

ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

РЕДКОЛЛЕГИЯ

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**Sharp constants in the classical weak form
of the John–Nirenberg inequality**

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ABSTRACT:

I would like to present the Bellman function related with the classical weak form of the John–Nirenberg inequality. This result was obtained about four years ago, this text was written that time as the beginning of a paper. The result was already presented several times on conferences, moreover, new papers based on this result appear (see, e.g. [5]). By this reason, after minimal changes in the list of references, I decided to make this old text accessible to everybody.

Key words: Bellman function, BMO space, John–Nirenberg inequality

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1. INTRODUCTION

A crucial property of elements of BMO-space, the exponential decay of their distribution function, was established in the classical paper [1]; it is known as the John–Nirenberg inequality.

For an interval I , and a real-valued function $\varphi \in L^1(I)$, let $\langle \varphi \rangle_I$ be the average of φ over I , i.e.,

$$\langle \varphi \rangle_I = \frac{1}{|I|} \int_I \varphi,$$

where $|I|$ stands for Lebesgue measure of I . For $1 \leq p < \infty$, let

$$\text{BMO}(J) = \{ \varphi \in L^1(J) : \langle |\varphi - \langle \varphi \rangle_I|^p \rangle_I \leq C^p < \infty, \forall I \subset J \} \quad (1.1)$$

with the best (smallest) such C being the corresponding “norm” of φ . For $\varepsilon \geq 0$, let

$$\text{BMO}_\varepsilon(J) = \{ \varphi \in \text{BMO}(J) : \|\varphi\| \leq \varepsilon \}.$$

The classical definition of John and Nirenberg uses $p = 1$; it is known that the norms are equivalent for different p 's. For every $\varphi \in \text{BMO}(J)$ and every $\lambda \in \mathbb{R}$ the classical John–Nirenberg inequality consists in the following assertion.

Theorem (John, Nirenberg; weak form)

$$\frac{1}{|J|} |\{s \in J : |\varphi(s) - \langle \varphi \rangle_J| \geq \lambda\}| \leq c_1 e^{-c_2 \lambda / \|\varphi\|_{\text{BMO}(J)}}.$$

I refer to this statement as to the weak form of the John–Nirenberg inequality to distinguish it from the following equivalent assertion.

Theorem (John, Nirenberg; integral form) *There exists $\varepsilon_0 > 0$ such that for every ε , $0 \leq \varepsilon < \varepsilon_0$, there is $C(\varepsilon) > 0$ such that for any function φ , $\varphi \in \text{BMO}_\varepsilon(J)$, the following inequality holds*

$$\langle e^\varphi \rangle_J \leq C(\varepsilon) e^{\langle \varphi \rangle_J}.$$

The sharp constants in the integral form were found in [9] and [7]. In the second paper the dyadic analog BMO^d is considered as well, for which every subinterval I of J in definition (1.1) is an element of the dyadic lattice rooted in J . It appears that the constants in the dyadic case and the usual one are different.

The mentioned constants were found by using so called Bellman function method (see, [3], [4]). Namely, the Bellman function of the corresponding extremal problem (the definition see below) was found explicitly. This function carries all the information about the problem: not only the sharp constants, but, for example, construction of extremal test functions (*extremizers*). The Bellman function corresponding to the integral John–Nirenberg inequality was found by solving the boundary value problem for the Bellman equation. In that case the

Bellman equation was a second order PDE with two variables, and due to a natural homogeneity of the problem, the Bellman PDE was reduced to an ordinary differential equation, which was successfully solved. The corresponding Bellman equation for the weak John–Nirenberg inequality has an additional parameter λ preventing a similar reducing of the Bellman PDE to an ordinary differential equation.

The Bellman equations for all these problems are in fact partial cases of the Monge–Ampère equation. After finding possibility to solve this type of equation explicitly (see [6], [10]) we are able to find the Bellman function (and therefore, the sharp constants) for the weak John–Nirenberg inequality as well. And this solution is described in the present paper.

We shall work with L^2 -based BMO-norm, i.e., $p = 2$ will be chosen in (1.1). For the classical case $p = 1$, Korenovskii [2] established the exact value $c_2 = 2/e$ using the equimeasurable rearrangements of the test function and the “sunrise lemma”. But to apply the Bellman function method the L^2 -based BMO-norm is more appropriate. Some Bellman-type function (so-called supersolution) for the weak John–Nirenberg inequality was proposed by Tao in [8], where there was no attempt to find true Bellman function and sharp constants. In the present paper it will be proved that for $p = 2$ the sharp constant are $c_1 = \frac{4}{e^2}$ and $c_2 = 1$.

2. DEFINITIONS AND STATEMENTS OF THE MAIN RESULTS

2.1. Bellman functions. Now the main subject of the paper will be introduced, the Bellman function corresponding to the John–Nirenberg inequality. First of all we define the following set of test functions

$$S_\varepsilon(x) = S(x_1, x_2; \varepsilon) = \{\varphi \in \text{BMO}(J) : \langle \varphi \rangle_J = x_1, \langle \varphi^2 \rangle_J = x_2, \langle |\varphi - \langle \varphi \rangle_J|^2 \rangle_J \leq \varepsilon^2 \forall I \subset J\}.$$

For any test function φ the point $x = (x_1, x_2) = (\langle \varphi \rangle_J, \langle \varphi^2 \rangle_J)$ belongs to the parabolic strip

$$\Omega_\varepsilon = \{x = (x_1, x_2) : x_1^2 \leq x_2 \leq x_1^2 + \varepsilon^2\}.$$

Indeed, the left inequality $x_1^2 \leq x_2$ is simply the Cauchy–Bunyakovsky–Schwartz inequality, but the right one $x_2 \leq x_1^2 + \varepsilon^2$ follows from the fact that $\varphi \in \text{BMO}_\varepsilon(J)$:

$$x_2 - x_1^2 = \langle \varphi^2 \rangle_J - \langle \varphi \rangle_J^2 = \langle |\varphi - \langle \varphi \rangle_J|^2 \rangle_J \leq \varepsilon^2.$$

Now we define the Bellman **B** function corresponding to the weak John–Nirenberg inequality:

$$\mathbf{B}(x; \lambda, \varepsilon) \stackrel{\text{def}}{=} \frac{1}{|I|} \sup \{|\{s \in I : |\varphi(s)| \geq \lambda\}| : \varphi \in S_\varepsilon(x)\}. \quad (2.1)$$

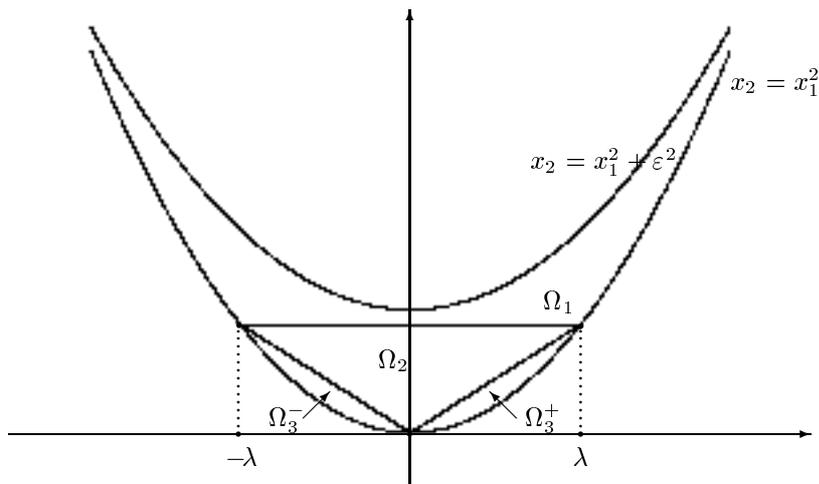


FIGURE 1

This function is defined on Ω and it supplies us with the sharp estimate of the distribution function

$$\frac{1}{|J|} |\{s \in J: |\varphi(s) - \langle \varphi \rangle_J| \geq \lambda\}| \leq \sup_{\xi \in [0, \varepsilon^2]} \mathbf{B}(0, \xi; \lambda) \quad \forall \varphi \in \text{BMO}_\varepsilon. \quad (2.2)$$

To check this, we consider a new function $\tilde{\varphi} \stackrel{\text{def}}{=} \varphi + c$. If $\varphi \in S_\varepsilon(x)$, then $\tilde{\varphi} \in S_\varepsilon(\tilde{x})$, where $\tilde{x}_1 = x_1 + c$ and $\tilde{x}_2 = x_2 + 2cx_1 + c^2$. Therefore, by definition (2.1), we have

$$\frac{1}{|J|} |\{s \in J: |\tilde{\varphi}(s)| \geq \lambda\}| \leq \mathbf{B}(\tilde{x}; \lambda).$$

If we take now $c = -\langle \varphi \rangle_J = -x_1$, we get $\tilde{x}_1 = 0$, $\tilde{x}_2 = x_2 - x_1^2$, and the latter inequality turns into

$$\frac{1}{|J|} |\{s \in J: |\varphi(s) - \langle \varphi \rangle_J| \geq \lambda\}| \leq \mathbf{B}(0, \tilde{x}_2; \lambda) \leq \sup_{\xi \in [0, \varepsilon^2]} \mathbf{B}(0, \xi; \lambda).$$

So, to find the sharp constants in the weak John–Nirenberg inequality we prove the following theorem.

Theorem 1. For $0 \leq \lambda \leq \varepsilon$ split Ω in three subdomains (see Fig. 1):

$$\begin{aligned} \Omega_1 &= \{x \in \Omega: x_2 \geq \lambda^2\}, \\ \Omega_2 &= \{x \in \Omega: \lambda|x_1| \leq x_2 \leq \lambda^2\}, \\ \Omega_3 &= \{x \in \Omega: x_2 < \lambda|x_1|\}, \end{aligned}$$

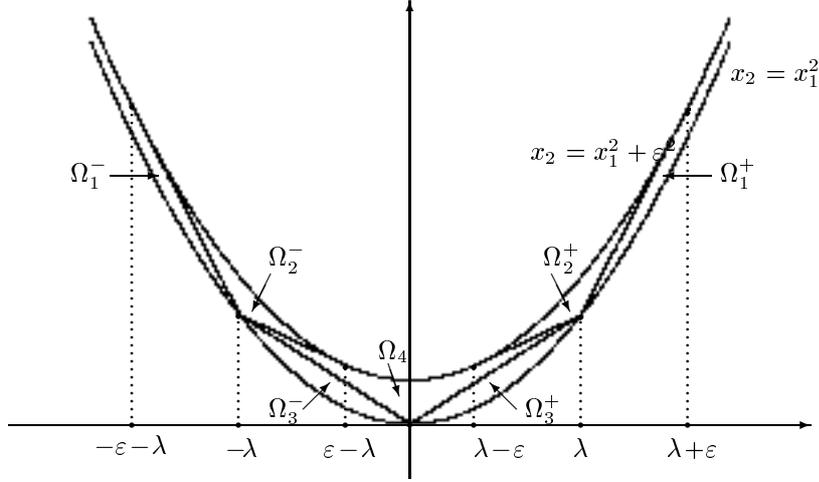


FIGURE 2

then

$$\mathbf{B}(x; \lambda, \varepsilon) = \begin{cases} 1, & x \in \Omega_1, \\ \frac{x_2}{\lambda^2}, & x \in \Omega_2, \\ \frac{x_2 - x_1^2}{x_2 + \lambda^2 - 2\lambda|x_1|}, & x \in \Omega_3. \end{cases}$$

For $\varepsilon < \lambda \leq 2\varepsilon$ split Ω in four subdomains (see Fig. 2):

$$\Omega_1 = \{x \in \Omega: |x_1| \geq \lambda \text{ and } x_2 \leq 2(\lambda + \varepsilon)|x_1| - \lambda^2 - 2\varepsilon\lambda \text{ for } |x_1| < \lambda + \varepsilon, \},$$

$$\Omega_2 = \{x \in \Omega: \lambda - \varepsilon \leq |x_1| \leq \lambda + \varepsilon, x_2 \geq \max\{2\lambda|x_1| - \lambda^2 \pm 2\varepsilon(|x_1| - \lambda)\}\},$$

$$\Omega_3 = \{x \in \Omega: x_2 < \lambda|x_1|\},$$

$$\Omega_4 = \{x \in \Omega: x_2 \geq \lambda|x_1| \text{ and } x_2 \leq 2(\lambda - \varepsilon)|x_1| - \lambda^2 + 2\varepsilon\lambda \text{ for } |x_1| > \lambda - \varepsilon\},$$

then

$$\mathbf{B}(x; \lambda, \varepsilon) = \begin{cases} 1, & x \in \Omega_1, \\ \frac{2(\lambda^2 - \varepsilon^2)|x_1| - (\lambda - \varepsilon)x_2 + \lambda(2\varepsilon^2 + \varepsilon\lambda - \lambda^2)}{2\varepsilon\lambda^2}, & x \in \Omega_2, \\ \frac{x_2 - x_1^2}{x_2 + \lambda^2 - 2\lambda|x_1|}, & x \in \Omega_3, \\ \frac{x_2}{\lambda^2}, & x \in \Omega_4. \end{cases}$$

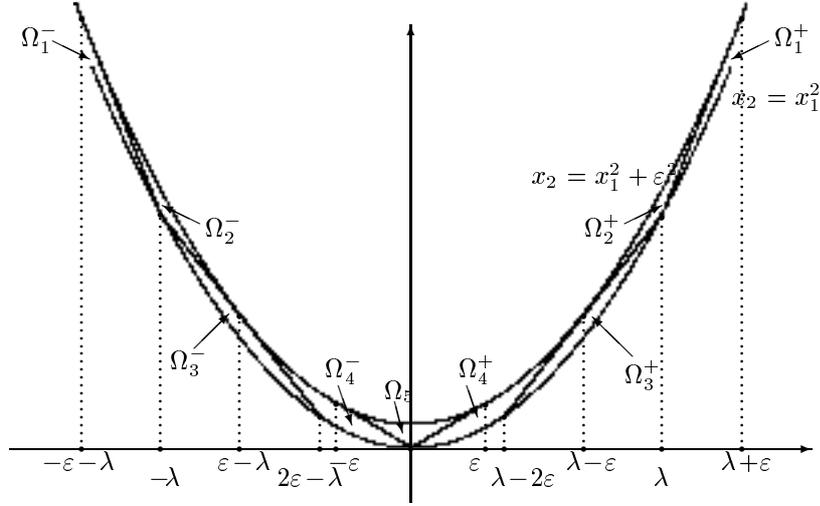


FIGURE 3

For $\lambda > 2\varepsilon$ split Ω in five subdomains (see Fig. 3):

$$\Omega_1 = \{x \in \Omega : |x_1| \geq \lambda \text{ and } x_2 \leq 2(\lambda + \varepsilon)|x_1| - \lambda^2 - 2\varepsilon\lambda \text{ for } |x_1| < \lambda + \varepsilon, \},$$

$$\Omega_2 = \{x \in \Omega : \lambda - \varepsilon \leq |x_1| \leq \lambda + \varepsilon, x_2 \geq \max\{2\lambda|x_1| - \lambda^2 \pm 2\varepsilon(|x_1| - \lambda)\}\},$$

$$\Omega_3 = \{x \in \Omega : x_2 < 2(\lambda - \varepsilon)|x_1| - \lambda^2 + 2\varepsilon\lambda, \},$$

$$\Omega_4 = \{x \in \Omega : x_2 \geq 2(\lambda - \varepsilon)|x_1| - \lambda^2 + 2\varepsilon\lambda \text{ and } x_2 \leq \varepsilon|x_1| \text{ for } |x_1| < \varepsilon, \},$$

$$\Omega_5 = \{x \in \Omega : x_2 \geq \varepsilon|x_1|\},$$

then

$$\mathbf{B}(x; \lambda, \varepsilon) = \begin{cases} 1, & x \in \Omega_1, \\ 1 - \frac{x_2 - 2(\lambda + \varepsilon)|x_1| + \lambda^2 + 2\varepsilon\lambda}{8\varepsilon^2}, & x \in \Omega_2, \\ \frac{x_2 - x_1^2}{x_2 + \lambda^2 - 2\lambda|x_1|}, & x \in \Omega_3, \\ \frac{e}{2} \left(1 - \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}}\right) \exp \left\{ \frac{|x_1| - \lambda}{\varepsilon} + \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}} \right\}, & x \in \Omega_4, \\ \frac{x_2}{4\varepsilon^2} \exp \left\{ 2 - \frac{\lambda}{\varepsilon} \right\}, & x \in \Omega_5. \end{cases}$$

Corollary. *If $\varphi \in \text{BMO}_\varepsilon(I)$, then*

$$\frac{1}{|I|} |\{s \in I: |\varphi(s) - \langle \varphi \rangle_I| \geq \lambda\}| \leq \begin{cases} 1, & \text{if } 0 \leq \lambda \leq \varepsilon, \\ \frac{\varepsilon^2}{\lambda^2} & \text{if } \varepsilon \leq \lambda \leq 2\varepsilon, \\ \frac{e^2}{4} e^{-\lambda/\varepsilon} & \text{if } 2\varepsilon \leq \lambda, \end{cases}$$

and this bound is sharp.

Proof. According to formula (2.2) it is sufficient to calculate

$$\sup_{\xi \in [0, \varepsilon^2]} \mathbf{B}(0, \xi; \lambda, \varepsilon).$$

Since $\mathbf{B}(0, x_2; \lambda, \varepsilon)$ is an increasing function in x_2 , this supremum is just the value $\mathbf{B}(0, \varepsilon^2; \lambda, \varepsilon)$, what yields the stated formula. \square

Before we start to prove Theorem 1, where the Bellman function has two singularities on the boundary at the points $x = (\pm\lambda, \lambda^2)$, let us consider the simplest possible extremal problem with one singularity. We shall consider two extremal problems simultaneously: one estimate from above and the second estimate from below. So, we define two Bellman functions: \mathbf{B}_{\max} and \mathbf{B}_{\min} .

$$\mathbf{B}_{\max}(x; \lambda, \varepsilon) \stackrel{\text{def}}{=} \frac{1}{|I|} \sup \{ |\{s \in I: \varphi(s) \geq \lambda\}| : \varphi \in S_\varepsilon(x) \},$$

$$\mathbf{B}_{\min}(x; \lambda, \varepsilon) \stackrel{\text{def}}{=} \frac{1}{|I|} \inf \{ |\{s \in I: \varphi(s) \geq \lambda\}| : \varphi \in S_\varepsilon(x) \},$$

For these function the following formula will be proved:

Theorem 2. *Split Ω in the following five subdomains (see Fig. 4):*

$$\Omega_1 = \{x \in \Omega: x_1 \geq \lambda + \varepsilon, x_2 \geq 2(\lambda + \varepsilon)x_1 - \lambda^2 - 2\varepsilon\lambda\},$$

$$\Omega_2 = \{x \in \Omega: x_2 \leq 2(\lambda + \varepsilon)x_1 - \lambda^2 - 2\varepsilon\lambda\},$$

$$\Omega_3 = \{x \in \Omega: \lambda - \varepsilon \leq x_1 \leq \lambda + \varepsilon, x_2 \geq 2\lambda x_1 - \lambda^2 + 2\varepsilon|x_1 - \lambda|\},$$

$$\Omega_4 = \{x \in \Omega: x_2 \leq 2(\lambda - \varepsilon)x_1 - \lambda^2 + 2\varepsilon\lambda\},$$

$$\Omega_5 = \{x \in \Omega: x_1 \leq \lambda - \varepsilon, x_2 \geq 2(\lambda - \varepsilon)x_1 - \lambda^2 + 2\varepsilon\lambda\}.$$

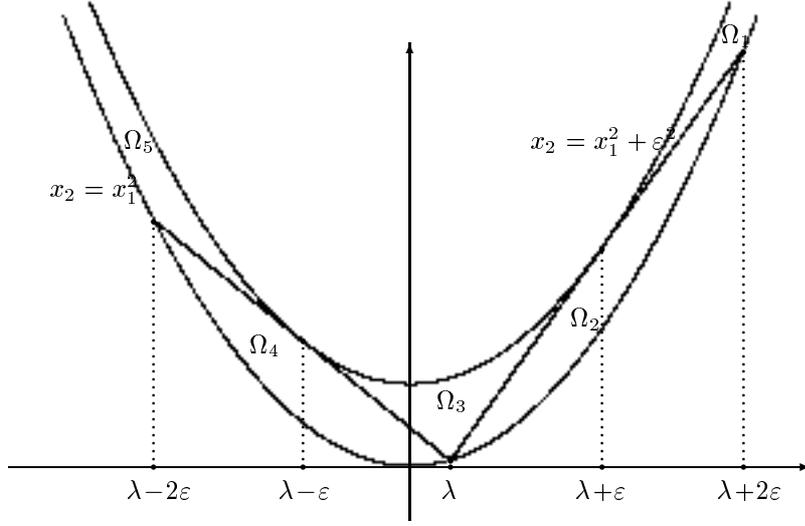


FIGURE 4

Then

$$\mathbf{B}_{\max}(x; \lambda, \varepsilon) = \begin{cases} 1, & x \in \Omega_1 \cup \Omega_2, \\ 1 - \frac{x_2 - 2(\lambda + \varepsilon)x_1 + \lambda^2 + 2\varepsilon\lambda}{8\varepsilon^2}, & x \in \Omega_3, \\ \frac{x_2 - x_1^2}{x_2 + \lambda^2 - 2\lambda x_1}, & x \in \Omega_4, \\ \frac{e}{2} \left(1 - \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}} \right) \exp \left\{ \frac{x_1 - \lambda}{\varepsilon} + \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}} \right\}, & x \in \Omega_5, \end{cases}$$

and

$$\mathbf{B}_{\min}(x; \lambda, \varepsilon) = \begin{cases} 0, & x \in \Omega_5 \cup \Omega_4, \\ \frac{x_2 - 2(\lambda - \varepsilon)x_1 + \lambda^2 - 2\varepsilon\lambda}{8\varepsilon^2}, & x \in \Omega_3, \\ 1 - \frac{x_2 - x_1^2}{x_2 + \lambda^2 - 2\lambda x_1}, & x \in \Omega_2, \\ 1 - \frac{e}{2} \left(1 - \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}} \right) \exp \left\{ \frac{\lambda - x_1}{\varepsilon} + \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}} \right\}, & x \in \Omega_1. \end{cases}$$

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