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ALGEBRAIC ANALOGUE OF ATIYAH'S THEOREM

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Abstract

In topology there is a well known theorem of Atyiah which states that for a connected Lie group G there is an isomorphism $\widehat{R(G)} \cong K_0(BG)$ where BG is the classifying space of G. In the present paper we consider an algebraic analogue of this theorem. In the paper of B.Totaro [8] there is a computation of $\varprojlim K_0(BG_i)$ for specially selected sequence BG_i. However, to compute $K_0(BG)$ one needs to prove that $\varprojlim^1 K_1(BG_i)$ vanishes. For split reductive groups we present another approach and prove that the Borel construction induces a ring isomorphism $\widehat{R(G)}_{I(G)} = K_0(BG)$, where BG is an étale classifying space introduced by Voevodsky and Morel in [6]. Our approach makes possible to compute $K_i(BG)$, which we expect to provide in a next preprint.

Key words: Atyiah's theorem, classifying space, K-theory, equivariant K-theory

1. Introduction

Morel and Voevodsky constructed the etale classifying space of a linear group G in the form $BG = \bigcup BG_k$ where $BG_k = EG_k/G$. EG_k are smooth algebraic varieties connected by a sequence of G-equivariant closed embeddings i_k.

$$\cdots \xrightarrow{i_{k-1}} (EG)_k \xrightarrow{i_k} (EG)_{k+1} \xrightarrow{i_{k+1}} \cdots$$

The motivic space $EG = \bigcup EG_k$ is $\mathbb{A}^1$ - contractible with a free G-action. We consider a split reductive affine algebraic group G. We prove that $K_0(BG) = \varprojlim K_0(BG_k)$. The Borel construction sends a representation V to the vector bundle $V_K = (V \times EG_k)/G$. It induces ring morphisms $R(G) \to K_0(BG_k)$ and consequently $R(G) \to K_0(BG)$. Here $R(G)$ is the representation ring of k-rational G-representations. Let I_G be an ideal of zero-dimensional representations in $R(G)$. After the I_G-adic completion we get a ring-morphism : $\widehat{R(G)} \to \widehat{K_0(BG)}$ The main theorem of recent paper states that the Borel morphism becomes an isomorpism after the completion: $\widehat{R(G)} \overset{\cong}{\to} \widehat{K_0(BG)}$. Also we prove that $K_0(BG)$ is complete. So we get $\widehat{R(G)} \cong K_0(BG)$.
The main idea of the proof is the reduction to the Borel subgroup B of G. For the Borel subgroup B the rings $K_0(BB)$ and $R(B)$ can be computed explicitly. It results in

Theorem 1. *The Borel construction induces an isomorphism*

$$\widehat{R(B)}_{I_B} \xrightarrow{\widehat{Borel}_B} \widehat{K_0(BB)}_{I_B} \xleftarrow{\cong} K_0(BB)$$

To make the reduction to Theorem 1 we prove

Theorem 2. *The following diagram commutes:*

(1.1)
$$\begin{tikzcd}
\widehat{R(G)}_{I_G} \arrow[r, "\widehat{Borel}_G"] \arrow[d, "res"'] & \widehat{K_0(BG)}_{I_G} \arrow[d, "\widehat{p^*}"'] & K_0(BG) \arrow[l] \arrow[d, "p^*"'] \\
\widehat{R(B)}_{I_B} \arrow[r, "\widehat{Borel}_B"] \arrow[d, "ind"'] & \widehat{K_0(BB)}_{I_B} \arrow[d, "\widehat{p_*}"'] & K_0(BB) \arrow[l, "\cong"'] \arrow[d, "p_*"'] \\
\widehat{R(G)}_{I_G} \arrow[r, "\widehat{Borel}_G"] & \widehat{K_0(BG)}_{I_G} & K_0(BG) \arrow[l]
\end{tikzcd}$$

With $Ind \circ Res = id$ and $p_* \circ p^* = id$.

Corollary(Main result) Borel construction induces a ring isomorphism

$$\widehat{R(G)}_{I_G} \xrightarrow{\widehat{Borel}_G} \widehat{K_0(BG)}_{I_G} \xleftarrow{\cong} K_0(BG)$$

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2. Auxiliary results

In this section we prove some properties of pullback and pushforward for K_0^G functor. Thomason in [3] developed G-equivariant K-theory. It is convinient to us to cite Merkurjev's paper [2].

Definition 1. Let X be a G-variety. We consider an action $\mu_x : G \times X \to X$ and a projection $p_x : G \times X \to X$. Let M be an $\mathcal{O}_X$-module. Following [2] we will call M a G-module if there is an isomorphism of $\mathcal{O}_{G\times X}$-modules $\alpha : \mu_X^*(M) \to p_X^*(M)$ such that the cocycle condition holds:

$$p_{23}^*(\alpha) \circ (id_G \times \mu_x)^*(\alpha) = (m \times id_X)^*(\alpha)$$

where $p_{23} : G \times G \times X \to G \times X$ is a projection and $m : G \times G \to G$ is a product morphism.

Lemma 1. Let $f : X \to Y$ be an equivariant morphism and let M be a G-module on Y Then f^*M has a structure of G-module on X.
Proof:
Consider the following diagram:

$$\begin{tikzcd}
G \times X \arrow[r, "id_G \times f"] \arrow[d, shift right, "p_X"'] \arrow[d, shift left, "\mu_X"] & G \times Y \arrow[d, shift right, "p_Y"'] \arrow[d, shift left, "\mu_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

We construct α as a composition of isomorphisms:

$$\begin{tikzcd}
p_X^* f^* M & (id_G \times f)^* p_Y^* M \arrow[l, "\cong"'] \\
\mu_X^* f^* M \arrow[u, "\alpha"] \arrow[r, "\cong"] & (id_G \times f)^* \mu_Y^* \arrow[u, "(id_G \times f)^*\beta"']
\end{tikzcd}$$

Here β is a G-module structure on M. The cocylce condition for α immediately follows from the cocycle condition for β.

Remark 1. Let X,Y be G-varieties, f be a G-morphism, M be an $\mathcal{O}_X$-module, N be an $\mathcal{O}_{G\times X}$-module, F=$id_G \times f$. Consider the diagram:

$$\begin{tikzcd}
G \times G \times X \arrow[r, "id_G \times F"] \arrow[d, shift right, "m \times id_X"'] \arrow[d, shift left, "id_G \times \mu_X"] & G \times G \times Y \arrow[d, shift right, "m \times id_Y"'] \arrow[d, shift left, "id_G \times \mu_Y"] \\
G \times X \arrow[r, "F"] \arrow[d, "\mu_X"] & G \times Y \arrow[d, "\mu_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \times X \arrow[r, "id_G \times F"] \arrow[d, shift right, "m \times id_X"'] \arrow[d, shift left, "p_{23_X}"] & G \times G \times Y \arrow[d, shift right, "m \times id_Y"'] \arrow[d, shift left, "p_{23_Y}"] \\
G \times X \arrow[r, "F"] \arrow[d, "p_X"] & G \times Y \arrow[d, "p_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Notation 1:
Since all vertical arrows are flat, we have natural isomorphisms ([1] Prop. 9.3):
$hh_\mu(M) : \mu_Y^* R^i f_* M \to R^i F_* \mu_X^* M;$
$hh_p(M) : p_Y^* R^i f_* M \to R^i F_* p_X^* M$
$hh_{m\times id}(N) : (m \times id_Y)^* R^i F_* N \to R^i(id_G \times F)_*(m \times id_X)^* N;$
$hh_{p_{23}}(N) : p_{23_Y}^* R^i F_* N \to R^i(id_G \times F)_* p_{23_X}^* N;$
$hh_{id_G\times\mu}(N) : (id_G \times \mu_Y)^* R^i F_* N \to R^i(id_G \times F)_*(id_G \times \mu_X)^* N;$

Note that since $\mu_Y \circ (m \times id_Y) = \mu_Y \circ (id_G \times \mu_Y)$, two isomorphisms coincide:
$hh_{\mu, id_G \times \mu}(M) : (id \times \mu_Y)^* \mu_Y^* R^i f_* M \to R^i (id_G \times F)_* (id_G \times \mu_X)^* \mu_X^* M$ and
$hh_{\mu, m \times id}(M) : (m \times id_Y)^* \mu_Y^* R^i f_* M \to R^i (id_G \times F)_* (m \times id_X)^* \mu_X^* M$
Similarly, there is another pair of equal isomorphisms:
$hh_{p, p_{23}}(M) : p_{23_Y}^* p_Y^* R^i f_* M \to R^i (id_G \times F)_* p_{23_X}^* p_X^* M$ and
$hh_{p, m \times id}(M) : (m \times id_Y)^* p_Y^* R^i f_* M \to R^i (id_G \times F)_* (m \times id_X)^* p_X^* M$

We need the following lemma about compostion of this isomorphisms.

Lemma 2. Consider the following diagram:

$$\begin{tikzcd}
X_3 \arrow[r, "f_3"] \arrow[d, "T"'] & Y_3 \arrow[d, "Q"'] \\
X_2 \arrow[r, "f_2"] \arrow[d, "t"'] & Y_2 \arrow[d, "q"'] \\
X_1 \arrow[r, "f_1"] & Y_1
\end{tikzcd}$$

Here q and Q are flat, $X_2 = X_1 \times_{Y_1} Y_2$, $X_3 = X_2 \times_{Y_2} Y_3$. Let M be an $\mathcal{O}_{X_1}$-module. Define
$hh_1 : q^* R^i f_{1*} \to R^i f_{2*} t^*$
$hh_{12} : Q^* q^* R^i f_{1*} \to R^i f_{3*} T^* t^*$
$hh_2 : Q^* R^i f_{2*} \to R^i f_{3*} T^*$ to be natural isomorphisms given by Prop. 9.3 [1]. Then the following diagram commutes:

$$\begin{tikzcd}
Q^* q^* R^i f_{1*} M \arrow[rr, "Q^* hh_1(M)"] \arrow[dr, "hh_{12}(M)"'] & & Q^* R^i f_{2*} t^* M \arrow[dl, "hh_2(t^* M)"] \\
& R^i f_{3*} T^* t^* M &
\end{tikzcd}$$

Proof:
Since the statement is local on Y_i, we consider the case when all Y_i are affine, $Y_i = Spec A_i$. If F is R- module, we will denote by $\widetilde{F}$ the corresponding sheaf on $Spec\ R$. Recall the construction of hh_1. Let M be an $\mathcal{O}_{X_1}$-module. Then
$R^i f_*(M) = \widetilde{H^i(X_1, M)}$; $q^* R^i f_{1*} M = A_2 \otimes_{A_1} \widetilde{H^i(X_1, M)}$; $R^i f_{2*} t^* M = \widetilde{H^i(X_2, t^* M)}$.
Let U_i be an affine covering of X_1. Denote by $K = \check{C}(X_1, M)$ the corrsponding Chech complex. Since Y_1 and Y_2 are affine, $t^{-1}(U_i)$ is the affine covering of X_2. For this covering we have that $A_2 \otimes_{A_1} K$ is a Chech complex of X_2-module $t^* M$. Then hh_1 is an obvious morphism

$$A_2 \otimes_{A_1} H^i(K) \to H^i(A_2 \otimes_{A_1} K)$$

which becomes an isomorphism since A_2 is flat over A_1. In similar way one can construct hh_{12} and hh_2. Then one can rewrite the diagram as

$$\begin{tikzcd}
A_3 \otimes_{A_2} A_2 \otimes_{A_1} H^i(K) \arrow[rr, "id \otimes hh_1"] \arrow[dr, "hh_{12}(M)"'] & & A_3 \otimes_{A_2} H^i(A_2 \otimes_{A_1} K) \arrow[dl, "hh_2(t^* M)"] \\
& H^i(A_3 \otimes_{A_1} K) &
\end{tikzcd}$$

Which is trivially commutative.

Lemma 3. Let $f : X \to Y$ be an equivariant morphism and M be a G-module on X Then for any i $R^i f_* M$ has a structure of G-module on Y.
Proof:
Let $\beta : \mu_x^* M \longrightarrow p_X^* M$ be the G-structure on M. Consider the following base-change diagram:

$$\begin{tikzcd}
G \times X \arrow[r, shift left, "\mu_X"] \arrow[r, shift right, "p_X"'] \arrow[d, "id_G \times f"] & X \arrow[d, "f"] \\
G \times Y \arrow[r, shift left, "\mu_Y"] \arrow[r, shift right, "p_Y"'] & Y
\end{tikzcd}$$

Since μ_Y and p_Y are flat, we use Proposition 9.3 from [1]. Sheaf isomorphisms $hh_\mu(M)$ and $hh_p(M)$ are described in Notation 1. Define α to be the unique isomorphism such that the following diagram commutes :

$$\begin{tikzcd}
\mu_Y^* R^i f_* M \arrow[r, "hh_\mu(M)"] \arrow[d, "\alpha"] & R^i(id \times f)_* \mu_X^* M \arrow[d, "R^i(id \times f)_* \beta"] \\
p_Y^* R^i f_* M \arrow[r, "hh_p(M)"] & R^i(id \times f)_* p_X^* M
\end{tikzcd}$$

Now we have to check the cocycle condition for α :
$p_{23} * (\alpha) \circ (id_G \times \mu_Y)^*(\alpha) = (m \times id_Y)^*(\alpha)$
This means commutativity of this diagfam:

$$\begin{tikzcd}
(id_G \times \mu_Y)^* \mu_Y^* R^i f_* M \arrow[r, "p_{23} * (\alpha) \circ (id_G \times \mu_Y)^*(\alpha)"] \arrow[d, equal] & p_{23}^* p_Y^* R^i f_* M \arrow[d, equal] \\
(m \times id_Y)^* \mu_Y^* R^i f_* M \arrow[r, "(m \times id_Y)^*(\alpha)"] & (m \times id_Y)^* p_Y^* R^i f_* M
\end{tikzcd}$$

Let $F = id_G \times f$. Subdivide this diagram into the following blocks:

$$\begin{tikzcd}
(id_G \times \mu_Y)^* \mu_Y^* R^i f_* M \arrow[r, "1"'] \arrow[d, "hh_{\mu, id \times \mu}"', "\cong"] & p_{23_Y}^* p_Y^* R^i f_* M \arrow[d, "hh_{p, p_{23}}"', "\cong"] \\
R^i(id_G \times F)_* (id_G \times \mu_X)^* \mu_X^* M \arrow[r, "2"'] \arrow[d, equal] & R^i(id_G \times F)_* p_{23_X}^* p_X^* M \arrow[d, equal] \\
R^i(id_G \times F)_* (m \times id_X)^* \mu_X^* M \arrow[r, "3"'] \arrow[d, "hh^{-1}_{m \times id}(\mu_X^* M)"', "\cong"] & R^i(id_G \times F)(m \times id_X)^* p_X^* M \arrow[d, "hh^{-1}_{m \times id}(p_X^* M)"', "\cong"] \\
(m \times id_Y)^* R^i(id_G \times f)_* \mu_X^* M \arrow[r, "4"'] \arrow[d, "(m \times id_Y)^* hh_\mu^{-1}(M)"', "\cong"] & (m \times id_Y)^* R^i(id_G \times f)_* p_X^* M \arrow[d, "(m \times id_Y)^* hh_p^{-1}(M)"', "\cong"] \\
(m \times id_Y)^* \mu_Y^* R^i f_* M \arrow[r, "(m \times id_Y)^*(\alpha)"] & (m \times id_Y)^* p_Y^* R^i f_* M
\end{tikzcd}$$

Square 2 is an image of cocycle diagram for M and therefore commutative.
Square 3 arises from functor isomorphism

$R^i(id_G \times id_G \times f)_*(m \times id_X)^* \cong (m \times id_Y)^* R^i(id_G \times f)_*$ ([1], Prop. 9.3) applied to G-module structure $\beta : \mu_X^* M \to p_X^* M$ So, it commutes.
Square 4 is commutative by definition of α.
It remains to show the commutativity of square 1. Let $\widetilde{F} = id_G \times F$. Rewrite square 1 as follows:

$$\begin{tikzcd}
(id \times \mu_Y)^* \mu^* R^i f_* M \arrow[r, "1.1"'] \arrow[d, "(id\times\mu)^* hh_\mu"', "\cong"] & (id \times \mu_Y)^* p_Y^* R^i f_* M \arrow[r, equal, "1.2"'] \arrow[d, "(id_G\times\mu_Y)^* hh_p"', "\cong"] & p_{23_Y}^* \mu_Y^* R^i f_* M \arrow[r, "1.3"'] \arrow[d, "p_{23_Y}^* hh_\mu"', "\cong"] & p_{23_Y}^* p_Y^* R^i f_* M \arrow[d, "p_{23_Y}^* hh_p"', "\cong"] \\
(id \times \mu_Y)^* R^i F_* \mu_X^* M \arrow[r, "1.4"'] \arrow[d, "hh_{id\times\mu}(\mu^* M)"', "\cong"] & (id \times \mu_Y) R^i F_* p_X^* M \arrow[d, "hh_{id\times\mu}(p_X^* M)"', "\cong"] & p_{23_Y}^* R^i F_* \mu_X^* M \arrow[r, "1.5"'] \arrow[d, "hh_{p_{23}}(\mu_X^* M)"', "\cong"] & p_{23_Y}^* R^i F_* p_X^* M \arrow[d, "hh_{p_{23}}(p_X^* M)"', "\cong"] \\
R^i \widetilde{F}_* (id \times \mu_X)^* \mu_X^* M \arrow[r] & R^i \widetilde{F}_* (id \times \mu_X)^* p_X^* M \arrow[r, equal] & R^i \widetilde{F}_* p_{23_X}^* \mu_X^* M \arrow[r] & R^i \widetilde{F}_* p_{23_X} p_X^* M
\end{tikzcd}$$

Square 1.1 is an image of functor $(id_G \times \mu_Y)^*$ applied to the diagram that defines α. Thus it is commutative. Commutativity of 1.2 follows from Lemma 2 and Prop. 9.3[1] applied to the base-change diagram

$$\begin{tikzcd}
G \times G \times X \arrow[r, "id_G \times id_G \times f"] \arrow[d, "p_Y \circ (id \times \mu_Y) = \mu_Y \circ p_{23_Y}"'] & G \times G \times Y \arrow[d, "p_Y \circ (id \times \mu_X) = \mu_X \circ p_{23_X}"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Square 1.3 is an image of functor $p_{23_Y}^*$ applied to the diagram defining α. and therefore commutes.
Prop 9.3 [1] gives us an isomorphsm of functors $(id_G \times \mu_Y)^* R^i F_* \cong R^i \widetilde{F}(id_G \times \mu_X)^*$. Applying this isomorphism to $\beta : \mu_X^* M \to p_X^* M$ we get commutativity of the square 1.4.
In a similar way we get commutativity of the square 1.5.
So, commutativity of 1-4 is proved. According to Lemma 2 the composition of vertical arrows is the identity. So, α satisfies the cocycle condition.
Corollary. If f is projective we can define the pushforward map $f_* : K_0^G(X) \to K_0^G(Y)$ by sending M to the alternating sum of $R^i f_*(M)$.
Also we need an equivariant version of Proposition 9.3 from [1].

Lemma 4. Consider the base change diagram

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "Q"'] & B \arrow[d, "q"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where X, Y, A, B are G-varieties; f, F, Q, q are G-morphisms; f is flat.
Let M be a G-module on B. Then there is a natural G-module isomorphism on X :

$$f^* R^i q_* M \to R^i Q_* F^* M.$$

Proof:
By Propostion 9.3 from [1] we have a natural isomorphsim of $\mathcal{O}_X$-modules

$hh_{X,Y,A,B} : f^*R^iq_*M \to R^iQ_*F^*M$. We need to check that $hh_{X,Y,A,B}$ is a G-morphism. That means commutativity of the following diagram:

$$\begin{tikzcd}
\mu_X^* f^* R^i q_* M \arrow[r, "G-structure"'] \arrow[d, "\mu_X^* hh_{X,Y,A,B}"] & p_X^* f^* R^i q_* M \arrow[d, "p_X^* hh_{X,Y,A,B}"] \\
\mu_X^* R^i Q_* F^* M \arrow[r, "G-structure"'] & p_X^* R^i Q_* F^* M
\end{tikzcd}$$

Consider the diagram:

$$\begin{tikzcd}
G\times A \arrow[rr, "id\times F"] \arrow[dr, shift left, "p_A"] \arrow[dr, shift right, "\mu_A"'] \arrow[dd, "id\times Q"'] & & G\times B \arrow[dr, shift left, "p_B"] \arrow[dr, shift right, "\mu_B"'] \arrow[dd, "id\times q"'] & \\
& A \arrow[rr, "F"'] \arrow[dd, "Q"] & & B \arrow[dd, "q"] \\
G\times X \arrow[rr, "id\times f"'] \arrow[dr, shift left, "p_X"] \arrow[dr, shift right, "\mu_X"'] & & G\times Y \arrow[dr, shift left, "p_Y"] \arrow[dr, shift right, "\mu_Y"'] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

For any square in this cube denote by hh (with corresponding subscript) the isomorphism arising from prop. 9.3[1], applied to this square. We rewrite the G-structure diagram:

$$\begin{tikzcd}
\mu_X^* f^* R^i q_* M \arrow[r, "1"'] \arrow[d, equal] & p_X^* f^* R^i q_* M \arrow[d, equal] \\
(id\times f)^*\mu_Y^* R^i q_* M \arrow[r, "2"'] \arrow[d, "(id\times f)^* hh^{\mu}_{G\times Y,Y,G\times B,B}(M)"'] & (id\times f)^* p_Y^* R^i q_* M \arrow[d, "(id\times f)^* hh^{p}_{G\times Y,Y,G\times B,B}(M)"'] \\
(id\times f)^* R^i (id\times q)_* \mu_B^* M \arrow[r, "3"'] \arrow[d, "hh_{G\times X,G\times Y,G\times A,G\times B}(\mu_B^* M)"'] & (id\times f)^* R^i (id\times q)_* p_B^* M \arrow[d, "hh_{G\times X,G\times Y,G\times A,G\times B}(p_B^* M)"'] \\
R^i(id\times Q)_*(id\times F)^*\mu_B^* M \arrow[r, "4"'] \arrow[d, equal] & R^i(id\times Q)_*(id\times F)^* p_B^* M \arrow[d, equal] \\
R^i(id\times Q)_*\mu_A^* F^* M \arrow[r, "5"'] \arrow[d, "hh^{\mu}_{G\times X,X,G\times A,A}"'] & R^i(id\times Q)_* p_A^* F^* M \arrow[d, "hh^{p}_{G\times X,X,G\times A,A}"'] \\
\mu_X^* R^i Q_* F^* M \arrow[r] & p_X^* R^i Q_* F^* M
\end{tikzcd}$$

Square 1 is commutative because of definition of the G-structure on pullback.
Square 2 is an $(id \times f)^*$ image of the G-structure diagram for R^iq_*M. Thus it commutes.
Square 3 arises from the functor isomorphism $(id\times f)^*R^i(id\times q)_* \to R^i(id\times Q)_*(id\times F)^*$ applied to the G-structure isomorphism $\mu_B^*M \to p_B^*M$. SO, it commutes.

Square 4 is commutative because of the definition of the G-structure on pullback.
Square 5 is commutative by the definition of the G-structure on $R^iQ_*F^*M$.
By lemma 2 compositions of vertical arrows are equal to $\mu_X^* hh_{X,Y,A,B}$ and $p_X^* hh_{X,Y,A,B}$.
This concludes the proof of Lemma 4.

Lemma 5. Let X, Y be smooth G-varieties, G - a smooth reductive affine algebraic group and $\pi : X \times Y \to Y$ a projection. Moreover let X be projective and Y be connected
Denote by $\mathcal{P}_\pi(G; X \times Y)$ the full subcategory of $\mathcal{P}(G; X \times Y)$ consisting of locally free G-modules P such that $R^k\pi_*P = 0$ for $k > 0$.
Then any G- module M possesses a finite length resolution of the form
$M \to P^0 \to P^1 \to \ldots \to P^N \to 0$ with $P^i \in OB(\mathcal{P}_\pi(G; X \times Y))$
Proof:
First we prove that for every M there is an embedding $M \hookrightarrow P^0$. We will construct P^0 in the form of $M(n)$ for a large enough n. To do this, we construct a very ample G-equivariant sheaf $\mathcal{O}_X(1)$ and an G-equivariant embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}}(1)$. Let L be a very ample line bundle. By corollary 1.6 of [5] $L^{\otimes k}$ is G-equivariant for some k. Then it defines the action of G on $V = \Gamma(X, L^{\otimes k})$ and equivariant morphism $i : X \to \mathbb{P}(V)$ which is an embedding since $L^{\otimes k}$ is very ample. Then we set $\mathcal{O}_X(1) = L^{\otimes k}$.
The standard embedding of the tautological bundle $\tau_{\mathbb{P}(V)} \hookrightarrow V \times \mathbb{P}(V)$ gives us a G-equivariant embedding of locally free sheaves $\mathcal{O}_{\mathbb{P}(V)}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}(V)} \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}(V)}$. After twisting by $\mathcal{O}_{\mathbb{P}}(1)$ we have $\mathcal{O}_{\mathbb{P}(V)} \hookrightarrow \mathcal{O}_{\mathbb{P}(V)}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}(V)}(1)$. Inductively we have the G-equivariant embedding $\mathcal{O}_{\mathbb{P}(V)} \hookrightarrow \mathcal{O}_{\mathbb{P}(V)}(n) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}(V)}(n)$. Applying i^* we get

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(n) \oplus \ldots \oplus \mathcal{O}_X(n).$$

Define $\mathcal{O}_{X \times Y}(1) = \pi^*\mathcal{O}_X(1)$. Applying π^* we get an equivariant embedding

$$M \hookrightarrow M(n) \oplus \ldots \oplus M(n).$$

for an arbitrary locally free G-module M. Clearely it's cokernel is G-equivariant. It's easy to check that it is a locally free sheaf. Then for every locally free G-module there is a resolution consisisting of direct sums of modules of the form $M(n)$
Let us show that $M(n)$ lies in $\mathcal{P}_\pi(G; X \times Y)$ for a large enough n. $R^k\pi_*M(n)$ is associated to a presheaf $V \mapsto H^k(X \times V, M(n))$. Consider a finite affine covering V_i of Y. By Serre's theorem $H^k(X \times V_i, M(n))$ equals zero for $n > n_i$. Thus $R^k\pi_*M(n) = 0$ for $n > n_M = max\{n_i\}$.
It remains to show that this resolution ends at some finite step. Let $N = dim X \times Y$. Let C^0 be a cokernel or the first resolution step: $0 \to M \to P^0 \to C^0 \to 0$. Then we have the exact sequence

$$0 = R^N\pi_*P^0 \to R^N\pi_*C^0 \to R^{N+1}\pi_*M = 0.$$

So, $R^N\pi_*C^0 = 0$. For the second cokernel C^1 we have the exact sequence $0 \to C^0 \to P^1 \to C^1 \to 0$. Then

$$0 = R^{N-1}\pi_*P^1 \to R^{N-1}\pi_*C^1 \to R^N\pi_*C^0 = 0.$$

So, $R^{N-1}\pi_*C^{N-1} = 0$. By induction we have all $R^k\pi_*C^N = 0$. Then $C^N \in Ob(\mathcal{P}_\pi(G; X \times Y))$.

Lemma 6. Under the notation of Lemma 5, we have a commutative up to an isomorphism diagram of exact functors.

$$
(2.1) \qquad \begin{tikzcd}
\mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B) \arrow[d, "\pi_{EG_j *}"'] & \mathcal{P}_{\pi_{EG_{j+1}}}(G; EG_{j+1} \times G/B) \arrow[l, "(i_j \times id)^*"] \arrow[d, "\pi_{EG_{j+1} *}"] \\
\mathcal{P}(G; EG_j) & \mathcal{P}(G; EG_{j+1}) \arrow[l, "i_j^*"]
\end{tikzcd}
$$

Proof:
To simplfy notation let $\pi_j = \pi_{EG_j}$ and $\mathcal{P}_j = \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$ Let us prove that $\mathcal{P}_{j+1}$ is mapped to $\mathcal{P}_j$ under $(i_j \times id)^*$ Let $M \in Ob(\mathcal{P}_{j+1})$. Let $dim(EG_j \times G/B) = N$. Then $R^{N+1}\pi_{j*}(i_j \times id)^* M = 0$. By corollary 2§5 of [4]

$$R^N \pi_{j*}(i_j \times id)^* M \otimes_{\mathcal{O}_{EG_j}} k(y) = H^N(EG_j \times \{y\}, (i_j \times id)^* M) = H^N(EG_j \times \{y\}, M) = 0$$

Then $R^N \pi_{j*}(i_j \times id)^* M = 0$. By induction we obtain that all $R^k \pi_{j*} i_j^* M = 0$ for $k > 0$. Then $i_j^* M \in Ob(\mathcal{P})$. Now we prove the commutativity of the diagram 2.1 up to a natural isomorphism. By remark 9.3.1 of [1] we have a natural morphism $hh : i_j^* \pi_{j+1*} M \to \pi_{j*}(i_j \times id)^* M$. One can easily see that for any $y \in EG_j$ the following diagram commutes:

$$
\begin{tikzcd}
\pi_{j*}(i_j \times id)^* M \otimes k(y) \arrow[d, "(1)"'] & i_j^* \pi_{j+1*} M \otimes k(y) \arrow[l, "hh \otimes k(y)"] \arrow[d, equal] \\
\Gamma(y \times G/B, (i_j \times id)^* M) \arrow[d, equal] & \pi_{j+1*} M \otimes k(y) \arrow[d, "(2)"] \\
\Gamma(y \times G/B, M) \arrow[r, equal] & \Gamma(y \times G/B, M)
\end{tikzcd}
$$

Here the arrows (1) and (2) are natural isomorphisms given by corollary 2 §5 of [4]. So, $hh \otimes k(y)$ is an isomorphism for any point y of EG_j. Therefore hh is a natural isomorphism. So, the diagram (2.1) is commutative.

Lemma 7. Under the notation of Lemma 5, for each $j \geqslant 0$ the functor

$$\pi_j^* : \mathcal{P}(G; EG_j) \to \mathcal{P}(G; EG_j \times G/B)$$

takes values in the subcategory $\mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$. As a consequence the following diagram of exact functors commutes up to a natural isomorphism.

$$
(2.2) \qquad \begin{tikzcd}
\mathcal{P}(G; EG_j) \arrow[d, "\pi_j^*"'] & \mathcal{P}(G; EG_{j+1}) \arrow[l, "i_j^*"] \arrow[d, "\pi_{j+1}^*"] \\
\mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B) & \mathcal{P}_{\pi_{EG_{j+1}}}(G; EG_{j+1} \times G/B) \arrow[l, "(i_j \times id)^*"]
\end{tikzcd}
$$

Proof:
To simplfy notation let $\pi_j = \pi_{EG_j}$ and $\mathcal{P}_j = \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$ First we prove that π_j^* maps $\mathcal{P}(G; EG_j)$ to $\mathcal{P}_j$. Let M be an object of $\mathcal{P}(G; EG_j)$. Then $R^k \pi_{j*} \pi_j^* M$ is associated to the presheaf $V \mapsto H^k(V \times G/B, \pi_j^* M)$. Let V be an affine

open subset of EG_j. Let $\{U_n\}$ be an affine covering of G/B. For any intersection $W = U_{n_1} \cap \ldots \cap U_{n_k}$. we have

$$\pi_j^* M(V \times W) = M(V) \otimes_{\mathcal{O}_{EG_j}(V)} \mathcal{O}_{EG_j \times G/B}(V \times W) = M(V) \otimes_k \mathcal{O}_{G/B}(W).$$

Then Čhech complex $\check{C}(\{V \times U_n\}, \pi_j^* M)$ equals $M(V) \otimes_k \check{C}(\{U_n\}, \mathcal{O}_{G/B})$. Consequently, $H^k(V \times G/B, \pi_j^* M) = M(V) \otimes_k H^k(G/B, \mathcal{O}_{G/B})$.
By proposition 4.5 from [7] $H^k(G/B, \mathcal{O}_{G/B}) = 0$ for $k > 0$. Then $\pi_{j*} M \in Ob(\mathcal{P}_j)$.
The commutativity of (2.2) trivally follows from the equality $\pi_{j+1} \circ (i_j \times id) = i_j \circ \pi_j$.

Proposition 1. *There is a commutative diagram with* $\pi_{EG_i *} \circ \pi^*_{EG_i} = id_{K_0^G(EG_i)}$, $\pi_{pt*}\pi^*_{pt} = id_{K_0^G(pt)}$

$$\begin{tikzcd}
K_0^G(pt) \arrow[r, "\pi^*_{pt}"] \arrow[d, "\pi^*_{pt}"'] & K_0^G(EG_i) \arrow[d, "\pi^*_{EG_i}"] \\
K_0^G(G/B) \arrow[r, "\pi^*_{G/B}"] \arrow[d, "\pi_{pt*}"'] & K_0^G(EG_i \times G/B) \arrow[d, "\pi_{EG_i *}"] \\
K_0^G pt \arrow[r, "\pi^*_{pt}"] & K_0^G(EG_i)
\end{tikzcd}$$

Proof:
Commutativity of the upper square is trivial.
Consider the lower square: Let M be a G-module on G/B, $[M] \in K_0^G(G/B)$. Then

$$\pi_{EG_i *} \circ \pi^*_{G/B}[M] = \sum (-1)^k [R^k \pi_{EG_i *}(\pi^*_{G/B} M)];$$

$$\pi^*_{pt} \circ \pi_{pt*}[M] = \pi^*_{pt}(\sum (-1)^k [R^k \pi_{pt*} M]) = \sum (-1)^k [\pi^*_{pt} R^k \pi_{pt*} M].$$

By Lemma 4 summands are isomorphic, so $\pi_{EG_i *} \circ \pi^*_{G/B}[M] = \pi^*_{pt} \circ \pi_{pt*}[M]$.
Prove that $\pi_{EG_i *}(1) = 1$. $R^k \pi_{EG_i *} \mathcal{O}_{EG_i \times G/B}$ is a sheaf associated to the presheaf $V \mapsto H^i(V \times G/B, \mathcal{O}_{V \times G/B})$. Let V be affine. Consider U_i an affine covering of G/B. Then $V \times U_i$ is an affine covering of $EG_i \times G/B$. Let $\check{C}(V \times U_i, \mathcal{O}_{EG_i \times G/B}|_{V \times G/B})$ be a chech complex for this covering. Since $\Gamma(V \times U_{i_1..i_k}, \mathcal{O}_{EG_i \times G/B}) = \Gamma(V, \mathcal{O}_{EG_i}) \otimes_k \Gamma(G/B, \mathcal{O}_{G/B})$, we have the isomorphism $H^k(V \times G/B, O) \to \Gamma(V, \mathcal{O}_V) \otimes_k H^k(G/B, \mathcal{O}_{G/B})$.
It is easy to verify that this isomorphism commutes with the restriction maps:

$$\begin{tikzcd}
H^k(V \times G/B, \mathcal{O}_{V \times G/B}) \arrow[r] \arrow[d, "res_{U,V}"'] & \Gamma(V, \mathcal{O}_V) \otimes_k H^k(G/B, \mathcal{O}_{G/B}) \arrow[d, "id \otimes res_{U,V}"] \\
H^k(U \times G/B, \mathcal{O}_{U \times G/B}) \arrow[r] & \Gamma(U, \mathcal{O}_U) \otimes_k H^k(G/B, \mathcal{O}_{G/B})
\end{tikzcd}$$

So, we get the sheaf isomorphism $R^k \pi_{EG_i *} \mathcal{O}_{EG_i \times G/B} \cong \mathcal{O}_{EG_i} \otimes_k H^k(G/B, \mathcal{O}_{G/B})$.
Therefore $\pi_{EG_i *}(1) = \chi(G/B, \mathcal{O}_{G/B}) \cdot 1$. By proposition 4.5 [7] we have $\chi(G/B, \mathcal{O}_{G/B}) = 1$, so $\pi_{EG_i *}(1) = 1$.
This shows that $\pi_{EG_i *} \circ \pi^*_{EG_i}$ is an $K_0^G(EG_i)$-module endomorphism (by projection formula) sending 1 to 1. Whence $\pi_{EG_i *} \circ \pi^*_{EG_i} = id_{K_0^G(EG_i)}$. By the same reasons $\pi_{pt*} \circ \pi^*_{pt} = id_{K_0^G(pt)}$. This concludes the proof.

Remark In particular, we get a well-known fact that the natural ring map $R(G) \to R(B)$ is injective.

Proposition 2. *The I_B-adic topology of $R(B)$ coincides with the $I_G \cdot R(B)$-adic topology.*
Proof:
Let T be a maximal torus in G. Then $R(B) = R(T)$ and $I_B = I_T$, where I_T is the ideal of zero-dimensional representations of T. We will prove that $\sqrt{I_G \cdot R(T)} = I_T$. Denote by $W = N_G(T)/T$ the Weil group of G. The group W acts by conjugation on $R(T)$. It is known that W is a finite group and $R(G)$ is the ring of invariants of W: $R(G) = R(T)^W$. We prove the following statement:

If q is a prime ideal of $R(T)$ and $q \cap R(G) \supseteq I_G$. Then $q \supseteq I_T$.

Let $x \in I_T$. Let $n = |W|$ and $W = \{\sigma_1, \ldots, \sigma_n\}$. For any symmetric polynomial f we have that $f(x^{\sigma_1} \ldots x^{\sigma_n})$ is invariant under W-action. Then $f(x^{\sigma_1} \ldots x^{\sigma_n}) \in R(G) \cap I_T = I_G \subseteq R(G) \cap q$. Then $f(x^{\sigma_1} \ldots x^{\sigma_n}) \in q$. Denote by $f_1 \ldots f_n$ the elementary symmetric polinomials. It is easy to see that x is a root of polynomial

$$\prod_{i=1}^{n}(t - x^{\sigma_i}) = t^n - f_1(x^{\sigma_1} \ldots x^{\sigma_n})t^{n-1} + \ldots + (-1)^n f_n(x^{\sigma_1} \ldots x^{\sigma_n}).$$

So we have $x^n - f_1(x^{\sigma_1} \ldots x^{\sigma_n})x^{n-1} + \ldots + (-1)^n f_n(x^{\sigma_1} \ldots x^{\sigma_n}) = 0$.
Then $x^n = -(-f_1(x^{\sigma_1} \ldots x^{\sigma_n})x^{n-1} + \ldots + (-1)^n f_n(x^{\sigma_1} \ldots x^{\sigma_n})) \in q$. So $x^n \in q$. Since q is prime, $x \in q$. This ends the proof of the statement.
Consider $A = \{p \mid p - prime, p \supseteq I_G \cdot R(T)\}$ Our statement implies that I_T is a minimal element of A. So,

$$\sqrt{I_G \cdot R(T)} = \bigcap_{p \in A} = I_T.$$

Since $R(B) = R(T)$ and $I_B = I_T$, we get $\sqrt{I_G \cdot R(B)} = I_B$. Since $R(B)$ is noetherian, it implies that $I_B^m \subseteq I_G \cdot R(B)$ for some m. Then I_B and $I_G \cdot R(B)$ determine the same topology on $R(B)$.

Proposition 3. $K_0(BG) = \varprojlim K_0(BG_i)$
Proof:
By [6] we have the following exact sequence:

$$0 \to \varprojlim{}^1 K_1(BG_i) \to K_0(BG) \to \varprojlim K_0(BG_i) \to 0$$

Let us show that $\varprojlim{}^1 K_1(BG_i) = 0$.
We prove that the sequence $K_1(BG_i)$ is a direct summand of the sequence $K_1(BB_i)$. By proposition 1 of [2] We have $K_1(BG_i) = K_1^G(EG_i)$ Since we can choose EG_i as a model for EB_i, we obtain $K_1(BB_i) = K_1^B(EB_i) = K_1^B(EG_i) = K_1^G(EG_i \times G/B)$. So, in fact, we prove that the sequence $K_1^G(EG_i)$ is a direct summand of the sequence $K_1^G(EG_i \times G/B)$.

To simplify the notation denote $\mathcal{P}_j = \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$. By lemmas 6 and 7 we obtain a commutative diagram with exact arrows:

(2.3)
$$\begin{tikzcd}
\mathcal{P}(G;EG_j) \arrow[d, "\pi_j^*"'] & \mathcal{P}(G;EG_{j+1}) \arrow[l, "(i_j\times id)^*"'] \arrow[d, "\pi_{j+1}^*{}^*"] \\
\mathcal{P}_j \arrow[d, "\pi_{j*}"'] & \mathcal{P}_{j+1} \arrow[l, "(i_j\times id)^*"'] \arrow[d, "\pi_{j*}"] \\
\mathcal{P}(G;EG_j) & \mathcal{P}(G;EG_{j+1}) \arrow[l, "i_j^*"']
\end{tikzcd}$$

Consider the composition of vertical arrows :

$$\mathcal{P}(G;EG_j) \xrightarrow{\pi_j^*} \mathcal{P}_j \xrightarrow{\pi_{j*}} \mathcal{P}(G;EG_j)$$

For any locally free G-equivariant sheaf M we are going to prove that $\pi_{j*}\pi_j^* M$ is naturally isomorphic to M, that is th fucntor $\pi_{j*}\pi_j^*$ is isomorphic to the identity functor. The sheaf $\pi_{j*}\pi_j^* M$ is associated to presheaf $V \mapsto \pi_j^*(M)(V \times G/B)$. Since $\pi_j^* M$ is a sheaf associated to $W \mapsto M(\pi_j(W))$ we see that $\pi_{j*}\pi_j^* M$ is associated to the presheaf $V \mapsto M(V)$. So, in category of presheaves $\pi_{j*}\pi_j^* \cong id$. Applying the sheaffication functor to this isomorphism, we get a natural isomorphism $\pi_{j*}\pi_j^* M \cong M$.

In the proof of Lemma 6 it is checked that $(i_j \times id)^*(\mathcal{P}_{j+1}) \subseteq \mathcal{P}_j$ By Lemma 5, each G-module in $\mathcal{P}(G; EG_j \times G/B)$ has a finite resolution consisting of sheaves from $\mathcal{P}_j$. Then by the Quillen's theorem we get the isomorphisms α_j such that the following diagram of groups commutes:

(2.4)
$$\begin{tikzcd}
K_1(\mathcal{P}_j) \arrow[d, "\alpha_j"'] & K_1(\mathcal{P}_{j+1}) \arrow[l, "(i_j\times id)^*"'] \arrow[d, "\alpha_{j+1}"] \\
K_1^G(EG_j \times G/B) & K_1^G(EG_{j+1} \times G/B) \arrow[l, "(i_j\times id)^*"']
\end{tikzcd}$$

Define $\Pi_{j*} : K_1^G(EG_j \times G/B) \to K_1^G(EG_j)$ as the composition of

$$K_1^G(EG_j \times G/B) \xrightarrow{\alpha_j^{-1}} K_1(\mathcal{P}_j) \xrightarrow{\pi_{j*}} K_1^G(EG_j)$$

Commutativity of the diagrams (2.3) and (2.4) gives us a commutative diagram:

(2.5)
$$\begin{tikzcd}
K_1^G(EG_j) \arrow[d, "\pi_j^*"'] & K_1^G(EG_{j+1}) \arrow[l, "(i_j\times id)^*"'] \arrow[d, "\pi_{j+1}^*{}^*"] \\
K_1^G(EG_j \times G/B) \arrow[d, "\Pi_{j*}"'] & K_1^G(EG_{j+1} \times G/B) \arrow[l, "(i_j\times id)^*"'] \arrow[d, "\Pi_{j+1*}"] \\
K_1^G(EG_j) & K_1^G(EG_{j+1}) \arrow[l, "i_j^*"']
\end{tikzcd}$$

As we have shown, compositions of vertical arrows are identity, so $K_1^G(EG_j)$ is a direct summand of sequence $K_1^G(EG_i \times G/B) = K_1(BB_j)$. Since $\varprojlim^1(K_1(BB_j)) = 0$ we get $\varprojlim^1(K_1^G(EG_j)) = 0$. This concludes the proof.

Remark 3.1. By the same reasons, there is an analogous retraction diagram for K_0^G functor:

(2.6)
$$\begin{tikzcd}
K_0^G(EG_j) \arrow[d, "\pi_j^*"'] & & K_0^G(EG_{j+1}) \arrow[ll, "(i_j\times id)^*"'] \arrow[d, "\pi_{j+1}^*{}^*"] \\
K_0^G(EG_j \times G/B) \arrow[d, "\Pi_{j*}"'] & & K_0^G(EG_{j+1} \times G/B) \arrow[ll, "(i_j\times id)^*"'] \arrow[d, "\Pi_{j*}"] \\
K_0^G(EG_j) & & K_0^G(EG_{j+1}) \arrow[ll, "i_j^*"']
\end{tikzcd}$$

3. PROOF OF MAIN RESULT

Theorem 1. The Borel construction induces an isomorphism

$$\widehat{R(B)}_{I_B} \xrightarrow{\widehat{Borel_B}} \widehat{K_0(BB)}_{I_B} \xleftarrow{\cong} K_0(BB)$$

Proof:

Let T be a maximal torus of G. Then the restriction map $R(B) \to R(T)$ and projection pullback $K_0(BT) \to K_0(BB)$ are isomorphisms. So, it suffices to prove this theorem for maximal torus T. Since G is split, $T = \mathbb{G}_m \times \ldots \times \mathbb{G}_m$ (n times). Let us compute $R(T)$ and $\widehat{R(T)}_{I_T}$.

$R(T) = \mathbb{Z}[\lambda_1 \ldots \lambda_n, t]/(\lambda_1 \cdot \ldots \cdot \lambda_n \cdot t = 1)$. $I_T = (1-\lambda_1, \ldots, 1-\lambda_n, 1-t)$. So, we have:

$$\widehat{R(T)}_{I_T} = \varprojlim \mathbb{Z}[\lambda_1, \ldots, \lambda_n, t]/((\Pi\lambda_i \cdot t - 1), (1-\lambda_1)^k, \ldots, (1-\lambda_n)^k, (1-t)^k) =$$

$$= \varprojlim \mathbb{Z}[1-\lambda_1, \ldots, 1-\lambda_n, 1-t]/((\Pi\lambda_i \cdot t - 1), (1-\lambda_1)^k, \ldots, (1-\lambda_n)^k, (1-t)^k) =$$

$$= \mathbb{Z}[[1-\lambda_1, \ldots, 1-\lambda_n, 1-t]]/(\Pi\lambda_i \cdot t - 1) = \mathbb{Z}[[\mu_1, \ldots, \mu_n, 1-t]]/(\Pi(1-\mu_i)\cdot t - 1)$$

Since $\frac{1}{1-\mu_i} = 1 + \mu_i + \mu_i^2 + \mu_i^3 + \ldots$ it follows that $t = \prod(1 + \mu_i + \mu_i^2 + \ldots)$ Therefore we have $1 - t = 1 - (1 + \mu_1 + \ldots + \mu_n + \ldots) = -(\mu_1 + \ldots + \mu_n + \ldots)$. Then

$$\widehat{R(T)}_{I_T} = \mathbb{Z}[[\mu_1, \ldots, \mu_n]].$$

Let us compute $K_0(BT)$.

We can choose by ET the space $\mathbb{A}^\infty \backslash \{0\} \times \ldots \times \mathbb{A}^\infty \backslash \{0\}$ This is contractible space with free $T-$action. Then $ET_k = \mathbb{A}^{k+1} \backslash \{0\} \times \ldots \mathbb{A}^{k+1} \backslash \{0\}$ and $BT_k = \mathbb{P}^k \times \ldots \times \mathbb{P}^k$. Then $K_0(BT_k) = \mathbb{Z}[x_1 \ldots x_n]/(x_1^k = 0, \ldots, x_n^k = 0)$. So we have $BT = \mathbb{P}^\infty \times \ldots \times \mathbb{P}^\infty$, and it is known that $K_0(BT) = \varprojlim K_0(BT_k) = \mathbb{Z}[[x_1 \ldots x_n]]$.

Borel construction $R(T) \to K_0(BT_k)$ works as follows:

$\lambda_i \mapsto 1 - x_i$

$t \mapsto \frac{1}{(1-x_1)\ldots(1-x_n)} = (1 + x_1 + \ldots + x_1^{k-1}) \ldots (1 + x_1 + \ldots + x_1^{k-1})$

Then on $\widehat{R(T)}_{I_T}$. Borel construction induces an isomorphism $\mu_i \mapsto x_i$ This completes the proof of theorem 1.

Theorem 2. The following diagram commutes

(3.1)
$$\begin{tikzcd} \widehat{R(G)}_{I_G} \arrow[r, "\widehat{Borel_G}"] \arrow[d, "res"'] & \widehat{K_0(BG)}_{I_G} \arrow[d, "\widehat{p^*}"] & K_0(BG) \arrow[l] \arrow[d, "p^*"] \\ \widehat{R(B)}_{I_B} \arrow[r, "\widehat{Borel_B}"] \arrow[d, "ind"'] & \widehat{K_0(BB)}_{I_B} \arrow[d, "\widehat{p_*}"] & K_0(BB) \arrow[l, "\cong"'] \arrow[d, "p_*"] \\ \widehat{R(G)}_{I_G} \arrow[r, "\widehat{Borel_G}"] & \widehat{K_0(BG)}_{I_G} & K_0(BG) \arrow[l] \end{tikzcd}$$

with $Ind \circ Res = id$ and $p_* \circ p^* = id$.
Proof:
To construct (3.1) We use the following interpretation: $R(G) = K_0^G(pt)$, $R(B) = K_0^B(pt)$ Since $EG_i \to BG_i$ is a G-torsor, $K_0(BG_i) = K_0^G(EG_i)$. (by Proposition 1 of[2]) EG can be chosen as a model for the contractible space EB Proposition 1 of [2] allows us express all these objects in terms of G-equivariant K-theory: $K_0^B(pt) \cong K_0^G(G/B)$ $K_0^B(EG_j) = K_0^G(EG_j \times G/B)$
So, first we construct :

(3.2)
$$\begin{tikzcd} K_0^G(pt) \arrow[r, "\pi_{pt}^*"] \arrow[d, "\pi_{pt}^*"'] & K_0^G(EG_i) \arrow[d, "\pi_{EG_i}^*"] \\ K_0^G(G/B) \arrow[r, "\pi_{G/B}^*"] \arrow[d, "\pi_{pt*}"'] & K_0^G(EG_i \times G/B) \arrow[d, "\pi_{EG_i*}"] \\ K_0^G(pt) \arrow[r, "\pi_{pt}^*"] & K_0^G(EG_i) \end{tikzcd}$$

Proposition 1 proves that this diagram commutes and the bottom arrow is a retract of the middle one. By the isomorphism construction of corollary 1 [2] one can check that horizontal arrows in this diagram coincide with the Borel morphisms. We consider (3.2) as a $K_0^G(pt)$-module diagram ($K_0^G(pt)$-module structure is induced by pullback morphisms) Proposition 2 tells us that I_B-adic topology on $R(B) = K_0^G(G/B)$ coincides with $I_G \cdot R(B)$-adic topology. So we consider I_G completion of (3.2) as $K_0^G(pt)$-modules:

$$\begin{tikzcd} \widehat{K_0^G(pt)}_{I_G} \arrow[r, "\pi_{pt}^*"] \arrow[d, "\pi_{pt}^*"'] & \widehat{K_0^G(EG_i)}_{I_G} \arrow[d, "\pi_{EG_i}^*"] \\ K_0^G\widehat{(G/B)}_{I_G \cdot R(B)} \arrow[r, "\pi_{G/B}^*"] \arrow[d, "\pi_{pt*}"'] & K_0^G\widehat{(EG_i \times G/B)}_{I_G \cdot R(B)} \arrow[d, "\pi_{EG_i*}"] \\ \widehat{K_0^G(pt)}_{I_G} \arrow[r, "\pi_{pt}^*"] & \widehat{K_0^G(EG_i)}_{I_G} \end{tikzcd}$$

It remains to show that $K_0^G(EG)$ is complete in I_G-adic topology. To show that we consider the diagram:

(3.3)

$$\begin{tikzcd}
\widehat{K_0^G(pt)}_{I_G} \arrow[r, "\widehat{\pi_{pt}^*}"] \arrow[d, "\widehat{\pi_{pt}^*}"'] & \widehat{K_0^G(EG_i)}_{I_G} \arrow[d, "\widehat{\pi_{EG_i}^*}"'] & K_0^G(EG_i) \arrow[l, "\cong"] \arrow[d, "\pi_{EG_i}^*"'] \\
\widehat{K_0^G(G/B)}_{I_G\cdot R(B)} \arrow[r, "\widehat{\pi_{G/B}^*}"] \arrow[d, "\widehat{\pi_{pt*}}"'] & \widehat{K_0^G(EG_i \times G/B)}_{I_G\cdot R(B)} \arrow[d, "\widehat{\pi_{EG_i*}}"'] & K_0^G(EG_i \times G/B) \arrow[l, "\cong"] \arrow[d, "\pi_{EG_i*}"'] \\
\widehat{K_0^G(pt)}_{I_G} \arrow[r, "\widehat{\pi_{pt}^*}"] & \widehat{K_0^G(EG_i)}_{I_G} & K_0^G(EG_i) \arrow[l, "\cong"]
\end{tikzcd}$$

By Remark 3.1 the diagram (3.3) commutes.
Recall that $K_0^G(EG_j) = K_0(BG_j)$, $K_0^G(EG_j \times G/B) = K_0(BB_j)$, $K_0^G(pt) = R(G)$ and $K_0^G(G/B) = R(B)$.
Therefore we can rewrite the diagram (3.3) in the terms of non-equivariant K-theory:
(3.4)

$$\begin{tikzcd}
\widehat{R(G)}_{I_G} \arrow[r, "\widehat{\pi_{pt}^*}"] \arrow[d, "\widehat{\pi_{pt}^*}"'] & \widehat{K_0(BG_i)}_{I_G} \arrow[d, "\widehat{\pi_{EG_i}^*}"'] & K_0(BG_i) \arrow[l, "\cong"] \arrow[d, "\pi_{EG_i}^*"'] \\
\widehat{R(B)}_{I_G\cdot R(B)} \arrow[r, "\widehat{\pi_{G/B}^*}"] \arrow[d, "\widehat{\pi_{pt*}}"'] & \widehat{K_0(BG_i \times G/B)}_{I_G\cdot R(B)} \arrow[d, "\widehat{\pi_{EG_i*}}"'] & K_0(BG_i \times G/B) \arrow[l, "\cong"] \arrow[d, "\pi_{EG_i*}"'] \\
\widehat{R(G)}_{I_G} \arrow[r, "\widehat{\pi_{pt}^*}"] & \widehat{K_0(BG_i)}_{I_G} & K_0(BG_i) \arrow[l, "\cong"]
\end{tikzcd}$$

By Proposition 3 after taking the projective limit we get the diagram :

(3.5)

$$\begin{tikzcd}
\widehat{R(G)}_{I_G} \arrow[r, "\widehat{\pi_{pt}^*}"] \arrow[d, "\widehat{\pi_{pt}^*}"'] & \widehat{K_0(BG)}_{I_G} \arrow[d, "\varprojlim \widehat{\pi_{EG_i}^*}"'] & K_0(BG) \arrow[l, "\cong"] \arrow[d, "\varprojlim \pi_{EG_i}^*"'] \\
\widehat{R(B)}_{I_G\cdot R(B)} \arrow[r, "\widehat{\pi_{G/B}^*}"] \arrow[d, "\widehat{\pi_{pt*}}"'] & \widehat{K_0(BB)}_{I_G\cdot R(B)} \arrow[d, "\varprojlim \widehat{\pi_{EG_i*}}"'] & K_0(BB) \arrow[l, "\cong"] \arrow[d, "\varprojlim \pi_{EG_i*}"'] \\
\widehat{R(G)}_{I_G} \arrow[r, "\widehat{\pi_{pt}^*}"] & \widehat{K_0(BG)}_{I_G} & K_0(BG) \arrow[l, "\cong"]
\end{tikzcd}$$

Here the completion morphism $\widehat{K_0(BG)}_{I_G} \leftarrow K_0(BG)$ is an isomorphism since it is a retract of the completion morphism $\widehat{K_0(BB)}_{I_G\cdot R(B)} = \widehat{K_0(BB)}_{I_B} \leftarrow K_0(BB)$ which is the isomorphism by Theorem 1. By proposition 1 and 3 we have $\pi_{pt}^* \circ \pi_{pt*} = id$ and $\pi_{EG}^* \circ \pi_{EG*} = id$. So, we have constructed the diagram (1.1) in the form (3.5). The theorem is proved.

Corollary. The Borel construction induces the isomorphism.

$$\widehat{R(G)}_{I_G} \xrightarrow{\widehat{Borel_G}} \widehat{K_0(BG)}_{I_G} \xleftarrow{completion_G} K_0(BG)$$

Proof:

Theorem 2 states that $\widehat{Borel_G}$ and $completion_G$ are retracts of $\widehat{Borel_B}$ and $completion_B$ which are isomorphisms by theorem 1. Then $\widehat{Borel_G}$ and $completion_G$ are also isomorphisms.

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