent parts. At first, we consider the corresponding non-homogeneous problems

$$
\begin{aligned}
&\boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(y,t), \quad \nabla\cdot\boldsymbol{v} = f(y,t) = \nabla\cdot\boldsymbol{F}(y,t), \quad y\in\mathcal{F}_1,\\
&\Pi_0 S(\boldsymbol{v})\boldsymbol{N} = \Pi_0\boldsymbol{d}(y,t),\\
&-p + \nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N} + \sigma B_0\rho = \boldsymbol{d}\cdot\boldsymbol{N},\\
&\rho_t - \Big(\boldsymbol{v} - |\Omega_0|^{-1}\int_{\mathcal{F}_1}\boldsymbol{v}(y,t)dy\Big)\cdot\boldsymbol{N} = g(y,t), \quad y\in S_{R_0},\\
&\boldsymbol{v}(y,t) = 0, \quad y\in\Sigma,\\
&\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y\in\mathcal{F}_1, \quad \rho(x,0) = \rho_0(y), \quad y\in S_{R_0};
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
&\mu_1\boldsymbol{H}_t + \alpha^{-1} rotrot\boldsymbol{H} = \boldsymbol{G}(y,t), \quad \nabla\cdot\boldsymbol{H} = 0, \quad x\in\mathcal{F}_1,\\
&rot\boldsymbol{H} = \boldsymbol{j}(y,t), \quad \nabla\cdot\boldsymbol{H} = 0, \quad x\in\mathcal{F}_2,\\
&[\mu\boldsymbol{H}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{H}_\tau] = \boldsymbol{a}(y,t), \quad y\in S_{R_0},\\
&\boldsymbol{H}\cdot\boldsymbol{n} = 0, \quad y\in S\cup\Sigma, \quad rot_\tau\boldsymbol{H} = 0, \quad y\in\Sigma,\\
&\boldsymbol{H}(y,0) = \boldsymbol{H}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2.
\end{aligned}
\tag{2.2}
$$

The main result of the present paper is the weighted estimates in the norms of Sobolev-Slobodetskii spaces. Let us remind the definition of these norms. Let $\Omega\in\mathbb{R}^n$, the norm in the Sobolev space $W_2^l(\Omega)$ for non-integer l is defined by the formula

$$\| u \|^2_{W_2^l(\Omega)} = \| u \|^2_{W_2^{[l]}(\Omega)} + \sum_{|\alpha|=[l]}\int_\Omega\int_\Omega \big|D^\alpha u(x) - D^\alpha u(y)\big|^2 \frac{dxdy}{|x-y|^{n+2(l-[l])}},$$

where

$$\| u \|^2_{W_2^{[l]}(\Omega)} = \sum_{0\le|\alpha|\le[l]}\int_\Omega \big|D^\alpha u(x)\big|^2 dx$$

is the norm in the space $W_2^{[l]}$. The anisotropic Sobolev-Slobodetskii space $W_2^{l,l/2}(Q_T)$ in the cylindrical domain $Q_T = \Omega\times(0,T)$ can be defined as the space $L_2((0,T),W_2^l(\Omega))\cap W_2^{l/2}((0,T),L_2(\Omega))$ with the following norm

$$\| u \|^2_{W_2^{l,l/2}(Q_T)} = \int_0^T \| u(\cdot,t) \|^2_{W_2^l(\Omega)} dt + \int_\Omega \| u(x,\cdot) \|^2_{W_2^{l/2}(0,T)} dx.$$

There exists many other equivalent norms in $W_2^{l,l/2}(Q_T)$. Sobolev spaces of functions defined on the smooth surfaces are introduced in a standard way, with the help of local maps and partition of unity.

Theorem 2.1. *Assume that $l\in[0,3/2)$, and that the data of the problem (2.1) possess the following regularity properties: $\boldsymbol{f}\in W_2^{l,l/2}(Q_T^1)$, $f\in W_2^{l+1,0}(Q_T^1)$, $f(x,t) = \nabla\cdot\boldsymbol{F}(x,t)$, $\boldsymbol{F}_t\in W_2^{0,l/2}(Q_T^1)$, $\boldsymbol{d}\cdot\boldsymbol{N}\in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))$, $\boldsymbol{d}-\boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})\in$*

$W_2^{l+1/2,l/2+1/4}(G_T)$, $g \in W_2^{l+3/2,l/2+3/4}(G_T)$, $\boldsymbol{v}_0 \in W_2^{l+1}(\mathcal{F}_1)$, $\rho_0 \in W_2^{l+2}(S_{R_0})$ *where* $T < \infty$, $Q_T^1 = \mathcal{F}_1 \times (0,T)$, $G_T = S_{R_0} \times (0,T)$. *Moreover, let the compatibility conditions*

$$\nabla \cdot \boldsymbol{v}_0(x) = f(x,0), \quad x \in \mathcal{F}_1, \quad \Pi_0 S(\boldsymbol{v}_0)\boldsymbol{N} = \Pi_0 \boldsymbol{d}(x,0), \quad x \in S_{R_0} \tag{2.3}$$

be satisfied. Then the problem (2.1) has a unique solution $\boldsymbol{v}, p, \rho$ *such that* $\boldsymbol{v} \in W_2^{l+2,l/2+1}(Q_T^1)$, $\nabla p \in W_2^{l,l/2}(Q_T^1)$, $p \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{1/2}(S_{R_0}))$, $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$, $\rho \in W_2^{l+5/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{5/2}(S_{R_0}))$, *and the solution satisfies the inequality*

$$\begin{aligned}
&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + \|p\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} \\
&+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \\
&\leq c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)} + \|f\|_{W_2^{l+1,0}(Q_T^1)} + \|\boldsymbol{F}_t\|_{W_2^{0,1+l/2}(Q_T^1)} \\
&+ \|\Pi_0 \boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} \\
&+ \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\Big) \equiv c(T)K(T).
\end{aligned} \tag{2.4}$$

Proof. Similar result for a problem arising in linearization of a free boundary problem with a fixed barycenter is proved in [2]. Problem (2.1) differs from the one considered in [2] by the term

$$|\Omega_0|^{-1} \int_{\mathcal{F}_1} \boldsymbol{v}(y,t)dy \cdot \boldsymbol{N},$$

which we have as a consequence of the non-zero velocity of the motion of the barycenter. This term is of the lower order in comparison with the others in the boundary condition, and can be estimated by the interpolation inequality. Indeed, applying to (2.1) the coercive estimate proved in [2], we arrive at

$$\begin{aligned}
&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + \|p\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} \\
&+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \\
&\leq c(T)\Big(K(T) + |\Omega_0|^{-1} \, \Big\| \int_{\mathcal{F}_1} \boldsymbol{v}(y,t)dy \cdot \boldsymbol{N} \, \Big\|_{W_2^{l+3/2,l/2+3/4}}\Big) \\
&\leq c_1(T)\Big(K(T) + \int_{\mathcal{F}_1} \| \, \boldsymbol{v}(y,\cdot) \, \|_{W_2^{l/2+3/4}(0,T)} \, dy\Big).
\end{aligned} \tag{2.5}$$

By the interpolation inequality [3], we have

$$\| \, \boldsymbol{v} \, \|_{W_2^{0,l/2+3/4}(Q_T^1)} \leq \varepsilon \, \| \, \boldsymbol{v} \, \|_{W_2^{0,l/2+1}} + C(\varepsilon) \, \| \, \boldsymbol{v} \, \|_{L_2(Q_T^1)} \tag{2.6}$$

Estimates (2.5) and (2.6) imply

$$\| \, \boldsymbol{v} \, \|_{W_2^{l+2,l/2+1}(Q_T^1)} \leq c_2(T)\Big(K(T) + \| \, \boldsymbol{v} \, \|_{L_2(Q_T^1)}\Big).$$

Clearly, this estimate is true also for any $t \le T$. In particular,

$$\begin{aligned}
J(t) &= \int_{\mathcal{F}_1} |\boldsymbol{v}(y,t)|^2 dy = \int_0^t \tfrac{d}{d\tau}\Big(\int_{\mathcal{F}_1} |\boldsymbol{v}(y,\tau)|^2 dy\Big) d\tau + \| \boldsymbol{v}_0 \|^2_{L_2(\mathcal{F}_1)} \\
&\le \int_0^t \int_{\mathcal{F}_1} |\boldsymbol{v}_\tau(y,\tau)|^2 dy d\tau + \int_0^t \int_{\mathcal{F}_1} |\boldsymbol{v}(y,\tau)|^2 dy d\tau + \| \boldsymbol{v}_0 \|^2_{L_2(\mathcal{F}_1)} \\
&\le c_3(t)\Big(K(t) + \int_0^t J(\tau) d\tau\Big). \qquad (2.7)
\end{aligned}$$

With the help of the Gronwall lemma, we deduce from (2.7) the estimate of the L_2 norm:

$$\| \boldsymbol{v} \|_{L_2(Q^1_t)} = \int_0^t \int_{\mathcal{F}_1} |\boldsymbol{v}(y,\tau)|^2 dy d\tau \le c_4(t) K(t), \qquad t \le T. \qquad (2.8)$$

Collecting (2.5), (2.6), (2.8), we obtain the desired estimate (2.4). □

Remark. *Estimate (4.22) [2], estimate (2.5), and interpolation inequalities imply that if we add to the right-hand side of (2.4) the week norm of* $\boldsymbol{v}$*, the constant* $c(T)$ *remains uniformly bounded. It means that under the assumptions of Theorem 2.1 the following estimate*

$$\begin{aligned}
&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q^1_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q^1_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + \|p\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} \\
&+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \\
&\le C\Big(\| \boldsymbol{v} \|_{L_2(Q^1_T)} + \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q^1_T)} + \|f\|_{W_2^{l+1,0}(Q^1_T)} + \|\boldsymbol{F}_t\|_{W_2^{0,1+l/2}(Q^1_T)} \qquad (2.9) \\
&+ \|\Pi_0 \boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} \\
&+ \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\Big).
\end{aligned}$$

holds with the constant C *independent of* T.

The unique solvability of problem (2.2) and estimates of the solution in Sobolev norms follows from the results obtained in [1].

Theorem 2.2. *Assume that the data of the problem (2.2) possess the following properties:* $\boldsymbol{G} \in W_2^{l,l/2}(Q^1_T)$, $\boldsymbol{H}_0 \in W_2^{l+1}(\mathcal{F}_1) \cap W_2^{l+1}(\mathcal{F}_2)$, $\boldsymbol{j} = \boldsymbol{j}^{(2)} \in W_2^{l+1,(l+1)/2}(Q^2_T)$, $\boldsymbol{a} \in W_2^{l+3/2,l/2+3/4}(G_T)$*, moreover,*

$$\boldsymbol{j}^{(2)} = rot \boldsymbol{J}^{(2)}, \quad x \in \mathcal{F}_2, \quad \boldsymbol{a}(x,t) = [\boldsymbol{A}(x,t)], \quad x \in S_{R_0},$$

with $\boldsymbol{J}^{(2)} \in W_2^{l+2,l/2+1}(Q^2_T)$, $\boldsymbol{J}^{(2)}_t \in W_2^{l/2}(0,T;W_2^{-1/2}(S_{R_0}))$, $\boldsymbol{A}^{(i)} \in W_2^{l+2,l/2+1}(Q^i_T)$, $\boldsymbol{A}^{(i)}_t \in W_2^{l/2}(0,T;W_2^{-1/2}(S_{R_0}))$, $i = 1,2$, *and the compatibility conditions*

$$\begin{aligned}
&\nabla\cdot\boldsymbol{G}(x,t) = 0, \quad x \in \mathcal{F}_1, \quad \nabla\cdot\boldsymbol{H}_0(x) = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad rot \boldsymbol{H}_0(x) = \boldsymbol{j}^{(2)}(x,0), \quad x \in \mathcal{F}_2, \\
&[\mu \boldsymbol{H}_0 \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{H}_{0\tau}] = \boldsymbol{a}(x,0), \quad \boldsymbol{a}\cdot\boldsymbol{N} = \boldsymbol{A}^{(i)}\cdot\boldsymbol{N} = 0, \quad x \in S_{R_0}, \\
&\boldsymbol{H}_0 \cdot \boldsymbol{N} = 0, \quad rot_\tau \boldsymbol{H}_0 = 0, \quad x \in \Sigma, \quad \boldsymbol{A}^{(2)}\Big|_S = 0, \quad \boldsymbol{A}^{(1)}\Big|_\Sigma = 0, \quad \boldsymbol{J}^{(2)}\Big|_S = 0,
\end{aligned}$$

are satisfied. Then the problem (2.2) has a unique solution

$$\boldsymbol{H} \in W_2^{l+2,l/2+1}(Q_T^1) \cap W_2^{l+2,l/2+1}(Q_T^2) \quad \textit{with} \quad \boldsymbol{H}_t^{(i)} \in W_2^{l/2}(0,T; W_2^{-1/2}(S_{R_0})), \quad i = 1, 2,$$

and

$$\sum_{i=1}^{2} \Big(\|\boldsymbol{H}^{(i)}\|_{W_2^{l+2,l/2+1}(Q_T^i)} + \|\boldsymbol{H}_t^{(i)}\|_{W_2^{l/2}(0,T;W_2^{-1/2}(S_{R_0}))} \Big) \leq c\Big(\|\boldsymbol{G}\|_{W_2^{l,l/2}(Q_T^1)} + \|\boldsymbol{H}_0\|_{W_2^{l+1}(\mathcal{F}_1)}$$
$$+ \|\boldsymbol{a}\|_{W_2^{l+3/2,0}(G_T)} + \sup_{t<T} \|\boldsymbol{a}(\cdot,t)\|_{W_2^{l+1/2}(S_{R_0})} + \|\boldsymbol{j}\|_{W_2^{l+1,0}(Q_T^2)} + \sup_{t<T} \|\boldsymbol{j}\|_{W_2^{l}(\mathcal{F}_2)} + \|\boldsymbol{J}_t^{(2)}\|_{W_2^{0,l/2}(Q_T^2)}$$
$$+ \|\boldsymbol{J}_t^{(2)}\|_{W_2^{l/2}(0,T;W_2^{-1/2}(S_{R_0}))} + \|\boldsymbol{A}_t\|_{W_2^{0,l/2}(Q_T)} + \sum_{i=1}^{2} \|\boldsymbol{A}_t^{(i)}\|_{W_2^{l/2}(0,T;W_2^{-1/2}(S_{R_0}))}\Big).$$

Now we consider the homogeneous linear problems:

$$\begin{aligned}
&\boldsymbol{v}_t - \nu \nabla^2 \boldsymbol{v} + \nabla p = 0, \quad \nabla \cdot \boldsymbol{v} = 0, \quad y \in \mathcal{F}_1, \\
&\Pi_0 S(\boldsymbol{v}) \boldsymbol{N} = 0, \\
&-p + \nu \boldsymbol{N} \cdot S(\boldsymbol{v}) \boldsymbol{N} + \sigma B_0 \rho = 0, \\
&\rho_t = \big(\boldsymbol{v} - |\Omega_0|^{-1} \int_{\mathcal{F}_1} \boldsymbol{v}(y,t)dy\big) \cdot \boldsymbol{N}, \quad y \in S_{R_0}, \\
&\boldsymbol{v}(y,t) = 0, \quad y \in \Sigma, \\
&\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y \in \mathcal{F}_1, \quad \rho(x,0) = \rho_0(y), \quad y \in S_{R_0}
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\mu_1 \boldsymbol{H}_t + \alpha^{-1} rot rot \boldsymbol{H} = 0, \quad \nabla \cdot \boldsymbol{H} = 0, \quad x \in \mathcal{F}_1, \\
&rot \boldsymbol{H} = 0, \quad \nabla \cdot \boldsymbol{H} = 0, \quad x \in \mathcal{F}_2, \\
&[\mu \boldsymbol{H} \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad y \in S_{R_0}, \\
&\boldsymbol{H} \cdot \boldsymbol{n} = 0, \quad y \in S \cup \Sigma, \quad rot_\tau \boldsymbol{H} = 0, \quad y \in \Sigma, \\
&\boldsymbol{H}(y,0) = \boldsymbol{H}_0(y), \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2,
\end{aligned} \tag{2.11}$$

Linearization of conditions (1.5) leads to

$$\int_{S_{R_0}} \rho_0(y) dS = 0, \quad \int_{S_{R_0}} y_i \rho_0(y) dS = 0, \quad i = 1, 2, 3. \tag{2.12}$$

It is easily seen that (1.3) implies the same conditions for $\rho(y,t)$:

$$\int_{S_{R_0}} \rho(y,t) dS = 0, \quad \int_{S_{R_0}} y_i \rho(y,t) dS = 0, \quad i = 1, 2, 3. \tag{2.13}$$

This follows from

$$\begin{aligned}
&\frac{d}{dt} \int_{S_{R_0}} \rho(y,t) dS = \int_{S_{R_0}} \boldsymbol{v} \cdot \boldsymbol{N} dS - \frac{1}{|\Omega_0|} \int_{\mathcal{F}_1} \boldsymbol{v} dy \cdot \int_{S_{R_0}} \boldsymbol{N}(y) dS = 0, \\
&\frac{d}{dt} \int_{S_{R_0}} y_i \rho(y,t) dS = \int_{\mathcal{F}_1} v_i(y,t) dy - \frac{1}{|\Omega_0|} \int_{\mathcal{F}_1} \boldsymbol{v} dy \cdot \int_{S_{R_0}} \boldsymbol{N}(y) y_i dS = 0.
\end{aligned}$$

Now we formulate the main result of the present paper:

Theorem 2.3 *For arbitrary* $\boldsymbol{v}_0 \in W_2^{1+l}(\mathcal{F}_1)$ *and* $\rho_0 \in W_2^{2+l}(S_{R_0})$, $l \in [0, 3/2)$, *satisfying the compatibility conditions*

$$\nabla \cdot \boldsymbol{v}_0(y) = 0, \quad y \in \mathcal{F}_1, \quad \Pi_0 S(\boldsymbol{v}_0)\boldsymbol{N}(y) = 0, \quad y \in S_{R_0}$$

and the conditions (2.12), the problem (2.10) has a unique solution, and

$$\begin{aligned}
&\|e^{at}\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|e^{at}\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|e^{at}p\|_{W_2^{l+1/2,0}(G_T)} + \|e^{at}p\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))}\\
&+ \|e^{at}\rho\|_{W_2^{l+5/2,0}(G_T)} + \|e^{at}\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \|e^{at}\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\\
&+ \sup_{t<T}\|e^{at}\boldsymbol{v}_0(\cdot,t)\|_{W_2^{l+1}(\mathcal{F}_1)} + \sup_{t<T}\|e^{at}\rho(\cdot,t)\|_{W_2^{l+2}(S_{R_0})}\\
&\le c\big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\big),
\end{aligned} \tag{2.14}$$

where $a > 0$, *the constant* c *is independent of* T.

Proof. At first we prove the energy estimate. We multiply the first equation in (2.10) by $\boldsymbol{v}$, integrate over $\mathcal{F}_1$, and integrate by parts. We arrive at

$$\frac{1}{2}\frac{d}{dt} \parallel \boldsymbol{v}(\cdot,t) \parallel^2_{L_2(\mathcal{F}_1)} + \frac{\nu}{2} \parallel S(\boldsymbol{v}) \parallel^2_{L_2(\mathcal{F}_1)} + \int\limits_{\partial\mathcal{F}_1} \big(-\nu\nabla\boldsymbol{v}\cdot\boldsymbol{N}\cdot\boldsymbol{v} + p\boldsymbol{v}\cdot\boldsymbol{N}\big)ds = 0. \tag{2.15}$$

Due to the boundary conditions in (2.10), the boundary integral takes the form

$$\int\limits_{S_{R_0}} \sigma\boldsymbol{B}_0\rho\big(\rho_t + \frac{1}{|\Omega_0|}\int\limits_{\mathcal{F}_1}\boldsymbol{v}(y,t)dy\cdot\boldsymbol{N}\big)ds = \int\limits_{S_{R_0}} \sigma\rho_t\boldsymbol{B}_0\rho ds + \sigma\int\limits_{S_{R_0}} \boldsymbol{B}_0\rho\xi'(t)\cdot\boldsymbol{N}ds, \tag{2.16}$$

where

$$\boldsymbol{B}_0\rho = -\Delta_{S_{R_0}}\rho - 2\rho.$$

The second term in (2.16) is equal to zero due to the condition (2.13), while the first term takes the form

$$-\frac{\sigma}{R_0^2}\int_{S_1}\big(\Delta_{S_1}\rho + 2\rho\big)\rho_t ds = \frac{\sigma}{2R_0^2}\frac{d}{dt}\int_{S_1}\big(|\nabla_\omega\rho|^2 - 2\rho^2\big)ds = \frac{1}{2}\frac{d}{dt}M(t).$$

As a result, (2.15) reads

$$\frac{1}{2}\frac{d}{dt}\Big(\parallel \boldsymbol{v}(\cdot,t) \parallel^2_{L_2(\mathcal{F}_1)} + M(t)\Big) + \frac{\nu}{2} \parallel S(\boldsymbol{v}) \parallel^2_{L_2(\mathcal{F}_1)} = 0. \tag{2.17}$$

It can be demonstrated in the same way as in [4], [5], that conditions (2.13) imply that $M(t)$ is positively defined. Really, due to (2.13), ρ is orthogonal to the first and the second eigenfunctions of Laplace-Beltarmi operator Δ_{S_1}. Consequently, if we decompose $\rho = \sum\limits_{n=2}^{+\infty} Y_n$, where Y_n are linear combinations of eigenfunctions, corresponding to the eigenvalue $\lambda_n = n(n+1)$, we see that

$$M(t) = \frac{\sigma}{R_0^2}\int\limits_{S_1}\sum_{n=2}^{+\infty}\Big(n(n+1)Y_n - 2Y_n\Big)\sum_{n=2}^{+\infty}Y_n ds \ge \frac{\sigma}{2R_0^2}\int\limits_{S_1}|\nabla_\omega\rho|^2 d\omega + \frac{\sigma}{R_0^2}\int\limits_{S_1}\rho^2 d\omega,$$

hence,

$$M(t) \geq C \parallel \rho(\cdot,t) \parallel^2_{W^1_2(S_1)} . \tag{2.18}$$

In (2.17) there is no dissipative term for ρ. To add this term, we use the so-called "free energy" method (see for example [6], [7], [8]).

Lemma1([6],[8]) *For any function $\rho \in W_2^{l+1/2,0}(G_T)$ such that $\rho_t \in L_2(G_T)$, and satisfying the orthogonality condition*

$$\int\limits_{S_{R_0}} \rho(y,t)ds = 0,$$

there exists a vector field

$$\boldsymbol{w}(\cdot,t) \in W^1_2(\mathcal{F}_1), \qquad \boldsymbol{w}_t(\cdot,t) \in L_2(\mathcal{F}_1),$$

which is a solution to the following problem

$$\begin{aligned} \nabla \cdot \boldsymbol{w} = 0, \qquad & y \in \mathcal{F}_1, \qquad t > 0, \\ \boldsymbol{w}\big|_{\Sigma} = 0, \qquad & \boldsymbol{w} \cdot \boldsymbol{N}\Big|_{S_{R_0}} = \rho, \end{aligned}$$

and satisfies the estimates

$$\begin{aligned} &\parallel \boldsymbol{w}(\cdot,t) \parallel_{W^1_2(\mathcal{F}_1)} \leq c \parallel \rho(\cdot,t) \parallel_{W^{1/2}_2(S_{R_0})}, \\ &\parallel \boldsymbol{w}(\cdot,t) \parallel_{L_2(\mathcal{F}_1)} \leq c \parallel \rho(\cdot,t) \parallel_{L_2(S_{R_0})}, \\ &\parallel \boldsymbol{w}_t(\cdot,t) \parallel_{L_2(\mathcal{F}_1)} \leq c \parallel \rho_t(\cdot,t) \parallel_{L_2(S_{R_0})} . \end{aligned} \tag{2.19}$$

We multiply the first equation in (2.10) by $\boldsymbol{w}$ and integrate over $\mathcal{F}_1$, we arrive at

$$\int\limits_{\mathcal{F}_1} \boldsymbol{v}_t \cdot \boldsymbol{w} dx + \nu \int\limits_{\mathcal{F}_1} \nabla^2 \boldsymbol{v} \cdot \boldsymbol{w} dx + \int\limits_{\mathcal{F}_1} \nabla p \cdot \boldsymbol{w} dx = 0.$$

Then we integrate by parts and take into account the boundary conditions in (2.10), we have

$$\begin{aligned} \frac{d}{dt} \int\limits_{\mathcal{F}_1} \boldsymbol{v} \cdot \boldsymbol{w} dx + \nu \int\limits_{\mathcal{F}_1} S(\boldsymbol{v}) : S(\boldsymbol{w}) dx - \int\limits_{\mathcal{F}_1} \boldsymbol{v} \cdot \boldsymbol{w}_t dx \\ + \int\limits_{S_{R_0}} \sigma B_0 \rho \boldsymbol{w} \cdot \boldsymbol{N} ds = 0. \end{aligned} \tag{2.20}$$

Due to the condition $\boldsymbol{w} \cdot \boldsymbol{N}\Big|_{S_{R_0}} = \rho$, we see that the boundary integral in (2.20) can be written in the form

$$\frac{\sigma}{R_0^2} \int\limits_{S_{R_0}} \big(|\nabla_\omega \rho|^2 - 2\rho^2\big) ds = M(t).$$

We multiply (2.20) by a small positive number γ and add to (2.17), we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(E(t) + \gamma E_1(t) + M(t)\Big) + D(t) + \gamma D_1(t) + \gamma M(t) = 0, \tag{2.21}$$

where

$$E(t)=\| \boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)},\quad E_1(t)=2\int\limits_{\mathcal{F}_1}\boldsymbol{v}\cdot\boldsymbol{w}dx,$$
$$D(t)=\frac{\nu}{2}\parallel S(\boldsymbol{v})\parallel^2_{L_2(\mathcal{F}_1)},\quad D_1(t)=\nu\int\limits_{\mathcal{F}_1}S(\boldsymbol{v}):S(\boldsymbol{w})dx-\int\limits_{\mathcal{F}_1}\boldsymbol{v}\cdot\boldsymbol{w}_t dx,$$
$$M(t)=\frac{\sigma}{R_0^2}\int\limits_{S_1}\big(|\nabla_\omega\rho(\cdot,t)|^2-2\rho^2(\cdot,t)\big)ds.$$

Making use of $(2.19)_1$, we estimate $E_1(t)$ in the following way

$$\big|E_1(t)\big|\le 2\parallel\boldsymbol{v}\parallel_{L_2(\mathcal{F}_1)}\parallel\boldsymbol{w}\parallel_{L_2(\mathcal{F}_1)}\le 2\parallel\boldsymbol{v}\parallel_{L_2(\mathcal{F}_1)}\parallel\rho\parallel_{L_2(S_{R_0})}\le\big(\parallel\boldsymbol{v}\parallel^2_{L_2(\mathcal{F}_1)}+\parallel\rho\parallel^2_{L_2(S_{R_0})}\big).$$

For the sufficiently small γ, it leads to

$$E(t)+\gamma E_1(t)+M(t)\ge c\big(\parallel\boldsymbol{v}(\cdot,t)\parallel^2_{L_2(\mathcal{F}_1)}+\parallel\rho(\cdot,t)\parallel^2_{W_2^1(S_{R_0})}\big).\tag{2.22}$$

Similarly, with the help of (2.19) and the boundary condition $(2.10)_4$, we have

$$\begin{aligned}\big|D_1(t)\big|&\le\parallel\boldsymbol{v}(\cdot,t)\parallel_{W_2^1(\mathcal{F}_1)}\Big(\parallel\boldsymbol{w}(\cdot,t)\parallel_{W_2^1(\mathcal{F}_1)}+\parallel\boldsymbol{w}_t(\cdot,t)\parallel_{L_2(\mathcal{F}_1)}\Big)\\&\le c\parallel\boldsymbol{v}(\cdot,t)\parallel_{W_2^1(\mathcal{F}_1)}\Big(\parallel\rho(\cdot,t)\parallel_{W_2^{1/2}(S_{R_0})}+\parallel\rho_t(\cdot,t)\parallel_{L_2(S_{R_0})}\Big)\\&\le c\parallel\boldsymbol{v}(\cdot,t)\parallel_{W_2^1(\mathcal{F}_1)}\Big(\parallel\rho(\cdot,t)\parallel_{W_2^{1/2}(S_{R_0})}+\parallel\boldsymbol{v}(\cdot,t)\parallel_{L_2(S_{R_0})}\Big).\end{aligned}\tag{2.23}$$

By the Korn inequality

$$\parallel\boldsymbol{v}(\cdot,t)\parallel_{W_2^1(\mathcal{F}_1)}\le\parallel S(\boldsymbol{v}(\cdot,t))\parallel_{L_2(\mathcal{F}_1)},$$

and, for the sufficiently small γ, (2.18), (2.23) imply

$$D(t)+\gamma D_1(t)+M(t)\ge c\big(\parallel\boldsymbol{v}(\cdot,t)\parallel^2_{W_2^1(\mathcal{F}_1)}+\gamma\parallel\rho(\cdot,t)\parallel^2_{W_2^1(S_{R_0})}\big).\tag{2.24}$$

As a consequence of (2.21) (2.22), (2.24), we obtain the exponential decay in L_2 norms:

$$\parallel\boldsymbol{v}(\cdot,t)\parallel^2_{L_2(\mathcal{F}_1)}+\parallel\rho(\cdot,t)\parallel^2_{W_2^1(S_{R_0})}\le Ce^{-\beta t}\Big(\parallel\boldsymbol{v}_0\parallel^2_{L_2(\mathcal{F}_1)}+\parallel\rho_0\parallel^2_{W_2^1(S_{R_0})}\Big),\quad\beta>0.\tag{2.25}$$

Now we pass to the estimate (2.14). Let us introduce the notation:

$$\tilde{\boldsymbol{v}}=e^{at}\boldsymbol{v},\qquad\tilde{p}=e^{at}p,\qquad\tilde{\rho}=e^{at}\rho,\quad a>0.$$

If $\boldsymbol{v}$, p, ρ is a solution to problem (2.10), then $\tilde{\boldsymbol{v}}$, $\tilde{p}$, $\tilde{\rho}$ satisfy the following relations

$$\begin{aligned}&\tilde{\boldsymbol{v}}_t-\nu\nabla^2\tilde{\boldsymbol{v}}+\nabla\tilde{p}=-a\tilde{\boldsymbol{v}},\quad\nabla\cdot\tilde{\boldsymbol{v}}=0,\quad y\in\mathcal{F}_1,\\&\Pi_0S(\tilde{\boldsymbol{v}})\boldsymbol{N}=0,\\&-\tilde{p}+\nu\boldsymbol{N}\cdot S(\tilde{\boldsymbol{v}})\boldsymbol{N}+\sigma B_0\tilde{\rho}=0,\\&\tilde{\rho}_t=\big(\tilde{\boldsymbol{v}}-|\Omega_0|^{-1}\int_{\mathcal{F}_1}\tilde{\boldsymbol{v}}(y,t)dy\big)\cdot\boldsymbol{N}-a\tilde{\rho},\quad y\in S_{R_0},\\&\tilde{\boldsymbol{v}}(y,t)=0,\quad y\in\Sigma,\\&\tilde{\boldsymbol{v}}(y,0)=\boldsymbol{v}_0(y),\quad y\in\mathcal{F}_1,\quad\tilde{\rho}(x,0)=\rho_0(y),\quad y\in S_{R_0}.\end{aligned}\tag{2.26}$$

By estimate (2.9), we have

$$\begin{aligned}
&\|\tilde{\boldsymbol{v}}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla\tilde{p}\|_{W_2^{l,l/2}(Q_T^1)} + \|\tilde{p}\|_{W_2^{l+1/2,0}(G_T)} + \|\tilde{p}\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))}\\
&+ \|\tilde{\rho}\|_{W_2^{l+5/2,0}(G_T)} + \|\tilde{\rho}\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \|\tilde{\rho}_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\\
&\leq C\Big(\| \tilde{\boldsymbol{v}} \|_{L_2(Q_T^1)} + a\|\tilde{\boldsymbol{v}}\|_{W_2^{l,l/2}(Q_T^1)} + a\|\tilde{\boldsymbol{v}}\|_{W_2^{l+1,0}(Q_T^1)}\\
&+ a\|\tilde{\rho}\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\Big),
\end{aligned} \tag{2.27}$$

where the constant C is independent of T. We apply to the right-hand side of (2.27) interpolation inequalities, and then estimate the week norms $\| \tilde{\boldsymbol{v}} \|_{L_2(Q_T^1)}$, $\| \tilde{\rho} \|_{W_1^1(S_{R_0})}$ by (2.25). For $a < \beta$, (2.25) implies

$$\| e^{at}\boldsymbol{v}(\cdot,t) \|_{L_2(Q_T^1)} + \| e^{at}\rho(\cdot,t) \|_{W_1^1(S_{R_0})} \leq c\big(\| \boldsymbol{v}_0 \|_{L_2(Q_T^1)} + \| \rho_0 \|_{W_2^1(S_{R_0})} \big).$$

As a consequence, we arrive at (2.14). $\square$

Weighted estimates for solutions of problem (2.11) can be carried out by the same scheme.

Theorem 2.4 *For arbitrary $\boldsymbol{H}_0 \in W_2^{l+1}(\mathcal{F}_i)$, $i = 1, 2$, satisfying the compatibility conditions*

$$\begin{aligned}
&\nabla \cdot \boldsymbol{H}_0(x) = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad rot\boldsymbol{H}_0(x) = 0, \quad x \in \mathcal{F}_2,\\
&[\mu\boldsymbol{H}_0 \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{H}_{0\tau}] = 0, \quad x \in S_{R_0},\\
&\boldsymbol{H}_0\boldsymbol{n} = 0, \quad rot_\tau\boldsymbol{H}_0 = 0, \quad y \in \Sigma,\\
&\boldsymbol{H}_0 \cdot \boldsymbol{n} = 0, \quad y \in S,
\end{aligned} \tag{2.28}$$

the problem (2.2) has a unique solution, and the inequality

$$\sum_{i=1}^{2} \big(\|e^{at}\boldsymbol{H}^{(i)}\|_{W_2^{l+2,l/2+1}(Q_T^i)} + \|e^{at}\boldsymbol{H}_t^{(i)}\|_{W_2^{l/2}(0,T;W_2^{-1/2}(S_{R_0}))}\big) \leq c\sum_{i=1}^{2} \|\boldsymbol{H}_0^{(i)}\|_{W_2^{l+1}(\mathcal{F}_i)} \tag{2.29}$$

holds with a certain $a > 0$ and with the constant c independent of T.

3 Nonlinear problem.

Weighted estimates for linear problems give us the opportunity to prove the global solvability for the free boundary problem in magnetohydrodynamics under the assumptions that the initial data are small and the initial position of the free boundary is close to the sphere. Here we formulate the result for the nonlinear problem (1.3), the paper with the detailed proof is in preparation.

Theorem 3.1. *Let $\boldsymbol{u}_0 \in W_2^{l+1}(\mathcal{F}_1)$, $\rho_0 \in W_2^{l+2}(S_{R_0})$, $\boldsymbol{h}_0 \in W_2^{l+1}(\mathcal{F}_1)$ and let the compatibility conditions*

$$\begin{aligned}
&\nabla \cdot \boldsymbol{u}_0 = l_2(\boldsymbol{u}_0,\rho_0), \quad y \in \mathcal{F}_1, \quad \Pi_0 S(\boldsymbol{u}_0)\boldsymbol{N}(y) = \boldsymbol{l}_3(\boldsymbol{u}_0,\rho_0), \quad y \in \mathcal{G},\\
&\nabla \cdot \boldsymbol{h}_0^{(1)} = 0, \quad \nabla \cdot \boldsymbol{h}_0^{(2)} = 0, \quad rot\boldsymbol{h}_0^{(2)} = rot\boldsymbol{l}_8(\boldsymbol{h}_0^{(2)},\rho_0),\\
&[\mu\boldsymbol{h}_0 \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{h}_{0\tau}] = \boldsymbol{l}_9(\boldsymbol{h}_0,\rho_0), \quad x \in \mathcal{G},\\
&\boldsymbol{h}_0 \cdot \boldsymbol{N} = 0, \quad rot_\tau\boldsymbol{h}_0 = 0, \quad y \in \Sigma
\end{aligned} \tag{3.1}$$

and the smallness condition

$$\|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+3/2}(S_{R_0})} + \|\boldsymbol{h}_0\|_{W_2^{l+1}(\mathcal{F}_1)} \leq \epsilon \ll 1 \tag{3.2}$$

be satisfied. Then the problem (1.3) has a unique solution with the following regularity properties:

$$\boldsymbol{u} \in W_2^{2+l,1+l/2}(Q^1_\infty), \quad \nabla q \in W_2^{l,l/2}(Q^1_\infty), \quad q \in W_2^{l+5/2,0}(G_\infty) \cap W_2^{l/2}(0,\infty; W_2^{1/2}(S_{R_0})),$$
$$\rho \in W_2^{l+5/2,0}(G_\infty) \cap W_2^{l/2}(0,T; W_2^{5/2}(\mathcal{G})), \quad \rho_t \in W_2^{l+3/2,l/2+3/4}(G_\infty), \quad \boldsymbol{h}^{(i)} \in W_2^{l'+2,l'/2+1}(Q^i_\infty),$$

where $Q^i_\infty = \mathcal{F}_i \times (0,\infty)$, $G_\infty = S_{R_0} \times (0,\infty)$, $\boldsymbol{h}^{(i)} = \boldsymbol{h}|_{x\in\mathcal{F}_i}$, $i = 1,2$. *The solution satisfies the inequality*

$$\begin{aligned}
&\|e^{at}\boldsymbol{u}\|_{W_2^{l+2,l/2+1}(Q^1_\infty)} + \|e^{at}\nabla q\|_{\widehat{W}_2^{l,l/2}(Q^1_\infty)} + \|e^{at}q\|_{W_2^{l+1/2,0}(G_\infty)} + \|e^{at}q\|_{\widehat{W}_2^{l/2}(0,\infty;W_2^{1/2}(\mathcal{G}))} \\
&+ \|e^{at}\rho\|_{W_2^{l+5/2,0}(G_\infty)} + \|e^{at}\rho\|_{\widehat{W}_2^{l/2}(0,\infty;W_2^{5/2}(S_{R_0}))} + \|e^{at}\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_\infty)} \\
&+ \sup_{t>0} \|e^{at}\rho(\cdot,t)\|_{W_2^{l+2}(S_{R_0})} + \sum_{i=1}^{2} \big(\|e^{at}\boldsymbol{h}^{(i)}\|_{W_2^{l'+2,l'/2+1}(Q^i_\infty)} + \|e^{at}\boldsymbol{h}^{(i)}_t\|_{\widehat{W}^{l'/2}(0,\infty;W_2^{-1/2}(S_{R_0}))}\big) \\
&\leq c\big(\|\boldsymbol{u}_0\|_{W_2^{l+2}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})} + \sum_{i=1}^{2} \|\boldsymbol{h}^{(i)}_0\|_{W_2^{l'+1}(\mathcal{F}_i)}\big)
\end{aligned} \tag{3.3}$$

with a certain $a > 0$.

Estimate (3.3) shows that velocity, pressure and the magnetic field tend to zero exponentially and Γ_t tends to a sphere when $t \to +\infty$. The center of the limit sphere is located at the point

$$\boldsymbol{\xi}(+\infty) = \int_0^{+\infty} d\tau \int_{\Omega_{1\tau}} \boldsymbol{v}(x,\tau)dx = \int_0^{+\infty} d\tau \int_{\mathcal{F}_1} \boldsymbol{u}(y,\tau)L dy.$$

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