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BOUNDARIES OF $\mathbb{Z}^n$-FREE GROUPS

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Abstract

We study random walks on groups with free actions on $\mathbb{Z}^n$-trees ($\mathbb{Z}^n$*-free groups*) and describe the Poisson–Furstenberg boundaries of these groups.

Key words: random walk, Poisson boundary, $\mathbb{Z}^n$-trees, $\mathbb{Z}^n$-free groups

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1 Introduction

In this paper we study random walks on groups with free actions on $\mathbb{Z}^n$-trees ($\mathbb{Z}^n$-*free groups*) and the boundaries of these groups. This family of groups is a particularly nice and well-studied subclass of groups acting freely on Λ-trees (Λ-*free groups*), where Λ is an arbitrary ordered abelian group.

The theory of group actions on Λ-trees goes back to the work of Lyndon who introduced in [18] abstract length functions on groups, axiomatizing Nielsen cancellation method; he initiated the study of groups with real valued length functions. Later in [5], Chiswell related such length functions with group actions on $\mathbb{Z}$- and $\mathbb{R}$-trees, providing a construction of the tree on which the group acts. At about the same time, Tits in [22] gave the first formal definition of $\mathbb{R}$-tree, which is a geodesic metric space with a tree-like structure. Eventually, in [20], Morgan and Shalen introduced Λ-trees for an arbitrary ordered abelian group Λ and the general form of Chiswell's construction. A Λ-tree is a metric space whose metric takes values in Λ and is subject to certain tree axioms. The theory of group actions such objects was consistently developed by Alperin and Bass (see [1]), where authors state the fundamental problem: find the group theoretic information carried by a Λ-tree action, in particular, the structure of Λ-free groups. If the case of Archimedean free actions, that is, when $\Lambda = \mathbb{R}$, is basically closed by the Rips' Theorem that describes finitely generated $\mathbb{R}$-free groups (see [10, 3]), the general non-Archimedean case is still open, but a lot of progress was made (see [2, 11, 16, 15]) though.

Introduction of infinite Λ-words was one of the major recent developments in this theory. In [21] Myasnikov, Remeslennikov and Serbin showed that groups admitting faithful representations by Λ-words act freely on some Λ-trees, while Chiswell proved the converse [7]. This gives an equivalent approach to group actions on Λ-trees, in which one can work with group elements viewed as Λ-words in the same manner as with ordinary words in standard free groups. The method of infinite words becomes a very powerful tool in the case when $\Lambda = \mathbb{Z}^n$ with the right lexicographic order due to the natural combinatorial structure of $\mathbb{Z}^n$-trees. This structure was exploited in [14] and [17] to obtain a description of finitely generated $\mathbb{Z}^n$-free groups in terms of free products with amalgamation and HNN-extensions of a particular type (see also [2]).

The class of $\mathbb{Z}^n$-free groups is a natural generalization of free groups which contains limit groups, $\mathbb{R}$-free groups, etc. and which is closed under taking subgroups, free products, and amalgamated free products along maximal cyclic subgroups (n is not preserved in general). All these groups are hyperbolic relative to non-cyclic maximal abelian subgroups (see [8, 11]) (hyperbolic if all maximal abelian subgroups are cyclic), coherent, with nice algorithmic properties.

In our paper we represent elements of a $\mathbb{Z}^n$-free group by $\mathbb{Z}^n$-words which are certain (finite or infinite) configurations of letters from some alphabet on the $\mathbb{Z}^n$-lattice. To be more precise let us fix the *right lexicographic order on* $\mathbb{Z}^n$, that is, given two n-tuples $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{Z}^n$, we have $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ if there exists $i \in [1, n]$ such that $a_i < b_i$ and $a_j = b_j, j \in [i+1, n]$. Denote by $\mathbb{Z}^n_+ \subset \mathbb{Z}^n$ the subset of all positive elements in $\mathbb{Z}^n$ with respect to the order just defined. For $\alpha, \beta \in \mathbb{Z}^n$, such that $\alpha \leqslant \beta$, the *closed segment* $[\alpha, \beta]$ is the set $[\alpha, \beta] = \{\gamma \in \mathbb{Z}^n \mid \alpha \leqslant \gamma \leqslant \beta\}$ and we define $\mathbb{Z}^n$-words as functions on closed segments of the type $[1, \alpha]$, where $\alpha \in \mathbb{Z}^n$ and 1 stands for the n-tuple $(1, 0, \ldots, 0) \in \mathbb{Z}^n$. To this end let $X = \{x_i \mid i \in I\}$ be a set, and put $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^{\pm} = X \cup X^{-1}$.

Now, a $\mathbb{Z}^n$-*word* (or just an *infinite word*) is a function of the type

$$w : [1, \alpha_w] \to X^{\pm},$$

where $\alpha_w \in \mathbb{Z}^n$, $\alpha_w \geqslant 0$ is the natural length of w, denoted $|w|$. The set of all $\mathbb{Z}^n$-words over X is denoted by $W(\mathbb{Z}^n, X)$. Concatenation, the trivial word ε, and inversion are naturally defined as in the standard free group. An infinite word is *reduced* if it does not contain xx^{-1}, $x^{-1}x$ as a subword for $x \in X$ and the set of all reduced $\mathbb{Z}^n$-words is denoted by $R(\mathbb{Z}^n, X)$. Next, following [21], one can define multiplication "$*$" on $R(\mathbb{Z}^n, X)$ and the subset $CDR(\mathbb{Z}^n, X) \subset R(\mathbb{Z}^n, X)$ of infinite words without torsion. From the results of [21] and [7] it follows that $\mathbb{Z}^n$-free groups are exactly subgroups of $CDR(\mathbb{Z}^n, X)$ (subsets closed with respect to "$*$" and inversion of infinite words), where X may vary (for all the details on infinite words see Subsection 2.2.3).

Next, a *maximal* $\mathbb{Z}^n$-*word* is a function of the type $\omega : \mathbb{Z}^n_+ \to X^{\pm}$. By $\partial_n W(\mathbb{Z}^n, X)$ we denote the set of all maximal $\mathbb{Z}^n$-words over $X^{\pm}$ which is a topological space: the topology is generated by the "cones" U_w, $w \in W(\mathbb{Z}^n, X)$, where U_w is the set of all maximal $\mathbb{Z}^n$-words with initial prefix w. $\partial_n W(\mathbb{Z}^n, X)$ is not compact in general if $n > 1$. Similarly to $\mathbb{Z}^n$-words, a maximal $\mathbb{Z}^n$-word ω is *reduced* if $\omega(\beta+1) \neq \omega(\beta)^{-1}$ for each $\beta \in [1, |w| - 1]$ and we denote by $\partial_n R(\mathbb{Z}^n, X)$ the subset of all reduced maximal $\mathbb{Z}^n$-words.

Now, given a $\mathbb{Z}^n$-free group G, we can assume that it embeds into $CDR(\mathbb{Z}^n, X)$ for an appropriate set X. Moreover, we can assume that G is *irreducible* in $CDR(\mathbb{Z}^n, X)$ in the sense that it is not conjugate in $R(\mathbb{Z}^n, X)$ to a subgroup H of $CDR(\mathbb{Z}^k, X)$, $k < n$, and non-abelian. Denote by $\partial_n G \subseteq \partial_n W(\mathbb{Z}^n, X)$ the limit set of G (with respect to the topology on $\partial_n W(\mathbb{Z}^n, X)$). In fact, since $G \subseteq CDR(\mathbb{Z}^n, X)$, it follows that $\partial_n G \subset \partial_n R(\mathbb{Z}^n, X)$. $\partial_n G$ has natural topology inherited from $\partial_n W(\mathbb{Z}^n, X)$ and is not compact in general.

First of all, we prove that sample paths of a non-degenerate random walk on G converges to elements of $\partial_n G$. Then we show that, under certain conditions on a distribution μ on G, the space $\partial_n G$ with the corresponding measure can be identified with the Poisson boundary of (G, μ). Our main result can be formulated as follows.

Theorem 1. *Let G be a $\mathbb{Z}^n$-free group for some $n \in \mathbb{N}$ which embeds into $CDR(\mathbb{Z}^n, X)$ for a finite alphabet X. Let G be irreducible and non-abelian. If $\partial_n G \subset \partial_n R(\mathbb{Z}^n, X)$ is the limit set of G in the space of maximal $\mathbb{Z}^n$-words then for every non-degenerate probability measure μ on G, a. e. path of the random μ-walk converges to an element of $\partial_n G$. Moreover, if G is finitely generated and μ has finite first moment with respect to a finite word metric[1] on G, then the measure space $(\partial_n G, \nu_\mu)$, where ν_μ is the distribution of limits of paths of the random μ-walk, is the Poisson boundary of (G, μ).*

In the proof of the above theorem we use methods of [9] and [4].

2 Preliminaries

2.1 Poisson–Furstenberg boundaries

Let G be a countable group and let μ be a probability measure on G (we consider measures that are regular with respect to the discrete topology on the group).

[1] We consider an arbitrary word metric that is induced by a finite generating set.

We say that μ is *non-degenerate* if its support generates G as a semigroup. The *right-hand random walk on G with distribution μ* (or, briefly, *μ-walk*) is the time-homogeneous Markov chain whose state space is G, the transition probabilities are given by $P(g,h) = \mu(g^{-1}h)$, and the initial distribution is concentrated at the identity of the group. Realizations of this process are called *paths* of the random walk. We write P_μ for the associated Markov measure on the path space $G^{\mathbb{Z}_+}$.

An *action* of the group G on a topological space M is a homomorphism from G to the group $\mathrm{Homeo}(M)$ of all homeomorphisms of M. An action is said to be *minimal* if each orbit of the action is dense (or, equivalently, if the action has no proper closed invariant subsets). A space endowed with an action of the group G is called a *G-space*. We denote the space of all (regular with respect to the discrete topology) probability measures on G by $\mathcal{P}(G)$ and the space of all regular Borel probability measures on M by $\mathcal{P}(M)$. We endow $\mathcal{P}(M)$ with the weak topology (by duality with the space of all bounded continuous functions on M). Any action of G on M induces an action of G on $\mathcal{P}(M)$: $(g \cdot \theta)(E) = \theta(g^{-1} \cdot E)$. A measure is said to be a *Dirac measure* if it is concentrated at a point. We say that a measure is *continuous* if it takes zero value on each point.

Suppose that $\mu \in \mathcal{P}(G)$. A measure $\nu \in \mathcal{P}(M)$ is said to be *μ-stationary* if

$$\mu * \nu = \sum_{g \in G} (g \cdot \nu)\mu(g) = \nu.$$

Assertion 1 ([9]). *Let G be a countable group acting on a compact metric space M and let $\mu \in \mathcal{P}(G)$ be an arbitrary measure. Then the set of μ-stationary measures in $\mathcal{P}(M)$ is non-empty.*

Theorem 2 ([9, 13]). *Let G be a countable group acting on a compact metric space M. Take $\mu \in \mathcal{P}(G)$ and let $\nu \in \mathcal{P}(M)$ be a μ-stationary measure. Then for a. e. path $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$ of the right random μ-walk, the sequence $\{\tau_i \cdot \nu\}_{i \in \mathbb{Z}_+}$ converges to some measure $\lambda(\tau) \in \mathcal{P}(M)$. We also have*

$$\int \lambda(\tau) dP_\mu(\tau) = \nu.$$

Lemma 1 ([13]). *Let G be a countable group acting on a space M, μ a non-degenerate measure on G and $\nu \in \mathcal{P}(M)$ be a μ-stationary measure. Let $E \subset M$ be a measurable subset such that for every $g \in G$ we have either $g \cdot E = E$, or $(g \cdot E) \cap E = \varnothing$. Suppose further that there is an infinite family of pairwise-disjoint sets of the form $g \cdot E$, where $g \in G$. Then $\nu(E) = 0$. In particular, if the orbit $G \cdot x$ of every point $x \in M$ is infinite, then the μ-stationary measure ν is continuous.*

Lemma 2. *Let G be a countable group acting minimally on a topological space M, μ a non-degenerate measure on G and $\nu \in \mathcal{P}(M)$ be a μ-stationary measure. Let $E \subset M$ be a non-empty open subset. Then $\nu(E) > 0$.*

Proof. Observe that by definition, the μ-stationary measure ν is μ^{*k}-stationary for every $k \in \mathbb{N}$. Consequently, for every $k \in \mathbb{N}$ we have

$$\nu(E) = \sum_{g \in G} \mu^{*k}(g)(g \cdot \nu)(E) = \sum_{g \in G} \mu^{*k}(g)\nu(g^{-1} \cdot E). \tag{1}$$

Suppose that $\nu(E) = 0$. Then (1) implies that $\nu(g^{-1} \cdot E) = 0$ whenever there exists $k \in \mathbb{N}$ such that $\mu^{*k}(g) > 0$. Hence, $\nu(g^{-1} \cdot E) = 0$ for each $g \in G$ because μ is

non-degenerate. On the other hand, since G acts on M minimally, and E is open and non-empty, it follows that for each $x \in M$ there exists $g \in G$ such that $g \cdot x \in E$. Hence, $\bigcup_{g \in G} g^{-1} \cdot E = \bigcup_{g \in G} g \cdot E = M$. Since G is countable, it follows that

$$1 = \nu(M) = \nu\left(\bigcup_{g \in G} g^{-1} \cdot E\right) \leqslant \sum_{g \in G} \nu(g^{-1} \cdot E) = 0.$$

This contradiction proves that $\nu(E) > 0$. □

Suppose that M is a topological G-space, $\mu \in \mathcal{P}(G)$, and $\nu \in \mathcal{P}(M)$ is a μ-stationary measure. The pair (M, ν) is called a *μ-boundary* for G if for a. e. path $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$ of the right random μ-walk, the sequence of measures $\{\tau_i \cdot \nu\}_{i \in \mathbb{Z}_+}$ converges to some Dirac measure $\delta_{\omega(\tau)}$, where $\omega(\tau) \in M$.

Usually, it makes sense to consider compact μ-boundaries only. In our case with $\mathbb{Z}^n$-free groups, we treat non-compact boundaries also. We give the following simple lemma here in order to shed some light on the non-compact case.

Lemma 3. *Let M be a compact metric G-space, $\mu \in \mathcal{P}(G)$, and $\nu \in \mathcal{P}(M)$ a μ-stationary measure. Assume that (M, ν) is a μ-boundary for G, and $M' \subset M$ is a measurable G-invariant subspace with $\nu(M') = 1$. Then (M', ν'), where ν' is the restriction of ν on M', is also a μ-boundary for G.*

Proof. Let $\Omega_1 \subset G^{\mathbb{Z}_+}$ be the set of sequences $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$ for which the sequence of measures $\{\tau_i \cdot \nu\}_{i \in \mathbb{Z}_+}$ converges to some Dirac measure $\delta_{\omega(\tau)}$, $\omega(\tau) \in M$. Set

$$\Omega_2 = \{\tau \in \Omega_1 \mid \omega(\tau) \in M'\}.$$

Since (M, ν) is a μ-boundary for G, we have $P_\mu(\Omega_1) = 1$. Since $\nu(M') = 1$, it follows by Theorem 2 that $P_\mu(\Omega_2) = 1$.

Observe that in an arbitrary metric space K, a sequence $\{\lambda_i\}_{i \in \mathbb{Z}_+}$ of measures in $\mathcal{P}(K)$ converges to a Dirac measure δ_x, $x \in K$ if and only if for each open subset U containing x we have $\lim_{i \to \infty} \lambda_i(U) = 1$. This clearly implies that for every $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+} \in \Omega_2$, the sequence $\{\tau_i \cdot \nu'\}_{i \in \mathbb{Z}_+}$ weakly converges to $\delta_{\omega(\tau)}$, where $\omega(\tau) \in M'$. In other words, for a. e. path $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$ of the right random μ-walk (since a. e. path of the right random μ-walk is in Ω_2), the sequence $\{\tau_i \cdot \nu'\}_{i \in \mathbb{Z}_+}$ converges to some Dirac measure $\delta_{\omega(\tau)}$, where $\omega(\tau) \in M'$. This precisely means that (M', ν') is a μ-boundary for G. □

We say that a topological G-space M is a *Furstenberg boundary* for G if for every non-degenerate measure $\mu \in \mathcal{P}(G)$ there is a μ-stationary measure in $\mathcal{P}(M)$, and for every μ-stationary measure $\nu \in \mathcal{P}(M)$ the pair (M, ν) is a μ-boundary.

Assertion 2 ([19], Lemma 3.1)**.** *Let G be a countable group acting on a compact metric space M and let $\mu \in \mathcal{P}(G)$ be an arbitrary probability measure. Suppose that for every μ-stationary measure $\nu \in \mathcal{P}(M)$ the pair (M, ν) is a μ-boundary. Then the μ-stationary measure on M is unique.*

Thus, if a compact metric G-space M is a Furstenberg boundary for G, then for every non-degenerate measure $\mu \in \mathcal{P}(G)$ there is a unique μ-stationary measure in $\mathcal{P}(M)$.

A μ-boundary (M, ν) of the pair (G, μ) is called a *Poisson–Furstenberg boundary* if it is maximal, that is, if every μ-boundary (M_1, ν_1) of the pair (G, μ) is a quotient

space of (M,ν) (as a measure space endowed with a group action, disregarding the topology).

If (M,ν) is a μ-boundary for G, it is natural to consider the partially defined mapping $\mathrm{bnd}: G^{\mathbb{Z}_+} \to M$ that sends a path $\tau = \{\tau_i\}_{i\in\mathbb{Z}_+} \in G^{\mathbb{Z}_+}$ to a point $w \in M$ if $\{\tau_i \cdot \nu\}_{i\in\mathbb{Z}_+}$ converges to δ_w. The mapping bnd is thus defined for P_μ-a. e. path. Suppose that $E \subset M$ is a measurable subset and τ a path of random μ-walk. Then Theorem 2 implies that $\mathrm{bnd}(\tau)$ lies in E with probability $\nu(E)$.

In order to establish the maximality of boundaries we use the Strip Criterion due to V. A. Kaimanovich. An increasing sequence $\mathcal{G} = (\mathcal{G}_k)$ of sets exhausting a countable group G is called a *gauge* on G. By $|\cdot|_{\mathcal{G}} = \min\{k \mid g \in \mathcal{G}_k\}$ we denote the corresponding *gauge function*. If μ is a measure on a group G, we denote by $\check{\mu}$ the *reflected measure* defined by $\check{\mu}(g) = \mu(g^{-1})$.

Theorem 3 (Strip Criterion [12]). *Let μ be a probability measure with finite entropy $H(\mu)$ on a countable group G, and let (B_+,λ_+) and (B_-,λ_-) be μ- and $\check{\mu}$-boundaries, respectively. If there exists a gauge $\mathcal{G} = (\mathcal{G}_k)$ on the group G with gauge function $|\cdot|_{\mathcal{G}}$ and a measurable G-equivariant map S assigning to pairs of points $(b_-,b_+) \in B_- \times B_+$ non-empty "strips" $S(b_-,b_+) \subset G$ such that for all $g \in G$ and $(\lambda_- \times \lambda_+)$-almost every $(b_-,b_+) \in B_- \times B_+$*

$$\frac{1}{i}\log\left|S(b_-,b_+)g \cap \mathcal{G}_{|\tau_i|_{\mathcal{G}}}\right| \xrightarrow[i\to\infty]{} 0$$

in probability with respect to the measure P_μ in the space of sample paths $\tau = \{\tau_i\}_{i\in\mathbb{Z}_+}$, then (B_+,λ_+) is a Poisson–Furstenberg boundary of the pair (G,μ).

Corollary 1 ([12]). *Let G be a finitely generated group with a finite word metric $|\cdot|$. Let μ be a probability measure on G with finite first moment $\sum |g|\mu(g)$. Let (B_+,λ_+) and (B_-,λ_-) be μ- and $\check{\mu}$-boundaries, respectively. If there exists a measurable G-equivariant map S assigning to pairs of points $(b_-,b_+) \in B_- \times B_+$ non-empty "strips" $S(b_-,b_+) \subset G$ such that for $(\lambda_- \times \lambda_+)$-almost every $(b_-,b_+) \in B_- \times B_+$ we have*

$$\frac{1}{i}\log|\{g \in S(b_-,b_+) : |g| \leqslant i\}| \xrightarrow[i\to\infty]{} 0, \tag{2}$$

then (B_+,λ_+) is a Poisson–Furstenberg boundary of the pair (G,μ).

Remark 1. *Note that, under the Strip Criterion, the "strips" $S(b_-,b_+)$ are required to be*

(i) *all non-empty,*

(ii') *$(\lambda_- \times \lambda_+)$-almost surely "thin".*

Clearly, since the strips are allowed to meet the "thinness" requirement $(\lambda_- \times \lambda_+)$-almost surely (not surely), we can handle the "non-emptiness" property in the same way. In other words, in order to use the Strip Criterion it suffices to construct a (measurable, equivariant) map $S' : B_- \times B_+ \to 2^G$ with strips, which are

(i') *$(\lambda_- \times \lambda_+)$-almost surely non-empty,*

(ii') *$(\lambda_- \times \lambda_+)$-almost surely thin.*

This is clear, because we can pass from S' to a map S with the property (i) by setting $S(b_-,b_+) = G$ if $S'(b_-,b_+) = \varnothing$ and $S(b_-,b_+) = S'(b_-,b_+)$ otherwise.

Note also that, having a map with the properties (i') *and* (ii'), *we can replace all non-thin strips by empty ones and thus obtain a map that has the property* (i) *and the property*

(ii) *all strips are "thin".*

A real function $\phi : G \to \mathbb{R}$ on a group G is called a *seminorm* if for every $g, h \in G$: (a) $\phi(g) \geqslant 0$ and $\phi(1) = 0$, (b) $\phi(gh) \leqslant \phi(g) + \phi(h)$, (c) $\phi(g) = \phi(g^{-1})$. By virtue of Remark 1 the Strip Criterion as follows.

Corollary 2. *Let G, $|\cdot|$, μ, (B_+, λ_+), (B_-, λ_-) be as in Corollary 1, and let $\hbar : G \to \mathbb{R}$ be a seminorm. Assume that there exists a measurable G-equivariant map $S : B_- \times B_+ \to 2^G$ sending pairs of points $(b_-, b_+) \in B_- \times B_+$ to "strips" $S(b_-, b_+) \subset G$ such that for $(\lambda_- \times \lambda_+)$-almost every pair $(b_-, b_+) \in B_- \times B_+$ we have*

(i) *the strip $S(b_-, b_+)$ is non-empty,*

(ii) $\frac{1}{i} \log |\{g \in S(b_-, b_+) \,:\, \hbar(g) \leqslant i\}| \xrightarrow[i\to\infty]{} 0.$

Then (B_+, λ_+) is a Poisson–Furstenberg boundary of the pair (G, μ).

Proof. Define the map $S' : B_- \times B_+ \to 2^G$ by setting $S'(b_-, b_+) = G$ if $S(b_-, b_+) = \varnothing$ and $S'(b_-, b_+) = S(b_-, b_+)$ otherwise. Then S' is still measurable and G-equivariant, and all its "strips" are non-empty. Observe also, that for $(\lambda_- \times \lambda_+)$-a. e. pair (b_-, b_+) we have

$$\frac{1}{i} \log \left|\left\{g \in S'(b_-, b_+) \,:\, \hbar(g) \leqslant i\right\}\right| \xrightarrow[i\to\infty]{} 0, \tag{3}$$

because $S'(b_-, b_+) = S(b_-, b_+)$ for $(\lambda_- \times \lambda_+)$-a. e. pair (b_-, b_+). Now, let us prove that for $(\lambda_- \times \lambda_+)$-a. e. pair (b_-, b_+) the strip $S'(b_-, b_+)$ meets the condition (2) of Corollary 1.

Recall that, for our metric $|\cdot|$, there exists a (unique) subset $Q \subset G \smallsetminus \{1\}$ such that $|g| = \min\{d \in \mathbb{N}_0 \,:\, g \in Q^d\}$ for all $g \in G$ (the set Q is finite, since we consider "finite" word metrics). Since Q is finite, there exists $k_0 \in \mathbb{N}$ such that $\hbar(q) \leqslant k_0 \cdot |q| = k_0$ for each $q \in Q$. Then for any $g \in G \smallsetminus \{e\}$ with $g = q_1 \cdots q_{|g|}$, $q_j \in Q$, we have (by the triangle inequality for seminorms)

$$\hbar(g) \leqslant \hbar(q_1) + \cdots + \hbar(q_{|g|}) \leqslant k_0 \cdot |q_1| + \cdots + k_0 \cdot |q_{|g|}| = k_0 \cdot |g|.$$

Since $\hbar(1) = |1| = 0$, it follows that $\hbar(g) \leqslant k_0 \cdot |g|$ for all $g \in G$. Consequently, for any $i \in \mathbb{N}$ and $S \subset G$ we have

$$\{g \in S \,:\, |g| \leqslant i\} \subset \{g \in S \,:\, \hbar(g) \leqslant k_0 i\},$$

whence

$$\frac{1}{i} \log |\{g \in S \,:\, |g| \leqslant i\}| \leqslant k_0 \frac{1}{k_0 i} \log |\{g \in S \,:\, \hbar(g) \leqslant k_0 i\}|,$$

The last inequality clearly shows that every strip $S'(b_-, b_+)$ meeting the condition (3) above, also meets the condition (2) of Corollary 1.

Thus, the map S satisfies all the requirements of Corollary 1 and hence (B_+, λ_+) is a Poisson–Furstenberg boundary of the pair (G, μ). □

2.2 $\mathbb{Z}^n$-free groups

Here we give some basic definitions in the theory of Λ-trees.

2.2.1 Λ-trees

A set Λ equipped with addition "+" and a partial order "$\leqslant$" is called a *partially ordered* abelian group if

(1) $\langle \Lambda, + \rangle$ is an abelian group,

(2) $\langle \Lambda, \leqslant \rangle$ is a partially ordered set,

(3) for all $\alpha, \beta, \gamma \in \Lambda$, $\alpha \leqslant \beta$ implies $\alpha + \gamma \leqslant \beta + \gamma$.

An abelian group Λ is called *orderable* if there exists a linear order $\leqslant$ on Λ, satisfying the condition (3) above. In general, the ordering on Λ is not unique.

An ordered abelian group Λ is called *discretely ordered* if Λ has a minimal positive element. With a slight abuse of notation we denote it by 1, but it is going to be clear from the context if 1 represents a natural number, or an element of Λ. In this event, for any $\alpha \in \Lambda$ the following hold:

(1) $\alpha + 1 = \min\{\beta \mid \beta > \alpha\}$,

(2) $\alpha - 1 = \max\{\beta \mid \beta < \alpha\}$.

For elements $\alpha, \beta \in \Lambda$ the *closed segment* $[\alpha, \beta]$ is defined by

$$[\alpha, \beta] = \{\gamma \in \Lambda \mid \alpha \leqslant \gamma \leqslant \beta\}.$$

Now a subset $C \subset \Lambda$ is called *convex* if for every $\alpha, \beta \in C$ the set C contains $[\alpha, \beta]$. In particular, a subgroup C of Λ is convex if $[0, \beta] \subset C$ for every positive $\beta \in C$.

Let X be a non-empty set and Λ an ordered abelian group. A Λ-*metric on* X is a mapping $p : X \times X \longrightarrow \Lambda$ such that for all $x, y, z \in X$

(M1) $p(x, y) \geqslant 0$,

(M2) $p(x, y) = 0$ if and only if $x = y$,

(M3) $p(x, y) = p(y, x)$,

(M4) $p(x, y) \leqslant p(x, z) + p(y, z)$.

A Λ-*metric space* is a pair (X, p), where X is a non-empty set and p is a Λ-metric on X. If (X, p) and (X', p') are Λ-metric spaces, an *isometry* from (X, p) to (X', p') is a mapping $f : X \to X'$ such that $p(x, y) = p'(f(x), f(y))$ for all $x, y \in X$.

A *segment* in a Λ-metric space is the image of an isometry $\alpha : [a, b]_\Lambda \to X$ for some $a, b \in \Lambda$ and $[a, b]_\Lambda$ is a segment in Λ. The endpoints of the segment are $\alpha(a), \alpha(b)$.

We call a Λ-metric space (X, p) *geodesic* if for all $x, y \in X$, there is a segment in X with endpoints x, y and (X, p) is *geodesically linear* if for all $x, y \in X$, there is a unique segment in X whose set of endpoints is $\{x, y\}$. We denote such a segment by $[x, y]$.

A Λ-*tree* is a Λ-metric space (X, p) with $X \neq \emptyset$ such that

(T1) (X, p) is geodesically linear,

(T2) if $x, y, z \in X$ then $[x, y] \cap [x, z] = [x, w]$ for some $w \in X$; this w is unique and we write $w = Y(y, x, z)$,

(T3) if $x, y, z \in X$ then $[x, y] \cap [y, z] = \{y\}$ then $[x, y] \cup [y, z] = [x, z]$.

Let X be a Λ-tree. We call $e \in X$ an *end point* of X if, whenever $e \in [x, y] \subset X$ either $e = x$ or $e = y$. A *linear subtree from* $x \in X$ or a *geodesic ray* $x \in X$ is any linear subtree L of X having x as an end point. L carries a natural linear ordering with x as least element. If $y \in L$ then $L_y = \{z \in L \mid y \leqslant z\}$ is a linear subtree from y.

A maximal linear subtree from x is an *X-ray from x*. Observe that (Proposition 2.22 [1]) that if L, L' are X-rays from x, x' respectively such that $L \cap L' \neq \emptyset$ then $L \cap L'$ is either a closed segment, or $L \cap L' = L_v$ for some $v \in X$. In fact, we call X-rays L and L' *equivalent* if $L \cap L' = L_v$ for some $v \in X$. The equivalence classes of X-rays for this relation are called *ends* of X.

We say that *group G acts on a Λ-tree X* if every element $g \in G$ defines an isometry $g : X \to X$.

Note, that every group has a trivial action on a Λ-tree, that is, all its elements act as identity.

Let a group G act as isometries on a Λ-tree X. $g \in G$ is called *elliptic* if it has a fixed point. $g \in G$ is called an *inversion* if it does not have a fixed point, but g^2 does. If g is not elliptic and not an inversion then it is called *hyperbolic*. For a hyperbolic element $g \in G$ define a characteristic set

$$Axis(g) = \{p \in X \mid [g^{-1} \cdot p, p] \cap [p, g \cdot p] = \{p\}\},$$

which is called the *axis of g*. $Axis(g)$ meets every $\langle g \rangle$-invariant subtree of X.

A group G acts *freely* and *without inversions* on a Λ-tree X if for all $1 \neq g \in G$, g acts as a hyperbolic isometry. In this case we also say that G is Λ-free.

2.2.2 Length functions

Let G be a group and Λ be an ordered abelian group. Then a function $l : G \to \Lambda$ is called a *(Lyndon) length function* on G if the following conditions hold

(L1) $\forall\ x \in G :\ l(x) \geqslant 0$ and $l(1) = 0$,

(L2) $\forall\ x \in G :\ l(x) = l(x^{-1})$,

(L3) $\forall\ x, y, z \in G :\ c(x, y) > c(x, z) \to c(x, z) = c(y, z)$,
where $c(x, y) = \frac{1}{2}(l(x) + l(y) - l(x^{-1}y))$.

It is not difficult to derive the following two properties of Lyndon length functions from the axioms (L1)–(L3)

- $\forall\ x, y \in G :\ l(xy) \leqslant l(x) + l(y)$,
- $\forall\ x, y \in G :\ 0 \leqslant c(x, y) \leqslant min\{l(x), l(y)\}$.

The axiom below helps to describe the connection between Λ-valued Lyndon length functions and actions on Λ-trees.

(L4) $\forall\ x \in G :\ c(x, y) \in \Lambda$.

Theorem 4 ([5]). *Let G be a group and $l : G \to \Lambda$ a Lyndon length function satisfying* (L4). *Then there are a Λ-tree (X, p), an action of G on X and a point $x \in X$ such that $l = l_x$.*

2.2.3 Infinite words

Let Λ be an ordered abelian group, that is, there is a linear order "$\leqslant$" on Λ such that for all $a, b, c \in \Lambda$, $a \leqslant b$ implies $a + c \leqslant b + c$. Recall that for elements $\alpha, \beta \in \Lambda$ the *closed segment* $[\alpha, \beta]$ is defined by

$$[\alpha, \beta] = \{\gamma \in \Lambda \mid \alpha \leqslant \gamma \leqslant \beta\}.$$

Now a subset $C \subset \Lambda$ is called *convex* if for every $\alpha, \beta \in C$ the set C contains $[\alpha, \beta]$. In particular, a subgroup C of Λ is convex if $[0, \beta] \subset C$ for every positive $\beta \in C$.

An ordered abelian group Λ is called *discretely ordered* if Λ has a minimal positive element. With a slight abuse of notation we denote it by 1, but it is going to be clear from the context if 1 represents a natural number, or an element of Λ. In this event, for any $\alpha \in \Lambda$ the following hold:

(1) $\alpha + 1 = \min\{\beta \mid \beta > \alpha\}$,

(2) $\alpha - 1 = \max\{\beta \mid \beta < \alpha\}$.

Observe that if Λ is any ordered abelian group then $\mathbb{Z} \oplus \Lambda$ is discretely ordered with respect to the right lexicographic order ($(a_1, b_1) < (a_2, b_2) \Leftrightarrow b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$).

Let Λ be a discretely ordered abelian group and let $X = \{x_i \mid i \in I\}$ be a set. Put $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^{\pm} = X \cup X^{-1}$. An Λ-*word* is a function of the type

$$w : [1, \alpha_w] \to X^{\pm},$$

where $\alpha_w \in \Lambda$, $\alpha_w \geqslant 0$. The element α_w is called the *length* $|w|$ of w.

By $W(\Lambda, X)$ we denote the set of all Λ-words over $X^{\pm}$. Observe, that $W(\Lambda, X)$ contains an empty Λ-word which we denote by ε.

Concatenation uv of two Λ-words $u, v \in W(\Lambda, X)$ is a Λ-word of length $|u| + |v|$ and such that:

$$(uv)(\alpha) = \begin{cases} u(\alpha) & \text{if } 1 \leqslant \alpha \leqslant |u| \\ v(\alpha - |u|) & \text{if } |u| < \alpha \leqslant |u| + |v| \end{cases}$$

Next, for any Λ-word w we define an *inverse* w^{-1} as an Λ-word of the length $|w|$ and such that

$$w^{-1}(\beta) = w(|w| + 1 - \beta)^{-1} \quad (\beta \in [1, |w|]).$$

A Λ-word w is *reduced* if $w(\beta + 1) \neq w(\beta)^{-1}$ for each $1 \leqslant \beta < |w|$. We denote by $R(\Lambda, X)$ the set of all reduced Λ-words. Clearly, $\varepsilon \in R(\Lambda, X)$. If the concatenation uv of two reduced Λ-words u and v is also reduced then we write $uv = u \circ v$.

For $u \in W(\Lambda, X)$ and $\beta \in [1, |u|]$ by u_β we denote the restriction of u on $[1, \beta]$. If $u \in R(\Lambda, X)$ and $\beta \in [1, |u|]$ then

$$u = u_\beta \circ \tilde{u}_\beta,$$

for some uniquely defined $\tilde{u}_\beta$.

An element $\mathrm{com}(u, v) \in R(\Lambda, X)$ is called the (*longest*) *common initial segment* of reduced Λ-words u and v if

$$u = \mathrm{com}(u, v) \circ \tilde{u}, \quad v = \mathrm{com}(u, v) \circ \tilde{v}$$

for some (uniquely defined) Λ-words $\tilde{u}$, $\tilde{v}$ such that $\tilde{u}(1) \neq \tilde{v}(1)$. Note that $com(u, v)$ does not always exist, if it does then denote

$$c(u, v) = |com(u, v)|.$$

Now, we can define the product of two Λ-words. Let $u, v \in R(\Lambda, X)$. If $\mathrm{com}(u^{-1}, v)$ is defined then

$$u^{-1} = \mathrm{com}(u^{-1}, v) \circ \tilde{u}, \quad v = \mathrm{com}(u^{-1}, v) \circ \tilde{v},$$

for some uniquely defined $\tilde{u}$ and $\tilde{v}$. In this event put

$$u * v = \tilde{u}^{-1} \circ \tilde{v}.$$

The product $*$ is a partial binary operation on $R(\Lambda, X)$.

An element $v \in R(\Lambda, X)$ is termed *cyclically reduced* if $v(1)^{-1} \neq v(|v|)$. We say that an element $v \in R(\Lambda, X)$ admits a *cyclic decomposition* if $v = c^{-1} \circ u \circ c$, where $c, u \in R(\Lambda, X)$ and u is cyclically reduced. Observe that a cyclic decomposition is unique (whenever it exists). We denote by $CDR(\Lambda, X)$ the set of all words from $R(\Lambda, X)$ which admit a cyclic decomposition.

Below we refer to Λ-words as *infinite words* usually omitting Λ whenever it does not produce any ambiguity.

The following result establishes the connection between infinite words and length functions.

Theorem 5 ([21]). *Let Λ be a discretely ordered abelian group and X be a set. Then any subgroup G of $CDR(\Lambda, X)$ has a free Lyndon length function with values in Λ – the restriction $L|_G$ on G of the standard length function L on $CDR(\Lambda, X)$.*

The converse of the theorem above was obtained by Chiswell [7].

Theorem 6 ([7]). *Let G have a free Lyndon length function $L : G \to A$, where Λ is a discretely ordered abelian group. Then there exists a set X and a length preserving embedding $\phi : G \to CDR(\Lambda, X)$, that is, $|\phi(g)| = L(g)$ for any $g \in G$.*

Corollary 3 ([7]). *Let G have a free Lyndon length function $L : G \to \Lambda$, where Λ is an arbitrary ordered abelian group. Then there exists an embedding $\phi : G \to CDR(\Lambda', X)$, where $\Lambda' = \mathbb{Z} \oplus \Lambda$ is discretely ordered with respect to the right lexicographic order and X is some set, such that, $|\phi(g)| = (0, L(g))$ for any $g \in G$.*

2.3 Universal trees

Let G be a subgroup of $CDR(\Lambda, X)$ for some discretely ordered abelian group Λ and a set X. We assume G, Λ, and X to be fixed for the rest of this section.

Briefly recall (see [16] for details) how one can construct a universal Λ-tree Γ_G for G. Every element $g \in G$ is a function

$$g : [1, |g|] \to X^{\pm},$$

with the domain $[1, |g|]$ which is a closed segment in Λ. Since Λ can be viewed as a Λ-metric space then $[1, |g|]$ is a geodesic connecting 1 and $|g|$, and every $\alpha \in [1, |g|]$ can be represented as a pair (α, g). Let

$$S_G = \{(\alpha, g) \mid g \in G, \alpha \in [0, |g|]\}.$$

Since for every $f, g \in G$ the word $\mathrm{com}(f, g)$ is defined, one can introduce an equivalence relation on S_G as follows: $(\alpha, f) \sim (\beta, g)$ if and only if $\alpha = \beta \in [0, c(f, g)]$. Now, let $\Gamma_G = S_G/\sim$ and $\epsilon = \langle 0, 1\rangle$, where $\langle \alpha, f\rangle$ is the equivalence class of (α, f). It was shown in [16] that Γ_G is a Λ-tree with a designated vertex ϵ and a metric $d : \Gamma_G \times \Gamma_G \to \Lambda$, on which G acts by Λ-isometries so that for every $g \in G$ the distance $d(\epsilon, g \cdot \epsilon)$ is exactly $|g|$. Moreover, Γ_G is equipped with the labeling function $\xi : (\Gamma_G \setminus \{\epsilon\}) \to X^{\pm}$, where $\xi(v) = g(\alpha)$ if $v = \langle \alpha, g\rangle$.

Next, for every $v_0, v_1 \in \Gamma_G$ such that $d(v_0, v_1) = 1$ we call the ordered pair (v_0, v_1) the *edge* from v_0 to v_1. Here, if $e = (v_0, v_1)$ then denote $v_0 = o(e)$, $v_1 = t(e)$ which are respectively the *origin* and *terminus* of e. There exists a natural orientation, with respect to ϵ, of edges in Γ_G, where an edge (v_0, v_1) is *positive* if $d(\epsilon, v_1) = d(\epsilon, v_0) + 1$, and *negative* otherwise. Denote by $E(\Gamma_G)$ the set of edges in Γ_G. If $e \in E(\Gamma_G)$ and $e = (v_0, v_1)$ then the pair (v_1, v_0) is also an edge and denote $e^{-1} = (v_1, v_0)$. Obviously, $o(e) = t(e^{-1})$. Because of the orientation, we have a natural splitting

$$E(\Gamma_G) = E(\Gamma_G)^+ \cup E(\Gamma_G)^-,$$

where $E(\Gamma_G)^+$ and $E(\Gamma_G)^-$ denote respectively the sets of positive and negative edges. Now, we can define a function $\sigma : E(\Gamma_G)^+ \to X^{\pm}$ as follows: if $e = (v_0, v_1) \in E(\Gamma_G)^+$ then $\sigma(e) = \xi(v_1)$. Next, σ can be extended to $E(\Gamma_G)^-$ (and hence to $E(\Gamma_G)$) by setting $\sigma(f) = \sigma(f^{-1})^{-1}$ for every $f \in E(\Gamma_G)^-$.

Example 1. *Let $F = F(X)$ be a free group on X. Hence, F embeds into (coincides with) $CDR(\mathbb{Z}, X)$ and Γ_F with the labeling σ defined above is just the Cayley graph of F with respect to X. That is, Γ_F is a labeled simplicial tree.*

The action of G on Γ_G induces the action on $E(\Gamma_G)$ as follows $g \cdot (v_0, v_1) = (g \cdot v_0, g \cdot v_1)$ for each $g \in G$ and $(v_0, v_1) \in E(\Gamma_G)$. It is easy to see that $E(\Gamma_G)^+$ is not closed under the action of G but the labeling is equivariant as the following lemma shows.

Lemma 4. *If $e, e' \in E(\Gamma_G)$ belong to one G-orbit then $\sigma(e) = \sigma(e')$.*

Proof. Let $e = (v_0, v_1) \in E(\Gamma_G)^+$. Hence, there exists $g \in G$ such that $v_0 = \langle \alpha, g\rangle$, $v_1 = \langle \alpha + 1, g\rangle$. Let $f \in G$ and consider the following cases.

Case 1. $c(f^{-1}, g) = 0$

Then $f * g = f \circ g$. Assume that $\alpha = 0$ (the case when $\alpha \neq 0$ is considered similarly). Then $f \cdot v_0 = \langle |f|, f\rangle = \langle |f|, f \circ g\rangle$, and $f \cdot v_1 = \langle |f| + 1, f \circ g\rangle$. Hence, $f \cdot e \in E(\Gamma_G)^+$ and $\sigma(f \cdot e) = \xi(f \cdot v_1) = g(1) = \xi(v_1) = \sigma(e)$.

Case 2. $c(f^{-1}, g) > 0$

(a) $\alpha + 1 \leqslant c(f^{-1}, g)$

Then $f \cdot v_0 = \langle |f| + \alpha - 2\alpha, f\rangle = \langle |f| - \alpha, f\rangle$ and $f \cdot v_1 = \langle |f| - (\alpha + 1), f\rangle$. So, $d(\epsilon, f \cdot v_1) < d(\epsilon, f \cdot v_0)$ and $f \cdot e \in E(\Gamma_G)^-$. Now,

$$\sigma(f \cdot e) = \sigma((f \cdot e)^{-1})^{-1} = \sigma((f \cdot v_1, f \cdot v_0))^{-1} = \xi(f \cdot v_0)^{-1} = f(|f| - \alpha)^{-1}$$

$$= g(\alpha + 1) = \xi(v_1) = \sigma(e).$$

(b) $\alpha = c(f^{-1}, g)$

We have $f \cdot v_0 = \langle |f| - \alpha, f\rangle$ and $f \cdot v_1 = \langle |f| + (\alpha + 1) - 2c(f^{-1}, g), f * g\rangle = \langle |f| - \alpha + 1, f * g\rangle$. It follows that $f \cdot e \in E(\Gamma_G)^+$ and $\sigma(f \cdot e) = \xi(f \cdot v_1) =$

$(f*g)(|f|-\alpha+1)$. At the same time, $f*g = f_1 \circ g_1$, where $|f_1| = |f| - c(f^{-1}, g) = |f| - \alpha$, $g = g_0 \circ g_1$, $|g_0| = \alpha$, so, $(f*g)(|f| - \alpha + 1) = g_1(1) = g(\alpha+1)$ and $\sigma(f \cdot e) = g(\alpha+1) = \xi(\langle \alpha+1, g\rangle) = \xi(v_1) = \sigma(e)$.

(c) $\alpha > c(f^{-1}, g)$

Hence, $f \cdot v_0 = \langle |f| + \alpha - 2c(f^{-1}, g), f*g\rangle$ and $f \cdot v_1 = \langle |f| + \alpha + 1 - 2c(f^{-1}, g), f*g\rangle$. Obviously, $f \cdot e \in E(\Gamma_G)^+$ and

$$\sigma(f \cdot e) = \xi(f \cdot v_1) = (f*g)(|f| + \alpha + 1 - 2c(f^{-1}, g)) = g_1(\alpha + 1 - c(f^{-1}, g))$$

$$= g(\alpha+1) = \xi(v_1) = \sigma(e),$$

where $f*g = f_1 \circ g_1$, $|f_1| = |f| - c(f^{-1}, g) = |f| - \alpha$, $g = g_0 \circ g_1$, $|g_0| = \alpha$.

Thus, in all possible cases we got $\sigma(f \cdot e) = \sigma(e)$ and the required statement follows. □

Let v, w be two points of Γ_G. Since Γ_G is a Λ-tree there exists a unique geodesic connecting v to w, which can be viewed as a "path" in the following sense. A *path from v to w* is a sequence of edges $p = \{e_\alpha\}$, $\alpha \in [1, d(v,w)]$ such that $o(e_1) = v$, $t(e_{d(v,w)}) = w$ and $t(e_\alpha) = o(e_{\alpha+1})$ for every $\alpha \in [1, d(v,w) - 1]$. In other words, a path is an "edge" counter-part of a geodesic and usually, for the path from v to w (which is unique since Γ_G is a tree) we are going to use the same notation as for the geodesic between these points, that is, $p = [v, w]$. In the case when $v = w$ the path p is empty. The *length* of p we denote by $|p|$ and set $|p| = d(v,w)$. Now, the *path label* $\sigma(p)$ for a path $p = \{e_\alpha\}$ is the function $\sigma : \{e_\alpha\} \to X^{\pm}$, where $\sigma(e_\alpha)$ is the label of the edge e_α.

Lemma 5. *Let v, w be points of Γ_G and p the path from v to w. Then $\sigma(p) \in R(\Lambda, X)$.*

Proof. From the definition of Γ_G it follows that the statement is true when $v = \epsilon$. Let $v_0 = Y(\epsilon, v, w)$ and let p_v and p_w be the paths from ϵ respectively to v and w. Also, let p_1 and p_2 be the paths from v_0 to v and w. Since $\sigma(p_v), \sigma(p_w) \in R(\Lambda, X)$ then $\sigma(p_1), \sigma(p_2) \in R(\Lambda, X)$ as subwords. Hence, $\sigma(p) \notin R(\Lambda, X)$ implies that the first edges e_1 and e_2 correspondingly of p_1 and p_2 have the same label. But this contradicts the definition of Γ_G because in this case $t(e_1) \sim t(e_2)$, but $t(e_1) \neq t(e_2)$. □

As usual, if p is a path from v to w then its *inverse* denoted p^{-1} is a path from w back to v. In this case, the label of p^{-1} is $\sigma(p)^{-1}$, which again an element of $R(\Lambda, X)$.

Define

$$V_G = \{v \in \Gamma_G \mid \exists\, g \in G : \ v = \langle |g|, g\rangle\},$$

which is a subset of points in Γ_G corresponding to the elements of G. Also, for every $v \in \Gamma_G$ let

$$path_G(v) = \{\sigma(p) \mid \ p = [v, w] \text{ where } w \in V_G\}.$$

The following lemma follows immediately.

Lemma 6. *Let $v \in V_G$. Then $path_G(v) = G \subset CDR(\Lambda, X)$.*

The action of G on $E(\Gamma_G)$ extends to the action on all paths in Γ_G, hence, Lemma 4 extends to the case when e and f are two G-equivalent paths in Γ_G.

3 From $\mathbb{Z}^n$-trees to metric spaces

Let G be a $\mathbb{Z}^n$-free group and Γ_G the associated universal $\mathbb{Z}^n$-tree described in Subsection 2.3. In this section we construct a metric space X_G, which is a compactification of Γ_G, equipped with an action of G without a global fixed point.

3.1 Compactification of a simplicial tree

Let T be a locally finite simplicial tree (that is, $\mathbb{Z}$-tree) and let d_T be a usual metric on T. That is, if $x, y \in T$ then there exists a unique geodesic path $p(x, y)$ connecting them and hence $d_T(x, y) = |p(x, y)|$, where $|p(x, y)|$ is the length of the path $p(x, y)$. It is easy to see that d_T is an ultrametric on T.

For $x, y, z \in T$ let $(x \cdot y)_z = \frac{1}{2}(d_T(x, z) + d_T(y, z) - d_T(x, y))$.

Let $x \in T$. Recall (see Subsection 2.2.1) that a T-ray, or a *geodesic ray* from x is an isometry $r : \mathbb{Z}^+ \to T$ such that $r(1) = x$. Two geodesic rays r_1, r_2 are equivalent ($r_1 \sim r_2$) if the intersection $r_1 \cap r_2$ is also a geodesic ray. It is easy to check that "$\sim$" is an equivalence relation on the set of all geodesic rays in T; let ∂T be the set of equivalent classes. For $x \in T$ and $a \in \partial T$ denote by $[x, a)$ the geodesic ray from x to a.

Similarly, for $a, b \in \partial T$ a *geodesic line from a to b*, denoted (a, b), is an isometry $r : \mathbb{Z} \to T$ such that for every $n \in \mathbb{Z}$ the restriction $r|_{(-\infty,n]}$ is a geodesic ray from $r(n)$ to a, and the restriction $r|_{[n,\infty)}$ is a geodesic ray from $r(n)$ to b.

Let $w \in T$. It is easy to see that every $a \in \partial T$ can be put in correspondence with a unique infinite geodesic ray $w, w_1, w_2, \ldots,\ w_i \in T$ from w. Now, let $a, b \in \partial T$ corresponding to rays $w, x_1, x_2, \ldots$ and $w, y_1, y_2, \ldots$ and define

$$(a \cdot b)_w = \lim_{n\to\infty} (x_i \cdot y_i)_w.$$

Similarly, for $a \in \partial T$ represented by a geodesic ray $w, x_1, x_2, \ldots$ and $y \in T$ we define

$$(a \cdot y)_w = \lim_{n\to\infty} (x_i \cdot y)_w.$$

Consider $\overline{T} = T \cup \partial T$ and define a function $d' : \overline{T} \times \overline{T} \to \mathbb{R}$ as follows

$$d'(x, y) = e^{-(x\cdot y)_w},\ \ x, y \in \overline{T}.$$

Lemma 7. *For every $x, y, z \in \overline{T}$, the function d' satisfies the following properties*

(a) $d'(x, y) \geqslant 0$,

(b) $d'(x, y) = 0 \quad \Leftrightarrow \quad x = y \in \partial T$,

(c) $d'(x, y) = d'(y, x)$,

(d) $d'(x, y) \leqslant \max\{d'(x, z), d'(y, z)\}$.

Proof. (a), (b) and (c) follow immediately from the definition.

To prove (d) notice that if $x \neq y \in \partial T$ are represented by the geodesic rays $w, x_1, x_2, \ldots$ and $w, y_1, y_2, \ldots$ then there exists $N > 0$ such that $(x \cdot y)_w = (x_n \cdot y_m)_w$ for every $n, m > N$ (otherwise $w, x_1, x_2, \ldots \ \sim\ w, y_1, y_2, \ldots$ and $x = y$). The same holds for $x \in \partial T$ and $y \in T$, that is, there exists $N > 0$ such that $(x \cdot y)_w = (x_n \cdot y)_w$ for every $n > N$. In other words it is enough to check (d) for $x, y, z \in T$, that is, we have to show that

$$e^{-(x\cdot y)_w} \leqslant \max\{e^{-(x\cdot z)_w}, e^{-(y\cdot z)_w}\}.$$

In particular, the inequality holds if $(x \cdot y)_w = (x \cdot z)_w = (y \cdot z)_w$.

Since T is a tree then the following holds for every $x, y, z \in T$

$$(x \cdot y)_w > (x \cdot z)_w \;\Rightarrow\; (x \cdot z)_w = (y \cdot z)_w.$$

Thus we can consider the following cases

(1) $(x \cdot y)_w > (x \cdot z)_w$ or $(x \cdot y)_w > (y \cdot z)_w$.
Immediately $e^{-(x \cdot y)_w} < e^{-(x \cdot z)_w} = e^{-(y \cdot z)_w}$.

(2) $(x \cdot z)_w > (x \cdot y)_w$ or $(x \cdot z)_w > (y \cdot z)_w$.
Then $(x \cdot y)_w = (y \cdot z)_w$ and $e^{-(x \cdot y)_w} = e^{-(y \cdot z)_w} = \max\{e^{-(x \cdot z)_w}, e^{-(y \cdot z)_w}\}$.

(3) $(y \cdot z)_w > (x \cdot y)_w$ or $(y \cdot z)_w > (x \cdot z)_w$.
Then $(x \cdot y)_w = (x \cdot z)_w$ and $e^{-(x \cdot y)_w} = e^{-(x \cdot z)_w} = \max\{e^{-(x \cdot z)_w}, e^{-(y \cdot z)_w}\}$.

□

Eventually, define $d : \overline{T} \times \overline{T} \to \mathbb{R}$ as follows

$$d(x, y) = \begin{cases} d'(x, y) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{4}$$

Lemma 8. *d is an ultrametric on $\overline{T}$.*

Proof. Immediately follows from Lemma 7. □

It is easy to see that $d(x, y) \leqslant 1$ for every $x, y \in \overline{T}$. In particular, $d(x, y) = 1$ if either $x \neq y \in \partial T$ and (x, y) is a geodesic line from x to y which passes through w, or $x \in T,\ y \in \partial T$ and the geodesic ray $[x, y)$ contains w.

Let $x \in T$. It is easy to see that for every $\varepsilon > 0$ the ball $B_\varepsilon(x)$ of radius ε centered at x is a subset of $\overline{T}$ of the following type

$$B_\varepsilon(x) = \{x\} \cup \{y \in \overline{T} \mid (x \cdot y)_w \geqslant -\ln \varepsilon\}.$$

Hence, if z is the closest to w vertex of the segment $[w, x]$ such that $(x \cdot z)_w = d(w, z) \geqslant -\ln \varepsilon$ then $B_\varepsilon(x) = T_z \cup \partial T_z$. If such z does not exist, that is, if $d(w, x) < -\ln \varepsilon$ for every $t \in [w, x]$ then $B_\varepsilon(x) = \{x\}$.

Similarly, for $x \in \partial T$. Here, for every $\varepsilon > 0$ we have

$$B_\varepsilon(x) = \{y \in \overline{T} \mid (x \cdot y)_w \geqslant -\ln \varepsilon\} = T_z \cup \partial T_z,$$

where z is the closest to w vertex of the geodesic ray $[w, x)$ such that $(x \cdot z)_w = d(w, z) \geqslant -\ln \varepsilon$. Observe that $B_\varepsilon(x)$ contains x.

Using the above considerations we prove the following lemma.

Lemma 9. *$\overline{T}$ is compact in the topology induced by the metric d.*

Proof. It is enough to show that $\overline{T}$ is totally bounded and complete.

Let $\varepsilon > 0$. Let K be a collection of points $x \in T$ such that $d(w, x) \geqslant -\ln \varepsilon$ and x is the closest to w point with this property. Then K is finite and $T_x \cup \partial T_x = B_\varepsilon(x)$ for every $x \in K$. Moreover, $T_0 = \overline{T} - \bigcup_{x \in K}(T_x \cup \partial T_x)$ is also finite and for every $y \in T_0$ one can choose $\varepsilon_y < \varepsilon$ such that $B_{\varepsilon_y}(y) = \{y\}$. It follows that $\overline{T}$ can be covered by finitely many $B_\varepsilon(x),\ x \in K$ together with finitely many $B_{\varepsilon_y}(y),\ y \in T_0$. Hence, $\overline{T}$ is totally bounded.

To prove completeness of d consider a Cauchy sequence $\{x_n\} \subset \overline{T}$, that is, for every $\varepsilon > 0$ there exists $N > 0$ such that for every $n, m \geqslant N$ we have $d(x_m, x_n) < \varepsilon$, or $(x_m \cdot x_n)_w \geqslant -\ln \varepsilon$. It follows that for every $\varepsilon > 0$ there exists $N > 0$ such that for every $n \geqslant N$ we have $d(x_n, w) \geqslant -\ln \varepsilon$ and $x_n \in T_{x_N} \cup \partial T_{x_N}$. It follows that $\{T_{x_N} \bigcup \partial T_{x_N}\}$ is a nested sequence of subsets of $\overline{T}$. Hence, take $a = \bigcap_{N=1}^{\infty} (T_{x_N} \bigcup \partial T_{x_N}) \in \overline{T}$ and it follows that $x_n \to a$ as $n \to \infty$. So, $\overline{T}$ is complete and thus compact.

□

Observe that the metric d defined in (4) works for every simplicial tree.

3.2 The case of a $\mathbb{Z}^2$-tree

Let Γ be a $\mathbb{Z}^2$-tree with the metric $d_\Gamma : \Gamma \times \Gamma \to \mathbb{Z}^2$. $v, w \in \Gamma$ are $\mathbb{Z}$-*equivalent* ($v \sim_{\mathbb{Z}} w$) if $d(v, w) \in \mathbb{Z}$, that is, $d(v, w) = (a, 0)$. From metric axioms it follows that "$\sim_{\mathbb{Z}}$" is an equivalence relation and every equivalence class defines a $\mathbb{Z}$-*subtree* (which is a simplicial tree) of Γ. For each $\mathbb{Z}$-subtree T of Γ denote by $Ends(T)$ the set of ends of T.

Let $\Delta = \Gamma / \sim_{\mathbb{Z}}$. Δ naturally inherits the tree-structure from Γ and in fact Δ is a simplicial tree (not locally finite, in general). Denote by d_Δ the induced metric on Δ. For a $\mathbb{Z}$-subtree T of Γ denote by v_T the corresponding vertex in Δ.

Let us fix a $\mathbb{Z}$-subtree T_0 of Γ. Then T_0 corresponds to a vertex v_{T_0} in Δ and now one can define the levels of $\mathbb{Z}$-subtrees of Γ using Δ as follows. A $\mathbb{Z}$-subtree T belongs to the *k-level (with respect to T_0)* of $\mathbb{Z}$-subtrees of Γ if $d_\Delta(v_{T_0}, v_T) = k$. Also, a $\mathbb{Z}$-subtree T' of Γ is *adjacent* to T if $d_\Delta(v_{T'}, v_T) = 1$. Observe that the edge between $v_{T'}$ and v_T in Δ corresponds to the pair of ends $(\alpha_{T'}, \alpha_T)$ in the adjacent trees T' and T.

Similarly to simplicial trees, one can define the set of ends of Γ. This set consists of the ends of each $\mathbb{Z}$-subtree of Γ together with the ends of $\mathbb{Z}^2$-type, that is, the equivalence classes of maximal linear subtrees of Γ (each maximal liner subtree of Γ is isometric to $[0, \infty)_{\mathbb{Z}^2} = \{a \in \mathbb{Z}^2 \mid a \geqslant 0\}$) defined exactly as in the case of simplicial trees. Hence, we have

$$End_{\mathbb{Z}}(\Gamma) = \bigcup \{Ends(T) \mid T \text{ is a } \mathbb{Z}-\text{subtree of } \Gamma\},$$

$$End_{\mathbb{Z}^2}(\Gamma) = \{\text{maximal linear subtrees of } \Gamma\} / \sim,$$

and

$$Ends(\Gamma) = End_{\mathbb{Z}}(\Gamma) \bigcup End_{\mathbb{Z}^2}(\Gamma).$$

Now we construct a compactification $\overline{\Gamma}$ of Γ as follows.

$$\overline{\Gamma} = \Gamma \bigcup (Ends(\Gamma) / \sim),$$

where $\alpha_{T'} \sim \alpha_T$ if $\alpha_{T'} \in Ends(T')$, $\alpha_T \in Ends(T)$ and T', T are adjacent $\mathbb{Z}$-subtrees of Γ. In other words

$$\overline{\Gamma} = \left(\bigcup \{\overline{T} \mid T \text{ is a } \mathbb{Z}-\text{subtree of } \Gamma\} \right) / \sim .$$

For each $\mathbb{Z}$-subtree T of Γ, the compactification $\overline{T}$ embeds into $\overline{\Gamma}$ so we can identify $\overline{T}$ with its image in $\overline{\Gamma}$. Thus, if T and T' are adjacent $\mathbb{Z}$-subtrees then $\overline{T} \cap \overline{T'} = \partial T \cap \partial T' = \{a\}$, where a is the identified $\alpha_{T'}$ and α_T.

The next step is to introduce a metric on $\overline{\Gamma}$. Using the construction of Subsection 3.1 we define a metric on the compactification of each $\mathbb{Z}$-subtree of Γ as follows. Here we assume that d is the metric on the compactification of an arbitrary simplicial tree T defined in (4). Let $\mathcal{T}_1$ be the set of all $\mathbb{Z}$-subtrees of the 1-level in Γ. $\mathcal{T}_1$ is countable and we can enumerate its elements in an arbitrary way. Thus we set

$$d_{\overline{T}_0}(x,y) = d(x,y),\ \ x,y \in \overline{T}_0$$

and

$$d_{\overline{T}_n}(x,y) = \frac{1}{2^n} d(x,y),\ \ x,y \in \overline{T}_n$$

for each $T_n \in \mathcal{T}_1$. Now we continue by induction on the level of $\mathbb{Z}$-subtrees in Γ. Suppose that the metric is defined on the compactifications of $\mathbb{Z}$-subtrees of the k-level. Let S be a $\mathbb{Z}$-subtree of Γ of the k-level, so, by the induction hypothesis the metric $d_{\overline{S}} = \frac{1}{2^N} d$ for some $N \in \mathbb{N}$ is already constructed on S. Let $\mathcal{T}_S$ be the set of all $\mathbb{Z}$-subtrees of the $k+1$-level in Γ attached to S. Again $\mathcal{T}_S$ is countable and we can enumerate its elements in an arbitrary way. Hence, we set

$$d_{\overline{T}_n}(x,y) = \frac{1}{2^{N+n}} d(x,y),\ \ x,y \in \overline{T}_n$$

for each $T_n \in \mathcal{T}_S$.

Thus having defined the metric on each $\overline{T}$ we define a function $d_{\overline{\Gamma}} : \overline{\Gamma} \times \overline{\Gamma} \to \mathbb{R}$ as follows. Let $x, y \in \overline{\Gamma}$ and we consider three cases.

(i) If $x, y \in \overline{\Gamma} - End_{\mathbb{Z}^2}(\Gamma)$ then there exists a sequence $T_1, \ldots, T_n$ of $\mathbb{Z}$-subtrees of Γ such that $x \in \overline{T}_1,\ y \in \overline{T}_n$ and T_i is adjacent to T_{i+1} for each $i \in [1, n-1]$. Denote $a_i = \overline{T}_i \cap \overline{T}_{i+1},\ i \in [1, n-1]$. Observe that the sequence $T_1, \ldots, T_n$ corresponds to a geodesic $v_{T_1}, \ldots, v_{T_n}$ in Δ. Now set

$$d_{\overline{\Gamma}}(x,y) = d_{\overline{T}_1}(x, a_1) + \sum_{i=1}^{n-2} d_{\overline{T}_{i+1}}(a_i, a_{i+1}) + d_{\overline{T}_n}(a_{n-1}, y).$$

(ii) If $x \in \overline{\Gamma} - End_{\mathbb{Z}^2}(\Gamma),\ y \in End_{\mathbb{Z}^2}(\Gamma)$ then there exists an infinite sequence $T_1, \ldots, T_n, \ldots$ of $\mathbb{Z}$-subtrees of Γ with T_i adjacent to T_{i+1} for each $i \geqslant 1$ such that $x \in \overline{T}_1$ and the sequence of linear subtrees $[x, a_k)$ converges to the linear subtree $[x, y)$ as $k \to \infty$ (here $a_i = \overline{T}_i \cap \overline{T}_{i+1},\ i \geqslant 1$). Now set

$$d_{\overline{\Gamma}}(x,y) = d_{\overline{T}_1}(x, a_1) + \sum_{i=1}^{\infty} d_{\overline{T}_{i+1}}(a_i, a_{i+1}).$$

(iii) If $x, y \in End_{\mathbb{Z}^2}(\Gamma)$ then there exists an infinite sequence $\ldots, S_{-n}, \ldots, S_0, \ldots, S_n, \ldots$ of $\mathbb{Z}$-subtrees of Γ with S_i is adjacent to S_{i+1} for each i such that the sequence of linear subtrees (a_{-k}, a_k) converges to the linear subtree (x, y) as $k \to \infty$ (here $a_i = \overline{S}_i \cap \overline{S}_{i+1},\ i \in \mathbb{N}$). Now set

$$d_{\overline{\Gamma}}(x,y) = \sum_{i=-\infty}^{\infty} d_{\overline{S}_{i+1}}(a_i, a_{i+1}).$$

Observe that $d_{\overline{\Gamma}}(x,y) \leqslant 4$ for every $x, y \in \overline{\Gamma}$, it follows from the construction. Indeed, the maximal possible value of $d_{\overline{\Gamma}}(x,y)$ is attained when $x, y \in End_{\mathbb{Z}^2}(\Gamma)$ and the infinite sequence of $\mathbb{Z}$-subtrees of Γ connecting x and y contains T_0 - in this case $d_{\overline{\Gamma}}(x,y) \leqslant 2\sum_{n=0}^{\infty} \frac{1}{2^n} = 4$.

Lemma 10. *$d_{\overline{\Gamma}}$ is a metric on $\overline{\Gamma}$.*

Proof. The properties

(a) $d_{\overline{\Gamma}}(x,y) \geqslant 0$,

(b) $d_{\overline{\Gamma}}(x,y) = 0 \quad \Leftrightarrow \quad x = y$,

(c) $d_{\overline{\Gamma}}(x,y) = d_{\overline{\Gamma}}(y,x)$,

follow immediately from the definition. Let us prove

(d) $d_{\overline{\Gamma}}(x,y) \leqslant d_{\overline{\Gamma}}(x,z) + d_{\overline{\Gamma}}(y,z)$.

Let $x,y,z \in \overline{\Gamma}$. Then each of these points either belongs to the compactification of some $\mathbb{Z}$-subtree of Γ, or it is an element of $End_{\mathbb{Z}^2}(\Gamma)$, and x,y,z are pairwise connected by linear subtrees (possibly infinite) in Γ. The corresponding vertices (or ends) in Δ form a (possibly infinite) tripod in Δ whose sides are $K_{x,y},\ K_{x,z},\ K_{y,z}$. If $q = K_{x,y} \cap K_{x,z} \cap K_{y,z}$ then there exists a $\mathbb{Z}$-subtree T of Γ such that $q = v_T$. It follows that

$$d_{\overline{\Gamma}}(x,y) = A + d_T(a,b) + B,\ d_{\overline{\Gamma}}(x,z) = A + d_T(a,c) + C,\ d_{\overline{\Gamma}}(y,z) = B + d_T(b,c) + C,$$

where $a,b,c \in \partial T,\ A = d_{\overline{\Gamma}}(x,a),\ B = d_{\overline{\Gamma}}(y,b),\ C = d_{\overline{\Gamma}}(z,c)$, and the triangle inequality for the triple x,y,z reduces to one for the triple a,b,c which holds since d_T is an ultrametric on T.

□

Lemma 11. *$\overline{\Gamma}$ is compact in the topology induced by the metric $d_{\overline{\Gamma}}$.*

Proof. It is enough to show that $\overline{\Gamma}$ is totally bounded and complete.

Let $\varepsilon > 0$. $\overline{T}_0$ is compact so it can be covered by finitely many balls of radius $\frac{\varepsilon}{2}$ and centered at some $y_1,\dots,y_k \in \overline{T}_0$. Hence, all points of intersection of $\overline{T}_0$ with the compactifications of the adjacent trees are covered too. Let T_N be adjacent to T_0 so that $\frac{1}{2^N} < \frac{\varepsilon}{2}$ and let $a_N = \overline{T}_0 \cap \overline{T}_N \in B_{\varepsilon/2}(y_i)$. Then $\overline{\Gamma}_{a_N} \subseteq B_{\varepsilon/2}(y_i)$. Indeed, for each $y \in \overline{\Gamma}_{a_N}$ we have $d_{\overline{\Gamma}}(x,y) < \sum_{n=0}^{\infty} \frac{1}{2^{N+n}} = \frac{1}{2^N} < \frac{\varepsilon}{2}$ and we get the required from the triangle inequality. It follows that for all but finitely many points $a = \overline{T}_0 \cap \overline{T}$, where T is adjacent to T_0, the subtree $\overline{\Gamma}_a$ is in one of finitely many balls of radius ε. If $S_1,\dots,S_m$ is the list of the adjacent to T_0 $\mathbb{Z}$-subtrees (all of which belong to the 1-level) then the procedure above can be repeated for each of them producing another finite list of the 2-level subtrees which are not covered etc. But with the growth of level the diameters of these subtrees decrease and eventually become smaller than ε. Hence, if S is such a $\mathbb{Z}$-subtree of radius $\frac{1}{2^L}$ then for every point $x \in \overline{S}$ we have $S \cup \overline{\Gamma}_x \subseteq B_x(\varepsilon)$. Thus all but finitely many $\mathbb{Z}$-subtrees of Γ can be covered by finitely many balls of radius ε. Now we are done since each of the balls left is compact, that is, $\overline{\Gamma}$ is totally bounded.

To prove completeness of $\overline{\Gamma}$ let $\{x_n\} \subset \overline{\Gamma}$ be a Cauchy sequence, that is, for every $\varepsilon > 0$ there exists $N > 0$ such that for every $n,m \geqslant N$ we have $d_{\overline{\Gamma}}(x_m,x_n) < \varepsilon$. Let Γ_0 be the subtree of Γ spanned by $\{x_n\} \cup \{x\}$, where $x \in T_0$ and consider the following cases.

(a) There exists a $\mathbb{Z}$-subtree T of Γ such that $\Gamma_0 \cap \partial T$ is infinite.

In this case there exists a subsequence $\{y_k\}$ of $\{x_n\}$ such that every segment $[x,y_k]$ intersects ∂T. If $\{a_k\} \subseteq \partial T$ then $\{a_k\}$ is a Cauchy sequence, hence it

converges to some $b \in \partial T$. But $d_{\overline{\Gamma}}(y_k, a_k) \to 0$ as $k \to \infty$, so $d_{\overline{\Gamma}}(y_k, b) \to 0$ as $k \to \infty$. It follows that $\{y_k\}$ converges to $b \in \overline{\Gamma}$ and since $\{x_n\}$ is a Cauchy sequence then $\{x_n\}$ also converges to b.

(b) The intersection $\Gamma_0 \cap \partial T$ is finite for every $\mathbb{Z}$-subtree T of Γ.

It follows that Γ_0 is locally finite and infinite, so there exists an infinite branch which converges to some $b \in End_{\mathbb{Z}^2}(\Gamma)$. Here, since the level of S_n such that $x_n \in \overline{S}_n$ grows to infinity, it follows that $d_{\overline{\Gamma}}(x_n, b) \to 0$ as $n \to \infty$ and $\{x_n\}$ also converges to b.

So, $\overline{T}$ is complete and thus compact. □

3.3 General case

Let Γ be a $\mathbb{Z}^n$-tree. Hence, similarly to the case $n = 2$ one can introduce the notion of $\mathbb{Z}^{n-1}$-subtree of Γ, the set of ends of $\mathbb{Z}^k$-type of Γ for $k \leqslant n$ etc., and then use the induction on n. That is, one can assume that the compactification on an arbitrary $\mathbb{Z}^{n-1}$-tree T is already constructed and then using rescaling argument introduce the compactification $\overline{\Gamma}$ of Γ with respect to the metric $d_{\overline{\Gamma}}$.

3.4 Action of G on $\overline{\Gamma}_G$

Let G be a $\mathbb{Z}^n$-free group and let Γ_G be the corresponding minimal tree with the base point ϵ. Let $\overline{\Gamma}_G$ be the compactification of Γ_G with respect to the metric $d_{\overline{\Gamma}_G}$ constructed in Subsection 3.3.

Theorem 7. *Let G be a non-abelian $\mathbb{Z}^n$-free group.*

(i) The action of G on Γ_G induces an action of G on $\overline{\Gamma}_G$ without a global fixed point. Moreover, no point of $\overline{\Gamma}_G$ has a finite orbit under the action of G.

(ii) The action of G on $\overline{\Gamma}_G$ is continuous with respect to the metric $d_{\overline{\Gamma}_G}$, that is, if a sequence $\{x_n\} \subset \overline{\Gamma}_G$ converges to $b \in \overline{\Gamma}_G$ then for each $g \in G$ we have $g \cdot x_n \to g \cdot b$ as $n \to \infty$.

Proof. (i) The action of G on Γ_G is minimal and free. Since every end of $\mathbb{Z}^k$-type of Γ, where $k \leqslant n$, belongs to some geodesic ray from ϵ then the action of G on Γ considered as a collection of geodesic rays from ϵ naturally extends to the action on $\overline{\Gamma}_G$. Let α be an end of $\mathbb{Z}^k$-type of Γ, where $k \leqslant n$. Then α is fixed by $g \in G$ only if $\alpha \in Axis(g)$. Moreover, if (α, β) is a pair of ends of $\mathbb{Z}^k$-type of Γ so that $\alpha \in \partial T$, $\beta \in \partial S$ and T, S are adjacent, then either both α and β are fixed by g, or both are moved by g. It follows that no end of Γ_G and hence no point of $\overline{\Gamma}_G$ is fixed by every element of G unless G is abelian and Γ_G is a linear subtree, which is impossible by the assumption.

Similarly, since the action of G on Γ_G is free, it follows that no point of Γ_G can have a finite orbit under the action of G. If $x \in \overline{\Gamma}_G \smallsetminus \Gamma_G$ then x is an end of Γ_G and, since G is infinite, it can have a finite G-orbit only if x is fixed by all elements of G. But in which case G is abelian and Γ_G is a linear subtree – a contradiction

(ii) Let $g \in G$. Without loss of generality we can assume that g does not stabilize any $\mathbb{Z}^k$-subtree of Γ for $k < n$. Let T_0 be the $\mathbb{Z}^{n-1}$-subtree containing ϵ and let $T = g \cdot T_0$. Then $d_T = \frac{1}{2^N} d_{T_0}$ and $d_{g \cdot S} = \frac{1}{2^N} d_S$ for every x_n such that $x_n \in \overline{S}$, where S is a $\mathbb{Z}^{n-1}$-subtree of Γ_G. That is, the action of g is equivalent to rescaling the metric by $\frac{1}{2^N}$ and the required statement follows. □

Recall that an action of a group on a topological space is said to be *minimal* if each orbit of the action is dense (or, equivalently, if the action has no proper closed invariant subsets)

Theorem 8. *Let G be an irreducible non-abelian $\mathbb{Z}^n$-free group. Let $\partial_n\overline{\Gamma}_G$ be the set of all ends of $\mathbb{Z}^n$-type in the compactification $\overline{\Gamma}_G$. Then the induced action of G on $\partial_n\overline{\Gamma}_G$ is minimal.*

Proof. Let $p\partial_n\overline{\Gamma}_G$. We are going to show that the closure of $G\cdot p$ coincides with $\partial_n\overline{\Gamma}_G$. Since p is an end of $\mathbb{Z}^n$-type of Γ_G, it follows that there exists a sequence $\{g_i\}$ of elements of G such that $g_i\cdot\epsilon\to p$ as $i\to\infty$.

Let $q\partial_n\overline{\Gamma}_G$. Again, since q is an end of $\mathbb{Z}^n$-type of Γ_G, there exists a sequence $\{h_j\}$ of elements of G such that $h_j\cdot\epsilon\to q$ as $j\to\infty$.

(to be completed)

□

Remark 2. (a) *If $n\geqslant 2$ then $\partial_n\overline{\Gamma}_G$ may be neither open nor closed in $\overline{\Gamma}_G$. Of course, $\partial_n\overline{\Gamma}_G$ is Borel-measurable in $\overline{\Gamma}_G$.*

(b) *Observe that the action of G on the "boundary" $\overline{\Gamma}_G\smallsetminus\Gamma_G=\bigcup_{k\in[1,n]}\partial_k\overline{\Gamma}_G$ may not be minimal. This is the case when the closure of $\partial_n\overline{\Gamma}_G$ does not coincide with $\overline{\Gamma}_G\smallsetminus\Gamma_G$. For example, let $X=\{a,b\}$ and let $u\in CDR(\mathbb{Z}^2,X)$ be such that $|u|=(0,1)$ and $u(\alpha)=a$ for every $\alpha\in[1,|u|]$. Now, consider $G=\langle a,b,u^5\rangle\subset CDR(\mathbb{Z}^2,X)$. Note that $u\notin G$. Then the $\mathbb{Z}$-type end of Γ_G represented by the sequence $\{ua^k\}$ is an element of $\partial_1\overline{\Gamma}_G$ which does not belong to the closure of $\partial_2\overline{\Gamma}_G$ (nor to the closure of $G\cdot\epsilon$)).*

Proposition 1. *Let G be an irreducible non-abelian $\mathbb{Z}^n$-free group, Γ_G the associated universal $\mathbb{Z}^n$-tree, $\overline{\Gamma}_G$ the compactification constructed above, and $\partial_n\overline{\Gamma}_G\subset\overline{\Gamma}_G$ be the set of ends of $\mathbb{Z}^n$-type. Let μ be a non-degenerate measure on G and ν a μ-stationary measure on $\overline{\Gamma}_G$. Then*

(i) *ν is continuous,*

(ii) *$\nu(\partial_n\overline{\Gamma}_G)=1$,*

(iii) *$\nu(E)>0$ for every open non-empty set $E\subset\overline{\Gamma}_G$ with $E\cap\partial_n\overline{\Gamma}_G\neq\varnothing$.*

Proof. (i) The continuity of ν follows from Lemma 1, because the action of G on $\overline{\Gamma}_G$ has no finite orbits (see Theorem 7).

(ii) In the case $n=1$ the set $\overline{\Gamma}_G\smallsetminus\partial_n\overline{\Gamma}_G=\Gamma_G$ is countable. Then, since ν is continuous, we have $\nu(\Gamma_G)=0$. Hence, $\nu(\partial_n\overline{\Gamma}_G)=1$.

Assume $n\geqslant 2$. Let Ξ_{n-1} be the set of all $\mathbb{Z}^{n-1}$ subtrees of Γ_G and let $\widehat{\Xi}_{n-1}=\{\overline{S}\subset\overline{\Gamma}_G\mid S\in\Xi_{n-1}\}$ be the set of closures of these subtrees in $\overline{\Gamma}_G$. Observe that $\widehat{\Xi}_{n-1}$ is countable, that the action of G on $\overline{\Gamma}_G$ induces an action of G on $\widehat{\Xi}_{n-1}$, and that the orbits of the latter action are countable.

Let $K=\bigcup_{S\neq T\in\widehat{\Xi}_{n-1}}(S\cap T)$. Then K is a countable subset of $\overline{\Gamma}_G$, because $\widehat{\Xi}_{n-1}$ is countable and for any $S\neq T\in\widehat{\Xi}_{n-1}$ the set $S\cap T$ consists of at most one point. Consequently, $\nu(K)=0$ (since ν is continuous).

Now, let $\widehat{\Xi}'_{n-1}$ be the family of subsets in $\overline{\Gamma}_G$ of the form $S\smallsetminus K$, where $S\in\widehat{\Xi}_{n-1}$. By construction, we obviously have:

(1) $\widehat{\Xi}'_{n-1}$ is countable,

(2) the action of G on $\overline{\Gamma}_G$ induces an action of G on $\widehat{\Xi}_{n-1}$, and that the orbits of the latter action are countable,

(3) for any $S \in \widehat{\Xi}_{n-1}$ the set $S \smallsetminus K \in \widehat{\Xi}'_{n-1}$ is a measurable subset of $\overline{\Gamma}_G$, because S is closed and K is countable,

4) for any $g \in G$, $S' \in \widehat{\Xi}'_{n-1}$ we have either $(g \cdot S') = S'$, or $(g \cdot S') \cap S' = \varnothing$.

It then follows by Lemma 1 that $\nu(S') = 0$ for each $S' \in \widehat{\Xi}'_{n-1}$, whence $\nu(\bigcup_{\widehat{\Xi}'_{n-1}} S') = 0$. Thus, we have $\nu(\bigcup_{\widehat{\Xi}_{n-1}} S) = \nu(\bigcup_{\widehat{\Xi}'_{n-1}} S') + \nu(K) = 0$. Consequently

$$\nu(\partial_n \overline{\Gamma}_G) = \nu(\overline{\Gamma}_G) - \nu(\bigcup_{\widehat{\Xi}_{n-1}} S) = 1 - 0 = 1.$$

(iii) This follows from (ii) by Lemma 2, because G is countable and acts on $\partial_n \overline{\Gamma}_G$ minimally (see Theorem 8). □

3.5 Additional constructions concerning $\overline{\Gamma}_G$

Let G be a $\mathbb{Z}^n$-free group with $n \geqslant 2$ and let Γ_G be the associated universal $\mathbb{Z}^n$-tree (with the base point ϵ) described in Subsection 2.3. Let $\overline{\Gamma}_G$ be the compactification of Γ_G constructed in Subsections 3.1–3.3.

Definition 1. *For a pair $a, b \in \overline{\Gamma}_G$, we denote by $[a, b]$ the geodesic in Γ_G connecting a and b. If $a \neq b$, we denote by $[\![a, b]\!]$ the closure of $[a, b]$ in the space $\overline{\Gamma}_G$. We also set $[\![a, a]\!] = \{a\}$ for every $a \in \overline{\Gamma}_G$.*

Lemma 12. *Let $a, b \in \overline{\Gamma}_G$. Then there exists $c \in \overline{\Gamma}_G$ such that $[\![\epsilon, a]\!] \cap [\![\epsilon, b]\!] = [\![\epsilon, c]\!]$. If $a, b \in \Gamma_G$, then $c \in \Gamma_G$. If $a \notin [\![\epsilon, b]\!]$ and $b \notin [\![\epsilon, a]\!]$, then $c \in \Gamma_G$.*

Definition 2. *For a pair $a, b \in \overline{\Gamma}_G$ with $[\![\epsilon, a]\!] \cap [\![\epsilon, b]\!] = [\![\epsilon, c]\!]$ we set*

$$Y(\epsilon, a, b) = c.$$

Let Ξ_{n-1} be the set of all $\mathbb{Z}^{n-1}$ subtrees of Γ_G and let

$$\widehat{\Xi}_{n-1} = \{\overline{S} \subset \overline{\Gamma}_G \mid S \in \Xi_{n-1}\}$$

be the set of closures of these subtrees in $\overline{\Gamma}_G$. (In the case $n = 1$, the sets Ξ_{n-1} and $\widehat{\Xi}_{n-1}$ both coincide with the set of vertices of the usual tree Γ_G.) Let $\partial_{n-1}\overline{\Gamma}_G$ be the set of those points in $\overline{\Gamma}_G$ that correspond to $\mathbb{Z}^{n-1}$-ends in Γ_G. (In the case $n = 1$, the set $\partial_{n-1}\overline{\Gamma}_G$ is empty.)

Definition 3. *For a pair of points $a \neq b \in \overline{\Gamma}_G$, we define*

$$\hbar_n(a, b) = \#\{S \in \Xi_{n-1} \mid S \cap [a, b] \neq \varnothing\} + \frac{\#\{\partial_{n-1}\overline{\Gamma}_G \cap \{a, b\}\}}{2} - 1,$$

where $\#\{S \in \Xi_{n-1} \mid S \cap [a, b] \neq \varnothing\}$ is the number of distinct $\mathbb{Z}^{n-1}$-subtrees of Γ_G intersecting with $[a, b]$ and $\#\{\partial_{n-1}\overline{\Gamma}_G \cap \{a, b\}\}/2$ is a number from $\{0, 1/2, 1\}$. If a or b is an end of $\mathbb{Z}^n$-type, then $\hbar_n(a, b) = \infty$. We set also $\hbar_n(a, a) = 0$ for each $a \in \overline{\Gamma}_G$.

For a point $p \in \overline{\Gamma}_G$, we set

$$\hbar_n(p) = \hbar_n(\epsilon, p).$$

For an element $g \in G$, we set

$$\hbar_n(g) = \hbar_n(g \cdot \epsilon) = \hbar_n(\epsilon, g \cdot \epsilon).$$

Remark 3. *Consider the simplicial tree Δ_{n-1} of $\mathbb{Z}^{n-1}$-subtrees of Γ_G. Then each geodesic segment $[a,b] \subset \Gamma_G$, where $a \neq b \in \Gamma_G$, naturally projects to Δ_{n-1} forming a path there. The value $\hbar_n(a,b)$ equals to the length of this path.*

If $v \in \Gamma_G$ lies at the $\mathbb{Z}^{n-1}$-subtree of the k-level (with respect to the $\mathbb{Z}^{n-1}$-subtree T_0 containing ϵ) then $\hbar_n(v) = k$.

If $g \in G$ and $|g| = (z_1, \ldots, z_n) \in \mathbb{Z}^n$, then $\hbar_n(g) = z_n$. In particular, $\hbar_n : G \to \mathbb{Z}$ is a seminorm.

Lemma 13. *Let $a, b \in \overline{\Gamma}_G$ and $p \in [\![a,b]\!]$, then*

$$\hbar_n(a,b) = \hbar_n(a,p) + \hbar_n(p,b).$$

Proof. Easily follows from the definition. □

Lemma 14 (convergence-1)**.** *Let $\{a_i\}$ and $\{b_i\}$ be two sequences of points in $\overline{\Gamma}_G$. Suppose that for each vertex $v \in \Gamma_G$, the set $\{i \in \mathbb{N} \mid v \in [a_i, b_i]\}$ is at most finite. Then $d_{\overline{\Gamma}_G}(a_i, b_i) \to 0$ as $i \to \infty$.*

Proof. (to be completed) □

Lemma 15 (convergence-2)**.** *Let $\{g_i\}$ be a sequence of elements from G, and let $y \in \partial_n \overline{\Gamma}_G$ be an end of $\mathbb{Z}^n$-type. Assume that there exist a vertex $v_0 \in \Gamma_G$ such that the sequence $\{g_i \cdot v_0\}$ converges to y. Then all the sequences $\{g_i \cdot v\}$, $v \in \Gamma_G$, converge to y.*

Proof. Consider the sequence of numbers $\{H_i\} = \{\hbar_n(g_i \cdot v_0)\}$. Since $\{g_i \cdot v_0\}$ converges to an end of $\mathbb{Z}^n$-type, it follows that $\{H_i\}$ tends to infinity. Let $H = \hbar_n(v_0, v)$. Observe that $\hbar_n(g \cdot v_0, g \cdot v) = \hbar_n(v_0, v) = H$ for every $g \in G$. Consequently, for any $i \in \mathbb{N}$ and $z \in [g_i \cdot v_0, g_i \cdot v]$ we have $\hbar_n(z) \geqslant H_i - H$. Since the sequence $\{H_i - H\}$ tends to infinity, it follows that for each vertex $w \in \Gamma_G$ the set $\{i \in \mathbb{N} \mid w \in [g_i \cdot v_0, g_i \cdot v]\}$ is at most finite. The result then follows by Lemma 14. □

Corollary 4 (convergence-3)**.** *Let $\{a_i\}$ and $\{b_i\}$ be two sequences of points in $\overline{\Gamma}_G$. Suppose that $\hbar_n(Y(\epsilon, a_i, b_i)) \to \infty$ as $i \to \infty$. Then $d_{\overline{\Gamma}_G}(a_i, b_i) \to 0$ as $i \to \infty$. In particular, $\{a_i\}$ converges to a point $\omega \in \overline{\Gamma}_G$ if and only if $\{b_i\}$ converges to ω.*

Proof. Observe that if $x \in \overline{\Gamma}_G$ and $y \in [\![\epsilon, x]\!]$, then $\hbar_n(y) \leqslant \hbar_n(z)$ for each $z \in [\![y, x]\!]$ (Lemma 13). If $a, b \in \overline{\Gamma}_G$ and $c = Y(\epsilon, a, b)$, then we have $c \in [\![\epsilon, a]\!]$, $c \in [\![\epsilon, b]\!]$, and $[\![a, b]\!] = [\![c, a]\!] \cup [\![c, b]\!]$; it then follows that

$$\min_{z \in [\![a,b]\!]} \hbar_n(z) = \hbar_n(Y(\epsilon, a, b)).$$

Therefore, we have $\min_{z \in [\![a_i, b_i]\!]} \hbar_n(z) = \hbar_n(Y(\epsilon, a_i, b_i))$ for all $i \in \mathbb{N}$. Since we assume $\hbar_n(Y(\epsilon, a_i, b_i)) \to \infty$ as $i \to \infty$, it follows that for each $v \in \Gamma_G$ the set $\{i \in \mathbb{N} \mid v \in [a_i, b_i]\}$ is at most finite (because we have $v \notin [a_i, b_i]$ whenever $\min_{z \in [\![a_i, b_i]\!]} \hbar_n(z) > \hbar_n(v)$). Then the result follows from Lemma 14. □

Definition 4. *Let v_1, v_2 be a pair of vertices in Γ_G. We define the* branch $\mathrm{Branch}(v_1, v_2)$ *to be the subset $\{w \in \overline{\Gamma}_G \mid [v_1, w] \ni v_2\}$.*

Lemma 16. *Let v_1, v_2 be a pair of vertices in Γ_G. Then $\mathrm{Branch}(v_1, v_2)$ is open and closed in $\overline{\Gamma}_G$.*

Proof. If $v_1 = v_2$, then $\mathrm{Branch}(v_1, v_2) = \overline{\Gamma}_G$. Suppose $v_1 \neq v_2$. Let v_0 be the vertex of Γ_G lying on the geodesic $[v_1, v_2]$ and adjacent to v_2. Then $\mathrm{Branch}(v_1, v_2) = \mathrm{Branch}(v_0, v_2)$. Observe that

$$\overline{\Gamma}_G \smallsetminus \mathrm{Branch}(v_0, v_2) = \mathrm{Branch}(v_2, v_0),$$

and that for any $a \in \mathrm{Branch}(v_2, v_0)$, $b \in \mathrm{Branch}(v_0, v_2)$ the geodesic $[a, b]$ contains the edge (v_0, v_2). Consequently, there exist $\delta > 0$ such that $d_{\overline{\Gamma}_G}(a, b) > \delta$ whenever $a \in \mathrm{Branch}(v_2, v_0) = \overline{\Gamma}_G \smallsetminus \mathrm{Branch}(v_0, v_2)$ and $b \in \mathrm{Branch}(v_0, v_2)$. Therefore, $\bigcup_{p \in \mathrm{Branch}(v_1, v_2)} \{x \in \overline{\Gamma}_G \mid d_{\overline{\Gamma}_G}(x, p) < \delta\} = \mathrm{Branch}(v_1, v_2)$, which means that $\mathrm{Branch}(v_1, v_2)$ is open. Similar argument shows that $\overline{\Gamma}_G \smallsetminus \mathrm{Branch}(v_1, v_2) = \mathrm{Branch}(v_2, v_1)$ is open, hence $\mathrm{Branch}(v_1, v_2)$ is closed. □

4 The Poisson–Furstenberg boundary

In this section, we prove the following theorem, which implies our main Theorem 1.

Theorem 9. *Let n be a positive integer, let $G \subset CDR(\mathbb{Z}^n, X)$ be an irreducible non-abelian $\mathbb{Z}^n$-free group over a finite alphabet X, Γ_G the associated universal $\mathbb{Z}^n$-tree with the base point ϵ described in Subsection 2.3, $\overline{\Gamma}_G$ the compactification of Γ_G constructed in Section 3, and $\partial_n \overline{\Gamma}_G$ the set of ends of $\mathbb{Z}^n$-type in $\overline{\Gamma}_G$. Then, for every non-degenerate probability measure μ on G, the following hold true:*

(0) *There exists a unique μ-stationary measure ν_μ on $\overline{\Gamma}_G$ (by Proposition 1, this measure is continuous, $\nu_\mu(\partial_n \overline{\Gamma}_G) = 1$, and $\nu_\mu(E) > 0$ for every open non-empty subset $E \subset \overline{\Gamma}_G$ with $E \cap \partial_n \overline{\Gamma}_G \neq \varnothing$).*

(1) *The pair $(\overline{\Gamma}_G, \nu_\mu)$ is a μ-boundary of G, i. e., for a. e. path $\tau = \{\tau_i\}$ of the random μ-walk, the sequence of measures $\{\tau_i \cdot \nu_\mu\}$ converges to a (random) Dirac measure $\delta_{\omega(\tau)}$, where $\omega(\tau) \in \overline{\Gamma}_G$; moreover, with probability $1 = \nu_\mu(\partial_n \overline{\Gamma}_G)$ we have[2] $\omega(\tau) \in \partial_n \overline{\Gamma}_G$.*

(2) *If $\tau = \{\tau_i\}$ is a path of the random μ-walk such that $\omega(\tau) \in \partial_n \overline{\Gamma}_G$, then the sequence $\{\tau_i \cdot v\}$ converges to $\omega(\tau)$ for each vertex $v \in \Gamma_G$. (Since the point $\omega(\tau)$ is defined and lies in $\partial_n \overline{\Gamma}_G$ for a. e. path $\tau = \{\tau_i\}$, it follows that the sequences $\{\tau_i \cdot v\}$, $v \in \Gamma_G$, converge to $\omega(\tau)$ for a. e. path $\tau = \{\tau_i\}$.)*

(3) *If G is finitely generated and μ has finite first moment with respect to a word metric[3] on G, then the measure space $(\overline{\Gamma}_G, \nu_\mu)$ is a Poisson–Furstenberg boundary of (G, μ).*

The structure of this section is as follows.
In Subsection 4.1, assertions (0) and (1) of Theorem 9 are proved.
In Subsection 4.2, assertion (2) of Theorem 9 is proved.
In Subsection 4.3, assertion (3) of Theorem 9 is proved.
Note that assertion (2) is not used in our proof of assertion (3).
In Subsection 4.4, we explain why Theorem 1 follows from Theorem 9.

[2] The fact that for a. e. path $\tau = \{\tau_i\}$ of the random μ-walk, the limit point $\omega(\tau)$ is an end of $\mathbb{Z}^n$-type, follows from Theorem 2 and Proposition 1. (By Theorem 2, the distribution of limit measures $\delta_{\omega(\tau)}$ is given by ν_μ, while by Proposition 1 we have $\nu_\mu(\partial_n \overline{\Gamma}_G) = 1$.)

[3] We consider an arbitrary word metric that is induced by a finite generating set.

4.1 The G-space $\overline{\Gamma}_G$ is a Furstenberg boundary of G

In this subsection, we prove the following theorem, which implies assertions 0) and 1) of Theorem 9.

Theorem 10. *Under assumptions of Theorem 9, the G-space $\overline{\Gamma}_G$ is a Furstenberg boundary of G.*

Assertion 0) of Theorem 9 follows from Theorem 10 by Assertion 2 (the last one is applicable, because $\overline{\Gamma}_G$ is compact and metrizable). The main statement of assertion 1) is equivalent to Theorem 10 by definition.

The standard pattern to prove that a (compact metrizable) $\mathcal{G}$-space $\mathcal{X}$ of a countable group $\mathcal{G}$ is a Furstenberg boundary of $\mathcal{G}$ is as follows. We consider an arbitrary μ-stationary measure ν on $\mathcal{X}$ (for an arbitrary non-degenerate measure μ on $\mathcal{G}$) and show that a. e. path $\{\tau_i\}$ of the random μ-walk contains a subsequence $\{\tau_{i_j}\}$ such that the sequence $\{\tau_{i_j}\cdot\nu\}$ converges to some Dirac measure. By combining this with Theorem 2, we see that for a. e. path $\{\tau_i\}$ the sequence $\{\tau_i\cdot\nu\}$ converges to some measure, and at the same time has a subsequence converging to a Dirac measure. It follows that the limit of $\{\tau_i\cdot\nu\}$ is this Dirac measure, and the proof is done (see the definition of Furstenberg boundary).

In order to prove Theorem 10, we will follow the above pattern. We will show that there exists a subset $\mathfrak{S} \subset G^{\mathbb{Z}_+}$ such that, for any nondegenerate measure μ on G and any μ-stationary measure ν on $\overline{\Gamma}_G$,

1) A. e. path of the random μ-walk contains a subsequence from $\mathfrak{S}$,

2) For each $\{g_i\} \in \mathfrak{S}$, the sequence $\{g_i\cdot\nu\}$ converges to a Dirac measure.

We define $\mathfrak{S}$ to be the subset in $G^{\mathbb{Z}_+}$ consisting of all sequences $\{g_i\}$ with the following three properties (here, ϵ is the base point in Γ_G):

i) $\{g_i\cdot\epsilon\}$ converges to a point of $\overline{\Gamma}_G \smallsetminus \Gamma_G$,

ii) $\{g_i^{-1}\cdot\epsilon\}$ converges to a point of $\overline{\Gamma}_G \smallsetminus \Gamma_G$,

iii) $\{\hbar_n(g_i)\}$ tends to infinity as $i \to \infty$, where $\hbar_n$ is the seminorm described in Subsection 3.5 (Definition 3).

Remark 4. For some groups and spaces, in order to apply the above pattern, it suffices to consider the set of sequences with property i) only. In fact, property ii) is redundant in our case too; we include it due to the local tactic of our proof. However, property iii) (or some analogue of it) is indispensable in our case. Note that property iii) does not imply that the ends where $\{g_i\cdot\epsilon\}$ and $\{g_i^{-1}\cdot\epsilon\}$ converge to are ends of $\mathbb{Z}^n$-type.

Proposition 2. *Let G, Γ_G, $\overline{\Gamma}_G$, μ be as in Theorem 9. Then a. e. path $\{\tau_i\}$ of the random μ-walk contains a subsequence $\{\tau_{i_j}\} \in \mathfrak{S}$.*

Proof. First, we prove that a. e. path $\{\tau_i\}$ of the random μ-walk contains a subsequence with property iii) from the definition of $\mathfrak{S}$. Let $d \in \mathbb{N}$. Since G is an irreducible $\mathbb{Z}^n$-free group (see the definition of irreducibility in the introduction and Lemma **??**), it follows that there exists an element $g_{2d} \in G$ such that $\hbar_n(g_{2d}) > 2d$. Since μ is a non-degenerate measure, there exists $k \in \mathbb{N}$ such that $\mu^{*k}(g_{2d}) > 0$. It follows that for a. e. path $\{\tau_i\}$ of the random μ-walk there exists $m \in \mathbb{N}$ such that

$\tau_{m+k} = \tau_m g_{2d}$. (To see this, use the ergodicity of the Bernoully shift in the space of sequences of μ^{*k}-equidistributed random variables $\{\tau_{jk}^{-1}\tau_{jk+k}\}_{j\in\mathbb{Z}_+}$, or observe that the measure P_μ of the set of all μ-walk paths with $\tau_{jk+k} \neq \tau_{jk}g_{2d}$ for each $j \in \mathbb{N}$ equals $\lim_{s\to\infty}(1-\varepsilon)^s = 0$, where $\varepsilon := \mu^{*k}(g_{2d}) > 0$.) Then

$$\hbar_n(\tau_m^{-1}\tau_{m+k}) = \hbar_n(g_{2d}) > 2d.$$

Since $\hbar_n : G \to \mathbb{Z}$ is a seminorm (see Remark 3; in particular, it satisfies the triangle inequality, and $\hbar_n(h) = \hbar_n(h^{-1})$ for each $h \in G$) it follows that

$$\min\{\hbar_n(\tau_{m+k}), \hbar_n(\tau_m)\} > d.$$

We have thus showed that for any $d \in \mathbb{N}$ the path $\{\tau_i\}$ a. s. has an element τ_i such that $\hbar_n(\tau_i) > d$. It obviously follows that $\{\tau_i\}$ a. s. has a subsequence $\{\tau_i'\}$ such that $\{\hbar_n(\tau_i')\}$ tends to infinity (property iii)).

Since $\overline{\Gamma}_G$ is compact, $G{\cdot}\epsilon$ is discrete, and $\{\tau_i'\}$ has no infinite constant subsequences, it follows that $\{\tau_i'\}$ has a subsequence $\{\tau_i''\}$ such that $\{\tau_i''{\cdot}\epsilon\}$ converges to a point of $\overline{\Gamma}_G \smallsetminus \Gamma_G$ (property i)). Similar argument shows that $\{\tau_i''\}$ has a subsequence $\{\tau_i'''\}$ such that $\{(\tau_i''')^{-1}{\cdot}\epsilon\}$ converges to a point of $\overline{\Gamma}_G \smallsetminus \Gamma_G$ (property ii)). Therefore, $\{\tau_i'''\}$ is a subsequence of $\{\tau_i\}$ and $\{\tau_i'''\} \in \mathfrak{S}$. □

Lemma 17. *Let $\{g_i\}$ be a sequence from $\mathfrak{S}$. Let ω and $\check{\omega}$ be points*[4] *of $\overline{\Gamma}_G$ the sequences $\{g_i{\cdot}\epsilon\}$ and $\{g_i^{-1}{\cdot}\epsilon\}$ converge to. Then for each $p \in \overline{\Gamma}_G \smallsetminus \check{\omega}$ the sequence $\{g_i{\cdot}p\}$ converges to ω.*

Proof. In order to prove the lemma, we will show that for every $p \in \overline{\Gamma}_G \smallsetminus \check{\omega}$ we have

$$\hbar_n(\mathrm{com}(g_i{\cdot}\epsilon, g_i{\cdot}p)) \xrightarrow[i\to\infty]{} \infty \tag{5}$$

and then apply Corollary 4.

For each $i \in \mathbb{N}$, set

$$p_i := \mathrm{com}(g_i{\cdot}\epsilon, g_i{\cdot}p).$$

Observe that for each i we have

$$g_i^{-1}{\cdot}p_i \;=\; \mathrm{com}(g_i^{-1}{\cdot}\epsilon, p),$$

because

$$g_i^{-1}{\cdot}p_i \;=\; g_i^{-1}{\cdot}\,\mathrm{com}(g_i{\cdot}\epsilon, g_i{\cdot}p) \;=\; g_i^{-1}{\cdot}Y(\epsilon, g_i{\cdot}\epsilon, g_i{\cdot}p) \;= \\ Y(g_i^{-1}{\cdot}\epsilon, \epsilon, p) \;=\; \mathrm{com}(g_i^{-1}{\cdot}\epsilon, p).$$

(Here, $Y(a,b,c) := [a,b] \cap [b,c] \cap [c,a]$. It is clear that $Y(a,b,c)$ is a point of $\overline{\Gamma}_G$. For any $g \in G$, we have $g{\cdot}Y(a,b,c) = Y(g{\cdot}a, g{\cdot}b, g{\cdot}c)$. For any $x, y \in \overline{\Gamma}_G$, we have $\mathrm{com}(x,y) = Y(\epsilon, x, y)$.)

Let us prove that the sequence

$$\{\hbar_n(\mathrm{com}(g_i^{-1}{\cdot}\epsilon, p))\}$$

is bounded. Assume that it does not. Then it would have a subsequence $\{\hbar_n(\mathrm{com}(g_{i_j}^{-1}{\cdot}\epsilon, p))\}$ tending to infinity. Then, by Corollary 4, $\{g_{i_j}^{-1}{\cdot}\epsilon\}$ would converge to p, which contradicts the assumption that $g_i^{-1}{\cdot}\epsilon \to \check{\omega} \neq p$. Thus, our sequence is bounded, i. e., there exists $N \in \mathbb{N}$ such that

$$\hbar_n(\mathrm{com}(g_i^{-1}{\cdot}\epsilon, p)) < N \quad \text{for each} \quad i \in \mathbb{N}.$$

[4] We do not require that $\omega \neq \check{\omega}$.

Therefore, for each $i \in \mathbb{N}$ we have

$$\hbar_n(g_i{\cdot}\epsilon, p_i) = \hbar_n(\epsilon, g_i^{-1}{\cdot}p_i) \stackrel{\text{def}}{=} \hbar_n(g_i^{-1}{\cdot}p_i) = \hbar_n(\mathrm{com}(g_i^{-1}{\cdot}\epsilon, p)) < N.$$

Since $\{g_i\} \in \mathfrak{S}$ and hence $\hbar_n(\epsilon, g_i{\cdot}\epsilon) \stackrel{\text{def}}{=} \hbar_n(g_i{\cdot}\epsilon) \stackrel{\text{def}}{=} \hbar_n(g_i)$ tends to ∞ (property iii)), while $p_i \in [\epsilon, g_i{\cdot}\epsilon]$ and hence $\hbar_n(\epsilon, p_i) + \hbar_n(p_i, g_i{\cdot}\epsilon) = \hbar_n(\epsilon, g_i{\cdot}\epsilon)$, it follows that $\hbar_n(p_i) \to \infty$.

By Corollary 4 this implies that $\{g_i{\cdot}p\}$ converges to ω as $\{g_i{\cdot}\epsilon\}$ does. □

Lemma 17 obviously implies the following corollary.

Corollary 5. *Let $\{g_i\}$ be a sequence from $\mathfrak{S}$ with $\{g_i{\cdot}\epsilon\}$ converging to a point $\omega \in \overline{\Gamma}_G$. Then for any continuous Borel probability measure λ on $\overline{\Gamma}_G$, the sequence $\{g_i{\cdot}\lambda\}$ converges to the Dirac measure δ_ω.*

Proof of Theorem 10. We follow the pattern described above. Let μ be a non-degenerate probability measure on G, and ν a μ-stationary measure on $\overline{\Gamma}_G$. Let $\{\tau_i\}$ be a path of the random μ-walk. Then by Theorem 2 the sequence $\{\tau_i{\cdot}\nu\}$ a. s. converges to some limit. By Lemma 2, $\{\tau_i\}$ a. s. contains a subsequence $\{\tau_{i_j}\} \in \mathfrak{S}$. By Corollary 5, the subsequence $\{\tau_{i_j}{\cdot}\nu\}$ converges to a Dirac measure (since ν is continuous; see Proposition 1). Thus, the limit of $\{\tau_i{\cdot}\nu\}$ is a. s. the same Dirac measure. It remains to recall definitions of μ-boundaries and Furstenberg boundaries. □

4.2 Stability of paths in $\mathbb{Z}^n$-free groups

In this subsection, we prove the following proposition, which obviously implies assertion 2) of Theorem 9.

Proposition 3. *Let G, Γ_G, $\overline{\Gamma}_G$, $\partial_n\overline{\Gamma}_G$, μ, ν_μ be as in Theorem 9, and let $\omega \in \partial_n\overline{\Gamma}_G$ be an end of $\mathbb{Z}^n$-type. Assume that a sequence $\{g_i{\cdot}\nu_\mu\}$, $g_i \in G$, converges to the Dirac measure δ_ω. Then, for each $x \in \Gamma_G$, the sequence $\{g_i{\cdot}x\}$ converges to ω.*

Proof. Take two distinct[5] points ω_1, ω_2 in $\partial_n\overline{\Gamma}_G$, then take two distinct vertices v_1, v_2 (of Γ_G) on the geodesic $[\omega_1, \omega_2]$, and consider the branches[6] $B_1 := \mathrm{Branch}(v_1, v_2)$ and $B_2 := \mathrm{Branch}(v_2, v_1)$. Observe that B_1 and B_2 are open subsets of $\overline{\Gamma}_G$ (Lemma 16), and that the intersections $B_1 \cap \partial_n\overline{\Gamma}_G$ and $B_1 \cap \partial_n\overline{\Gamma}_G$ are nonempty (because our definition of B_1 and B_2 implies that B_1 containes one of the points ω_1, ω_2, while B_2 contains another one). It then follows by assertion (iii) of Proposition 1 that $\nu_\mu(B_1) > 0$ and $\nu_\mu(B_2) > 0$.

Let us show that the sequences $\{g_i{\cdot}v_1\}$ and $\{g_i{\cdot}v_2\}$ converge to ω. For this, it suffices to prove that for each vertex $v \in [\epsilon, \omega]$ the branch $B_v := \mathrm{Branch}(\epsilon, v)$ contains all but finitely many elements of our two sequences. Since $B_v \ni \omega$, the condition that $\{g_i{\cdot}\nu_\mu\}$ converges to δ_ω implies[7] that the sequence $\{(g_i{\cdot}\nu_\mu)(B_v)\}$ converges to $\delta_\omega(B_v) = 1$. Consequently, there exists $N_v \in \mathbb{N}$ such that $(g_j{\cdot}\nu_\mu)(B_v) >$

[5] Such points exist, because the set $\partial_n\overline{\Gamma}_G$ of ends of $\mathbb{Z}^n$-type is infinite. It is non-empty by Lemma **??** (because we assume that G is an irreducible $\mathbb{Z}^n$-free group). It consists of G-orbits (i. e., it is G-invariant). All G-orbits in $\overline{\Gamma}_G$ is infinite by Theorem 7.

[6] See Definition 4.

[7] To see this implication, recall that we consider the weak topology on the space of regular Borel probability measures. That is, the condition that $\{g_i{\cdot}\nu_\mu\}$ converges to δ_ω means that, for every bounded continuous function $f : \overline{\Gamma}_G \to \mathbb{R}$, the sequence $\{\int_{\overline{\Gamma}_G} f dg_i{\cdot}\nu_\mu\}$ converges to $\int_{\overline{\Gamma}_G} f d\delta_\omega = f(\omega)$. Since the subset $B_v \subset \overline{\Gamma}_G$ is open and closed (Lemma 16), it follows that the characteristic function $\mathbb{1}_{B_v}$, which is defined by $\mathbb{1}_{B_v}(x) = 1$ if $x \in B_v$ and $\mathbb{1}_{B_v}(x) = 0$ otherwise, is continuous. By substituting $f = \mathbb{1}_{B_v}$ in the above formulas, we see that the sequence $\{\int_{\overline{\Gamma}_G} \mathbb{1}_{B_v} dg_i{\cdot}\nu_\mu\} = \{(g_i{\cdot}\nu_\mu)(B_v)\}$ should converge to $\mathbb{1}_{B_v}(\omega) = 1$.

$1 - \min\{\nu_\mu(B_1), \nu_\mu(B_2)\}$ whenever $j > N_v$. In particular, since $(g_j{\cdot}\nu_\mu)(B_v) = \nu_\mu(g_j^{-1}{\cdot}B_v)$, for each $j > N_v$ we have $(g_j^{-1}{\cdot}B_v) \cap B_1 \neq \varnothing$ and $(g_j^{-1}{\cdot}B_v) \cap B_2 \neq \varnothing$, whence $B_v \cap (g_j{\cdot}B_1) \neq \varnothing$ and $B_v \cap (g_j{\cdot}B_2) \neq \varnothing$. Therefore, for each $j > N_v$ there exist points $p_1^{(j)} \in B_v \cap (g_j{\cdot}B_1)$ and $p_2^{(j)} \in B_v \cap (g_j{\cdot}B_2)$. Now, on the one hand, we have $[p_1^{(j)}, p_2^{(j)}] \subset B_v$ (because $p_1^{(j)}, p_2^{(j)} \in B_v$, while B_v is geodesically convex, being a branch). On the other hand, we have $[p_1^{(j)}, p_2^{(j)}] \supset [g_j{\cdot}v_1, g_j{\cdot}v_2]$ (because $p_1^{(j)} \in g_j{\cdot}B_1 = \mathrm{Branch}(g_j{\cdot}v_1, g_j{\cdot}v_2)$ and $p_2^{(j)} \in g_j{\cdot}B_2 = \mathrm{Branch}(g_j{\cdot}v_2, g_j{\cdot}v_1)$; see definition of branches and [6, Ch. 2, Lemma 1.5]). It follows that $g_j{\cdot}v_1, g_j{\cdot}v_2 \in B_v$ (for each $j > N_v$). We have thus proved that the sequences $\{g_i{\cdot}v_1\}$ and $\{g_i{\cdot}v_2\}$ converge to ω.

Since ω is an end of $\mathbb{Z}^n$-type, it then follows by Lemma 15 that $\{g_i{\cdot}x\}$ converge to ω for every $x \in \Gamma_G$. □

4.3 Maximality of the boundary $\overline{\Gamma}_G$

In this subsection, we verify assertion 3) of Theorem 9, which says that if our $\mathbb{Z}^n$-free group G is finitely generated and the distribution $\mu \in \mathcal{P}(G)$ has finite first moment (with respect to a finite word metric on G), then the space $\overline{\Gamma}_G$ with μ-stationary measure is a maximal Furstenberg boundary, that is, a Poisson–Furstenberg boundary of (G, μ). Our plan is to prove Proposition 4 below, which states the existence of maps $\overline{\Gamma}_G \times \overline{\Gamma}_G \to 2^G$ satisfying the requirements of the Kaimanovich Strip Criterion (see Theorem 3 and Corollaries 1, 2). Proposition 4 and Corollary 2 obviously imply assertion 3).

Proposition 4. *Let G, Γ_G, $\overline{\Gamma}_G$, $\partial_n\overline{\Gamma}_G$ be as in Theorem 9. Then there exists a measurable G-equivariant map $S : \overline{\Gamma}_G \times \overline{\Gamma}_G \to 2^G$, assigning to pairs of points $(b_-, b_+) \in \overline{\Gamma}_G \times \overline{\Gamma}_G$ certain "strips" $S(b_-, b_+) \subset G$, such that*

1) *For all pairs $(b_-, b_+) \in \overline{\Gamma}_G \times \overline{\Gamma}_G$ and for all $k \in \mathbb{N}$ we have*[8]

$$\mathrm{card}\,\{g \in S(b_-, b_+) \,:\, \hbar_n(g) \leqslant k\} \;\leqslant\; k.$$

2) *For every non-degenerate probability measure μ on G with μ-stationary measure*[9] *ν_+ on $\overline{\Gamma}_G$ and $\check{\mu}$-stationary*[10] *measure ν_- on $\overline{\Gamma}_G$, the strips $S(b_-, b_+)$ are $(\nu_- \times \nu_+)$-almost surely non-empty.*

In the proof of Proposition 4 we will use several additional notions and lemmas.

Definition 5. Let $w : [1_{\mathbb{Z}^n}, |w|] \to X^\pm$ be a $\mathbb{Z}^n$-word and $\beta \in \mathbb{Z}^n_+$. We say that w is *β-periodic* if $w(\alpha) = w(\alpha + \beta)$ for each $\alpha \in [1_{\mathbb{Z}^n}, |w| - \beta]$. (In particular, w is β-periodic if $|w| \geqslant \beta$.) We say that w is *faithfully periodic* if it is β-periodic for some $\beta \in \mathbb{Z}^n_+$ with $\beta + \beta \leqslant |w|$.

Lemma 18. *Let w_1 and w_2 be distinct $\mathbb{Z}^n$-words such that $|w_1| = |w_2| = |\,\mathrm{com}(w_1, w_2)| + 1_{\mathbb{Z}^n}$. Then at least one of w_1, w_2 is not faithfully periodic.*

Proof. Let $w := \mathrm{com}(w_1, w_2)$ and let $\gamma := |w_1| = |w_2| = |w| + 1_{\mathbb{Z}^n}$. Observe that for every $\alpha \in [1_{\mathbb{Z}^n}, |w|]$ we have $w_1(\alpha) = w_2(\alpha) = w(\alpha)$. Since $w_1 \neq w_2$ and $|w_1| = |w_2| = |w| + 1_{\mathbb{Z}^n}$, it then follows that $w_1(\gamma) \neq w_2(\gamma)$.

[8] See Subsection 3.5 for the definition of $\hbar_n$.

[9] In fact, the above-proved assertion 0) of the theorem says that μ-stationary measure is unique, but it is not of importance for this proposition.

[10] Recall that we denote by $\check{\mu}$ the reflected measure of μ defined by $\check{\mu}(g) := \mu(g^{-1})$.

Assume that w_1 and w_2 are both faithfully periodic. Then there exist $\beta_1 \in \mathbb{Z}^n_+$ and $\beta_2 \in \mathbb{Z}^n_+$ with $2 \cdot \beta_1 \leqslant |w_1| = \gamma$ and $2 \cdot \beta_2 \leqslant |w_2| = \gamma$ such that w_1 is β_1-periodic and w_2 is β_2-periodic.

Since w_1 is β_1-periodic, we have $w_1(\gamma) = w_1(\gamma - \beta_1)$. Since w_2 is β_2-periodic, we have $w_2(\gamma) = w_2(\gamma - \beta_2)$. Observe that $\beta_1 \neq \beta_2$, because otherwise we would have $w_1(\gamma) = w_1(\gamma - \beta_1) = w(\gamma - \beta_1) = w(\gamma - \beta_2) = w_2(\gamma - \beta_2) = w_2(\gamma)$.

The conditions $2 \cdot \beta_1 \leqslant \gamma$, $2 \cdot \beta_2 \leqslant \gamma$, and $\beta_1 \neq \beta_2$ imply that $\beta_1 + \beta_2 < \gamma$. Therefore, the elements $\gamma - \beta_1$, $\gamma - \beta_2$, and $\gamma - \beta_1 - \beta_2$ lie in $[1_{\mathbb{Z}^n}, |w|]$. Observe that w is α_1-periodic and α_2-periodic[11]. It follows that

$$w_1(\gamma - \alpha_1) = w(\gamma - \alpha_1) = w(\gamma - \alpha_1 - \alpha_2) = w(\gamma - \alpha_2) = w_2(\gamma - \alpha_2).$$

Therefore,

$$w_1(\gamma) = w_1(\gamma - \alpha_1) = w_2(\gamma - \alpha_2) = w_2(\gamma).$$

This contradiction proves the lemma. □

Definition 6. Recall that a measurable G-space X, with a (quasi-invariant) measure ν, is said to be *ergodic* if for each G-invariant measurable subset $Y \subset X$ we have $\nu(Y) \in \{0, 1\}$.

Lemma 19 ([12]). *Let G be a countable group, $\mu \in \mathcal{P}(G)$ a measure, and $\check{\mu} \in \mathcal{P}(G)$ the reflected measure $\check{\mu}(g) := \mu(g^{-1})$. Let (M_+, ν_+) and (M_-, ν_-) be μ-boundary and $\check{\mu}$-boundary for G, respectively. Then the action of G on the product $(M_- \times M_+, \nu_- \times \nu_+)$ is ergodic*[12].

Recall that the *characteristic set (or axis)* of an isometry $g : \Gamma_G \to \Gamma_G$ is the subset A_g of Γ_G defined by

$$A_g = \{p \in \Gamma_G \mid [g^{-1} \cdot p, p] \cap [p, g \cdot p] = \{p\}\}.$$

Lemma 20 ([6, Corollary 1.5 of Chapter 3]). *Let G, Γ_G be as above. Let $g \in G$, $x \in A_g$. Then for any $y \in \Gamma_G$ we have $d(x, g{\cdot}x) \leqslant d(y, g{\cdot}y)$.*

Lemma 21. *Let G, Γ_G be as above. Let $x \neq y \in \Gamma_G$, $g \in G$. Assume that $[x, g{\cdot}y] \supset [y, g{\cdot}x]$. Then $x, y \in A_g$.*

Proof. Following [6], we write $[x_0, x_k] = [x_0, x_1, \ldots, x_k]$ to mean that, if $\alpha : [0, d(x_0, x_k)]_{\mathbb{Z}^n} \to \Gamma_G$ is the unique isometry with $\alpha(0) = x_0$ and $\alpha(d(x_0, x_k)) = x_k$, then $x_i = \alpha(a_i)$, where $0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n = d(x_0, x_k)$. In other words, x_0, x_1, $\ldots$, x_k are elements of $[x_0, x_k]$ listed in order of increasing $\mathbb{Z}^n$-distance from x_0.

In this notation, since $[x, g{\cdot}y] \supset [y, g{\cdot}x]$, we have either

A) $[x, g{\cdot}y] = [x, y, g{\cdot}x, g{\cdot}y]$ or

B) $[x, g{\cdot}y] = [x, g{\cdot}x, y, g{\cdot}y]$.

In the case A) we have $[g^{-1}{\cdot}x, y] = [g^{-1}{\cdot}x, g^{-1}{\cdot}y, x, y]$. It is shown in [6, Ch. 2, Lemma 1.5] that we can *splice* the geodesics

$$[g^{-1}{\cdot}x, y] = [g^{-1}{\cdot}x, g^{-1}{\cdot}y, x, y]$$

[11] For the sake of clarity we remark that these conditions does not imply that w is faithfully periodic.

[12] In [12], the lemma is stated and proved in the case where (M_+, ν_+) and (M_-, ν_-) are maximal (Poisson) μ-boundary and $\check{\mu}$-boundary, respectively. The lemma and its proof hold verbatim for the general case.

and $[x, g{\cdot}y] = [x, y, g{\cdot}x, g{\cdot}y]$ by their common part $[x, y]$ to obtain a geodesic (here, we use the assumption that $x \neq y$).

In other words, we have $[g^{-1}{\cdot}x, g{\cdot}y] = [g^{-1}{\cdot}x, g^{-1}{\cdot}y, x, y, g{\cdot}x, g{\cdot}y]$.

Consequently, $[g^{-1}{\cdot}x, x] \cap [x, g{\cdot}x] = \{x\}$ and $[g^{-1}{\cdot}y, y] \cap [y, g{\cdot}y] = \{y\}$. This means that $x, y \in A_g$.

In the case B) we have $[g^{-1}{\cdot}x, y] = [g^{-1}{\cdot}x, x, g^{-1}{\cdot}y, y]$.

By using splicing similar to that one in the case A), we see that either

Ba) $[g^{-1}{\cdot}x, g{\cdot}y] = [g^{-1}{\cdot}x, x, g^{-1}{\cdot}y, g{\cdot}x, y, g{\cdot}y]$ or

Bb) $[g^{-1}{\cdot}x, g{\cdot}y] = [g^{-1}{\cdot}x, x, g{\cdot}x, g^{-1}{\cdot}y, y, g{\cdot}y]$.

In both cases (Ba and Bb) we clearly have $[g^{-1}{\cdot}x, x] \cap [x, g{\cdot}x] = \{x\}$ and $[g^{-1}{\cdot}y, y] \cap [y, g{\cdot}y] = \{y\}$, which means that $x, y \in A_g$ by definition of the characteristic set. □

Proof of Proposition 4. First, let us show that there exists a pair of distinct points ω_1, $\omega_2 \in \partial_n\overline{\Gamma}_G$ and a pair of vertices v_1, v_2 of Γ_G lying on the geodesic $[\omega_1, \omega_2]$ such that i) $\hbar_n(v_1, v_2) > 9$ and ii) $\sigma[v_1, v_2]$ is not faithfully periodic. Recall that $\partial_n\overline{\Gamma}_G$ is non-empty (because G is irreducible; see Lemma **??**) and hence infinite (because G has no finite orbits in $\partial_n\overline{\Gamma}_G \subset \overline{\Gamma}_G$; see Theorem 7). Consequently, we can pick up a triple of distinct points ω_1, ω_2, $\omega_3 \in \partial_n\overline{\Gamma}_G$. Since Γ_G is a universal $\mathbb{Z}^n$-tree for a $\mathbb{Z}^n$-free group, it follows that the geodesics $[\omega_1, \omega_2]$, $[\omega_1, \omega_3]$, and $[\omega_2, \omega_3]$ have a unique common vertex $w := [\omega_1, \omega_2] \cap [\omega_1, \omega_3] \cap [\omega_2, \omega_3]$. Let v_1 be a vertex lying on the geodesic $[\omega_1, w]$ such that $\hbar_n(v_1, w) > 9$ (a vertex with this property exists, because $\omega_1 \in \partial_n\overline{\Gamma}_G$ and $w \in \Gamma_G$, whence $\hbar_n(\omega_1, w) = \infty$). Let u_2, u_3 be vertices lying on the geodesics $[w, \omega_2]$ and $[w, \omega_3]$, respectively, such that $d(w, u_2) = d(w, u_3) = 1_{\mathbb{Z}^n}$ (that is, (w, u_2) and (w, u_3) are edges of Γ_G). Then Lemma 18 clearly implies that at least one of two $\mathbb{Z}^n$-words $\sigma[v_1, u_2]$, $\sigma[v_1, u_3]$ is not faithfully periodic. Without loss of generality, we can and will assume that $\sigma[v_1, u_2]$ is not faithfully periodic. Set $v_2 := u_2$. Thus, v_1 and v_2 lie on the geodesic $[\omega_1, \omega_2]$ (where ω_1, $\omega_2 \in \partial_n\overline{\Gamma}_G$), $\sigma[v_1, v_2]$ is not faithfully periodic, and $\hbar_n(v_1, v_2) = \hbar_n(v_1, w) > 9$ (because $d(v_1, v_2) = d(v_1, w) + 1_{\mathbb{Z}^n}$), as required.

Now, we fix the constructed above pair v_1, v_2 and define the map

$$S := S_{v_1,v_2} : \overline{\Gamma}_G \times \overline{\Gamma}_G \to 2^G$$

in the following manner:

$$S_{v_1,v_2}(b_-, b_+) := \{g \in G \ : \ (g{\cdot}[v_1, v_2] \subset [b_-, b_+]) \ \& \ ([b_-, g{\cdot}v_2] \ni g{\cdot}v_1)\}\,.$$

Here, the condition $g{\cdot}[v_1, v_2] \subset [b_-, b_+]$ is equivalent to the conditions $g{\cdot}v_1 \in [b_-, b_+]$, $g{\cdot}v_2 \in [b_-, b_+]$. The second condition $[b_-, g{\cdot}v_2] \ni g{\cdot}v_1$ says that g maps $[v_1, v_2]$ to $[b_-, b_+]$ preserving certain orientation.

Our S_{v_1,v_2} is obviously measurable and G-equivariant. Let us check that S_{v_1,v_2} has required properties 1) and 2). We start with property 2).

2) Let μ be a non-degenerate probability measure on G, ν_+ a μ-stationary measure on $\overline{\Gamma}_G$, and ν_- a $\check{\mu}$-stationary measure on $\overline{\Gamma}_G$. Then Theorem 10 implies that $(\overline{\Gamma}_G, \nu_+)$ is a μ-boundary for G and $(\overline{\Gamma}_G, \nu_+)$ is a $\check{\mu}$-boundary for G. In particular, the action of G on the space $\overline{\Gamma}_G \times \overline{\Gamma}_G$ is ergodic with respect to the product measure $\nu_- \times \nu_+$ (see Lemma 19).

Let $B_1 := \mathrm{Branch}(v_2, v_1)$ and $B_2 := \mathrm{Branch}(v_1, v_2)$, and consider the set

$$\Omega := \Omega_{v_1,v_2} := B_1 \times B_2 \subset \overline{\Gamma}_G \times \overline{\Gamma}_G.$$

Our definitions immediately imply that

$$S_{v_1,v_2}(b_-,b_+) = \left\{g \in G \ : \ g^{-1}\cdot(b_-,b_+) \in \Omega_{v_1,v_2}\right\}.$$

Observe that B_1 and B_2 are open subsets of $\overline{\Gamma}_G$ (Lemma 16), and that the intersections $B_1 \cap \partial_n\overline{\Gamma}_G$ and $B_1 \cap \partial_n\overline{\Gamma}_G$ are nonempty (because $B_1 \cap \partial_n\overline{\Gamma}_G \ni \omega_1$ and $B_2 \cap \partial_n\overline{\Gamma}_G \ni \omega_2$). It then follows by assertion (iii) of Proposition 1 that $\nu_-(B_1) > 0$ and $\nu_+(B_2) > 0$. Consequently, we have

$$(\nu_- \times \nu_+)(\Omega_{v_1,v_2}) = (\nu_- \times \nu_+)(B_1 \times B_2) = \nu_-(B_1)\cdot\nu_+(B_2) > 0.$$

Since the action of G on the space $(\overline{\Gamma}_G \times \overline{\Gamma}_G, \nu_- \times \nu_+)$ is ergodic (see Lemma 19) and $(\nu_- \times \nu_+)(\Omega_{v_1,v_2}) > 0$, it follows that $(\nu_- \times \nu_+)(G\cdot\Omega_{v_1,v_2}) = 1$. In particular, for $(\nu_- \times \nu_+)$-a. e. pair $(b_-,b_+) \in \overline{\Gamma}_G \times \overline{\Gamma}_G$ the set $\{g \in G : g^{-1}\cdot(b_-,b_+) \in \Omega_{v_1,v_2}\}$ is not empty. It remains to notice that (as it was already mentioned above) $S_{v_1,v_2}(b_-,b_+) = \{g \in G : g^{-1}\cdot(b_-,b_+) \in \Omega_{v_1,v_2}\}$.

Remark: a more accurate argument with, e. g., the Ergodic Theorem shows that $S_{v_1,v_2}(b_-,b_+)$, indeed, "looks like a strip" for $(\nu_- \times \nu_+)$-a. e. pair (b_-,b_+), but here we need to check only that $S_{v_1,v_2}(b_-,b_+)$ is a. s. non-empty. (Note also that $(\nu_- \times \nu_+)$-a. e. pair (b_-,b_+) lies in $\partial_n\overline{\Gamma}_G \times \partial_n\overline{\Gamma}_G \subset \overline{\Gamma}_G \times \overline{\Gamma}_G$.)

1) Let $(b_-,b_+) \in \overline{\Gamma}_G \times \overline{\Gamma}_G$ be fixed. We will prove two auxiliary claims.

Claim i): If $g_1, g_2 \in S_{v_1,v_2}(b_-,b_+)$ and $g_1 \neq g_2$, then $\hbar_n(g_1\cdot v_1, g_2\cdot v_1) \geqslant 5$.

Proof: Suppose that $\hbar_n(g_1\cdot v_1, g_2\cdot v_1) \leqslant 4$. Our definition of S_{v_1,v_2} implies that the vertices $g_1\cdot v_1$, $g_1\cdot v_2$, $g_2\cdot v_1$, $g_2\cdot v_2$ lie on the geodesic $[b_-,b_+]$. Let us use the following notation: if $x, y \in [b_-,b_+]$, we will write $x \leqslant y$ if and only if $x \in [b_-,y]$ (if $x \leqslant y$ and $x \neq y$, we write $x < y$). Under this notation, we have $g_1\cdot v_1 < g_1\cdot v_2$ and $g_2\cdot v_1 < g_2\cdot v_2$ (see the definition of S_{v_1,v_2} again). Without loss of generality, we can assume that $g_1\cdot v_1 \leqslant g_2\cdot v_1$. Then, since we have assumed that $\hbar_n(g_1\cdot v_1, g_2\cdot v_1) \leqslant 4$, while according to the choice of the pair v_1, v_2

$$\hbar_n(g_1\cdot v_1, g_1\cdot v_2) = \hbar_n(g_2\cdot v_1, g_2\cdot v_2) = \hbar_n(v_1, v_2) > 9,$$

it follows that

$$g_1\cdot v_1 \leqslant g_2\cdot v_1 \leqslant g_1\cdot v_2 \leqslant g_2\cdot v_2.$$

Now we consider the $\mathbb{Z}^n$-word $\sigma[g_1\cdot v_1, g_2\cdot v_2]$ and its subwords. Let

$$w_1 := \sigma[g_1\cdot v_1, g_2\cdot v_1],$$

$$w_2 := \sigma[g_2\cdot v_1, g_1\cdot v_2],$$

$$w_3 := \sigma[g_1\cdot v_2, g_2\cdot v_2].$$

Then $\sigma[g_1\cdot v_1, g_2\cdot v_2] = w_1w_2w_3$. Observe that

$$w_1w_2 = \sigma[g_1\cdot v_1, g_1\cdot v_2] = \sigma[v_1, v_2] = \sigma[g_2\cdot v_1, g_2\cdot v_2] = w_2w_3.$$

This equality shows that the $\mathbb{Z}^n$-word $\sigma[v_1, v_2] = w_1w_2 = w_2w_3$ is $|w_1|$-periodic[13]. Observe that $2\cdot|w_1| < |\sigma[v_1, v_2]|$, because

$$\hbar_n(|w_1|) = \hbar_n(g_1\cdot v_1, g_2\cdot v_1) \leqslant 4,$$

[13]We follow the definition of β-periodicity. Since $|\sigma[v_1, v_2]| - |w_1| = |w_1w_2| - |w_1| = |w_2|$, it follows that for each $\alpha \in [1_{\mathbb{Z}^n}, |\sigma[v_1, v_2]| - |w_1|]$ we have

$$w(\alpha) = (w_2w_3)(\alpha) = w_2(\alpha) = (w_1w_2)(|w_1| + \alpha) = w(|w_1| + \alpha),$$

as required by the definition.

while $\hbar_n(v_1, v_2) > 9$. Thus, $\sigma[v_1, v_2]$ is $|w_1|$-periodic and $2 \cdot |w_1| < |\sigma[v_1, v_2]|$. This means that $\sigma[v_1, v_2]$ is faithfully periodic (see Definition 5). Recall that v_1 and v_2 were chosen to have $\sigma[v_1, v_2]$ not faithfully periodic. The claim is proved (by contradiction).

Claim ii): If $g_1, g_2 \in S_{v_1,v_2}(b_-, b_+)$, then

$$\hbar_n(g_2 g_1^{-1}) \stackrel{\text{def}}{=} \hbar_n(g_1{\cdot}\epsilon, g_2{\cdot}\epsilon) \geqslant \hbar_n(g_1{\cdot}v_1, g_2{\cdot}v_1).$$

Proof: In the case $n = 1$, the desired inequality follows from the fact that $g_2 g_1^{-1} : \Gamma_G \to \Gamma_G$ is a hyperbolic isometry of Γ_G and $g_1{\cdot}v_1$, $g_2{\cdot}v_1$ lie on the axis of this isometry. This argument generalizes to the case of arbitrary n in the following way. Consider the isometry $g := g_2 g_1^{-1} : \Gamma_G \to \Gamma_G$, which sends $[g_1{\cdot}v_1, g_1{\cdot}v_2]$ to $[g_2{\cdot}v_1, g_2{\cdot}v_2]$, and let A_g be its characteristic set (see Definition **??**). Observe that $g_1{\cdot}v_1 \in A_g$ (in fact, $[g_2{\cdot}v_1, g_2{\cdot}v_2]$ and $[g_1{\cdot}v_1, g_1{\cdot}v_2]$ are in A_g). This follows from Lemma 21, because the definition of $S_{v_1,v_2}(b_-, b_+)$ clearly implies that $I := [g_1{\cdot}v_1, g_2{\cdot}v_2] = [g_1{\cdot}v_1, gg_1{\cdot}v_2]$ and $J := [g_1{\cdot}v_2, g_2{\cdot}v_1] = [g_1{\cdot}v_2, gg_1{\cdot}v_1]$ lie on the geodesic $[b_-, b_+]$ and we have either $I \subset J$ or $J \subset I$. It then follows by Lemma 20 that $d(g_1{\cdot}v_1, g_2{\cdot}v_1) \leqslant d(g_1{\cdot}\epsilon, g_2{\cdot}\epsilon)$, which implies the desired inequality. Claim ii) is proved.

Now we are ready to prove property 1). Let $k \in \mathbb{N}$ and let M_k denote the set $\{g \in S_{v_1,v_2}(b_-, b_+) : \hbar_n(g) \leqslant k\}$. We have to show that $\operatorname{card} M_k \leqslant k$. By Claim ii), we have[14]

$$\operatorname{diam}_{\hbar_n}(M_k{\cdot}v_1) \leqslant \operatorname{diam}_{\hbar_n}(M_k) \leqslant \operatorname{diam}_{\hbar_n}(\{g \in G : \hbar_n(g) \leqslant k\}) = 2k. \tag{6}$$

On the other hand, all the points of $M_k{\cdot}v_1$ lie on the geodesic $[b_-, b_+]$ and by Claim i) for any $g_1, g_2 \in M_k$ we have $\hbar_n(g_1{\cdot}v_1, g_2{\cdot}v_1) \geqslant 5$. This implies (use, e. g., Lemma 13) that

$$\operatorname{diam}_{\hbar_n}(M_k{\cdot}v_1) \geqslant 5(\operatorname{card} M_k - 1). \tag{7}$$

We combine (6) and (7) and get

$$5(\operatorname{card} M_k - 1) \leqslant 2k. \tag{8}$$

In our case, where $k \in \mathbb{N}$ and $\operatorname{card} M_k$ is integer, inequality (8) implies that $\operatorname{card} M_k \leqslant k$, as required. □

4.4 Proof of Theorem 1

Theorem 1 readily follows from Theorem 9. To see this, recall that we have a natural one-to-one correspondence $g \mapsto g{\cdot}\epsilon$ between the set of elements of our group G and the subset $G{\cdot}\epsilon$ in the universal $\mathbb{Z}^n$-tree Γ_G, where $\epsilon \in \Gamma_G$ is the basepoint. This correspondence is equivariant and induces an equivariant homeomorphism between the topological G-spaces $G \cup \partial_n G$ and $(G{\cdot}\epsilon) \cup \partial_n \overline{\Gamma}_G$. (The fact follows from our definitions in a routine way.) Under the homeomorphism

$$G \cup \partial_n G \cong (G{\cdot}\epsilon) \cup \partial_n \overline{\Gamma}_G, \tag{9}$$

[14] Here,

$$\operatorname{diam}_{\hbar_n}(X) := \sup_{x,y,\in X} \{\hbar_n(x, y)\} \quad \text{for} \quad X \subset \overline{\Gamma}_G,$$

$$\operatorname{diam}_{\hbar_n}(H) := \sup_{g,h \in H} \{\hbar_n(g^{-1}h)\} \quad \text{for} \quad H \subset G.$$

it is clear that the first assertion of Theorem 1 (saying that for every non-degenerate probability measure μ on G, a. e. path of the random μ-walk converges to a point in $\partial_n G$) is equivalent to the proven fact that for a. e. path $\tau = \{\tau_i\}$ of the random μ-walk, the sequence $\{\tau_i \cdot \epsilon\}$ converge to $\omega(\tau) \in \partial_n \overline{\Gamma}_G$ (see assertions 1 and 2 of Theorem 9).

Now, let μ be a non-degenerate probability measure on G. Then, by Theorem 9, there is a unique μ-stationary measure ν_μ on $\overline{\Gamma}_G$, the pair $(\overline{\Gamma}_G, \nu_\mu)$ is a μ-boundary of G, and ν_μ is concentrated on $\partial_n \overline{\Gamma}_G$. Since $\nu_\mu(\partial_n \overline{\Gamma}_G) = 1$ and $\partial_n \overline{\Gamma}_G$ is a measurable G-invariant subset in $\overline{\Gamma}_G$, it then follows that the pair $(\partial_n \overline{\Gamma}_G, \nu'_\mu)$, where ν'_μ is the measure in $\mathcal{P}(\partial_n \overline{\Gamma}_G)$ associated with the measure $\nu_\mu \in \mathcal{P}(\overline{\Gamma}_G)$, is also a μ-boundary of G (see Lemma 3). Consequently, the pair $(\partial_n G, \nu''_\mu)$, where ν''_μ is the measure in $\mathcal{P}(\partial_n G)$ that corresponds to the measure ν_μ under the homeomorphism $\partial_n G \cong \partial_n \overline{\Gamma}_G$, is (again) a μ-boundary of G. We have isomorphic measure spaces

$$(\overline{\Gamma}_G, \nu_\mu) \cong (\partial_n \overline{\Gamma}_G, \nu'_\mu) \cong (\partial_n G, \nu''_\mu). \tag{10}$$

Since the μ-boundaries $(\overline{\Gamma}_G, \nu_\mu)$ and $(\partial_n G, \nu''_\mu)$ are isomorphic as measure spaces, it follows that one of these boundaries is a Poisson–Furstenberg (maximal) boundary of (G, μ) if and only if the other one is so. Thus, the second (and last) statement of Theorem 1 is equivalent to assertion 3) of Theorem 9. (To take into account all the details of the second statement of Theorem 1, remark also that the measure ν''_μ is precisely the distribution of limits of paths of the random μ-walk. This can be seen from the homeomorphism (9) and isomorphisms (10), since Theorems 9 and 2 imply that the measure ν_μ determines the distribution of limits of sequences $\{\tau_i \cdot \epsilon\}$ for the paths $\tau = \{\tau_i\}$ of the random μ-walk, while the homeomorphism $(G \cdot \epsilon) \cup \partial_n \overline{\Gamma}_G \to G \cup \partial_n G$ turns limits of sequences $\{\tau_i \cdot \epsilon\}$ to limits of corresponding paths $\{\tau_i\}$ and sends $\nu'_\mu \cong \nu_\mu$ to ν''_μ.)

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