ПРЕПРИНТЫ ПОМИ РАН

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Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16 октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых коммуникаций

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

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http://www. pdmi.ras.ru /preprint/

Заведующая информационно-издательским сектором Симонова В.Н

PDMI Preprint 08/2011

On the averaged wave operators on the singular spectrum

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ABSTRACT:

We prove that for the class of unitary operators with singular spectral measure there exist no universal regular summation method allowing one to construct averaged wave operators even for the case of rank-two commutators. Also we discuss the closely related problem of constructing the Hilbert transform with respect to a singular measure on the unit circle.

Key words: wave operator, summation methods, singular spectral measure.

Partially supported by RFBR grant 11-01-00584-a.

Given a pair of unitary operators $U_1 : H_1 \to H_1$, $U_2 : H_2 \to H_2$ and an operator $X : H_1 \to H_2$, the wave operator is the limit of the sequence (X_n),

$$X_n = U_2^n X U_1^{-n}.$$

We consider the past wave operator corresponding to the limit as $n \to +\infty$. According to the classical scattering theory, the strong wave operator, which is the strong limit of the sequence (X_n), exists if the spectral measures of U_1, U_2 are absolutely continuous with respect to the Lebesgue measure and the commutator $U_2X - XU_1$ is of trace class. For the example where $H_1 = H_2 = \mathbb{C}$ is the one-dimensional space, $U_1 = I$, $U_2 = \omega I$ with $|\omega| = 1$, $\omega \neq 1$, the limit of X_n obviously does not exist unless $X = 0$. However, then the Cesàro means of X_n (namely, the operators $\frac{1}{m+1}\sum_{n=0}^{m} X_n$) do converge. One could also apply summation methods that are stronger than the method of Cesàro means. It was natural to conjecture [5] that, in the general case, *if the commutator* $K = U_2X - XU_1$ *is of finite rank, then the limit of the Abel means* $(1-r)\sum_{k=0}^{\infty} r^n X_n$ *of the sequence* (X_n) *exists in the weak operator topology*; our main result will disprove this conjecture.

One can prove that if U_1 has a complete family of eigenvectors, then the Cesàro means of X_n have a strong limit; if the spectral measures of U_1 and U_2 are mutually singular, then the Cesàro means of X_n weakly tend to zero, see, e.g., [5]. This reduces the problem to the case where U_1, U_2 have no eigenvectors and their spectral measures are singular with respect to the Lebesgue measure. In the case $\operatorname{rank} K = 1$ the above conjecture was essentially proved in [4], the explicit formulation is contained in [5]. For the case $\operatorname{rank} K = 2$, in [6] the problem was rewritten in terms of truncated Toeplitz operators (see below) and sufficient conditions for the existence of strong Cesàro wave operators were obtained.

One can think that $U_1 = U_2$, and from now on we work with a single unitary operator U. Consider the spectral decomposition of U. The space H can be written as the direct integral

$$H = \int \oplus H(z)\, d\mu(z),$$

U acts as multiplication by z. The measure μ admits the decomposition $\mu = \mu_{pp} + \mu_{ac} + \mu_{cs}$, where μ_{pp}, μ_{ac}, μ_{cs} are the purely point, absolutely continuous, continuous singular parts of μ, resp. In accordance with this decomposition, H splits into three direct summands:

$$H = H_{pp} \oplus H_{ac} \oplus H_{cs}.$$

Now we formulate the main result of this article. All necessary information about summation methods can be found in Section 1.

Theorem 0.1. *Assume that* $U : H \to H$ *is a unitary operator with nontrivial continuous singular part, i.e., such that* $H_{cs} \neq \{0\}$. *Then for any regular summation method, which is stronger than the Cesàro summation method, there exists an operator* X *on* H *with* $\operatorname{rank}(XU - UX) = 2$ *such that the averages of the sequence* $U^n X U^{-n}$ *do not have a limit in the weak operator topology.*

For an individual operator one can always find, for instance, a weakly convergent subsequence of the sequence of the Cesàro means, so it is natural to think of a "universal" summation method working for a certain class of operators. It is important that the class of operators with rank-two commutator is large. Namely, our proof is based on the fact that this class in a sense contains all truncated Toeplitz operators. We show that the averaged wave operator cannot exist for the whole class of operators determined by any singular measure μ in the spectral decomposition of H unless μ is a sum of point masses. Our result is only an existence theorem, while it would be interesting to find concrete examples and to describe the corresponding asymptotics of the sequence (X_n), which is bounded, but may thus have a very irregular behaviour.

1. Summation methods

Regular summation methods are determined by a directed set Ω of indices with order $\prec$ and a collection of nonnegative numbers $(p_{\omega,n})$ with $\omega \in \Omega$, $n \geqslant 0$, such that

- for any $\omega \in \Omega$ we have $\sum_n p_{\omega,n} = 1$;
- $p_{\omega,n} \to 0$ in ω for every n.

The averages of a sequence $(x_n)_{n\geqslant 0}$ are indexed by $\omega \in \Omega$ and have the form

$$\sum_n p_{\omega,n} x_n.$$

The standard examples are the Cesàro summation method with direction $m \to +\infty$, for which the averages are $\frac{1}{m+1}\sum_{n=0}^m x_n$, and the Abel summation method with direction $r \nearrow 1$ and averages $(1-r)\sum_n r^n x_n$.

We always assume that Ω has a countable cofinal subset, which means that there exists a sequence $(\omega_n) \subset \Omega$ indexed by nonnegative integers such that

- $n_1 < n_2$ implies $\omega_{n_1} \prec \omega_{n_2}$;
- for any $\omega \in \Omega$ there exists an integer n such that $\omega \prec \omega_n$.

The typical example of the set of indices for which this assumption is fulfilled, but which is not a sequence, is the interval $(0,1)$ with direction $r \nearrow 1$ used as the set of indices for the Abel summation method. If we consider only the sequence (ω_n) in place of the whole directed set Ω, we obtain another summation method. As soon as we prove the theorem for the new summation method, we immediately obtain a proof for the original one. This allows us to think that in Theorem 0.1 the set of indices is equivalent to the set of all nonnegative integers and the averages form a sequence. This assumption will be used when we apply the weak sequential completeness of L^1.

We impose the restriction that the summation method is stronger than the Cesàro summation method, which means that the corresponding averages converge to the same limit for every sequence whose Cesàro means converge, or, equivalently,

- $\frac{1}{m+1}\sum_{n=0}^m x_n \to 0 \implies \sum_n p_{\omega,n} x_n \to 0.$

Examples of such methods are iterations of the Cesàro averaging procedure and the Abel summation method. The property that the summation method is stronger than the Cesàro method is a regularity assumption. One can expect that the result holds without this assumption. However, we are interested only in the fact that one cannot get convergence by taking stronger averaging methods, and hence we do not consider "exotic" summation methods that are incomparable with the Cesàro method.

Also the assumption that the averaging method is stronger than the Cesàro summation method guarantees that if the averaged limit of the sequence $U^n X U^{-n}$ exists, then it commutes with U. Indeed, this follows from averaging the relation

$$\left(\frac{1}{m}\sum_{n=0}^{m-1} U^n X U^{-n}\right) U - U\left(\frac{1}{m}\sum_{n=0}^{m-1} U^n X U^{-n}\right) = \frac{1}{m}(X - U^m X U^{-m}),$$

where the right-hand side tends to 0 as $m \to \infty$. If a summation method is not stronger than the Cesàro method, e.g., takes a sequence to its subsequence, the limit operator may not commute with U (or intertwine U_1 and U_2 if $U_1 \neq U_2$: take $U = U_1 \oplus U_2$, and $\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$ in place of X for the reduction to the case of a single operator U). For instance, once again, consider operators in the one-dimensional space: $H = \mathbb{C}$, $U_1 = I$, $U_2 = -I$, $X = I$; let the summation method take a sequence (x_n) to the subsequence (x_{2n}). Then all the elements and hence also the limit of the sequence $U_2^{2n} X U_1^{-2n}$ are equal to I, but $IU_1 \neq U_2 I$.

2. The construction

Take a singular probability measure μ on the unit circle having no point masses. Construct the inner function θ in the unit disk by the formula

$$\frac{1+\theta(z)}{1-\theta(z)} = \int \frac{1+\bar{\xi}z}{1-\bar{\xi}z}\, d\mu(\xi). \tag{1}$$

Since μ is a probability measure, we have $\theta(0) = 0$. Consider the space $K_\theta = H^2 \ominus \theta H^2$ and the unitary operator U on K_θ,

$$Uh = zh + (1-\theta)(h, \bar{z}\theta). \tag{2}$$

In [3] it is proved that U is unitarily equivalent to the operator of multiplication by the independent variable in $L^2(\mu)$.

The truncated Toeplitz operator T_u on K_θ with symbol $u \in L^\infty$ is defined by

$$T_u f = P_\theta u f,$$

where P_θ is the orthogonal projection onto K_θ. Necessary information about truncated Toeplitz operators can be found in [8], [1]. An important property of truncated Toeplitz operators is

$$\operatorname{rank}(T_u U - U T_u) \leqslant 2,$$

see also formula (4) below. If the symbol u is a bounded function, for any $f, g \in K_\theta$ we obviously have

$$(T_u f, g) = (uf, g) = \int u \cdot f\bar{g}. \tag{3}$$

The truncated Toeplitz operators A_λ, $|\lambda| < 1$, with bounded symbols $\frac{1}{1-|\lambda|^2}\bar{b}_\lambda\theta$, where $b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$, are rank-one operators, namely,

$$A_\lambda = (\cdot, k_\lambda)\tilde{k}_\lambda,$$

where

$$k_\lambda(z) = \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\bar{\lambda}z}, \qquad \tilde{k}_\lambda(z) = \frac{\theta(z)-\theta(\lambda)}{z-\lambda}.$$

The function k_λ is the reproducing kernel in K_θ, which means that

$$(f, k_\lambda) = f(\lambda)$$

for any $f \in K_\theta$. If we set

$$\tilde{g} = \bar{z}\theta\bar{g} \in K_\theta,$$

using the property of reproducing kernels we obtain

$$(A_\lambda f, g) = (f, k_\lambda)\cdot(\tilde{k}_\lambda, g) = (f, k_\lambda)\cdot\overline{(k_\lambda, \tilde{g})} = f(\lambda)\cdot\tilde{g}(\lambda).$$

3. Proof of Theorem 0.1

Proof. Take an arbitrary scalar singular probability measure μ on the unit circle having no point masses, construct the inner function θ by relation (1) and consider the operator U defined by (2). Then U is unitarily equivalent to the operator of multiplication by the independent variable in $L^2(\mu)$, see [3]. It suffices to prove the theorem for the unitary operator U.

Suppose that the theorem is not true and for some summation method which is stronger than the Cesàro method, for some $f, g \in K_\theta$, the averages of the sequence $(U^nT_uU^{-n}f, g) = (T_uU^{-n}f, U^{-n}g)$ have a limit for every truncated Toeplitz operator T_u with bounded symbol u. Therefore, by formula (3), the averages of the sequence

$$\int u\cdot(U^{-n}f\cdot\overline{U^{-n}g}) = \int u\cdot z\bar{\theta}\cdot(U^{-n}f\cdot\widetilde{U^{-n}g}) = \int u\, z\bar{\theta}\cdot F_n,$$

where

$$F_n = U^{-n}f\cdot\widetilde{U^{-n}g} \in H^1,$$

have a limit for any $u \in L^\infty$. Since L^1 is weakly sequentially complete, the averages of F_n have a weak limit in L^1, which is a function from H^1; denote it by F.

For rank-one operators A_λ we have strong convergence of the Cesàro means of the sequence $U^nA_\lambda U^{-n}$ to the zero operator, see, e.g., [5]. Hence the Cesàro means of the numbers

$$(U^nA_\lambda U^{-n}f, g) = (A_\lambda U^{-n}f, U^{-n}g) = (U^{-n}f)(\lambda)\cdot(\widetilde{U^{-n}g})(\lambda) = F_n(\lambda)$$

tend to zero. The convergence to zero holds also for summation methods that are stronger than the Cesàro method, therefore $F(\lambda) = 0$.

Thus, $F \equiv 0$, and for any truncated Toeplitz operator T_u with bounded symbol u the averages of $(U^nT_uU^{-n}f, g)$ tend to 0. To get a contradiction, it suffices to consider the truncated Toeplitz operator T_u with symbol $u \equiv 1$ and any pair of functions $f, g \in K_\theta$ with $(f, g) \neq 0$. Then $T_u = I$, and we obtain the constant sequence,

$$(U^n T_u U^{-n} f, g) = (f, g),$$

whose averages and the limit are also equal to $(f, g) \neq 0$. $\square$

Remark. Another proof of the fact that the Cesàro means of the sequence (F_n) tend to zero is based on the duality between the classes of continuous functions and finite complex measures on the unit circle. Namely, if u is a continuous function on the circle, then the operator $T_{\theta u}$ is compact, and this implies the desired convergence. Similarly, the fact that $T_u - u(U)$ is a compact operator whenever u is continuous implies that the Cesàro means of the sequence $(z\bar{\theta}F_n) = (U^{-n}f \cdot \overline{U^{-n}g})$ $*$-weakly tend to the complex measure $f\bar{g}\,\mu$. Also the operator $T_{\bar{\theta}u}$ is compact and therefore the Cesàro means of $z^2\bar{\theta}^2 F_n$ $*$-weakly tend to 0.

4. The Hilbert transform

Let μ be a finite measure on the unit circle having no point masses. For $f \in L^2(\mu)$ define $H_r f \in L^2(\mu)$ by

$$(H_r f)(z) = \int \frac{f(z) - f(\xi)}{1 - r\bar{\xi}z} d\mu(\xi).$$

If the functions $H_r f$ converge in $L^2(\mu)$, the limit function will be called the *Hilbert transform* of f. The class of functions, for which the Hilbert transform is defined, contains all sufficiently smooth functions and is dense in $L^2(\mu)$. However, for every singular measure μ having no atoms, the Hilbert transform is not a bounded operator in $L^2(\mu)$ [4], moreover, it is not even closable [2].

For a singular probability measure μ, let θ be the inner function determined by relation (1). We have $\theta(0) = 0$, which means that $1 \in K_\theta$. Define the unitary operator U on K_θ by formula (2). An operator T on K_θ is a truncated Toeplitz operator if and only if the commutator $TU - UT$ has the form

$$TU - UT = (\cdot, \bar{z}\theta)\varphi - (\cdot, \bar{z}\theta\bar{\varphi})1 \tag{4}$$

for some function $\varphi \in K_\theta$, see [6]. (In general, the symbol of a bounded truncated Toeplitz operator may belong to L^2; in this case the operator is initially defined on bounded functions from K_θ, and then extended to the whole space by continuity.)

Functions from K_θ have angular boundary values μ-almost everywhere, and the operator $V : K_\theta \to L^2(\mu)$ taking a function from K_θ to its boundary function is unitary, see [3], [7]. Given a truncated Toeplitz operator T on K_θ, define the function from $L^2(\mu)$ *associated* with T by

$$f = V\varphi,$$

where φ is determined (up to an additive constant) by relation (4).

The function 0 is associated with a truncated Toeplitz operator T if and only if T commutes with U, that is, $T = \gamma(U)$ for some $\gamma \in L^\infty(\mu)$. Therefore, if a function $f \in L^2(\mu)$ is associated with a truncated Toeplitz operator T, then the class of all truncated Toeplitz operators for which f is the associated function equals $\{T + \gamma(U) : \gamma \in L^\infty(\mu)\}$.

Proposition 4.1. *A function $f \in L^2(\mu)$ is associated with some truncated Toeplitz operator on K_θ if and only if*

$$\left\| \int \frac{f(z) - f(\xi)}{1 - r\bar{\xi}z} s(\xi) d\mu(\xi) \right\|_{L^2(\mu)} \leqslant C \cdot \|s\|_{L^2(\mu)}, \quad s \in L^2(\mu) \tag{5}$$

for some constant C depending on f.

Proof. We need some formulas from [5], [6], whose proofs will be briefly sketched here.

Denote by Z the operator of multiplication by z on $L^2(\mu)$; then $VU = ZV$. Suppose that f is associated with T, set $K = (\cdot, 1)f - (\cdot, \bar{f})1$.

It follows from formula (4) that

$$V(TU - UT)V^{-1} = (\cdot, \bar{z})f - (\cdot, \bar{z}\bar{f})1 = KZ.$$

We must prove that the norms of the operators X_r on $L^2(\mu)$,

$$(X_r s)(z) = \int \frac{f(z) - f(\xi)}{1 - r\bar{\xi}z} s(\xi) d\mu(\xi) = \sum_{n \geqslant 0} r^n z^n \int (f(z) - f(\xi))\bar{\xi}^n s(\xi) d\mu(\xi),$$

are bounded by a constant not depending on r. We have

$$\begin{aligned} X_r =& \sum_{n \geqslant 0} r^n Z^n K Z^{-n} = \sum_{n \geqslant 0} r^n Z^n V(TU - UT)V^{-1} Z^{-(n+1)} \\ =& V \left(\sum_{n \geqslant 0} r^n (U^n T U^{-n} - U^{n+1} T U^{-(n+1)}) \right) V^{-1} \\ =& V \left(T - (1 - r) \sum_{n \geqslant 1} r^n U^n T U^{-n} \right) V^{-1}, \end{aligned} \tag{6}$$

and one can see that the norm of X_r does not exceed $2 \cdot \|T\|$.

Conversely, take $f \in L^2(\mu)$ and suppose that (5) is fulfilled for all $s \in L^2(\mu)$. For $s \equiv 1$ find a sequence (r_n) tending to 1 such that the functions $\int \frac{f(z)-f(\xi)}{1-r_n\bar{\xi}z} d\mu(\xi)$ weakly converge. Let Y be the operator taking a function $s \in L^2(\mu)$ to the weak limit of the functions $\int \frac{f(z)-f(\xi)}{1-r_n\bar{\xi}z} s(\xi) d\mu(\xi)$. Y is defined on the set of $s \in L^2(\mu)$ for which the limit exists; by the construction of the sequence (r_n), $Y1$ is well defined. For $r<1$ we have

$$\begin{aligned}
&\int \frac{f(z)-f(\xi)}{1-r\bar{\xi}z}\xi s(\xi)d\mu(\xi) - z\int \frac{f(z)-f(\xi)}{1-r\bar{\xi}z}s(\xi)d\mu(\xi)\\
&= \int (f(z)-f(\xi))\frac{1-\bar{\xi}z}{1-r\bar{\xi}z}\xi s(\xi)d\mu(\xi)\\
&= \int (f(z)-f(\xi))\xi s(\xi)d\mu(\xi) - (1-r)\cdot z\int \frac{f(z)-f(\xi)}{1-r\bar{\xi}z}s(\xi)d\mu(\xi)\\
&\xrightarrow[r\nearrow 1]{} \int (f(z)-f(\xi))\xi s(\xi)d\mu(\xi) = (KZs)(z).
\end{aligned} \tag{7}$$

Therefore, Ys is defined if and only if $Y(Zs)$ is, and, moreover, $YZ-ZY=KZ$, where $K=(\cdot,1)f-(\cdot,\bar{f})1$. Thus Yz^n is defined for all integer n, and hence by linearity Y is defined on a dense subset of $L^2(\mu)$. By condition (5) we have $\|Y\|\leqslant C$, hence Y is a bounded operator defined on the whole space $L^2(\mu)$.

Now define the operator T on K_θ by $T=V^{-1}YV$. Then

$$TU-UT=V^{-1}(YZ-ZY)V=V^{-1}KZV,$$

which coincides with the right-hand side of (4) for $\varphi=V^{-1}f\in K_\theta$. □

It follows from relations (6), (7), and Proposition 4.1 that *for a truncated Toeplitz operator T on K_θ the limit of the Abel means of the sequence U^nTU^{-n} exists if and only if the limit of the functions H_rf exists in $L^2(\mu)$ as $r\nearrow 1$, where $f\in L^2(\mu)$ is associated with T.*

If T is a compact operator, then the Cesàro means of U^nTU^{-n} strongly tend to 0, see, e.g., [6]. As follows from the proof, if μ is a Menshov (or Rajchmann) measure, that is, $\int z^n d\mu(z)\to 0$ as $n\to\infty$, then even the sequence U^nTU^{-n} itself strongly tends to 0. In any case, this implies that if T is a sum of a compact truncated Toeplitz operator and an operator commuting with U, i.e., having the form $\gamma(U)$ with $\gamma\in L^\infty(\mu)$, then the Abel means of U^nTU^{-n} have a strong limit. Thus, *if $f\in L^2(\mu)$ is associated with a compact truncated Toeplitz operator, then the Hilbert transform of f is well defined, that is, the limit of H_rf exists in $L^2(\mu)$.* It would be interesting to know if this condition is also necessary for convergence of the Abel means, and to get any information about the asymptotic of the sequence U^nTU^{-n} if the convergence fails.

The class of compact truncated Toeplitz operators contains all operators of the form $T=u(U_\alpha)-u(U)$, where u is a continuous function on the unit circle, $|\alpha|=1$, U_α is the unitary operator defined by $U_\alpha h=zh+(\alpha-\theta)(h,\bar{z}\theta)$ (cf. formula (2): for $\alpha=1$ we have $U_1=U$). For the operator T having this form, the corresponding function $\varphi\in K_\theta$ (such that $\varphi=f$ μ-almost everywhere) coincides with a continuous function σ_α-almost everywhere, see [6], where the measure σ_α is determined by the relation $\frac{\alpha+\theta(z)}{\alpha-\theta(z)}=\int\frac{1+\bar{\xi}z}{1-\bar{\xi}z}d\sigma_\alpha(\xi)$ (for $\alpha=1$ we obtain $\sigma_1=\mu$, cf. (1)).

Sufficient conditions for the existence of the Hilbert transform can be formulated in terms of continuity, however, not for a function f itself, but for its unitary transplantation. R. Bessonov [2] constructed a measure μ and a continuous function f, for which (5) fails and hence by Proposition 4.1, f is not associated with any truncated Toeplitz operator. Also it follows from the results of [2] that if continuity of every f associated with a truncated Toeplitz operator implied the convergence, then the wave operators would exist in the general case of rank-two commutators. Since this contradicts our Theorem 0.1, we obtain the following result.

Theorem 4.2. *For any singular measure μ on the unit circle having no atoms, there exists a continuous function f on the unit circle such that* (5) *is fulfilled, but the functions $H_r f$ do not converge.*

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